TODAY : Existence of coarse moduli spaces

Theorem (The Keel–Mori Theorem). *A separated DM stack* $\mathcal{X}$ *admits a coarse moduli space* $\pi : \mathcal{X} \to X$ *where* X *is a separated algebraic space.*

Last time

① Quasi-coherent sheaves on DM stacks

② Local structure of DM stacks

§0. Review

Theorem (Local Structure of DM Stacks).
Let $\mathcal{X}$ be a separated DM stack and $x \in \mathcal{X}(k)$ be a geom. point with stabilizer G_x. Then $\exists$ an affine, étale map

$$f\colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$$

such that f induces an isom of stabilizer groups at w.

§1. Definition

alg stack　alg. space

Def A map $\mathcal{X} \xrightarrow{\pi} X$ is a coarse moduli space (or cms) if

(1) $\forall k = \bar{k}$, $\mathcal{X}(k)/\sim \xrightarrow{\sim} X(k)$

isom. classes of objects in $\mathcal{X}(k)$

(2) π is universal for maps to alg space, i.e

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"] \arrow[rd, bend left] & \\ X \arrow[r, dashed, "\exists!"] & Y \end{tikzcd}$$

alg. space

View X as the closest approximation to $\mathcal{X}$ which is an alg. space

Tradeoff

$\mathcal{X}$ univ. properties (eg $\exists$ univ. family) $\rightsquigarrow$ X more familiar

ideally, X is projective

Strategy to show existence of cms

① Special case: If $\mathcal{X} = [\operatorname{Spec} A/G]$

finite

Then

$$\mathcal{X} = [\operatorname{Spec} A/G] \xrightarrow{\text{cms}} \operatorname{Spec} A^G$$

② Use Local Structure Thm

$$\begin{tikzcd} {[\operatorname{Spec} A/G_x]} \arrow[r, "\text{ét}"] \arrow[d, "\text{cms}"] & \mathcal{X} \arrow[d, rightsquigarrow] \\ \operatorname{Spec} A^{G_x} \arrow[r, dashed] & X \end{tikzcd}$$

glue these in étale topology

Definition. A map $\pi\colon \mathcal{X} \to X$ from an algebraic stack to an algebraic space is a *coarse moduli space* if

(1) for all k alg. closed, $\mathcal{X}(k)/\!\sim\ \xrightarrow{\sim} X(k)$

(2) π is universal for maps to algebraic spaces.

In practice, we desire stronger properties.

For us, if $\mathcal{X}$ sep. DM stack, we will construct $\mathcal{X} \xrightarrow{\pi} X$ satisfying

$$\begin{tikzcd}
 & \mathcal{X} \arrow[r, "\pi"] & X \\
\mathcal{X}' \arrow[ur] \arrow[r, "\text{cms}"] \arrow[urr, phantom, "\square"] & X' \arrow[ur, "\text{flat}"] &
\end{tikzcd}$$

ⓐ stable under flat base change

ⓑ $\pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$

ⓒ π is proper (in part separated!)

ⓓ π univ. homeomorphism

$\implies |\mathcal{X}| \xrightarrow{\sim} |X|$

Rmk: (a) $\implies$ (b)

$$\begin{tikzcd}
\mathcal{X}' \arrow[r] \arrow[d, "\text{cms}"] & \mathcal{X} \arrow[d] \arrow[dr] & \\
X' \arrow[ur, phantom, ""] \arrow[r, "\text{ét}"] & X \arrow[r, dashed, "\exists!"] & \mathbb{A}^1
\end{tikzcd}$$

$\Gamma(X, \mathcal{O}_X) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$

$\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) = \Gamma(X', \mathcal{O}_{X'})$

$\quad\parallel$

$\Gamma(X' \xrightarrow{\text{ét}} X, \pi_* \mathcal{O}_{\mathcal{X}}) \implies \mathcal{O}_X \xrightarrow{\sim} \pi_* \mathcal{O}_{\mathcal{X}}$

Descent lemma

alg stack

alg space

Let $\mathcal{X} \xrightarrow{\pi} X$ be a map

If $\{X_i \xrightarrow{\text{ét}} X\}$ étale cover (even fppf) s.t. $\mathcal{X} \times_X X_i \to X_i$ cms, then $\mathcal{X} \to X$ is.

§2. Quotients by finite groups

Let G finite group $\circlearrowright$ Spec A

Define $A^G = \{a \in A \mid g.a = a \ \forall g\}$

G acts via R-alg hom. on A

Lemma 1 Let R noeth ring

If A fin. gen R-algebra, then $A^G \to A$ finite & A^G fin gen R-alg.

