LECTURE 14 : The stack of all curves

> **Thm.** *The moduli space $\overline{\mathrm{M}}_g$ of stable curves of genus $g \geq 2$ is a smooth, proper and irreducible Deligne–Mumford stack of dimension $3g-3$ which admits a projective coarse moduli space.*

Where are we?

- We've introduced stable curves
  $\rightsquigarrow \overline{\mathcal{M}}_g$ prestack

We almost know

- $\overline{\mathcal{M}}_g$ is DM stack smooth over $\operatorname{Spec} \mathbb{Z}$ of rel. dim $3g-3$] Need: $\overline{\mathcal{M}}_g$ algebraic
- $\exists\, \overline{\mathcal{M}}_g \xrightarrow{\text{cms}} \overline{M}_g$] Need: $\overline{\mathcal{M}}_g$ separated

§0. Six steps toward projective moduli

Setup: Let $\mathcal{X}$ be the moduli stack of interest. Let $\mathcal{X} \subset \mathcal{M}$ an "enlargement". Call $x \in \mathcal{M}(k)$ <u>stable</u> if $x \in \mathcal{X}(k)$.

<u>STEP 1</u> (Algebraicity)

$\mathcal{M}$ is an alg. stack loc. of f.type/base

<u>STEP 2</u> (Openness of stability)

Given $E \in \mathcal{M}(T)$, ($\Longrightarrow \mathcal{X}$ alg.)

$\{t \in T \mid E|_{\operatorname{Spec} k(t)} \text{ stable}\} \subset T$ open

<u>STEP 3</u> (Boundedness of stability)

$\mathcal{X}$ of f.type ($\Leftarrow$ quasi-compact)

<u>STEP 4</u> (Existence of coarse moduli space)

$\exists\, \mathcal{X} \xrightarrow{CMS} X$ (need $\mathcal{X}$ sep)

<u>STEP 5</u> (Stable reduction)

$\mathcal{X}$ proper ($\Longrightarrow X$ proper)

<u>STEP 6</u> (Projectivity) X projective

Example: Moduli of stable curves

$$\mathcal{X} = \overline{\mathcal{M}}_g \subset \mathcal{M} = \mathcal{M}_g^{all}$$

$\overline{\mathcal{M}}_g = \{\text{stable curves}\}$, $\mathcal{M}_g^{all} = \{\text{all curves}\}$

Today's goal

Follows from
- openness of nodal locus
- openness of stability for a family of nodal curves

Follow from the fact: $C \to S$ stable $\Longrightarrow \omega_{C/S}^{\otimes 3}$ very ample

Soon: $\overline{\mathcal{M}}_g$ proper

Final lecture: following Kollár

§ 1. Recap on stable curves

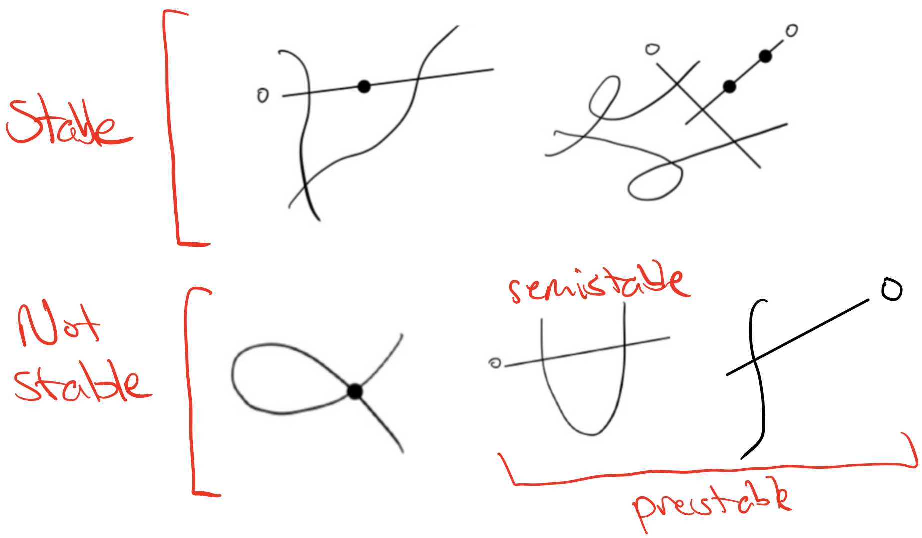
Def (Stable curves)**.** An n-pointed curve $(C, p_1, \ldots, p_n)$ over k is *stable* if C is a connected, nodal and projective curve, and $p_1, \ldots, p_n \in C$ are distinct smooth points such that

node or marked

(1) every smooth rational subcurve $\mathbb{P}^1 \subset C$ contains at least 3 special points, and

