Math 404A: Introduction to Modern Algebra (Spring 2021)
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Homework 3
Due: Friday, April 16

Problem 3.1. Let $f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$ be a polynomial with roots $\alpha_1, \ldots, \alpha_n$. The *discriminant* of $f(x)$ is defined as

$$\Delta = \prod_{i<j} (\alpha_i - \alpha_j)^2.$$

(a) Prove that this is a symmetric function.
(b) If $f(x) = x^3 + a_2 x + a_3$, express the discriminant Δ in terms of coefficients a_2, a_3.

Problem 3.2.
(a) Show that $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ are linearly independent over $\mathbb{Q}$.
(b) Show that $\{1, \sqrt[3]{2}, \sqrt[3]{4}\}$ are linearly independent over $\mathbb{Q}(i)$.

Problem 3.3. Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be an irreducible polynomial in $F[x]$, where F is a field. Let L be the extension field $F[x]/(f)$. Show that the dimension $\dim_F L$ of L as a vector space over F is equal to n.

Problem 3.4 (*Eisenstein's criterion with a twist*)**.**
(a) Let a be any integer. Prove that a polynomial $f(x) \in \mathbb{Z}[x]$ is irreducible iff $f(x+a) \in \mathbb{Z}[x]$ is irreducible.
(b) Use this trick to prove that $x^3 - 3x^2 + 9x - 5$ is irreducible.
(c) Use this trick to prove that, for any prime p, the polynomial $x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible.

Problem 3.5. Consider the field extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$.
(a) What is the degree $[\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}) : \mathbb{Q}]$ of this field extension?
(b) Prove that this is a simple field extension; that is, find an element α such that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$.

Challenge Problem 3.6. (10 points) Try to use a similar method to Lagrange's solution of the quartic to derive a solution to the cubic equation.

Hint: Search for a polynomial $f(\alpha_1, \alpha_2, \alpha_3)$ whose orbit under S_3 consists of only 2 elements (or in other words whose stabilizer subgroup has 3 elements).