Math 404A: Introduction to Modern Algebra (Spring 2021)
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Homework 1
Due: Friday, April 2

Problem 1.1. Let $f(x) = x^5 + x^4 + x^3 + x^2 + x - 5 \in \mathbb{Z}[x]$. Find a polynomial $g(x)$ such that

$$f(x) = (x-1)g(x).$$

Problem 1.2. Let k be a field. Recall that a non-constant polynomial $f(x) \in k[x]$ is *irreducible* if it cannot be written as the product $f(x) = g(x)h(x)$ where both $g(x)$ and $h(x)$ are non-constant polynomials. Show that that the following statements are equivalent for a polynomial $f(x) \in k[x]$:

(1) $f(x)$ is irreducible;
(2) the ideal $(f) \subset k[x]$ is prime; and
(3) the quotient ring $k[x]/(f)$ is a field.

Problem 1.3.

(a) Show that $\mathbb{R}[x]/(x^2 + x + 1) \cong \mathbb{C}$.
(b) How many elements does the quotient ring $\mathbb{F}_3[x, y]/(x^3, x^2y^2, y^3)$ have?

Notation: For a prime p, $\mathbb{F}_p$ denotes the field $\mathbb{Z}/p\mathbb{Z}$. Hungerford uses the notation $\mathbb{Z}_p$ but we will not use this notation.

Problem 1.4.

(a) Prove that $x^5 - x^2 + 1 \in \mathbb{Q}[x]$ is irreducible.
(b) Provide a factorization of $x^5 - x \in \mathbb{F}_5[x]$ into irreducibles.

Problem 1.5. Find the solution to the following cubic equations using the method introduced in class on Wednesday:

(a) $x^3 - 3x + 2 = 0$.
(b) $x^3 + 3x - 36 = 0$. Which solutions are real? rational?