(2) C is not of genus 1 without marked points.

Semistable : Replace 3 with 2 in (1)
Prestable : Drop (1) & (2)



Prop Let $(C, p_1, \ldots, p_n)$ n-pointed prestable. TFAE

① $(C, \{p_i\})$ stable

② $\mathrm{Aut}(C, \{p_i\})$ finite

③ $\omega_C(p_1 + \cdots + p_n)$ ample

Proposition. *Let $(C, p_1, \ldots, p_n)$ be an n-pointed stable curve of genus g over k. Then*

$$\dim_k \mathrm{Ext}^i(\Omega_C(\sum_i p_i), \mathcal{O}_C) = \begin{cases} 0 & \text{if } i = 0 \\ 3g - 3 + n & \text{if } i = 1 \\ 0 & \text{if } i = 2 \end{cases}$$

FAMILIES

Definition (Families)**.**

(1) A *family of n-pointed nodal curves* is a flat, proper and finitely presented morphism $\mathcal{C} \to S$ of schemes with n sections $\sigma_1, \ldots, \sigma_n : S \to \mathcal{C}$ such that every geometric fiber is a (reduced) connected nodal curve.

(2) A *family of n-pointed stable curves* is a family $\mathcal{C} \to S$ of n-pointed nodal curves such that every geometric fiber $(\mathcal{C}_s, \sigma_1(s), \ldots, \sigma_n(s))$ is stable.

Same for semistable, prestable

Proposition (Properties of Families of Stable Curves)**.** *Let $(\mathcal{C} \to S, \{\sigma_i\})$ be a family of n-pointed stable curves of genus g, and set $L := \omega_{\mathcal{C}/S}(\sum_i \sigma_i)$. If $k \geq 3$, then $L^{\otimes k}$ is relatively very ample and $\pi_* L^{\otimes k}$ is a vector bundle of rank $(2k-1)(g-1) + kn$.*

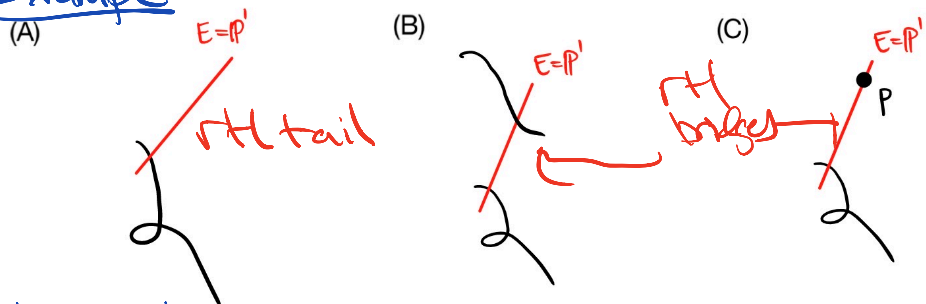
Proposition (Openness of Stability)**.** *Let $(\mathcal{C} \to S, \{\sigma_i\})$ be a family of n-pointed nodal curves. The locus of points $s \in S$ such that $(\mathcal{C}_s, \{\sigma_i(s)\})$ is stable is open.*

§2. More on stability: contraction morphisms

Definition (Rational tails and bridges). Let $(C, p_1, \dots, p_n)$ be an n-pointed prestable curve. We say that a smooth rational subcurve $E \cong \mathbb{P}^1 \subset C$ is

- a *rational tail* if $E \cap E^c = 1$ and E contains no marked points;
- a *rational bridge* if either $E \cap E^c = 2$ and E contains no marked points, or $E \cap E^c = 1$ and E contains one marked point.

