Math 427 Homework #3 Solutions

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October 22, 2018

Lemma 0.1. *Let f and g be complex-valued functions defined on a domain E, and $a \in \overline{E}$ that is not an isolated point of $\overline{E}$. Suppose* $\lim_{z\to a} f(z) = L$ *and* $\lim_{z\to a} g(z) = L'$*, then* $\lim_{z\to a} f(z) + g(z) = L + L'$.

Proof. Let $\epsilon > 0$ be given, by definition there exists $\delta_1, \delta_2 > 0$ such that $|f(z) - L| < \epsilon/2$ whenever $|z - a| < \delta_1$ and $|g(z) - L'| < \epsilon/2$ whenever $|z - a| < \delta_2$. Define $\delta := \min\{\delta_1, \delta_2\}$, then if $|z - a| < \delta$, we may use triangle inequality:

$$|(f(z) + g(z)) - (L + L')| \leq |f(z) - L| + |g(z) - L'| < \epsilon/2 + \epsilon/2 = \epsilon.$$

This proves $\lim_{z\to a} f(z) + g(z) = L + L'$. □

Problem 3.1. Let $E \subset \mathbb{C}$ be an open set and $f : E \to \mathbb{C}$ be a function. If f is differentiable at a point $z \in E$, show that f is also continuous at z.

Proof 1. (ϵ-δ proof) Since f is differentiable at z, by definition for any $\epsilon > 0$, we can find $\delta > 0$ such that

$$\left|\frac{f(w) - f(z)}{w - z} - f'(z)\right| < \epsilon \quad \text{whenever} \quad |w - z| < \delta.$$

Since $|w - z| \geq 0$, multiplying both sides of the inequality $|(f(w) - f(z))/(w - z) - f'(z)| < \epsilon$ by $|w - z|$ gives us

$$|w - z| \left|\frac{f(w) - f(z)}{w - z} - f'(z)\right| < |w - z| \, \epsilon.$$

Since $|\alpha\beta| = |\alpha| \, |\beta|$ for any $\alpha, \beta \in \mathbb{C}$, the inequality above is equivalent to

$$|f(w) - f(z) - f'(z)(w - z)| < |w - z| \, \epsilon.$$

By the triangle inequality, we have

$$|f(w) - f(z)| - |w - z| \, |f'(z)| \leq |f(w) - f(z) - f'(z)(w - z)| \, .$$

Therefore together we have

$$|f(w) - f(z)| \leq |w - z| \, |f'(z)| + |f(w) - f(z) - f'(z)(w - z)| < |w - z| \, (|f'(z)| + \epsilon).$$

Now let $\epsilon' > 0$ be given, we want to show there exists $\delta' > 0$ such that $|f(w) - f(z)| < \epsilon'$ whenever $|w - z| < \delta'$. By the argument above, if we put $\epsilon := \min\{\epsilon'/2 \, |f'(z)| \, , \sqrt{\epsilon'/2}\}$, then we may pick $\delta > 0$ such that

$$|f(w) - f(z)| < |w - z| \, (|f'(z)| + \epsilon) \quad \text{whenever} \quad |w - z| < \delta.$$

We then set $\delta' := \min\{\delta, \epsilon\}$. Then if $|w - z| < \delta'$,

$$|f(w) - f(z)| < \delta'(|f'(z)| + \epsilon) \leq \epsilon \, |f'(z)| + \epsilon^2 \leq \epsilon'/2 + \epsilon'/2 = \epsilon'.$$

This proves f is continuous at z. □

Proof 2. (using the other characterization of differentiability) Since f is differentiable at z, there exists $f'(z) \in \mathbb{C}$ and a function $\epsilon(\lambda)$ such that we may write

$$f(z+\lambda) - f(z) = f'(z)\lambda + \epsilon(\lambda),$$

and $\lim_{\lambda\to 0} \epsilon(\lambda)/\lambda = 0$. Therefore,

$$\lim_{\lambda\to 0} |f(z+\lambda) - f(z)| = \lim_{\lambda\to 0} |\lambda| \left| f'(z) + \frac{\epsilon(\lambda)}{\lambda} \right|.$$

