

TODAY : Existence of coarse moduli spaces

Theorem (The Keel–Mori Theorem). *A separated DM stack \mathcal{X} admits a coarse moduli space $\pi: \mathcal{X} \rightarrow X$ where X is a separated algebraic space.*

Last time

- ① Quasi-coherent sheaves on DM stacks
- ② Local structure of DM stacks

§0. Review

Theorem (Local Structure of DM Stacks).

Let \mathcal{X} be a separated DM stack and $x \in \mathcal{X}(k)$ be a geom. point with stabilizer G_x . Then \exists an affine, étale map

$$f: ([\text{Spec } A/G_x], w) \rightarrow (\mathcal{X}, x)$$

such that f induces an isom of stabilizer groups at w .

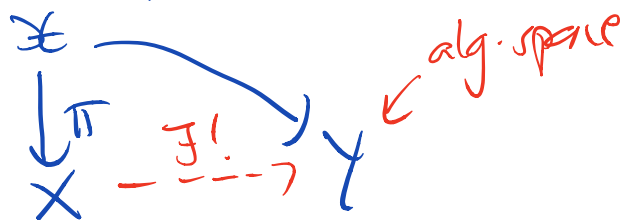
§1. Definition alg stack alg-space

Def A map $\mathcal{X} \xrightarrow{\pi} X$ is a coarse moduli space (or cms) if

$$(1) \forall k = \bar{k}, \mathcal{X}(k)/\sim \xrightarrow{\sim} X(k)$$

↑
isom. classes of
objects in $\mathcal{X}(k)$

(2) π is universal for maps to alg space, i.e



View \mathcal{X} as the closest approximation to X which is an alg. space

Tranbolt

\mathcal{X} univ. properties (eg \mathcal{I} univ. family) \rightsquigarrow X more familiar ideally, X is projective

Strategy to show existence of cms

① Special case: If $\mathcal{X} = [\text{Spec } A/G]$ then

$$\mathcal{X} = [\text{Spec } A/G] \xrightarrow{\text{cms}} \text{Spec } A^G$$

finite ↑

② Use Local Structure Thm

$$\begin{array}{ccc} [\text{Spec } A/G_x] & \xrightarrow{\text{ét}} & \mathcal{X} \\ \downarrow \text{cms} & & \downarrow \text{ } \\ \text{Spec } A^{G_x} & \dashrightarrow & X \end{array}$$

give these in étale topology

Definition. A map $\pi: \mathcal{X} \rightarrow X$ from an algebraic stack to an algebraic space is a *coarse moduli space* if

- (1) for all k alg. closed, $\mathcal{X}(k)/\sim \xrightarrow{\sim} X(k)$
- (2) π is universal for maps to algebraic spaces.

In practice, we desire stronger properties.

For us, if \mathcal{X} sep. DM stack, we will construct $\mathcal{X} \xrightarrow{\pi} X$ satisfying



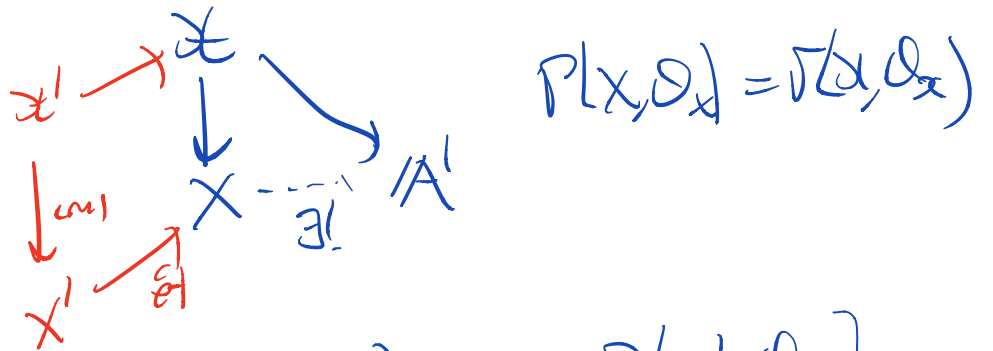
(a) stable under flat base change

(b) $\pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$

(c) π is proper (in part separated!)

