Math 404 HW 2 Solutions

Problem 2.1

For each of the following actions of a group G on a set X, determine the orbit $Gx = \{g \cdot x | g \in G\}$ and stabilizer $G_x = \{g \in G | g \cdot x = x\}$ for each element $x \in X$:

(a) (a) $G = S_3$ acting on $X = S_3$ via conjugation: for $g \in G$ and $x \in X$,the action is defined by $g \cdot x = gxg^{-1}$.

(b) (b) $G = (1), (12), (345), (354), (12)(345), (12)(354) \subset S_6$ acting on $X = \{1, 2, 3, 4, 5, 6\}$ via permutation.

(a) We are looking at the action of S_3 on itself by conjugation. Perhaps the most straightforward way to find Gx and G_x are by direct computation. Recall that we are using the convention of composition for calculating the product of cycles, so we are going right to left.

Geometric interpretation: The stabilizer is the set of all elements which do not "move" a point x away from itself under the action. It lives in the domain G. The orbit of x is the set of all possible images of x under the action of G, so it lives in the target space.

First, note that $S_3 = \{(1), (12), (13), (23), (123), (132)\}$. We will first compute the stabilizer of every element;, then, let $x = (12)$.

$$(1)(12)(1) = (12)$$

$$(12)(12)(12) = (12)$$
$$(13)(12)(13) = (23)(23)(12)(23) = (13)$$
$$(123)(12)(132) = (23)(132)(12)(123) = (23)$$
$$(132)(12)(123) = (23)$$

We can see directly from this that $G_{(12)} = \{(1), (12)\}$. By replicating this calculation for each of the elements in S_3, we get that:

$$G_{(13)} = \{(1), (13)\}$$

$$G_{(23)} = \{(1), (23)\}$$
$$G_{(123)} = \{(1), (123), (132)\}$$
$$G_{(132)} = \{(1), (123), (132)\}$$
$$G_{(1)} = \{(1)\}$$

Next, we compute the orbit of each element. As a note, because conjugation is an automorphism, it should preserve cycle type. We also have the given hint that $|G| = |Gx||G_x|$. Both of these can help you to check your work.

By using our earlier computation, we can simply collect all of the products of the conjugation $g(12)g^{-1}$, because those are the image of (12) under the action of G. Then,

$$G(12) = \{(12), (13), (23)\}$$
$$G(13) = \{(12), (23), (13)\}$$
$$G(23) = \{(12), (23), (13)\}$$
$$G(123) = \{(123), (132))\}$$
$$G(132) = \{(123), (132)\}$$
$$G(1) = \{(12), (23), (13), (123), (132)\} = S_3$$

We can see that the orbit set only includes cycles of the same type, like we mentioned earlier.

(b) Let $G = (1), (12), (345), (354), (12)(345), (12)(354) \subset S_6$ act on $X = 1, 2, 3, 4, 5, 6$ via permutation. What does it mean to act by permutation? In previous quarters, we learned that the symmetric group S_n is the collection of bijections between a set X and itself, where $|X| = n$. Symbolically, for $\sigma \in S_n$, $\sigma \cdot x = \sigma(x)$, and the image of x is whatever element σ sends x to. As an example, if $x = 4$ and $\sigma = (1423)$, then $\sigma(x) = 2$.

We can find the orbits and stabilizers again by direct calculation of where each element of G sends each element of $X = S_6$.

For $x = 1$, $G(1) = \{1, 2\}$, and $G_{(1)} = \{(1), (345), (354)\}$.

For $x = 2$, $G(2) = \{2, 1\}$, and $G_{(2)} = \{(1), (345), (354)\}$. For $x = 3$, $G(3) = \{3, 4, 5\}$, and $G_{(3)} = \{(1), (12)\}$. For $x = 4$, $G(4) = \{4, 5, 3\}$, and $G_{(4)} = \{(1), (12)\}$. For $x = 5$, $G(5) = \{5, 3, 4\}$, and $G_{(5)} = \{(1), (12)\}$. For $x = 6$, $G(6) = \{6\}$, and $G_{(6)} = G$. This is because 6 is not an element of any permutation on G.

