Math 427 Homework #6 Solutions

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Problem 6.1. Taylor 3.1.2.

Proof. First let $k \geq 0$ be fixed, we want to show for any $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for all $N > N_0$ we have $|\sin(x/N)| < \epsilon$ for all $x \in [0, k]$. Observe by increasing N, we are stretching up the graph $\sin(x)$ horizontally, hence the idea is to stretch it enough so that for any $x \in [0, k]$ the value of the function $\sin(x/N)$ is less than ϵ.

Formally, let $\epsilon > 0$ be given. If $\epsilon > 1$ we are done since we know $|\sin(x/N)| \leq 1$ for all x. Suppose now that $\epsilon \in (0, 1]$, pick natural number N_0 such that $N_0 > \max\{k/\arcsin(\epsilon), 2k/\pi\}$, since $\epsilon \in (0, 1]$, we may pick the value of the arcsin function from the interval $(0, \pi/2]$. Therefore, for any $N > N_0$ and $x \in [0, k]$, we have

$$N > N_0 > \frac{k}{\arcsin(\epsilon)} > \frac{x}{\arcsin(\epsilon)}.$$

Rearranging the inequality we have $\arcsin(\epsilon) > x/N$. Since the sin function is monotonically increasing on the interval $(0, \pi/2]$, applying it to both sides of the inequality we obtain $\epsilon > \sin(x/N)$. Since $N > N_0 > 2k/\pi$ and $x \in [0, k]$, we know $x/N \in [0, \pi/2]$, hence $\sin(x/N) \in [0, 1]$, therefore $|\sin(x/N)| = \sin(x/N) < \epsilon$. This shows the sequence of functions $\{\sin(x/n)\}_n$ converges uniformly on the interval $[0, k]$.

To show the sequence of functions $\{\sin(x/n)\}_n$ does not converge uniformly on $[0, \infty)$, we just need to notice for any $n \in \mathbb{N}$, we may pick $x := \pi n/2 \in [0, \infty)$ such that $\sin(x/n) = \sin(\pi/2) = 1$, which cannot be made arbitrarily small. □

Problem 6.2. Taylor 3.1.7.

Proof. Let $s > 1$ be fixed, suppose $z = x + iy \in \mathbb{C}$ with $x > s$ be given, then for each positive integer k we have $|k^{-z}| = |k^{-x}| / \left|k^{iy}\right| = |k^{-x}| < |k^{-s}| = k^{-s}$. Furthermore by p-series test, the series $\sum_{k=1}^{\infty} k^{-s}$ converges. Therefore it follows from Weierstraß's M Test that the series $\sum_{k=1}^{\infty} k^{-z}$ converges uniformly on each set of the form $\{z \in \mathbb{C} : Re(z) > s\}$, where $s > 1$. □

Problem 6.3. Taylor 3.1.10.

Solution: We want to compute

$$\left(\limsup_{k\to\infty} \left|2 + (-1)^k\right|\right)^{-1}.$$

Observe for each positive integer k and $n \geq k$, the value $(-1)^n$ is either 1 or -1, hence the supremum of the set $\{2+(-1)^n\}_{n\geq k}$ is 3 for each positive integer k. Hence the limit of the constant sequence $\{\sup_{n\geq k} 2+(-1)^n\}_k$ is 3. Therefore the radius of convergence is 1/3.

□

Problem 6.4. Taylor 3.1.16.

Solution: Recall the power series expansion of the function e^{-w^2} about 0 is $\sum_{k=0}^{\infty}(-1)^k w^{2k}/k!$ and the radius of convergence of it is ∞, hence it converges absolutely and uniformly on the entire complex plane. In particular, we may integrate the series term by term on the entire complex plane: for all $z \in \mathbb{C}$, we have

$$E(z) = \int_0^z \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} w^{2k} dw = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^z w^{2k} dw = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)} z^{2k+1}.$$

We claim the series converges absolutely on the entire complex plane. Let $z \in \mathbb{C}$ be given, consider the series of absolute values of the terms: $\sum_{k=0}^{\infty} |z|^{2k+1}/(k!(2k+1))$. The ratio between to consecutive terms is

$$\frac{|z|^{2k+3}}{(k+1)!(2k+3)} \cdot \frac{k!(2k+1)}{|z|^{2k+1}} = \frac{|z|^2}{k+1} \cdot \frac{2k+1}{2k+3}.$$

Since $\lim_{k\to\infty}(2k+1)/(2k+3) = 1$ and $\lim_{k\to\infty} |z|^2/(k+1) = 0$, it follows that the limit of the product as k approaches infinity is 0. Therefore we conclude by ratio test that the series of interest converges absolutely for all $z \in \mathbb{C}$. □

Problem 6.6. Establish the following:

1. For any integer n, $\lim_{k\to\infty}(k^n)^{1/k} = 1$.
2. Suppose that $\{a_k\}$ and $\{b_k\}$ are sequences of non-negative real numbers with $a = \lim_{k\to\infty} a_k$ and $b = \limsup_{k\to\infty} b_k$. Show that $ab = \limsup_{k\to\infty}(a_k b_k)$.