(d) π univ. homeomorphism $\Rightarrow |\mathcal{X}| \xrightarrow{\sim} |X|$

Rank: (a) \Rightarrow (b)



$\rho(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = \rho(\mathcal{A}', \mathcal{O}_{\mathcal{A}'})$

$\rho(\mathcal{A}', \mathcal{O}_{\mathcal{A}'}) = \rho(X', \mathcal{O}_{X'})$

$\rho(X', \mathcal{O}_{X'}) \xrightarrow{\cong} \rho(X, \pi_* \mathcal{O}_{\mathcal{X}}) \Rightarrow \mathcal{O}_X \xrightarrow{\cong} \pi_* \mathcal{O}_{\mathcal{X}}$

Descent lemma alg stack alg space

Let $\mathcal{X} \xrightarrow{\pi} X$ be a map

If $\{X_i \xrightarrow{f_i} X\}$ étale cover (even fpqc)

st. $\mathcal{X}_{X_i} \rightarrow X_i$ cong, then $\mathcal{X} \rightarrow X$ is.

§2. Quotients by finite groups

Let G finite group $\triangleright \text{Spec } A$

Define $A^G = \{a \in A \mid g \cdot a = a \forall g\}$
 G acts via R -alg hom. on A

Lemma 1 Let R noeth ring

If A fin gen R -algebra, then
 $A^G \rightarrow A$ finite & A^G fin gen R -alg.

Pf: $A^G \rightarrow A$ integral: if $a \in A$

$\prod_{g \in G} (x - ga) \in A^G[x]$ monic poly
 with a as a root

noeth fin gen $R \rightarrow A$ $\xrightarrow{\text{think}}$ $A^G \rightarrow A$ fin gen
 $\Rightarrow A^G \rightarrow A$ finite

Comm alg $\Rightarrow A^G$ fin gen R -alg.

Lemma 2 Let $A^G \rightarrow B$ ring map.

So G acts on $\text{Spec}(B \otimes_{A^G} A)$. *G -equiv*

Consider $\text{Spec}(B \otimes_{A^G} A) \rightarrow \text{Spec } A$
 $\downarrow \text{univ}$ \square $\downarrow \text{univ}$
 $\text{Spec}(B \otimes_{A^G} A) \xrightarrow{\chi} \text{Spec } B \rightarrow \text{Spec } A^G$

① $A^G \rightarrow B$ flat $\Rightarrow B \xrightarrow{\chi^*} (B \otimes_{A^G} A)^G$

② In general, χ integral univ. homeo.

Pf: ①

$0 \rightarrow A^G \rightarrow A \Rightarrow \prod_{g \in G} A$

- ② $B =$

$0 \rightarrow B \rightarrow A \otimes_{A^G} B \Rightarrow \prod_{g \in G} (A \otimes_{A^G} B)$
 exact

② exercise

Theorem. Let G be a finite group acting on an affine scheme $\text{Spec } A$ of finite type over a noeth ring R . Then

$$\pi: [\text{Spec } A/G] \rightarrow \text{Spec } A^G$$

is a coarse moduli space such that

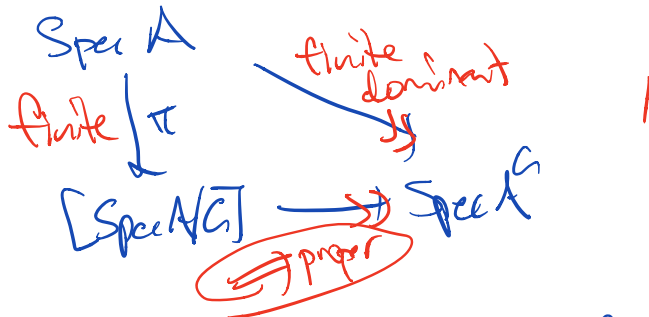
- ✓(1) A^G is finitely generated over R ,
- ✓(2) π is a proper universal homeomorphism, and
- (3) the base change of π along any flat map of noetherian algebraic spaces is a coarse moduli space.