Problem 2.2

(a) Express $f = x_1^4 + x_2^4 + x_3^4 + x_4^4$ as a polynomial combination of the elementary symmetric polynomials $s_1, \ldots, s_4$

(b) Let $\alpha_1, \ldots \alpha_4$ denote the roots of the polynomial

$$g = x^4 + x^3 + 2x^2 + 3x + 5.$$

Determine the number $\alpha_1^4 + \alpha_2^4 + \alpha_3^4 + \alpha_4^4$

(a) By degree considerations, we wish to find rational numbers A through E so that

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 = As_1^4 + Bs_1^2 s_2 + Cs_2^2 + Ds_1 s_3 + Es_4 \qquad (1)$$

For the rest of this problem, we use the fact that the fundamental theorem on symmetric functions guarantees that there exist unique values for A through E that make this equality hold. We simply figure out enough constraints to determine what the values must be, and then the fundamental theorem guarantees we must have the desired equality at the end. Or if values for the coefficients have been prescribed, the desired equality can always be checked quickly using mathematica or some equivalent piece of software. We'll present one solution, and a sketch of another.

0.1 Equating Coefficients

By investigating the coefficient of x_1^4 on both sides of (1), we can immediately deduce that $\boxed{A = 1}$.

Next we investigate the coefficient of $x_1^3 x_2$. On the left hand side of (1), we have zero. For the right hand side, note that the $s_2^2, s_1 s_3$, and s_4 terms all contribute zero to this monomial. For the $s_1^2 s_2$ term, if were to imagine expanding it out, we would get a sum consisting of products of every possible way of choosing two monomials from s_1, and one from s_2. To get $x_1^3x_2$, the only monomial from s_2 we can pick is x_1x_2, and this forces picking the x_1 term for both choices from s_1. For the s_1^4 term, we need to pick the x_1 term three times and the x_2 term once, so our choice is just which of the 4 copies of s_1 we pick the x_2 from (the multinomial theorem is good for this, if you want to look it up. That is, we have a coefficient of $\binom{4}{3,1,0,0}$ here).

That is to say, we get an equality

$$0 = 4A + B,$$

from equating the coefficients of $x_1^3x_2$ on both sides. Since $A = 1$, we get $\boxed{B = -4}$.

We can continue in this way analyzing the $x_1^2x_2^2$, $x_1^2x_2x_3$, and finally $x_1x_2x_3x_4$ terms, solving for the remaining coefficients B, C, and D one at a time. This choice of terms to analyze is motivated by moving up in the lexicographic order, and increasing the number of variables present one at a time. Admittedly, the combinatorics gets more complicated as we go. At the end, we obtain

$$\boxed{f = s_1^4 - 4s_1^2s_2 + 2s_2^2 + 4s_1s_3 - 4s_4,}$$

which can be checked by plugging both sides into Mathematica or some equivalent piece of software.

□

0.2 Successive evaluations

We are trying to solve

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 = As_1^4 + Bs_1^2s_2 + Cs_2^2 + Ds_1s_3 + Es_4 \tag{2}$$

This is an equality of polynomials. So if we had values A through E which yielded this equality, then the equality would persist after evaluating the x_i at some rational numbers. If we evaluate at 5 different sets of rational numbers for the x_i, then we'll get 5 different *linear* equations of rational numbers in the five unknowns A through E. If we chose these 5 sets of values generically enough, then linear algebra says that this system of equations has a unique solution, and we can find that solution by any standard method of solving linear systems of equations (for example, row reduction).

For example, if we evaluate both sides at $(x_1, x_2, x_3, x_4) = (1, 0, 0, 0)$ we get $1 = A$, as s_2, s_3 and s_4 all vanish when evaluated at $(1, 0, 0, 0)$. Evaluating at 4 more values (where some of these zeros become nonzero), we can get the 4 remaining unknowns. A nice next choice might be $(1, 1, 0, 0)$ as s_3 and s_4 vanish there.