Since $\lim_{\lambda\to 0} \epsilon(\lambda)/\lambda = 0$, by Lemma 0.1, $\lim_{\lambda\to 0} f'(z) + \epsilon(\lambda)/\lambda = f'(z)$. Therefore $\lim_{\lambda\to 0} |f'(z) + \epsilon(\lambda)/\lambda| = |f'(z)|$. Since in addition $\lim_{\lambda\to 0} |\lambda| = 0$, it follows

$$\lim_{\lambda\to 0} |\lambda| \left| f'(z) + \frac{\epsilon(\lambda)}{\lambda} \right| = 0,$$

which proves f is continuous at z. □

Problem 3.2. Prove the product formula: if f and g are complex functions that are differentiable at z, then fg is differentiable at z with derivative $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$.

Proof. Since f and g are differentiable at z, there exists $f'(z), g'(z) \in \mathbb{C}$ such that

$$\lim_{w\to z} \frac{f(w) - f(z)}{w - z} = f'(z), \quad \lim_{w\to z} \frac{g(w) - g(z)}{w - z} = g'(z).$$

By Problem 1, f is also continuous at z, we know $\lim_{w\to z} f(w) = f(z)$. Together with the fact $\lim_{w\to z} g(z) = g(z)$ and Problem 10 in homework 1, we know

$$\lim_{w\to z} \frac{(f(w) - f(z))g(z)}{w - z} = f'(z)g(z), \quad \lim_{w\to z} \frac{(g(w) - g(z))f(w)}{w - z} = g'(z)f(z).$$

We may apply Lemma 0.1 to get

$$\lim_{w\to z} \frac{(f(w) - f(z))g(z) + (g(w) - g(z))f(w)}{w - z} = f'(z)g(z) + g'(z)f(z).$$

The limit then simplifies to

$$\lim_{w\to z} \frac{f(w)g(w) - f(z)g(z)}{w - z} = f'(z)g(z) + g'(z)f(z),$$

which by definition means the function fg is differentiable at z with derivative $(fg)'(z) = f'(z)g(z) + g'(z)f(z)$. □

Problem 3.3. Use Problem 3.2 and induction to show that

$$\frac{d(z^n)}{dz} = nz^{n-1}$$

Proof. We first induction on $n \geq 0$.

1. Suppose $n = 0$, then by direct computation we have the derivative of $f(z) = z^0$ at z is $\lim_{w\to z}(1 - 1)/(w - z) = \lim_{w\to z} 0 = 0$, which equals the right hand side $nz^{n-1} = 0 \cdot z^{-1} = 0$.

   Remark: One could also do the base case with $n = 1$. In this case, by direct computation we have the derivative of $f(z) = z$ at z is $\lim_{w\to z}(w - z)/(w - z) = \lim_{w\to z} 1 = 1$, which equals the right hand side $1 \cdot z^0 = 1$.

2. Suppose $n > 1$, then by the product formula and the case $n = 1$ we have
$$\frac{d(z^n)}{dz} = \frac{d(z \cdot z^{n-1})}{dz} = \frac{d(z)}{dz} \cdot z^{n-1} + z \cdot \frac{d(z^{n-1})}{dz} = z^{n-1} + z \cdot \frac{d(z^{n-1})}{dz}.$$
The inductive hypothesis tells us $d(z^{n-1})/dz = (n-1)z^{n-2}$, we may simplify the equation above to $z^{n-1} + (n-1)z \cdot z^{n-2} = nz^{n-1}$. This finishes the induction.

Now let n be a positive integer, since $z^{-n} = 1/z^n$. We use Theorem 2.2.6 to conclude that z^{-n} is differentiable on $\mathbb{C} \setminus \{0\}$, with derivative
$$\frac{d(z^{-n})}{dz} = -\frac{nz^{n-1}}{z^{2n}} = (-n)z^{-n-1}.$$
This proves the statement is true for all integer n, when the derivative exists. $\square$

Problem 3.4. Taylor 2.2.8.