Know: $A^G \rightarrow A$ finite & A^G fin gen/ R

Step 1: π is a proper univ. homeo

($\Rightarrow \pi$ is bij. on geom pts)

Consider



Claim: π is injective on geom pts

Assume $R = \bar{k} = \bar{k}$

Let $x, x' \in \text{Spec } A$ closed pts with $Gx \neq Gx'$

$$Gx \cap Gx' = \emptyset \Rightarrow$$

$$\exists f \in A \text{ s.t. } f|_{Gx} = 1 \ \& \ f|_{Gx'} = 0$$

$$\Rightarrow f' = \prod_{g \in G} gf \in A^G \quad f'|_{Gx} = 1 \quad f'|_{Gx'} = 0$$

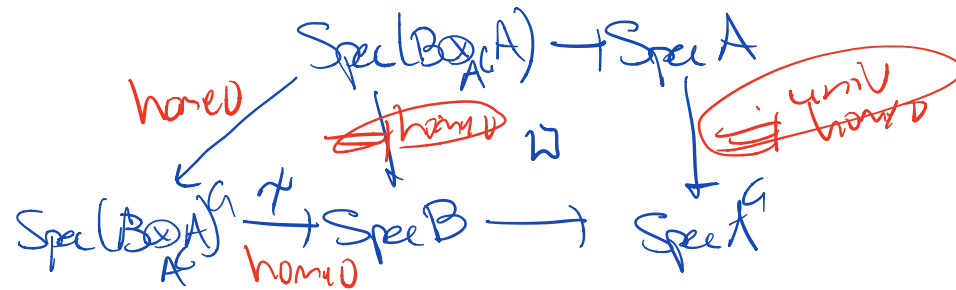
$$\Rightarrow \pi(x) \neq \pi(x')$$

$\Rightarrow \pi$ bij. on geom pts

Since π is proper, π^{-1} continuous

$\Rightarrow \pi$ is a homeo

For $A^G \rightarrow B$



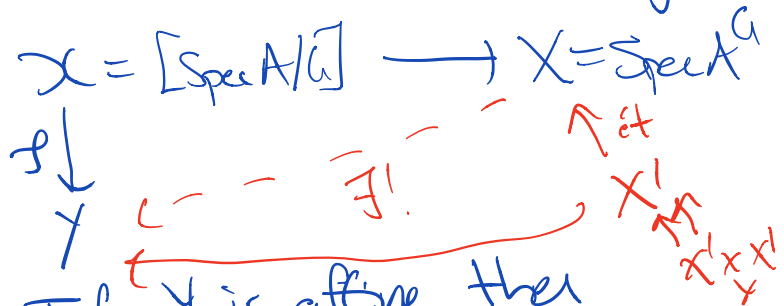
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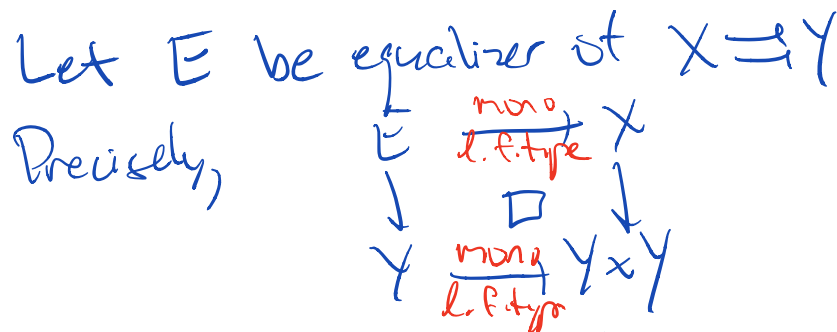
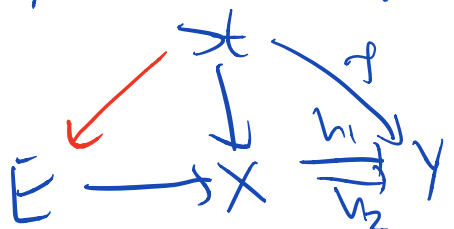
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Step 2 π is univ. for maps to alg. space



