Math 480A: Algebraic Complexity Theory, Spring Quarter 2019
Jarod Alper
Homework 4
Due: Friday, May 31

Problem 4.1. Assume that the characteristic of the field k is zero. Suppose that $\{g_n\}$ is a p-computable sequence of polynomials such that g_n has exactly n variables. Let $\{d_n\}$ is a polynomially bounded sequence of positive integers. Define the polynomial f_n of degree $\leq d_n$ as follows:

$$f_n(x_1, \ldots, x_n) = \sum_{I=(i_1,\ldots,i_n), \sum_j i_j = d_n} g_n(i_1, \ldots, i_n) x_1^{i_1} \cdots x_n^{i_n}.$$

(a) Show that $\{f_n\}$ is a p-projection of a sequence of polynomials

$$f'_n(x_1, \ldots, x_{m_n}) = \sum_{e_1,\ldots,e_{m_n} \in \{0,1\}} g'_n(e_1, \ldots, e_{m_n}) x_1^{e_1} \cdots x_{m_n}^{e_{m_n}}$$

where $\{g'_n\}$ is p-computable and m_n is an integer depending on n.

Hint: Write

$$x_1^{i_1} \cdots x_n^{i_n} = \underbrace{x_1^1 \cdots x_1^1}_{i_1 \text{ times}} \underbrace{x_1^0 \cdots x_1^0}_{d_n - i_1 \text{ times}} \cdots\cdots \underbrace{x_n^1 \cdots x_n^1}_{i_n \text{ times}} \underbrace{x_n^0 \cdots x_n^0}_{d_n - i_n \text{ times}}$$

and argue that $\{f_n\}$ is a p-projection of suitably defined polynomials $\{f'_n\}$ where f'_n has nd_n number of variables.

(b) Conclude that $\{f_n\}$ is p-definable.

Problem 4.2. Let G be directed labelled graph. A *Hamiltonian cycle* of G is a cycle which is also a cycle cover; that is, it is a cycle of G that passes through every vertex precisely once. Define

$$\mathrm{Ham}(G) = \sum_{\text{Hamiltonian cycles } \sigma} \mathrm{weight}(\sigma).$$

Let G_n be the directed labelled graph with n vertices where the label between vertex i and j is the variable $x_{i,j}$. Analogous to the sequence $\{\mathrm{perm}_n\}$, define the sequence of *Hamiltonian polynomials* $\{\mathrm{Ham}_n\}$ as

$$\mathrm{Ham}_n = \mathrm{Ham}(G_n).$$

(a) Compute $\mathrm{Ham}_2, \mathrm{Ham}_3$ and Ham_4.
(b) Show that $\{\mathrm{Ham}_n\}$ is p-definable.

Problem 4.3. Show that the following sequences of polynomials is p-definable:

(a)

$$f_n(x_1, \ldots, x_n) = \sum_{I, |I|=n} x^I$$

where as usual $I = (i_1, \ldots, i_n)$, $|I| = \sum_{k=1}^n i_k$ and $x^I = x_1^{i_1} \cdots x_n^{i_n}$.

(b)

$$f_n(x_{1,1}, \ldots, x_{n,n}) = \sum_{e_{i,j} \in \{0,1\}, \sum_{i,j} e_{i,j} = n} x_{1,1}^{e_{1,1}} \cdots x_{n,n}^{e_{n,n}}.$$

(c)

$$f_n(x_1, \ldots, x_n) = \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq n} x_1^{i_1} \cdots x_n^{i_n}.$$

Problem 4.4. Let G_1 and G_2 be acyclic directed labelled graphs each with distinguished vertices s (for 'source') and t (for 'target') such that the edge $t \mapsto s$ has label 1. Let $f_1 = \text{perm}(G_1)$ and $f_2 = \text{perm}(G_2)$.

(a) Construct an acyclic directed labelled graph G_3 with distinguished vertices s and t such that $f_1 + f_2 = \text{perm}(G_3)$.

(b) Construct an acyclic directed labelled graph G_3 with distinguished vertices s and t such that $f_1 f_2 = \text{perm}(G_3)$.

Problem 4.5. In lecture, we covered an analogous procedure of parts (a) and (b) of Problem 4.4 to find a determinantal expression for any polynomial.

(a) Find an explicit determinantal expression for $xy(x + y) + x + z$. You should write down the corresponding directed labelled graph **and** the matrix.

(b) Modify your answer to find a 'perminantal' expression of $xy(x + y) + x + z$; that is, find a directed labeled graph G such that $\text{perm}(G) = xy(x+y)+x+z$.

Problem 4.6. Repeat Problem 4.5(a) for the polynomial $x^2y + y^2z + xz^2 - 4xyz$. You **do not** need to write down the matrix.

Problem 4.7.

(a) The procedure from lecture allows us to construct a determinantal expression for perm_3. What is the size of this determinantal expression? This gives an upper bound for $\text{dc}(\text{perm}_3)$.

(b) Provide an upper bound for $\text{dc}(\text{perm}_n)$.

You do not need to explicitly write down the determinantal expressions.

Problem 4.8. Let $f = x^d + y^d$.

(a) Show that if the base field k is the complex numbers $\mathbb{C}$, then $\text{dc}(f) = d$.

(b) (Extra credit) Is the same true over $\mathbb{R}$ or $\mathbb{Q}$?