Pf: $A^G \to A$ integral: if $a \in A$

$$\prod_{g \in G} (x - ga) \in A^G[x] \text{ monic poly}$$

with a as a root

$$\begin{tikzcd} & A \\ R \arrow[ur, "\text{fin gen}"] \arrow[r] & A^G \arrow[u, "\text{finite}"'] \end{tikzcd}$$

noeth

$\Rightarrow A^G \to A$ fin gen

$\Rightarrow A^G \to A$ finite

Comm alg $\Rightarrow$ A^G fin gen as R-alg.

Lemma 2 Let $A^G \to B$ ring map.

So G acts on $\mathrm{Spec}(B \otimes_{A^G} A)$.

Consider

$$\begin{tikzcd} & \mathrm{Spec}(B\otimes_{A^G} A) \arrow[dl] \arrow[d, "G\text{-inv}"] \arrow[r, "G\text{-equiv}"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Spec}\,A \arrow[d, "G\text{-inv}"] \\ \mathrm{Spec}(B\otimes_{A^G} A)^G \arrow[r, "\psi"] & \mathrm{Spec}\,B \arrow[r] & \mathrm{Spec}\,A^G \end{tikzcd}$$

① $A^G \to B$ flat $\Rightarrow$ $B \xrightarrow[\sim]{\psi^*} (B \otimes_{A^G} A)^G$

② In general, ψ integral univ. homeo.

Pf: ①

$$0 \to A^G \to A \rightrightarrows \prod_{g \in G} A$$

$- \otimes_{A^G} B \Rightarrow$

$$0 \to B \to A \otimes_{A^G} D \rightrightarrows \prod_{g \in G} (A \otimes_{A^G} B)$$

exact

② exercise

Theorem. *Let G be a finite group acting on an affine scheme* $\mathrm{Spec}\, A$ *of finite type over a noeth ring R. Then*

$$\pi\colon [\mathrm{Spec}\, A/G] \to \mathrm{Spec}\, A^G$$

is a coarse moduli space such that

✓ *(1) A^G is finitely generated over R,*

✓ *(2) π is a proper universal homeomorphism, and*

(3) the base change of π along any flat map of noetherian algebraic spaces is a coarse moduli space.

Know: $A^G \to A$ finite & A^G fin gen/R

Step 1: π is a proper univ. homeo

($\Rightarrow$ π is bij. on geom pts)

Consider

$$\begin{tikzcd}
\mathrm{Spec}\, A \arrow[d, "\pi", "\text{finite}"'] \arrow[dr, two heads, "\text{finite dominant}"] & \\
{[\mathrm{Spec} A/G]} \arrow[r, two heads, "\Rightarrow \text{proper}"'] & \mathrm{Spec}\, A^G
\end{tikzcd}$$

Claim: π is injective on geom pts

Assume $R = k = \bar{k}$

Let $x, x' \in \mathrm{Spec}\, A$ closed pts

with $Gx \neq Gx'$

$Gx \cap Gx' = \emptyset \Longrightarrow$

$\exists\, f \in A$ s.t. $f|_{Gx} = 1$ & $f|_{Gx'} = 0$

$\Rightarrow f' = \prod_{g \in G} gf \in A^G \quad f'|_{Gx} = 1 \quad f'|_{Gx'} = 0$

$\Rightarrow \pi(x) \neq \pi(x')$

$\Rightarrow \pi$ bij. on geom pts

Since π is proper, π^{-1} continuous

$\Rightarrow \pi$ is a homeo

For $A^G \to B$

$$\begin{tikzcd}
& \mathrm{Spec}(B \otimes_{A^G} A) \arrow[dl, "\text{homeo}"'] \arrow[d, "\Rightarrow \text{homeo}"] \arrow[r] \arrow[dr, phantom, "\square"] & \mathrm{Spec}\, A \arrow[d, "\Rightarrow \text{univ homeo}"] \\
\mathrm{Spec}(B \otimes_{A^G} A)^G \arrow[r, "\not\sim", "\text{homeo}"'] & \mathrm{Spec}\, B \arrow[r] & \mathrm{Spec}\, A^G
\end{tikzcd}$$

Theorem. *Let G be a finite group acting on an affine scheme* Spec A *of finite type over a noeth ring R. Then*

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✓ *(1) A^G is finitely generated over R,*

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(3) the base change of π along any flat map of noetherian algebraic spaces is a coarse moduli space.