Remark: If Y is affine, then $\mathcal{P}(Y, \mathcal{O}_Y) \rightarrow \mathcal{P}(A, \mathcal{O}_A)$ defines $X \rightarrow Y$

(i) Uniqueness Suppose



But $\mathcal{E} \rightarrow X$ proper & sch. dominant
 $\Rightarrow E \rightarrow X$

So $E \rightarrow X$ mono, ét. type, proper, sch. dominant $\Rightarrow E \rightarrow X$ closed immersion $\Rightarrow E \cong X$

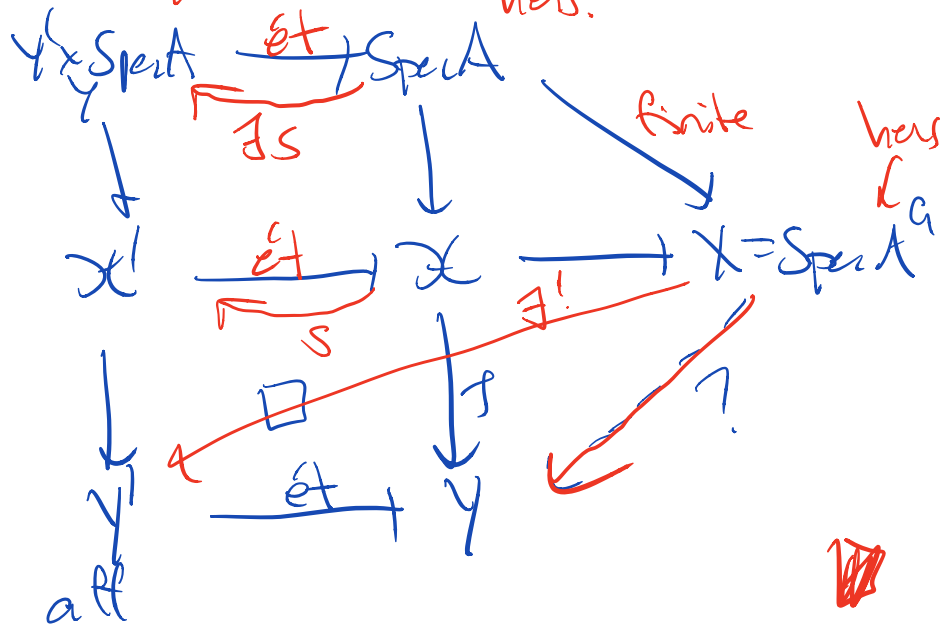
(ii) Existence

Question is étale-local on X

Reason: étale descent & univ. property

\Rightarrow Assume A^G is strictly henselian
 Can also assume Y is g. compact

A^G strictly henselian
 Y q. compact



Heuristically: $R = k = \bar{k}$

Let $x \in \text{Spec } A$ complete A^G at $\pi_A(x)$

FACT $(\hat{A})^{G_x} = \hat{A}^G$

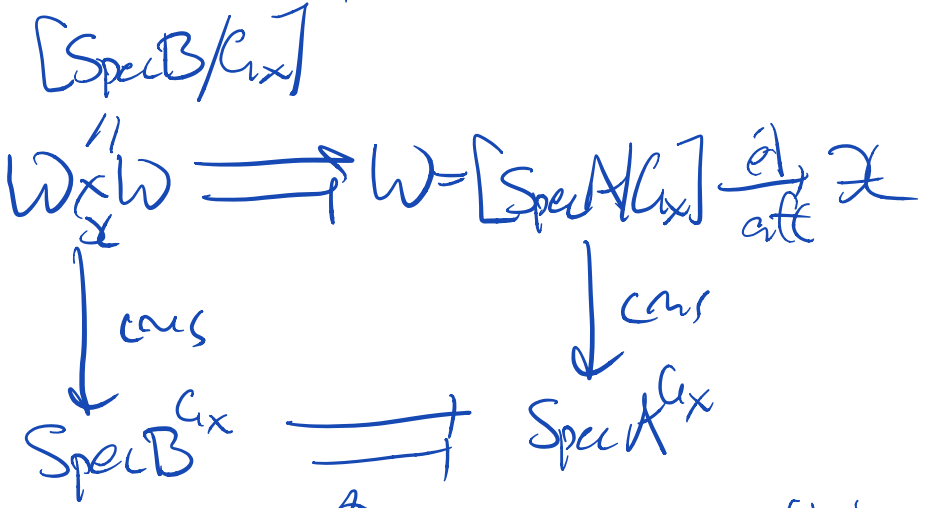
↑ completion of A at x

f étale at $x \implies \hat{A} \xrightarrow{\sim} \hat{B}$

\bar{f} étale at $\pi_A(x) \iff \hat{A}^G \xrightarrow{\sim} \hat{B}^G$
 $\iff \hat{A}^{G_x} \xrightarrow{\sim} \hat{B}^{G_{f(x)}}$

§3. Reducing to quotient stacks

Our strategy



Need: this is an étale equiv. relation