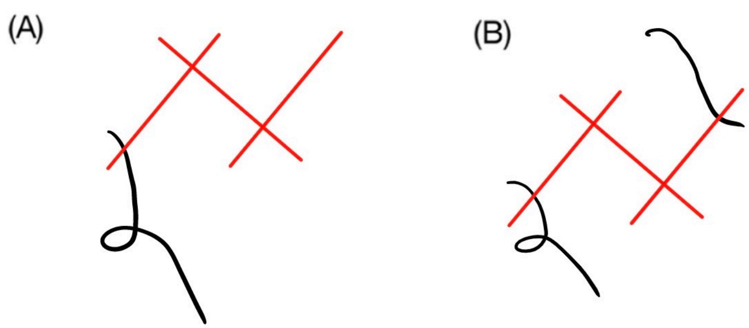
Example



Observation

- $(C, \{p_i\})$ stable $\iff$ $\nexists$ rtl tails or bridges
  semistable $\iff$ $\nexists$ rtl tails

Example (Chains of rtl tails & bridges)



Contraction of rational tails & bridges

If $(C, \{p_i\})$ prestable,

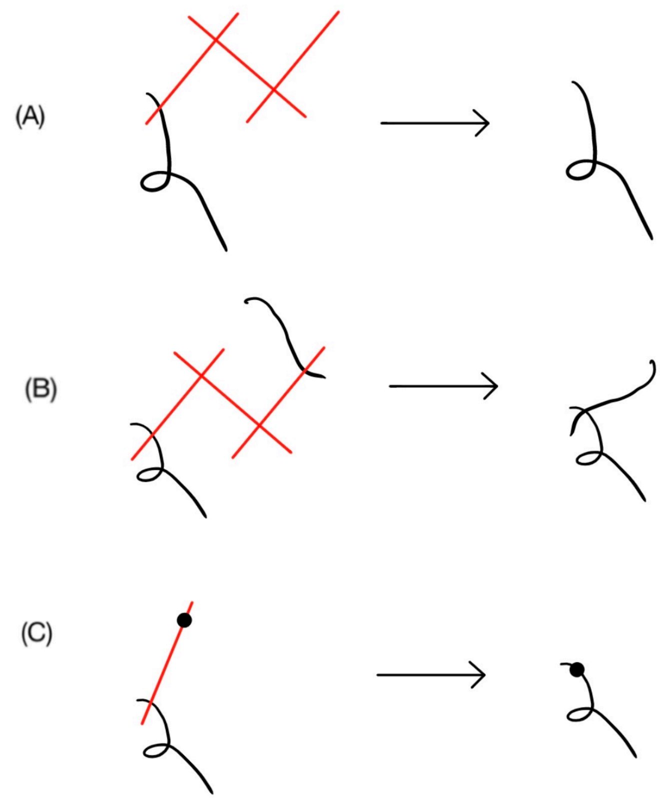
let C^{st} = proper curve obtained by removing all rtl tails & bridges E_i

i.e. $C^{st} = C \setminus \bigcup E_i$] not quite right

Let $C \xrightarrow{\pi} C^{st}$

$p_i \longmapsto p_i'$

Then $(C^{st}, p_1', \dots, p_n')$ is stable



Contraction in families

Proposition. *If $(\mathcal{C} \to S, \sigma_1, \dots, \sigma_n)$ is a family of prestable curves, $\exists!$ map $\pi\colon \mathcal{C} \to \mathcal{C}^{st}$ over S such that*

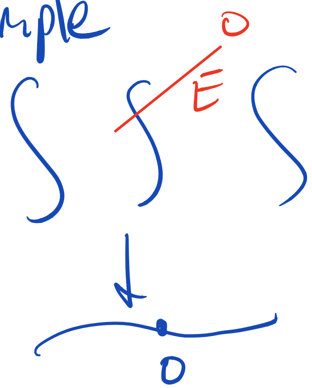
(1) $(\mathcal{C}^{st} \to S, \{\sigma'_i\})$ is a family of stable curves with $\sigma'_i = \pi \circ \sigma_i$;