Solution: First recall the log function is only defined on the punctured complex plane $\mathbb{C} \setminus \{0\}$, hence the function $\log(z)/z$ is only defined on $\mathbb{C} \setminus \{0\}$. We would like to analyze if it's differentiable anywhere on the set $\mathbb{C} \setminus \{0\}$. Observe the function $f(z) = z$ is nonzero and differentiable on the punctured plane $\mathbb{C} \setminus \{0\}$, therefore if $\log(z)$ is differentiable at z_0, so is the function $\log(z)/z$ by Theorem 2.2.6(c). By Example 2.2.11 in the book, we know the function $\log(z)$ is differentiable everywhere except on its cut line, that is the line of negative reals. Therefore we conclude the function $\log(z)/z$ is differentiable on $\mathbb{C} \setminus (-\infty, 0]$ (same in the book, we use the notation $(-\infty, 0]$ to denote the set $\{x + iy \in \mathbb{C} : y = 0, x \leq 0\}$).

From Example 2.2.11 we also know the derivative of $\log(z)$ at $z \in \mathbb{C} \setminus (-\infty, 0]$ is $1/z$. By Theorem 2.2.6 and Problem 3.3 we conclude the derivative of $\log(z)/z$ at $z \in \mathbb{C} \setminus (-\infty, 0]$ is
$$(\log(z) \cdot z^{-1})' = \log(z) \cdot \frac{-1}{z^2} + \frac{1}{z^2} = \frac{1 - \log(z)}{z^2}.$$
$\square$

Problem 3.5. Taylor 2.2.11.

Solution: Let $f(z) = u(x, y) + iv(x, y)$ where $z \in x + iy$ be a real function defined and analytic on $\mathbb{C}$. Then $v(x, y) = 0$ for all $z = x + iy \in \mathbb{C}$. Then by Cauchy-Riemann equations we conclude for all $z = x + iy \in \mathbb{C}$,
$$u_x(x, y) = v_y(x, y) = 0, \quad u_y(x, y) = -v_x(x, y) = 0.$$
We know from real analysis that a real-valued differentiable function $u : \mathbb{R}^2 \to \mathbb{R}$ satisfying above must be constant. Therefore any real-valued function that is analytic on $\mathbb{C}$ is constant. $\square$

Problem 3.6. Taylor 2.2.12.

Proof. Let $f = u + iv$ be a function defined on some domain E that is differentiable at $z_0 = r_0 e^{i\theta_0} \in E$. Using the change of variable $x = r\cos\theta$, $y = r\sin\theta$, we may define functions $\tilde{u}$ and $\tilde{v}$ defined on those points (r, θ) with $re^{i\theta} \in E$, such that
$$\tilde{u}(r, \theta) = u(r\cos\theta, r\sin\theta), \quad \tilde{v}(r, \theta) = v(r\cos\theta, r\sin\theta).$$
Since the maps $\phi(r, \theta) = r\cos\theta$ and $\psi(r, \theta) = r\sin\theta$ are differentiable at all those points (r, θ) such that $re^{i\theta} \in E$. We may apply the chain rule to $\tilde{u}$ and $\tilde{v}$ and obtain (the partials of u and v are all evaluated at $(r_0\cos\theta_0, r_0\sin\theta_0)$ below, and the partials of $\tilde{v}$ and $\tilde{v}$ are evaluated at (r_0, θ_0)):
$$\begin{bmatrix}\tilde{u}_r & \tilde{u}_\theta\end{bmatrix} = \begin{bmatrix}u_x & u_y\end{bmatrix}\begin{bmatrix}\cos\theta_0 & -r_0\sin\theta_0 \\ \sin\theta_0 & r_0\cos\theta_0\end{bmatrix} = \begin{bmatrix}u_x\cos\theta_0 + u_y\sin\theta_0 & r_0(-u_x\sin\theta_0 + u_y\cos\theta_0)\end{bmatrix}$$
$$\begin{bmatrix}\tilde{v}_r & \tilde{v}_\theta\end{bmatrix} = \begin{bmatrix}v_x & v_y\end{bmatrix}\begin{bmatrix}\cos\theta_0 & -r_0\sin\theta_0 \\ \sin\theta_0 & r_0\cos\theta_0\end{bmatrix} = \begin{bmatrix}v_x\cos\theta_0 + v_y\sin\theta_0 & r_0(-v_x\sin\theta_0 + v_y\cos\theta_0)\end{bmatrix}.$$