Ques: If $f: \text{Spec } A \rightarrow \text{Spec } B$

G -equiv. & étale, when is $\text{Spec } A^G \xrightarrow{\bar{f}} \text{Spec } B^G$ étale?

Upshot: If $G_x = G_{f(x)}$, we win

Proposition.

- Let G be a finite group.
- Let $f: \text{Spec } A \rightarrow \text{Spec } B$ be a G -equivariant map of schemes of finite type over a noetherian ring R .
- Let $x \in \text{Spec } A$ be a closed point.

Assume that

(a) f is étale at x and

(b) the map $G_x \xrightarrow{\sim} G_{f(x)}$ of stab groups is bijective.

Then there is open affine ngbd $W \subset \text{Spec } A^G$ of $\pi_A(x)$ such that $W \rightarrow \text{Spec } A^G \rightarrow \text{Spec } B^G$ is étale and the outer square in

$$\begin{array}{ccccc}
 \pi_A^{-1}(W) & \longrightarrow & [\text{Spec } A/G] & \xrightarrow{f} & [\text{Spec } B/G] \\
 \downarrow & & \downarrow \pi_A & & \downarrow \pi_B \\
 W \subset \text{Spec } A^G & \longrightarrow & \text{Spec } A^G & \longrightarrow & \text{Spec } B^G \\
 & \searrow & \text{ét} & \nearrow & \\
 & & & &
 \end{array}$$

is cartesian.