(2) $\forall s \in S$, $(\mathcal{C}_s, \{\sigma_i(s)\}) \to (\mathcal{C}^{st}_s, \{\sigma'_i(s)\})$ is the map contracting rational tails and bridges; and

(3) $\mathcal{O}_{\mathcal{C}^{st}} = \pi_*\mathcal{O}_{\mathcal{C}}$ and $R^1\pi_*\mathcal{O}_{\mathcal{C}} = 0$, and this remains true after base change;

(4) If $\mathcal{C} \to S$ is semistable, then $\omega_{C/S}(\sum_i \sigma_i) = \pi^*\omega_{\mathcal{C}^{st}/S}(\sum_i \sigma'_i)$.

Example



$\mathcal{C}$

$\downarrow$

S smooth curve

$\mathcal{C}_0 = E + E^c$

$0 = \mathcal{C}_0 \cdot E$

$= E\cdot E + E^c \cdot E$

E rtl tail $\Rightarrow E^2 = -1$

E rtl bridge $\Rightarrow E^2 = -2$

Vague sketch 1 (assume $\mathcal{C} \to S$ semistable)

$$\mathcal{C} \xrightarrow{\pi} \operatorname{Proj}_S \underbrace{\bigoplus_{k \geq 0} \pi_* \omega_{C/S}^{\otimes k}}_{\text{Need to show fin. gen}}$$

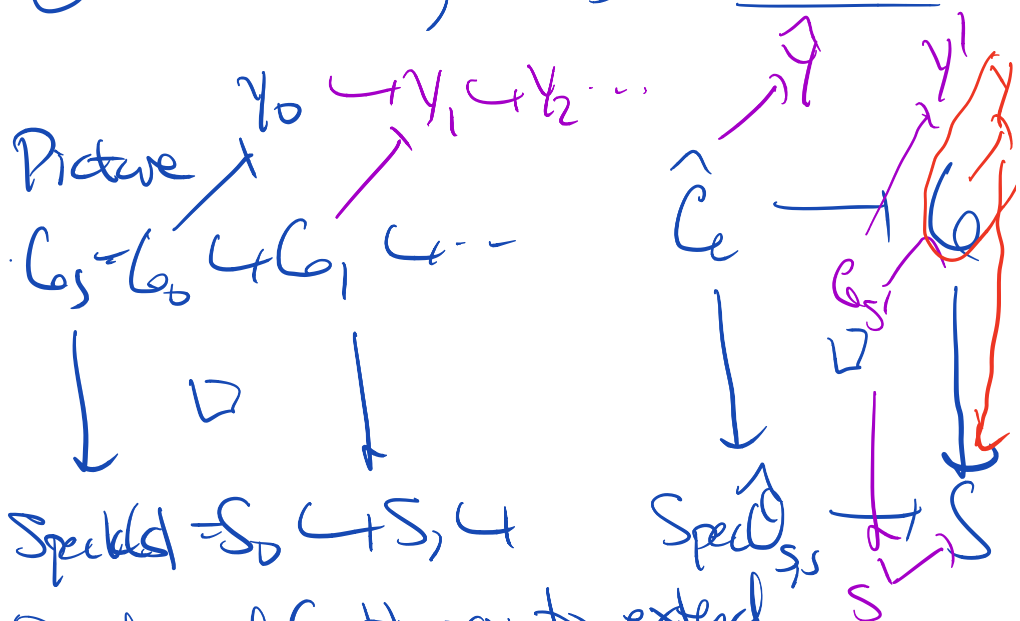
semistable π: $\mathcal{C} \to S$, $\operatorname{Proj}_S \to S$

Vague sketch 2 (see TAG 0E8A)

Local to global

0. Use Noeth approx to reduce to S fin type $/\mathbb{Z}$
1. Show uniqueness of $\mathcal{C}^{st} \to S$
2. Given $s \in S$, $\exists\, \mathcal{C}_s \to \mathcal{C}_s^{st} = Y_0$

Picture



3. Use def. theory to extend $\mathcal{C}_0 \to Y_0$ to $\mathcal{C}_n \to Y_n$
4. Algebraize to $\hat{\mathcal{C}} \to \hat{Y}$
5. Artin Approximation: $\exists S' \to S$ & $\mathcal{C}_{S'} \to Y'$
6. Use uniqueness to descend to $\mathcal{C} \to Y$

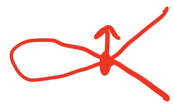
§3. Stack of all curves

Redefine a curve as a scheme C of f. type over a field k of dimension 1

Not assumed pure dim 1 or connected

Reason:

$[x,y] \mapsto [x^3, x^2y, xy^2, ty^3]$



Definition. Let S be a scheme.

