Math 404A: Introduction to Modern Algebra (Spring 2021)
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Homework 8
Due: Friday, May 21

Problem 8.1. Let ω be a primitive 7th root of unity. Show that $\mathbb{Q} \subseteq \mathbb{Q}(\omega)$ is a Galois extension and compute its Galois group.

Problem 8.2.

(a) Show that $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{5}, i)$ is a Galois extension and compute its Galois group.
(b) Give the complete correspondence between intermediate field extensions $\mathbb{Q} \subseteq E \subseteq \mathbb{Q}(\sqrt{5}, i)$ and subgroups $H \subseteq \mathrm{Gal}(\mathbb{Q}(\sqrt{5}, i)/\mathbb{Q})$.

Problem 8.3. Let L be the splitting field of $f(x) = x^3 - 2$ over $\mathbb{Q}$.

(a) Compute the Galois group $\mathrm{Gal}(L/\mathbb{Q})$.
(b) Give the complete correspondence between intermediate field extensions $\mathbb{Q} \subseteq E \subseteq L$ and subgroups $H \subseteq \mathrm{Gal}(L/\mathbb{Q})$.

Problem 8.4. Let L be the splitting field of $f(x) = x^4 - 2$ over $\mathbb{Q}$.

(a) Determine $\mathrm{Gal}(L/\mathbb{Q})$.
(b) Give the complete correspondence between intermediate field extensions $\mathbb{Q} \subseteq E \subseteq L$ and subgroups $H \subseteq \mathrm{Gal}(L/\mathbb{Q})$.

Problem 8.5. Let K be a field with $\mathrm{char}(K) \neq 3$. Suppose that $f(x) = x^3 - 3x + 1 \in K[x]$ is irreducible. Let $L = K(\alpha)$ where α is a root of f. Prove that f splits over L, and deduce that $K \subseteq L$ is a Galois extension with Galois group $\mathbb{Z}/3\mathbb{Z}$.

Hint: Factor f over L as $(x - \alpha)g$, and solve for the roots of g using the quadratic formula. Use the fact that $12 - 3\alpha^2 = (-4 + \alpha + 2\alpha^2)^2$ is a perfect square in $K(\alpha)$.