Since $f = u+iv$ is differentiable at z_0, the Cauchy-Riemann equations is satisfied at $z_0 = r_0\cos\theta_0 + ir_0\sin\theta_0$:

$$u_x = v_y, \quad u_y = -v_x.$$

Substitute these into the four relations above, we obtain

$$r_0\tilde{u}_r(r_0,\theta_0) = r_0u_x\cos\theta_0 + r_0u_y\sin\theta_0 = r_0v_y\cos\theta_0 - r_0u_x\sin\theta_0 = \tilde{v}_\theta,$$

and

$$\tilde{u}_\theta(r_0,\theta_0) = r_0(-v_y\sin\theta_0 - v_x\cos\theta_0) = (-r_0)\tilde{v}_r.$$

Which is the desired result if $r_0 \neq 0$. □

Problem 3.7. Taylor 2.2.13.

Proof. We consider the log function in the branch $I = (a, a+2\pi]$. After the change of coordinate $x = r\cos\theta$ and $y = r\sin\theta$, we have $\log_I(r,\theta) = u(r,\theta) + iv(r,\theta) = \log(r) + i\theta$. Observe u and v are differentiable for $r \neq 0$ and $\theta \in (a, a+2\pi)$. We would like to prove the Cauchy-Riemann equations for polar coordinates holds for $\log(r,\theta)$ for $r \neq 0$ and $\theta \in (a, a+2\pi)$. Direct computation of the partial derivatives of u and v yields

$$u_r(r,\theta) = 1/r, \quad u_\theta(r,\theta) = 0$$
$$v_r(r,\theta) = 0, \quad v_\theta(r,\theta) = 1.$$

This tells us the Cauchy-Riemann equations are satisfied for all $z = re^{i\theta}$ where $\theta \in (a, a+2\pi)$ and $r \neq 0$. Therefore the usual Cauchy-Riemann equations are satisfied for $\log(z)$ on the complex plane besides the cut line and the origin. We conclude the log function is analytic on the complex plane with its cut line and the origin removed. □

Problem 3.8. Taylor 2.2.15.

Proof. Write f as $f(z) = u(x,y) + iv(x,y)$ where $z = x+iy$, then $g(z) = \tilde{u}(x,y) + i\tilde{v}(x,y) = u(x,-y) - iv(x,-y)$ where $z = x+iy$. Let $z_0 = x_0 + iy_0 \in U$ be given, we would like to show g is differentiable at $\overline{z_0}$. We first obverse $\tilde{u}(x_0,-y_0) = u(x_0,y_0)$ and $\tilde{v}(x_0,-y_0) = -\tilde{v}(x_0,y_0)$, therefore $\tilde{u}$ and $\tilde{v}$ are both differentiable at $\overline{z_0}$. Computing the partial derivatives of $\tilde{u}$ and $\tilde{v}$ using chain rule yields,

$$\tilde{u}_x(x_0,-y_0) = \frac{\partial u(x,-y)}{\partial x}(x_0,-y_0) = u_x(x_0,y_0)$$
$$\tilde{u}_y(x_0,-y_0) = \frac{\partial u(x,-y)}{\partial y}(x_0,-y_0) = -u_y(x_0,y_0)$$
$$\tilde{v}_x(x_0,-y_0) = \frac{\partial(-v(x,-y))}{\partial x}(x_0,-y_0) = -v_x(x_0,y_0)$$
$$\tilde{v}_y(x_0,-y_0) = \frac{\partial(-v(x,-y))}{\partial y}(x_0,-y_0) = v_y(x_0,y_0).$$