Step 2 π is univ. for maps to alg. space

$$\begin{tikzcd}
\mathcal{X} = [\operatorname{Spec} A/G] \arrow[r] \arrow[d, "f"'] & X = \operatorname{Spec} A^G \arrow[dl, dashed, "\exists !"'] \\
Y & X' \arrow[u, "\text{ét}"'] \arrow[l, bend left] \\
& X' \times_X X' \arrow[u, shift left] \arrow[u, shift right]
\end{tikzcd}$$

Rmk: If Y is affine, then

$$\begin{tikzcd}
\Gamma(Y, \mathcal{O}_Y) \arrow[r] \arrow[dr] & \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \arrow[d, equal] \\
& \Gamma(X, \mathcal{O}_X)
\end{tikzcd}$$

defines $X \to Y$

(i) Uniqueness Suppose

$$\begin{tikzcd}
& \mathcal{X} \arrow[dl] \arrow[d] \arrow[dr, "f"] & \\
E \arrow[r] & X \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & Y
\end{tikzcd}$$

Let E be equalizer of $X \rightrightarrows Y$

Precisely,

$$\begin{tikzcd}
E \arrow[r, "\text{mono}", "\text{l.f.type}"'] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\
Y \arrow[r, "\text{mono}", "\text{l.f.type}"'] & Y \times Y
\end{tikzcd}$$

But $\mathcal{X} \to X$ proper & sch. dominant

$\Rightarrow$ $E \to X$

So $E \to X$ $\left.\begin{array}{l}\text{mono} \\ \text{l.f.type} \\ \text{proper}\end{array}\right] \Rightarrow E \to X$ closed imm

sch. dominant$\Big] \Rightarrow E \xrightarrow{\sim} X$ ✓

(ii) Existence

Question is étale-local on X

Reason: étale descent & univ. property

$\Rightarrow$ Assume A^G is strictly henselian

Can also assume Y is q. compact

A^G strictly henselian

Y q.compact

$$\begin{tikzcd}
Y' \times_Y \operatorname{Spec} A \arrow[r, "ét"] \arrow[d] & \operatorname{Spec} A \ \text{hens.} \arrow[l, bend left, "\exists s"] \arrow[d] \arrow[dr, "\text{finite}"] & \\
\mathcal{X}' \arrow[r, "ét"] \arrow[d] & \mathcal{X} \arrow[l, bend left, "s"] \arrow[r] \arrow[d, "f"] \arrow[dl, phantom, "\square"] & X = \operatorname{Spec} A^G \ \text{hens} \arrow[dll, "\exists !"] \arrow[dl, dashed, "?"] \\
Y' \arrow[r, "ét"] & Y & \\
\text{aff} & &
\end{tikzcd}$$

Heuristically: $R = k = \bar{k}$

Let $x \in \operatorname{Spec} A$

FACT $(\hat{A})^{G_x} = \widehat{A^G}$ ← complete A^G at $\pi_A(x)$

↑ completion of A at x

f étale at $x \implies \hat{A} \xrightarrow{\sim} \hat{B}$

$\bar{f}$ étale at $\pi_A(x) \iff \widehat{A^G} \xrightarrow{\sim} \widehat{B^G}$

$\iff \hat{A}^{G_x} \xrightarrow{\sim} \hat{B}^{G_{f(x)}}$

§3. Reducing to quotient stacks

Our strategy

$$\begin{tikzcd}
{[\operatorname{Spec} B/G_x]} \arrow[d, equal] & & \\
W \times_{\mathcal{X}} W \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\text{cms}"] & W = {[\operatorname{Spec} A/G_x]} \arrow[r, "ét"', "\text{aff}"] \arrow[d, "\text{cms}"] & \mathcal{X} \\
\operatorname{Spec} B^{G_x} \arrow[r, shift left] \arrow[r, shift right] & \operatorname{Spec} A^{G_x} &
\end{tikzcd}$$

Need: ↑ this is an étale equiv. relation

Ques: If $f : \operatorname{Spec} A \to \operatorname{Spec} B$ G-equiv. & étale, when is $\operatorname{Spec} A^G \xrightarrow{\bar{f}} \operatorname{Spec} B^G$ étale?

Upshot: If $G_x = G_{f(x)}$, we win

Proposition.

- *Let G be a finite group.*
- *Let $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ be a G-equivariant map of schemes of finite type over a noetherian ring R.*
- *Let $x \in \operatorname{Spec} A$ be a closed point.*

Assume that

(a) f is étale at x and

(b) the map $G_x \xrightarrow{\sim} G_{f(x)}$ of stab groups is bijective.