Proof.

1. Observe $k^{n/k} = e^{(n/k)\ln k}$. Since the function that maps $z \in \mathbb{C}$ to e^{nz} is continuous on $\mathbb{C}$, we have

$$\lim_{k\to\infty} e^{n(\ln k)/k} = e^{n \lim_{k\to\infty}(\ln k)/k}.$$

   To show the limit is 1, it suffices to show $\lim_{k\to\infty}(\ln k)/k = 0$. Observe for all $x \in \mathbb{R}$, $\ln(x) < x$, hence

$$0 \le \frac{\ln k}{k} = \frac{2\ln\sqrt{k}}{k} \le \frac{2\sqrt{k}}{k} = \frac{2}{\sqrt{k}}.$$

   We know $\lim_{k\to\infty} 2/\sqrt{k} = 0$, it follows by comparison that $\lim_{k\to\infty}(\ln x)/x = 0$.

2. Observe a and b cannot simultaneously be 0 and ∞ since the product $0 \cdot \infty$ does not make sense.

   First consider the case that $\limsup_{k\to\infty} b_k = b = \infty$ and $a \neq 0$, we want to show $\limsup_{k\to\infty} a_k b_k = \infty$ as well. Let $M \in \mathbb{R}$ and $N \in \mathbb{N}$ be given, we want to show there exists $K_0 > N$ such that $\sup_{n\ge K_0} a_n b_n > M$. Since $\lim_{k\to\infty} a_k = a$, there exists $K_1 \in \mathbb{N}$ such that for all $k > K_1$, $|a_k - a| < a/2$ (notice we need $a > 0$ here). Since $\limsup_{k\to\infty} b_k = \infty$, there exists $K_0 > \max\{N, K_1\}$ such that $\sup_{n\ge K_0} b_n > 2M/a$. Then for any $n \ge K_0$, we have $a/2 < a_n$, hence $ab_n/2 < a_n b_n$, hence after taking the supremum we have

$$\frac{a}{2} \sup_{n\ge K_0} b_n \le \sup_{n\ge K_0} a_n b_n.$$

   Since $\sup_{n\ge K_0} b_n > 2M/a$, we have

$$M = \frac{a}{2} \cdot \frac{2M}{a} < \frac{a}{2} \sup_{n\ge K_0} b_n \le \sup_{n\ge K_0} a_n b_n.$$

   This proves the sequence $\{\sup_{n\ge k} a_n b_n\}_k$ is unbounded, hence $\limsup_{k\to\infty} a_k b_k = \infty$ by definition.

Now suppose $b \neq \infty$. Since $a = \lim_{k\to\infty} a_k$ and $b = \limsup_{k\to\infty} b_k$, the product $a_k \sup_{n\geq k} b_n$ converges to ab. Therefore it suffices to show

$$\lim_{k\to\infty}\Big[a_k(\sup_{n\geq k} b_n) - \sup_{n\geq k} a_n b_n\Big] = 0.$$

Let $\epsilon > 0$ be given. Since $\lim_{k\to\infty} a_k = a$, there exists $K_1 \in \mathbb{N}$ such that for all $k > K_1$, $|a_k - a| < \epsilon/(2(b+1))$ (notice we need $b \neq \infty$ here). Since $\limsup_{k\to\infty} b_k = b$, there exists K_2 such that for all $k \geq K_2$, $\big|\sup_{n\geq k} b_n - b\big| < 1$. Let $K_0 := \max\{K_1, K_2\}$, then for all $k > K_0$,

$$a - \frac{\epsilon}{2(b+1)} < a_k < a + \frac{\epsilon}{2(b+1)}.$$

Since $b_n \geq 0$ for all n, $\sup_{n\geq k} b_n \geq 0$, hence multiplying each side by $\sup_{n\geq k} b_n$ gives us inequality

$$\Big(a - \frac{\epsilon}{2(b+1)}\Big)\Big(\sup_{n\geq k} b_n\Big) < a_k\Big(\sup_{n\geq k} b_n\Big) < \Big(a + \frac{\epsilon}{2(b+1)}\Big)\Big(\sup_{n\geq k} b_n\Big).$$

On the other hand, we may multiply each side by b_k and get

$$\Big(a - \frac{\epsilon}{2(b+1)}\Big)b_k < a_k b_k < \Big(a + \frac{\epsilon}{2(b+1)}\Big)b_k.$$

Rename k to n and take the supremum over $n \geq k$ gives us

$$\Big(a - \frac{\epsilon}{2(b+1)}\Big)\Big(\sup_{n\geq k} b_n\Big) < \sup_{n\geq k} a_n b_n < \Big(a + \frac{\epsilon}{2(b+1)}\Big)\Big(\sup_{n\geq k} b_k\Big).$$

Therefore the difference $\big|a_k(\sup_{n\geq k} b_n) - \sup_{n\geq k} a_n b_n\big|$ is upperbounded by the difference

$$\Big(a + \frac{\epsilon}{2(b+1)}\Big)\Big(\sup_{n\geq k} b_k\Big) - \Big(a - \frac{\epsilon}{2(b+1)}\Big)\Big(\sup_{n\geq k} b_n\Big),$$

which equals

$$\frac{\epsilon}{b+1}\Big(\sup_{n\geq k} b_k\Big).$$

Since $\sup_{n\geq k} b_k < b+1$, it follows the difference is upperbounded by

$$\frac{\epsilon}{b+1}(b+1) = \epsilon,$$

which is what we wanted.