Cor If in addition (a) & (b) hold at all closed pts, then

$$\begin{array}{ccc}
 [\text{Spec } A/G] & \longrightarrow & [\text{Spec } B/G] \\
 \downarrow & \square & \downarrow \\
 \text{Spec } A^G & \xrightarrow{\text{ét}} & \text{Spec } B^G
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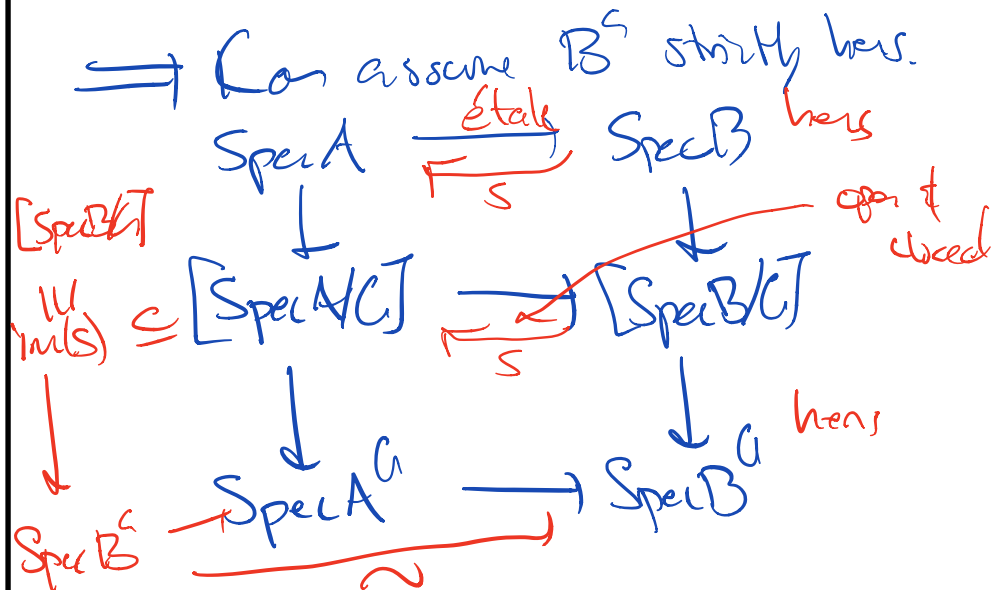
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 W & \longrightarrow & \text{Spec } A^G & \longrightarrow & \text{Spec } B^G \\
 & \searrow & & \nearrow & \\
 & & & \text{ét} &
 \end{array}$$

is cartesian.

PF: Question is étale-local around $\pi_B(y)$



§4. Keel-Mori

Theorem (Keel-Mori). Let \mathcal{X} be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S . Then there exists a coarse moduli space $\pi: \mathcal{X} \rightarrow X$ with $\mathcal{O}_X = \pi_* \mathcal{O}_{\mathcal{X}}$ such that

- (1) X is separated and of finite type over S ,
- (2) π is a proper universal homeomorphism, and
- (3) for any flat map $X' \rightarrow X$ of noetherian algebraic spaces, $\mathcal{X} \times_X X' \rightarrow X'$ is a coarse moduli space.

Special case

Allow us to reduce to special case

Three ingredients of proof

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Theorem (Local Structure of DM Stacks).

Let \mathcal{X} be a separated DM stack and $x \in \mathcal{X}(k)$ be a geometric point with stabilizer G_x . Then \exists an affine, étale map

$$f: ([\text{Spec } A/G_x], w) \rightarrow (\mathcal{X}, x)$$

such that f induces an isomorphism of stabilizer groups at w .

Prop. Let G be a finite group and $f: \text{Spec } A \rightarrow \text{Spec } B$ be a G -equivariant map of schemes of finite type over a noetherian ring R . Suppose that for all closed points $x \in \text{Spec } A$ that (a) f is étale at x and (b) $G_x \xrightarrow{\sim} G_{f(x)}$. Then $\text{Spec } A^G \rightarrow \text{Spec } B^G$ is étale and

$$\begin{array}{ccc} [\text{Spec } A/G] & \xrightarrow{f} & [\text{Spec } B/G] \\ \downarrow \pi_A & \square & \downarrow \pi_B \\ \text{Spec } A^G & \longrightarrow & \text{Spec } B^G \end{array}$$

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PF: Assume $S = \text{Spec } R$ affine
 • Question is Zariski-local on \mathcal{X}
 \Rightarrow Suffices to show that for a closed pt $x \in \mathcal{X}$, \mathcal{I} open neighborhood of x with a coarse moduli space