- A *family of curves over* S is a flat, proper and finitely presented morphism $\mathcal{C} \to S$ of algebraic spaces such that every fiber is a curve. alg. space
- A *family of n-pointed curves over* S is a family of curves $\mathcal{C} \to S$ with n sections $\sigma_1, \ldots, \sigma_n : S \to \mathcal{C}$.

arbitrary

Examples

① (Fulghesu) $\exists$ family of genus 0 nodal curves

$$\begin{tikzcd} \mathbb{P}^1_{k(s)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d] \\ \operatorname{Spec} k(s) \arrow[r] & S \end{tikzcd}$$

$\mathcal{C}$: smooth 3 dim'l alg. space, not a scheme

S: sm proj surface

② (Raynaud) $\exists$ family of smooth genus 1 curves

$$\begin{tikzcd} \mathcal{C} \arrow[d] \\ S \end{tikzcd}$$

$\mathcal{C}$: not a scheme

S: normal surface

Rmk: If $\mathcal{C} \to S$ stable family, $\omega_{\mathcal{C}/S}$ is ample $\Rightarrow$ $\mathcal{C}$ proj$/S$.

Prop If $\mathcal{C} \to S$ is a family of curves, then $\exists\, S' \twoheadrightarrow S$ étale cover s.t. $\mathcal{C}_{S'}$ proj$/S'$.

Prop If $\mathcal{C} \to S$ is a family of curves, $\exists\, S' \xrightarrow{\text{ét}} S$ cover s.t. $\mathcal{C}_{S'} \to S'$ proj.

We will sketch 2 approaches

$\boxed{\text{1st approach: Local to global}}$

$$\begin{tikzcd}
\mathcal{C}_s = \mathcal{C}_0 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C}_1 \arrow[r, hook] \arrow[d] & \cdots & \hat{\mathcal{C}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d] & \mathcal{C}_{S'} \arrow[l] \arrow[d] \\
\operatorname{Spec} k(s) \arrow[r, hook] & \operatorname{Spec} \hat{\mathcal{O}}_{S,s}/\mathfrak{m}_s^2 & \cdots & \operatorname{Spec} \hat{\mathcal{O}}_{S,s} \arrow[r] & S \ni s & S' \ni s' \arrow[l]
\end{tikzcd}$$

ample L_0, L_1, $L_2 \ldots$, $\hat{L}$, L'

Case 1: $\mathcal{C}_0$

Use sep 1 dim'l alg. space are schemes

& Proper 1 dim'l schemes are projective

Case 2 $\mathcal{C}_n = \mathcal{C} \times_S \operatorname{Spec} \mathcal{O}_{S,s}/\mathfrak{m}_s^{n+1}$

Defn theory says obstructions to deforming line bdl L_n on $\mathcal{C}_n$ to L_{n+1} on $\mathcal{C}_{n+1}$ live: $H^2(\mathcal{C}_0, \mathcal{O}_{\mathcal{C}_0}) = 0$

$\Rightarrow \exists\, L_n$ on $\mathcal{C}$

Case 3 $\hat{\mathcal{C}} \xrightarrow{\cancel{\text{proj}}} \operatorname{Spec} \hat{\mathcal{O}}_{S,s} \leftarrow$ complete local noeth

Use Grothendieck Existence Theorem

$\operatorname{Coh}(\hat{\mathcal{C}}) \xrightarrow{\sim} \varprojlim \operatorname{Coh}(\mathcal{C}_n)$