Then there is open affine ngbd $W \subset \operatorname{Spec} A^G$ of $\pi_A(x)$ such that $W \to \operatorname{Spec} A^G \to \operatorname{Spec} B^G$ is étale and the outer square in

$$\begin{tikzcd}
\pi_A^{-1}(W) \arrow[r] \arrow[d] & {[\operatorname{Spec} A/G]} \arrow[r, "f"] \arrow[d, "\pi_A"] & {[\operatorname{Spec} B/G]} \arrow[d, "\pi_B"] \\
W \arrow[r, hook] \arrow[rr, bend right, "\text{ét}"'] & \operatorname{Spec} A^G \arrow[r] & \operatorname{Spec} B^G
\end{tikzcd}$$

is cartesian.

Cor If in addition (a) & (b) hold at all closed pts, then

$$\begin{tikzcd}
{[\operatorname{Spec} A/G]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {[\operatorname{Spec} B/G]} \arrow[d] \\
\operatorname{Spec} A^G \arrow[r, "\text{ét}"] & \operatorname{Spec} B^G
\end{tikzcd}$$

Proposition.

- *Let G be a finite group.*
- *Let $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ be a G-equivariant map of schemes of finite type over a noetherian ring R.*
- *Let $x \in \operatorname{Spec} A$ be a closed point.*

Assume that

(a) f is étale at x and

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$$\begin{tikzcd}
\pi_A^{-1}(W) \arrow[r] \arrow[d] & {[\operatorname{Spec} A/G]} \arrow[r, "f"] \arrow[d, "\pi_A"] & {[\operatorname{Spec} B/G]} \arrow[d, "\pi_B"] \\
W \arrow[r, hook] \arrow[rr, bend right, "\text{ét}"'] & \operatorname{Spec} A^G \arrow[r] & \operatorname{Spec} B^G
\end{tikzcd}$$

is cartesian.

Pf: Question is étale-local around $\pi_B(y)$

$\implies$ Can assume B^G strictly hens.

$$\begin{tikzcd}
 & \operatorname{Spec} A \arrow[r, "\text{étale}"] \arrow[d] & \operatorname{Spec} B \arrow[l, bend left, "s"] \arrow[d] & \text{hens} \\
{[\operatorname{Spec} B/G] \cong \operatorname{Im}(s)} \arrow[r, phantom, "\subseteq"] \arrow[d] & {[\operatorname{Spec} A/G]} \arrow[r] \arrow[d] & {[\operatorname{Spec} B/G]} \arrow[l, bend left, "s"] \arrow[d] & \text{open \& closed} \\
\operatorname{Spec} B^G \arrow[r] \arrow[rr, bend right, "\sim"'] & \operatorname{Spec} A^G \arrow[r] & \operatorname{Spec} B^G & \text{hens}
\end{tikzcd}$$

§4. Keel-Mori

Theorem (Keel-Mori)**.** *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S. Then there exists a coarse moduli space $\pi\colon \mathcal{X} \to X$ with $\mathcal{O}_X = \pi_*\mathcal{O}_{\mathcal{X}}$ such that*

(1) X is separated and of finite type over S,

(2) π is a proper universal homeomorphism, and

(3) for any flat map $X' \to X$ of noetherian algebraic spaces, $\mathcal{X} \times_X X' \to X'$ is a coarse moduli space.

Special case

Allow us to reduce to special case

Three ingredients of proof

Theorem. *Let G be a finite group acting on an affine scheme* Spec *A of finite type over a noeth ring R. Then*

$$\pi\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$$

is a coarse moduli space such that

(1) A^G is finitely generated over R,

(2) π is a proper universal homeomorphism, and

(3) the base change of π along any flat map of noetherian algebraic spaces is a coarse moduli space.

Theorem (Local Structure of DM Stacks)**.** *Let $\mathcal{X}$ be a separated DM stack and $x \in \mathcal{X}(k)$ be a geom. point with stabilizer G_x. Then $\exists$ an affine, étale map*

$$f\colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$$

such that f induces an isom of stabilizer groups at w.