$\square$

(b) We recall that if we write

$$\begin{aligned} g &= x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \\ &= x_4 + x_3 + 2x^2 + 3x + 5 \end{aligned}$$

with roots $\alpha_1, \ldots, \alpha_4$, then we have

$$\begin{aligned} s_1(\alpha) &= -a_3 = -1 \\ s_2(\alpha) &= a_2 = 2 \\ s_3(\alpha) &= -a_1 = -3 \\ s_4(\alpha) &= a_0 = 5. \end{aligned}$$

Using the result of part (a), we obtain

$$\begin{aligned} \alpha_1^4 + \alpha_2^4 + \alpha_3^4 + \alpha_4^4 &= f(\alpha) \\ &= s_1^4(\alpha) - 4s_1^2s_2(\alpha) + 2s_2^2(\alpha) + 4s_1s_3(\alpha) - 4s_4(\alpha) \\ &= (-1)^4 - 4(-1)^2(2) + 2(2)^2 + 4(-1)(-3) - 4(5) \\ &= -7 \end{aligned}$$

$\square$

Problem 2.4

Let $f \in k[\alpha_1, \ldots, \alpha_n]$ be any polynomial. Let $f_1, \ldots, f_k$ be the orbit of f under the action of S_n.

(a) Show that $f_1 + \cdots + f_k$ is symmetric.

(b) If $s(x_1, \ldots, x_k)$ is a symmetric polynomial in $x_1, \ldots, x_k$, show that $s(f_1, \ldots, f_k)$ is a symmetric polynomial in $\alpha_1, \ldots, \alpha_n$

Throughout this problem, we will fix an element $\sigma \in S_n$. First is a lemma we will use in both parts of the problem.

Lemma 1. *There is an element $\tau \in S_k$ so that*

$$\sigma \cdot f_i = f_{\tau(i)}. \tag{3}$$

Proof. Since the f_i are all in the orbit of f, there exist $\rho_i \in S_n$ so that $\rho_i \cdot f = f_i$ (these ρ_i are not unique unless $k = n!$). Then we have

$$\begin{aligned} \sigma \cdot f_i &= \sigma \cdot (\rho_i \cdot f) \\ &= (\sigma\rho_i) \cdot f, \end{aligned}$$

by associativity of the group action. This shows that $\sigma \cdot f_i$ is in the orbit of f, so that we at least get some function $\tau : \{1, \ldots, k\} \to \{1, \ldots k\}$ so that (3) holds. We can use the same reasoning applied to the action of σ^{-1} to build an inverse for τ, so $\tau \in S_k$, as desired. □

Exercise Say you have a group G acting on a set X. Can you make a similar statement to this lemma about the action of G on the orbits? Can you prove it?

(a) Note that

$$\begin{aligned} \sigma \cdot (f_1 + \cdots + f_k) &= \sigma(f_1) + \cdots + \sigma(f_k) \\ &= f_{\tau(1)} + \cdots + f_{\tau(k)} \\ &= f_1 + \cdots + f_k. \end{aligned}$$

Since $\sigma \in S_n$ was arbitrary, the result holds.

□

(b) We'll present two solutions. One where we just show the result directly, and one where we first prove it for *elementary* symmetric polynomials in k variables, then deduce it for arbitrary symmetric polynomials.

Direct proof

I claim that we have a commutative diagram

$$\begin{array}{ccc} k[x_1,\dots,x_k] & \xrightarrow{\tau} & k[x_1,\dots,x_k] \\ \downarrow{\scriptstyle (f_1,\dots,f_k)} & & \downarrow{\scriptstyle (f_1,\dots,f_k)} \\ k[\alpha_1,\dots,\alpha_n] & \xrightarrow{\sigma} & k[\alpha_1,\dots,\alpha_n] \end{array} \qquad (4)$$

with the vertical arrows the homomorphisms defined by $x_i \mapsto f_i$. Commutativity of this diagram means that for any $g \in k[x_1,\dots,x_n]$ that $\sigma \cdot g(f_1,\dots,f_k) = (\tau \cdot g)(f_1,\dots,f_k)$. This equality follows from the equality (3) and the fact that homomorphisms out of $k[x_1,\dots,x_n]$ are uniquely determined by what happens to the field k and the images of the elements x_i.