Since u and v satisfies the Cauchy-Riemann equations at z_0, we also have

$$u_x(x_0,y_0) = v_y(x_0,y_0), \quad u_y(x_0,y_0) = -v_x(x_0,y_0).$$

Putting all these relations together, we have

$$\tilde{u}_x(x_0,-y_0) = u_x(x_0,y_0) = v_y(x_0,y_0) = \tilde{v}_y(x_0,-y_0)$$
$$\tilde{u}_y(x_0,-y_0) = -u_y(x_0,y_0) = v_x(x_0,y_0) = -\tilde{v}_x(x_0,-y_0)$$

as we wished. By Theorem 2.2.9, we may conclude g is differentiable at $\overline{z_0}$. □

Problem 3.9. For each of the following functions of z, express the function in the form $u(x,y) + iv(x,y)$ where $z = x+iy$:

(a) $z^3 + \overline{z}^3$

(b) z^2e^z

(c) $\cos(z)$

Solution:

(a) By binomial expansion we have

$$\begin{aligned} z^3 + \overline{z}^3 &= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 + x^3 + 3x^2(-iy) + 3x(-iy)^2 + (-iy)^3 \\ &= x^3 + 3x^2(iy) - 3xy^2 - iy^3 + x^3 + 3x^2(-iy) - 3xy^2 + iy^3 = 2x^3 - 6xy^2. \end{aligned}$$

(b)

$$\begin{aligned} z^2 e^z &= (x+iy)^2 e^{x+iy} = (x^2 - y^2 + 2ixy)e^x(\cos y + i \sin y) \\ &= e^x[(x^2-y^2)\cos y - 2xy \sin y] + ie^x[2xy\cos y + (x^2-y^2)\sin y] \end{aligned}$$

(c)

$$\begin{aligned} \cos(z) &= \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{1}{2}(e^{-y+ix} + e^{y-ix}) \\ &= \frac{1}{2}(e^{-y}\cos x + e^y \cos x) + i\frac{1}{2}(e^{-y}\sin x - e^y \sin x) \end{aligned}$$

□

Problem 3.10. For each Part (a)(c) of Problem 3.9, use the CauchyRiemann equations to determine if the function is analytic.

Proof.

1. Observe this function is real-valued, and not constant, hence is nowhere analytic by Problem 5.

2. This function is defined everywhere on $\mathbb{C}$, let $z = x + iy$ be given, by direct computation,

$$\begin{aligned} u_x &= e^x[(x^2-y^2)\cos y - 2xy\sin y + 2x\cos y - 2y\sin y] \\ u_y &= e^x[-y\cos y - (x^2-y^2)\sin y - 2x\sin y - 2xy\cos y] \\ v_x &= e^x[2xy\cos y + (x^2-y^2)\sin y + 2y\cos y + 2x\sin y] \\ v_y &= e^x[2x\cos y - 2xy\sin y - 2y\sin y + (x^2-y^2)\cos y]. \end{aligned}$$

   It's easy to see Cauchy-Riemann equations are satisfied, hence z^2e^z is analytic on the entire complex plane.

3. This function is defined everywhere on $\mathbb{C}$ since both e^{iz} and e^{-iz} are. Let $z = x + iy$ be given, by direct computation,

$$\begin{aligned} u_x &= \frac{1}{2}(-e^{-y}\sin x - e^y \sin x) \\ u_y &= \frac{1}{2}(-e^{-y}\cos x + e^y\cos x) \\ v_x &= \frac{1}{2}(e^{-y}\cos x - e^y\cos x) \\ v_y &= \frac{1}{2}(-e^{-y}\sin x - e^y\sin x). \end{aligned}$$

   Observe Cauchy-Riemann equations are satisfied, hence $\cos(z)$ is analytic on the entire complex plane.

□