□

Problem 6.6. Taylor 3.2.1.

Solution: Observe $(1-z)^{-2} = d(1-z)^{-1}/dz$ for all $z \in \mathbb{C} \setminus \{1\}$, and recall $(1-z)^{-1} = \sum_{n=0}^{\infty} z^n$ for $0 \leq |z| < 1$. Since the radius of convergence of the series $\sum_{n=0}^{\infty} z^n$ is 1, it follows by Corollary 3.1.8 that the series $\sum_{n=0}^{\infty} z^n$ converges uniformly on $\{z \in \mathbb{C} : 0 \leq |z| < 1\}$, hence we may differentiate term by term and obtain

$$\frac{d}{dz}\frac{1}{1-z} = \sum_{n=1}^{\infty} n z^{n-1}, \quad 0 \leq |z| < 1.$$

Therefore the power series $\sum_{n=1}^{\infty} n z^{n-1}$ and the function $(1-z)^{-2}$ differs by a constant for $0 \leq |z| < 1$. Since they both have value 0 when evaluated at $z = 0$, they coincides. We conclude that

$$\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1}, \quad 0 \leq |z| < 1.$$

□

Problem 6.7. Taylor 3.2.2.

Solution: Recall $\sqrt{1+z} = e^{(1/2)\log(1+z)}$ and in the principal branch, the function $\log(1+z)$ is analytic on $\mathbb{C} \setminus \{(-\infty, -1]\}$. Since the function that maps z to $e^{(1/2)z}$ is analytic on $\mathbb{C}$, it follows the function $\sqrt{1+z}$ is analytic on $\mathbb{C} \setminus \{(-\infty, -1]\}$. Therefore the radius of convergence of the power series expansion of $\sqrt{1+z}$ about 0 is 1 (it is the radius of the largest disk centered at 0 contained in $\mathbb{C} \setminus \{(-\infty, 0]\}$). Recall the function $\log(1+z)$ has power series expansion about 0

$$\log(1+z) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^n, \quad 0 \le |z| < 1,$$

and the function $e^{(1/2)z}$ has power series expansion about 0

$$e^{(1/2)z} = \sum_{m=0}^{\infty} \frac{(z/2)^m}{m!}, \quad z \in \mathbb{C}.$$

Therefore for $0 \le |z| < 1$,

$$\sqrt{1+z} = e^{(1/2)\log(1+z)} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \Big(\sum_{n=0}^{\infty} \frac{(-1)^n}{2n} z^n \Big)^m =: \sum_{k=0}^{\infty} c_k z^k, \quad 0 \le |z| < 1.$$

Where the coefficients c_k have explicit form

$$c_k = \sum_{l=1}^{\infty} \sum_{n_1 + \cdots + n_l = k} \frac{(-1)^l}{l!} \prod_{j=1}^{l} \frac{(-1)^{n_j}}{2n_j}.$$

$\square$

Problem 6.8. Find the power series expansion of

$$f(z) = \frac{1}{(z+1)(z+2)}$$

about $z = 0$, and find its radius of convergence.

Solution: By partial fraction we have

$$f(z) = \frac{1}{z+1} - \frac{1}{z+2}.$$

Since we know $1/(1-z)$ has power series expansion about 0 being $\sum_{n=0}^{\infty} z^n$. Then $1/(z+1) = 1/(1-(-z))$ has power series expansion about 0 being $\sum_{n=0}^{\infty} (-1)^n z^n$ with radius of convergence being 1. Similarly $1/(z+2) = (1/2)(1/(1-(-z/2)))$ has power series expansion about 0

$$\frac{1}{2} \sum_{n=0}^{\infty} \Big(-\frac{1}{2} \Big)^n z^n, \quad 0 \le |z| < 2.$$

Therefore $f(z)$ has power series expansion about 0

$$f(z) = \left(\sum_{n=0}^{\infty} (-1)^n z^n \right) - \left(\sum_{n=0}^{\infty} \Big(-\frac{1}{2^{n+1}} \Big) z^n \right) = \sum_{n=0}^{\infty} \Big[(-1)^n + \frac{1}{2^{n+1}} \Big] z^n, \quad 0 \le |z| < 1.$$

The radius of convergence of the series is indeed 1 because $f(z)$ is analytic on $\mathbb{C} \setminus \{-1, -2\}$, and the radius of the largest disk centered at 0 on which f is analytic is 1. $\square$