(Need to general to proper alg. space / complete local noeth

Need to first show

Chow's lemma: $\exists\, \bar{\mathcal{C}} \xrightarrow[\text{birtl}]{\text{proj}} \hat{\mathcal{C}}$, $\bar{\mathcal{C}} \xrightarrow{\text{proj}} S$, $\hat{\mathcal{C}} \to S$)

$\Rightarrow \exists\, \hat{L} \longmapsto (L_n)$

Case 4 S f.type $/\mathbb{Z}$

Apply Artin approx to

$Sch/S \to Sets$

$(T \to S) \mapsto \operatorname{Pic}(\mathcal{C}_T)$

Case 5 S general

Use Noeth approx

Prop If $\mathcal{C} \to S$ is a family of curves, $\exists\, S' \xrightarrow{\text{ét}} S$ cover s.t. $\mathcal{C}_{S'} \to S'$ proj.

$\boxed{\text{2nd approach: Explicitly extend line bundle}}$

$$\begin{tikzcd}
L_0 \text{ ample} \arrow[dr, dash] & & & \mathcal{C}^0 \arrow[dl, hook] \arrow[ddl, "\text{smooth}"] & \mathcal{C}^0_{S'} \arrow[d] \\
& \mathcal{C}_s \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] & & S' \arrow[u, bend right, "\sigma_i"'] \arrow[dll] \\
& \operatorname{Spec} k(s) \arrow[r] & S & &
\end{tikzcd}$$

Assume all fibers are gen. reduced ($\Rightarrow$ gen smooth)

Choose $p_1, \dots, p_n \in \mathcal{C}_s$ s.t. every irr. dim 1 comp contains a p_i

$$L_0 = \mathcal{O}_{\mathcal{C}_s}(p_1 + \dots + p_n)$$

Use étale-local structure of smooth

$\Rightarrow \exists\, S' \to S$ & sections

$\sigma_i : S' \to \mathcal{C}^0$ extending p_i

$\Rightarrow \mathcal{O}_{\mathcal{C}}(\sigma_1 + \dots + \sigma_n)$ ample in an open nbhd of $s \in S$.

§4. Algebraicity of the stack of all curves

Let $\mathcal{M}_{g,n}^{all}$ be prestack

- objects: $(C \to S, \sigma_1, \dots, \sigma_n)$ families of curves
- Morphisms are diagrams

$$\begin{tikzcd} C' \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d] \\ S' \arrow[r, "f"] \arrow[u, bend left, "\sigma_i'"] & S \arrow[u, bend left, "\sigma_i"] \end{tikzcd} \qquad g \circ \sigma_i' = \sigma_i \circ f$$

Lemma 1 $\mathcal{M}_{g,n}^{all}$ is a stack over $Sch_{ét}$

Handle $n=0$

ét. equiv. relation

$$\begin{tikzcd} R = p_1^* C' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2 \circ \alpha"'] \arrow[d] & C' \arrow[r, dashed] \arrow[d] & C'/R \arrow[d, dashed] \\ S' \times_S S' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & S' \arrow[r] & S \end{tikzcd}$$

$C'/R = C$: alg. space shown

Given $\alpha: p_1^* C' \xrightarrow{\sim} p_2^* C'$

Lemma 2 $\mathcal{M}_{g,n}^{all} \xrightarrow{\Delta} \mathcal{M}_{g,n}^{all} \times \mathcal{M}_{g,n}^{all}$ is representable

($\Longrightarrow$ any map $S \to \mathcal{M}_{g,n}^{all}$ from a scheme is representable)

Pf: ($n=0$)

$$\begin{tikzcd} \mathrm{Isom}_T(C_1, C_2) \arrow[r] \arrow[d] & T \arrow[d] \\ \mathcal{M}_g^{all} \arrow[r] & \mathcal{M}_g^{all} \times \mathcal{M}_g^{all} \end{tikzcd}$$

2 families $C_1 \to T$, $C_2 \to T$

Need to show $\mathrm{Isom}_T(C_1, C_2)$ is alg. space

Reduce to C_1, C_2 are projective

Know $\exists\, T' \xrightarrow{ét} T$ s.t. $C_{1,T}$ & $C_{2,T}$ are proj/T

scheme

$$\begin{tikzcd} \mathrm{Isom}_{T'}(C_{1,T'}, C_{2,T'}) \arrow[r] \arrow[d, "\text{ét, repn, surj}"] & T' \arrow[d, "ét"] \\ \mathrm{Isom}_T(C_1, C_2) \arrow[r] & T \end{tikzcd}$$