Let $\text{Spec } k \rightarrow \mathcal{X}$ be a rep of x
 w/ $k = \bar{k}$

Set $U = G_x$

Local Structure Thm \Rightarrow

$$[\text{Spec } A / G_x] \xrightarrow[\text{alt}]{\text{alt}} \mathcal{X}$$

$w \longmapsto x$

$$(*) \text{Aut}(w) \xrightarrow{\sim} \text{Aut}(x)$$

Need to show: $(*)$ holds in a neighborhood of w

$$\mathcal{X} \text{ sep} \Rightarrow \begin{array}{ccc} \mathcal{I}_a & \xrightarrow{f_n} & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{\text{finite}} & \mathcal{X} \times \mathcal{X} \end{array}$$

Consider

$$\begin{array}{ccc} \mathcal{I}_w & \xrightarrow[\text{open}]{\text{closed}} & W \times_{\mathcal{X}} \mathcal{I}_a \\ \downarrow \rho & \swarrow \text{finite} & \downarrow \\ W & \xrightarrow[\text{closed}]{\text{open}} & W \times_{\mathcal{X}} W \end{array}$$

map of group schemes
 For a $w \in W(k)$,
 the fiber is
 $\text{Aut}(w) \rightarrow \text{Aut}(f(w))$

$$W \rightarrow \mathcal{X} \text{ alt} \Rightarrow W \rightarrow W \times_{\mathcal{X}} W \text{ closed imm}$$

open imm

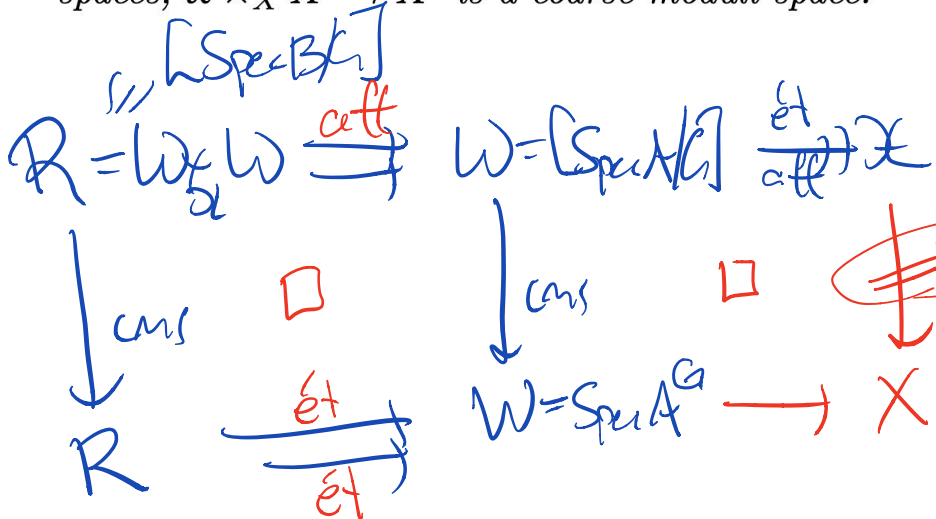
$$\rho^{-1}(\rho(W \times_{\mathcal{X}} \mathcal{I}_a) \setminus \mathcal{I}_w) \subseteq W$$

ρ is precisely where $W \rightarrow \mathcal{X}$ is not stabilizer preserving.

\Rightarrow Can arrange $W \rightarrow \mathcal{X}$ stab pres

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Since $W \rightarrow \mathcal{X}$ stable pre, so is $\mathcal{R} \neq W$

\Rightarrow Two square cartesian

$\Rightarrow \mathcal{R} \neq W$ étale groupoid of affine schemes

Check: $R \rightarrow W \times W$ mono

$\Rightarrow X = W/R$ alg. space quotient

Use étale descent

Why is X sep?

B/c $\mathcal{X} \xrightarrow{\text{proper}} X$
 $\text{sep} \xrightarrow{\text{ét}} \text{sep}$

G