The result of this problem will follow from investigating the image of $s \in k[x_1,\dots,x_k]$. Going along the bottom left composite, s goes to $\sigma \cdot s(f_1,\dots,f_k)$. Going along the top right compsite, s goes to $(\tau \cdot s)(f_1,\dots,f_k)$. But since s was symmetric by assumption, we have $\tau \cdot s = s$. Combining this, we have that $\sigma \cdot s(f_1,\dots f_k) = s(f_1,\dots,f_k)$, as desired.

□

Elementary first

First we prove a lemma:

Lemma 2. *Let $f \in k[x_1,\dots,x_n]$ with orbit $\{f_1,\dots,f_k\}$ under the standard S_n action. Let $s \in k[\alpha_1,\dots,\alpha_k]$ be an* ***elementary*** *symmetric polynomial in k variables. Then $s(f_1,\dots,f_k)$ is a symmetric polynomial in n variables.*

Proof. Let $\sigma \in S_n$ be arbitrary. For any integer $1 \leq \ell \leq k$ let $\binom{[k]}{\ell}$ be the collection of all m-element subsets of $\{1, \ldots, k\}$. By definition of elementary symmetric polynomials, we have that there is some integer $1 \leq \ell \leq k$ so that

$$s = \sum_{D \in \binom{[k]}{\ell}} \prod_{i \in D} \alpha_i.$$

Then we have that

$$\begin{aligned} s(f_1, \ldots, f_k) &= \sum_{D \in \binom{[k]}{\ell}} \prod_{i \in D} f_i, \text{ and} \\ \sigma \cdot s(f_1, \ldots, f_k) &= \sum_{D \in \binom{[k]}{\ell}} \prod_{i \in D} \sigma \cdot f_i \\ &= \sum_{D \in \binom{[k]}{\ell}} \prod_{i \in D} f_{\tau(i)}, \end{aligned}$$

with $\tau \in S_k$ the permutation given by lemma (1). These are the same sum and products written in a different order, as $i \in D$ if and only if $\tau(i) \in \tau(D)$, and τ permutes the ℓ-element subsets of $[k]$, that is, τ permutes $\binom{[k]}{\ell}$. Since addition and multiplication are commutative, we have that $\sigma \cdot s(f_1, \ldots, f_k) = s(f_1, \ldots, f_k)$, as desired.

□

Now for the main result, we want to use the fundamental theorem on symmetric functions, that any symmetric function $s \in k[\alpha_1, \ldots, \alpha_k]$ is a polynomial in the elementary symmetric functions. One way to do this is as follows.

Let $R \subset k[\alpha_1, \ldots, \alpha_k]$ be the set of all polynomials $r \in k[\alpha_1, \ldots, \alpha_k]$ so that $r(f_1 \ldots, f_k)$ is symmetric (this is a kind of "stabilizer"). We have shown that all the elementary symmetric polynomials $s_i \in R$. It's also immediate to see that the base field $k \subset R$. Thus, if we can show that R is a subring of $k[\alpha_1, \ldots, \alpha_k]$, then R will necessarily have to contain all polynomials in the s_i. Since every symmetric function is a polynomial in this s_i, this shows that the collection of all symmetric functions is contained in R, which is what the problem is asking.