~~alg space~~ étale presheaf

$\Longrightarrow$ alg. space

Lemma 2 $\mathcal{M}^{all}_{g,n} \xrightarrow{\Delta} \mathcal{M}^{all}_{g,n} \times \mathcal{M}^{all}_{g,n}$ is representable

Let

$$\begin{tikzcd} C_1 \arrow[dr, "proj"'] & & C_2 \arrow[dl, "proj"] \\ & T & \end{tikzcd}$$

Need to show $\underline{\mathrm{Isom}_T(C_1, C_2)}$ is alg. space

We have inclusions of functors

$$\underline{\mathrm{Isom}}_T(C_1, C_2) \subset \underline{\mathrm{Mor}}_T(C_1, C_2) \subset \mathrm{Hilb}(C_1 \times_T C_2 / T) \leftarrow \underline{\underline{\text{scheme}}} \leftarrow \underline{\underline{\text{projective!}}}$$

$$(C_1 \xrightarrow{\alpha} C_2) \longmapsto \text{graph}\left[C_1 \xrightarrow{\Gamma_\alpha} C_1 \times_T C_2\right]$$

Fact $\Longrightarrow$ repn open imm

FACT Given

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "pr"'] & & Y \arrow[dl, "pr"] \\ & T & \end{tikzcd}$$

diagram of schemes

$\exists\, T^0 \subset T$ open s.t. $\forall\, S \to T$ then

$X_S \xrightarrow{\sim} Y_S \iff S \to T$ factors through T_0

A subscheme $Z \subset C_1 \times_T C_2$ is the image of a map $C_1 \to C_2$

$$\iff \begin{tikzcd} Z \arrow[r, hook] \arrow[dr, "\sim"'] & C_1 \times_T C_2 \arrow[d, "p_1"] \\ & C_1 \end{tikzcd}$$

Fact $\Longrightarrow$ repn open imm

THM $\mathcal{M}_{g,n}^{all}$ is an algebraic stack locally of f. type $/\mathbb{Z}$

Sketch

Reductions

- Suffices to assume $n=0$

Reason: $\mathcal{M}_{g,n+1}^{all} \longrightarrow \mathcal{M}_{g,n}^{all}$ univ. family

- Suffices to show that $\forall$ proj. curves C_0/k

$$\exists \underset{\text{scheme}}{U} \xrightarrow{\text{sm repn}} \mathcal{M}_g^{all} \quad \text{w/ } [C_0] \text{ in image}$$

- Choose embedding $C_0 \hookrightarrow \mathbb{P}^N$ s.t.

$$\boxed{h^1(C_0, \mathcal{O}(1)) = 0}$$

Let $P(t)$ be Hilb. poly

- Consider Hilbert scheme $H := \mathrm{Hilb}^P(\mathbb{P}^N_{\mathbb{Z}})$ proj$/\mathbb{Z}$

& univ family

$$\begin{tikzcd} C_{h_0} = C_0 \arrow[r, hook] \arrow[d] & C_e \arrow[r, hook] \arrow[d] & \mathbb{P}^N_H \arrow[dl] \\ h_0 \in & H & \end{tikzcd}$$

- Coh & base change $\Longrightarrow$

$\exists H' \subset H$ open nbd of h_0 s.t.

$$\forall s \in H' \quad \boxed{h^1(C_{e_s}, \mathcal{O}(1)) = 0}$$

- We have a map

$$H' \longrightarrow \mathcal{M}_g^{all} \quad \text{repn}$$

$$[C \subset \mathbb{P}^N] \longmapsto [C]$$

Claim: $H' \longrightarrow \mathcal{M}_g^{all}$ smooth

Use formal lifting criteria

CLAIM $H' \longrightarrow \mathcal{M}_g^{all}$ smooth

Use formal lifting criteria

$A \twoheadrightarrow A_0$ local artinian w/ res field k s.t. $\ker(A \twoheadrightarrow A_0) = k$