Prop. *Let G be a finite group and $f\colon$ Spec $A \to$ Spec B be a G-equivariant map of schemes of finite type over a noetherian ring R. Suppose that for all closed points $x \in$ Spec A that (a) f is étale at x and (b) $G_x \xrightarrow{\sim} G_{f(x)}$. Then* Spec $A^G \to$ Spec B^G *is étale and*

$$\begin{tikzcd}
[\operatorname{Spec} A/G] \arrow[r, "f"] \arrow[d, "\pi_A"] \arrow[dr, phantom, "\square"] & {[\operatorname{Spec} B/G]} \arrow[d, "\pi_B"] \\
\operatorname{Spec} A^G \arrow[r] & \operatorname{Spec} B^G
\end{tikzcd}$$

Theorem (Keel-Mori). *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S. Then there exists a coarse moduli space $\pi\colon \mathcal{X} \to X$ with $\mathcal{O}_X = \pi_*\mathcal{O}_{\mathcal{X}}$ such that*

(1) X is separated and of finite type over S,

(2) π is a proper universal homeomorphism, and

(3) for any flat map $X' \to X$ of noetherian algebraic spaces, $\mathcal{X} \times_X X' \to X'$ is a coarse moduli space.

Pf: Assume $S = \operatorname{Spec} R$ affine

- Question is Zariski-local on $\mathcal{X}$

$\Rightarrow$ Suffices to show that for a closed pt $x \in |\mathcal{X}|$, $\exists$ open ngbd of x with a coarse mod space

Let $\operatorname{Spec} k \to \mathcal{X}$ be a rep of x w/ $k = \bar{k}$

Set $G = G_x$

Local Structure Thm $\Rightarrow$

$$[\operatorname{Spec} A / G_x] \xrightarrow[\text{aff}]{\text{ét}} \mathcal{X}$$

$$w \longmapsto x$$

$$(*) \quad \operatorname{Aut}(w) \xrightarrow{\sim} \operatorname{Aut}(x)$$

Need to show: $(*)$ holds in a ngbd of w

$\mathcal{X}$ sep $\Rightarrow$

$$\begin{tikzcd} I_{\mathcal{X}} \arrow[r, "\text{fin}"] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathcal{X} \arrow[r, "\text{finite}"] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

Consider

$$\begin{tikzcd} I_W \arrow[r, "\text{closed}", "\text{open}"'] \arrow[d] & W \times_{\mathcal{X}} I_{\mathcal{X}} \arrow[d] \arrow[dl, "p_1"', "\text{finite}"] \\ W \arrow[r, "\text{cl \& open}"] & W \times_{\mathcal{X}} W \end{tikzcd}$$

map of group sch/W

For a $w \in W(k)$, the fiber is $\operatorname{Aut}(w) \to \operatorname{Aut}(f(w))$

$W \to \mathcal{X}$ aff $\Rightarrow$ $W \to W \times_{\mathcal{X}} W$ closed imm

étale $\Rightarrow$ open imm

$$p_1(|W \times_{\mathcal{X}} I_{\mathcal{X}}| \setminus I_W) \subseteq W$$

is precisely where $W \to \mathcal{X}$ is <u>not</u> stabilizer preserving.

$\Rightarrow$ Can arrange $W \to \mathcal{X}$ stab pres

Theorem (Keel-Mori). *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S. Then there exists a coarse moduli space $\pi\colon \mathcal{X} \to X$ with $\mathcal{O}_X = \pi_*\mathcal{O}_{\mathcal{X}}$ such that*

(1) X is separated and of finite type over S,

(2) π is a proper universal homeomorphism, and

(3) for any flat map $X' \to X$ of noetherian algebraic spaces, $\mathcal{X} \times_X X' \to X'$ is a coarse moduli space.

$$\begin{tikzcd}
R = W \times_{\mathcal{X}} W \cong [\mathrm{Spec}\, B/G] \arrow[r, shift left, "\mathrm{aff}"] \arrow[r, shift right] \arrow[d, "\mathrm{cms}"] \arrow[dr, phantom, "\square"] & W = [\mathrm{Spec}\, A/G] \arrow[r, "\text{ét}", "\mathrm{aff}"'] \arrow[d, "\mathrm{cms}"] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\exists\ \mathrm{cms}"] \\
R \arrow[r, shift left, "\text{ét}"] \arrow[r, shift right, "\text{ét}"'] & W = \mathrm{Spec}\, A^G \arrow[r] & X
\end{tikzcd}$$

Since $W \to \mathcal{X}$ stab pres, so is $R \rightrightarrows W$

$\Rightarrow$ Two square cartesian

$\Rightarrow$ $R \rightrightarrows W$ étale groupoid of affine schemes

Check: $R \to W \times W$ mono

$\Rightarrow$ $X = W/R$ alg. space quotient

Use étale descent

Why is X sep?

B/c $\underset{\text{sep}}{\mathcal{X}} \xrightarrow{\text{proper}} \underset{\Rightarrow \text{sep}}{X}$

G