So all that needs to be done is show that if $r, r' \in R$, then $r + r'$ and $rr' \in R$. To see this, let $\sigma \in S_n$, and compute

$$\begin{aligned} \sigma \cdot (r + r')(f_1, \ldots, f_k) &= \sigma \cdot [r(f_1, \ldots, f_k) + r'(f_1, \ldots, f_k)] \\ &= \sigma \cdot r(f_1, \ldots, f_k) + \sigma \cdot r'(f_1, \ldots, f_k) \\ &= r(f_1, \ldots, f_k) + r'(f_1, \ldots, f_k) \qquad \text{(as } r, r' \in R\text{)} \\ &= (r + r')(f_1, \ldots, f_k), \end{aligned}$$

so $r + r' \in R$, as desired. Multiplication follows via the exact same argument, and so the result holds.

□

Problem 2.5

Let

$$f_1 = (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)$$
$$f_2 = (\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4)$$
$$f_3 = (\alpha_1 + \alpha_4)(\alpha_2 + \alpha_3)$$

(a) Express $f_1 + f_2 + f_3$ as a polynomial in terms of the elementary symmetric functions $s_1, \ldots, s_4$.

(b) Express $f_1 f_2 + f_1 f_3 + f_2 f_3$ as a polynomial in $s_1, \ldots, s_4$.

(c) Express $f_1 f_2 f_3$ as a polynomial in $s_1, \ldots, s_4$.

Solution: The formulas are:

$$f_1 + f_2 + f_3 = 2s_2$$
$$f_1 f_2 + f_1 f_3 + f_2 f_3 = s_2^2 + s_1 s_3 - 4s_4$$
$$f_1 f_2 f_3 = s_1 s_2 s_3 - s_4 s_1^2 - s_3^2$$

Problem 2.6

(a) Verify that Lagrange's method to solve the quartic yields the correct solutions to $x^4 + a_0 = 0$.

(b) Use Lagrange's method to solve $x^4 + x + 1 = 0$.

Solution: In general, if $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the roots of a quartic

$$f(x) = x^4 + a_2 x^2 + a_1 x + a_0$$

then

$$\begin{aligned} 0 = -a_3 &= s_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ a_2 &= s_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ -a_1 &= s_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ a_0 &= s_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4). \end{aligned}$$

where $s_1, \ldots, s_4$ are the elementary symmetric functions. The first equation gives us the identity $\alpha_3 + \alpha_3 = -(\alpha_1 + \alpha_2)$. Taking the orbit of $f_1 = (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)$ under the action of S_4 yields

$$\begin{aligned} f_1 &= (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) \\ f_2 &= (\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4) \\ f_3 &= (\alpha_1 + \alpha_4)(\alpha_2 + \alpha_3). \end{aligned}$$

The goal is to first find the f_i by solving the cubic equation

$$\begin{aligned} (y - f_1)(y - f_2)(y - f_3) &= y^3 - (f_1 + f_2 + f_3)y^2 + (f_1 f_2 + f_1 f_3 + f_2 f_3)y - f_1 f_2 f_3 \\ &= y^3 - 2s_2 y^2 + (s_2^2 + s_1 s_3 - 4s_4)y + (-s_1 s_2 s_3 + s_4 s_1^2 + s_3^2) \\ &= y^3 - 2a_2 y^2 + (a_2^2 - 4a_0)y + (a_1^2) \end{aligned}$$

where we've used the formulas from Challenge Problem 2.5 and the identity $a_3 = 0$. Solving for the variable y leads to the solutions f_i. The second step is to solve for the α_i: the main idea is to use the relation $\alpha_3 + \alpha_4 = -(\alpha_1 + \alpha_2)$ to get that

$$\begin{aligned} \alpha_1 + \alpha_2 &= -(\alpha_3 + \alpha_4) = \sqrt{-f_1} \\ \alpha_1 + \alpha_3 &= -(\alpha_2 + \alpha_4) = \sqrt{-f_2} \\ \alpha_1 + \alpha_4 &= -(\alpha_2 + \alpha_3) = \sqrt{-f_3} \end{aligned}$$

and $\alpha_3 + \alpha_4 = -\sqrt{-f_1}$. The α_i's are expressed using the f_i's as

$$\alpha_1 = \frac{\sqrt{-f_1} + \sqrt{-f_2} + \sqrt{-f_3}}{2}$$
$$\alpha_2 = \frac{\sqrt{-f_1} - \sqrt{-f_2} - \sqrt{-f_3}}{2}$$
$$\alpha_3 = \frac{-\sqrt{-f_1} + \sqrt{-f_2} - \sqrt{-f_3}}{2}$$
$$\alpha_4 = \frac{-\sqrt{-f_1} - \sqrt{-f_2} + \sqrt{-f_3}}{2}$$