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[d, hook] \arrow[rr, "{[C \subset \mathbb{P}^N_k]}"] & & H' \arrow[d] \\
\operatorname{Spec} A_0 \arrow[d] \arrow[rru, "{[C_0 \subset \mathbb{P}^N_{A_0}]}"] & & \\
\operatorname{Spec} A \arrow[rr, "{[C_e]}"'] \arrow[rruu, dashed, "{[C_e \subset \mathbb{P}^N_A]}"'] & & \mathcal{M}_g^{all}
\end{tikzcd}$$

Translates to $\longleftrightarrow$

$$\begin{tikzcd}
 & \mathbb{P}^n_k & & \mathbb{P}^n_{A_0} & & \mathbb{P}^n_A \\
C \arrow[ur, hook] \arrow[r, hook] \arrow[d] & C_0 \arrow[urr, hook] \arrow[rr] \arrow[d] & & C_e \arrow[urr, dashed, hook] \arrow[d] & & \\
\operatorname{Spec} k \arrow[r, hook] \arrow[ur, phantom, "\square"] & \operatorname{Spec} A_0 \arrow[rr, hook] \arrow[urr, phantom, "\square"] & & \operatorname{Spec} A & &
\end{tikzcd}$$

defined by $\mathcal{I}$

Simplifying assumption: C is local complete int

$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathbb{P}^n}|_C \to \Omega_C \to 0$

Apply $\operatorname{Hom}(-, \mathcal{O}_C)$ gives

$\operatorname{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_C) \to \operatorname{Ext}^1(\Omega_C, \mathcal{O}_C) \to \operatorname{Ext}^1(\Omega_{\mathbb{P}^n}|_C, \mathcal{O}_C) = H^1(C, T_{\mathbb{P}^n}|_C) = 0$

$\exists\, [C_e \subset \mathbb{P}^n_A] \longmapsto [C_e]$

Use inf. def. theory

$$\left\{ \text{ext} \quad \begin{tikzcd}
C_0 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & C_e \arrow[d, dashed, "\text{flat}"] \\
\operatorname{Spec} A_0 \arrow[r, hook] & \operatorname{Spec} A
\end{tikzcd} \right\} = \operatorname{Ext}^1(\Omega_C, \mathcal{O}_C)$$

$$\left\{ \text{exts} \quad \begin{tikzcd}
 & \mathbb{P}^n_{A_0} & & \mathbb{P}^n_A \\
C_0 \arrow[ur, hook] \arrow[rr, dashed, hook] \arrow[d] & & C_e \arrow[ur, hook] \arrow[d, dashed] & \\
\operatorname{Spec} A_0 \arrow[rr, hook] & & \operatorname{Spec} A &
\end{tikzcd} \right\} = \operatorname{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_C) = H^0(N_{C/\mathbb{P}^n})$$

Euler seq

$0 \to \mathcal{O}_C \to \mathcal{O}_C(1)^{\oplus(n+1)} \to T_{\mathbb{P}^n}|_C \to 0$

$h^2 = 0 \qquad h^1 = 0 \qquad \Rightarrow h^1 = 0$

THM $\mathcal{M}^{all}_{g,n}$ is an algebraic stack locally of f. type $/\mathbb{Z}$

Have inclusions

$$\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n} \subset \mathcal{M}^{ss}_{g,n} \subset \mathcal{M}^{pre}_{g,n} \subset \mathcal{M}^{\leq nodal}_{g,n} \subset \underbrace{\mathcal{M}^{all}_{g,n}}_{\text{algebraic}}$$

smooth locus

stable open

open locus $\omega_{C/S}$ nef

open locus where n sections are disjoint & smooth

open

Last time: we showed this is an open imm

Rmk: Contraction map gives us

$$\begin{tikzcd}
\mathcal{M}^{pre}_{g,n} \arrow[r] & \overline{\mathcal{M}}_{g,n} \\
\overline{\mathcal{M}}_{g,n} \arrow[u, hook, "\text{open}"] \arrow[ur, "id"'] &
\end{tikzcd}$$

not sep

univ. closed

proper

proper