To solve (1), observe that the above cubic becomes

$$y^3 - 4a_0 y = 0$$

and the solutions for y are $0, \pm 2\sqrt{a_0}$ where $\sqrt{a_0}$ denotes a choice of a square root (if $a_0 > 0$, we can arrange for this choice to be the positive real solution). Writing $f_1 = 0$, $f_2 = 2\sqrt{a_0}$ and $f_3 = -2\sqrt{a_0}$, we know that

$$0 = f_1 = (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) = -(\alpha_1 + \alpha_2)^2$$

and we get that $\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4 = 0$ so $\alpha_2 = -\alpha_1$ and $\alpha_4 = -\alpha_3$. We also have the equations

$$2\sqrt{a_0} = f_2 = (\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4) = -(\alpha_1 + \alpha_3)^2$$
$$-2\sqrt{a_0} = f_3 = (\alpha_1 + \alpha_4)(\alpha_2 + \alpha_3) = -(\alpha_1 - \alpha_3)^2$$

and we get $\alpha_1 + \alpha_3 = \sqrt{-2\sqrt{a_0}}$ and $\alpha_1 - \alpha_3 = \sqrt{2\sqrt{a_0}}$. Adding these equations and dividing by 2 yields

$$\begin{aligned}
\alpha_1 &= \frac{1}{2}\left(\sqrt{-2\sqrt{a_0}} + \sqrt{2\sqrt{a_0}}\right) \\
&= \frac{1}{2}\sqrt{2\sqrt{a_0}}(i+1) \\
&= \sqrt[4]{a_0}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) \\
&= \sqrt[4]{a_0}\sqrt[4]{-1} \\
&= \sqrt[4]{-a_0}
\end{aligned}$$

Solving the remaining α_i gives $\alpha_2 = -\sqrt[4]{-a_0}$, $\alpha_3 = \sqrt[4]{-a_0}i$ and $\alpha_4 = -\sqrt[4]{-a_0}i$. In the end, we get the expected solutions to $x^4 + a_0 = 0$!

To solve (2) where $a_3 = a_2 = 0$ and $a_1 = a_0 = 1$, the cubic becomes

$$y^3 - 4y + 1 = 0.$$

To solve the cubic, we use the substitution $y = z + \frac{4}{3z}$ which yields (after simplifying)

$$z^6 + z^3 + \frac{64}{27} = 0.$$

Solving for z^3 using the quadratic formula gives

$$z^3 = \frac{-1 \pm \sqrt{1 - 4(64)/27}}{2} = -\frac{1}{2} \pm \sqrt{\frac{229}{108}} i \approx -.5 \pm 1.449i$$

where one of the solutions is

$$z_0 = \sqrt[3]{-\frac{1}{2} + \sqrt{\frac{229}{108}} i} \approx 0.9304 + 0.6839i$$

and the others are $\omega z_0 \approx -1.05474 + 0.4638i$ and $\omega^2 z_0 \approx 0.1271 - 1.1477i$ where $\omega = e^{2\pi i/3}$ is a 3rd root of unity. This gives the following three solutions for $y = z + \frac{4}{3z}$

$$f_1 \approx 1.8608, f_2 \approx -2.1149, f_3 \approx 0.2541$$

which gives the four solutions

$$x \approx 0.7271 \pm 0.9371i, -0.7271 \pm 0.4300i$$

Alternatively, one can of course express these solutions using radicals.