Midterm 2 Solutions

Calculus I (Math 124)
Instructor: Jarod Alper
Fall 2019
November 19, 2019 **Name:** ______________________

Section: ______________________

Read all of the following information before starting the exam:

- You may use a Ti-30x IIS calculator.
- You may refer to your hand-written note sheet (8.5"x11", two-sided).
- You may not consult any other outside sources (phone, computer, textbook, other students, ...) to assist in answering the exam problems. All of the work will be your own!
- Simplify your answers as much as possible. Unless otherwise stated, give exact answers to questions. For example, 2ln(3)/pi and 1/3 are exact while 0.699 and 0.333 are approximations for the those numbers.
- Write clearly!! You need to write your solutions carefully and clearly in order to convince me that your solution is correct. Partial credit will be awarded.
- Good luck!

Problem	Points
1 (20 points)	______
2 (20 points)	______
3 (20 points)	______
4 (20 points)	______
5 (20 points)	______
Total (100 points)	

Problem 1.

(a) Find the derivative of $f(x) = \arctan(\sqrt{x})$. (*Recall that* $\arctan(x)$ *is the same function as the inverse tangent function* $\tan^{-1}(x)$.)

Solution: Using the formula $\frac{d}{dx}(\arctan(x)) = 1/(1+x^2)$, we obtain that

$$\begin{aligned} f'(x) &= \frac{1}{1+(\sqrt{x})^2}\frac{d(\sqrt{x})}{dx} \\ &= \frac{1}{1+x}(1/2x^{-1/2}) \\ &= \frac{1}{2(1+x)\sqrt{x}}. \end{aligned}$$

(b) If $y = \ln(x)^{\ln(x)}$, find an expression for $\frac{dy}{dx}$ in terms of x.

Solution: We will use logarithmic differentiation in addition to carefully applying the rule that $\ln(x^a) = a\ln(x)$. If we take the natural logarithm $\ln(-)$ of both sides, we obtain

$$\begin{aligned} \ln(y) &= \ln\big(\ln(x)^{\ln(x)}\big) \\ &= \ln(x)\ln\big(\ln(x)\big) \end{aligned}$$

and taking the derivative of both sides and using the chain and product rules yields

$$\begin{aligned} \frac{1}{y}\frac{dy}{dx} &= \frac{1}{x}\ln\big(\ln(x)\big) + \ln(x)\frac{1}{\ln(x)}\frac{1}{x} \\ &= \frac{1}{x}\big(\ln\big(\ln(x)\big)+1\big) \end{aligned}$$

Solving for $\frac{dy}{dx}$ and substituting $y = \ln(x)^{\ln(x)}$ gives

$$\frac{dy}{dx} = \frac{\ln(x)^{\ln(x)}}{x}\big(\ln\big(\ln(x)\big)+1\big)$$

.

Problem 2. Consider the curve defined by the equation $x^4 - 2x^2y^2 = -56$.
(a) Find the equation of the tangent line at the point $(2, 3)$.

Solution: (a) We will use implicit differentiation. Taking d/dx of both sides and using both the chain and product rule, we obtain that

$$4x^3 - 4xy^2 - 4x^2y\frac{dy}{dx} = 0$$

and solving for $\frac{dy}{dx}$ yields

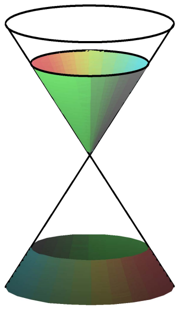
$$\frac{dy}{dx} = \frac{4x^3 - 4xy^2}{4x^2y} = \frac{x^2 - y^2}{xy}.$$

At $(2, 3)$, $dy/dx = -5/6$. So the tangent line is $y - 3 = -5/6(x - 2)$.

(b) Let P be the point on the curve near $(2, 3)$ with x-coordinate 2.1. Find an approximate value of the y-coordinate of P. (*Please round your answer to three digits after the decimal.*)

Solution: The linear approximation at $(2, 3)$ is $L(x) = -5/6(x-2)+3$. Evaluating at 2.1, we get $L(2.1) = -5/6(.1) + 3 = 3 - 5/60 = 2.917$.

Problem 3. An hourglass is made up of two glass cones connected at their tips (as in the diagram below). Both cones have radius 1 cm and height 2 cm. When the hourglass is flipped over, sand starts falling to the lower cone.



(a) When the sand remaining in the *upper cone* has height y cm, give a formula for its volume A in terms of y.

Solution: If R denotes the radius of the cone containing the sand remaining in the upper cone, then the ratio of R to y must the same as the ratio of 1 cm (the radius of the entire glass cone) to 2 cm (the height of the entire glass cone). Therefore, $R/y = 1/2$ or in other words $R = y/2$. The volume A is therefore

$$A = \frac{1}{3}\pi R^2 y = \frac{\pi}{12}y^3.$$

(b) When the sand in the *lower cone* has reached a height of h cm, give a formula for its volume B in terms of h.

Solution: The volume B can be expressed as the difference of the volume of the entire bottom glass cone subtracted by the volume of the bottom cone *not* containing sand. The height of this bottom cone not containing sand is $2 - h$ and therefore its radius is $(2 - h)/2$ (using again that the ratio of the height to the radius should be $2/1$ as above).

Therefore,

$$\begin{aligned} B &= \frac{1}{3}\pi(1)^2(2) - \frac{1}{3}\pi r^2 h \\ &= \frac{2\pi}{3} - \frac{\pi}{12}(2-h)^3. \end{aligned}$$

(c) Assume that the total volume of sand is $2\pi/3$ cm^3 and that the height of the sand in the upper cone is decreasing at a rate of 1 cm/sec. At the instant that the sand in the lower cone is 1 cm high, determine the rate at which the height of the sand in the lower cone is increasing.

Solution: We have that $2\pi/3 = A + B$ or in other words

$$\frac{2\pi}{3} = \frac{\pi}{12}y^3 + \frac{2\pi}{3} - \frac{\pi}{12}(2-h)^3$$

which simplifies to

$$y^3 = (2-h)^3. \tag{1}$$

When $h = 1$, this gives $y = 1$. By differentiating (1) with respect to time t, we have $3y^2\frac{dy}{dt} = 3(2-h)^2(-\frac{dh}{dt})$ or in other words

$$y^2\frac{dy}{dt} = (2-h)^2(-\frac{dh}{dt}).$$

We are told that $\frac{dy}{dt} = -1$. Therefore, so $\frac{dh}{dt} = 1$. The rate at which the height of the sand in the lower cone is increasing is 1 cm/sec.

Problem 4. Beginning at time $t = 0$ seconds, an ant crawls according to the equations

$$x(t) = t^3 + 45t + 1 \qquad \text{and} \qquad y(t) = -12t^2.$$

(a) At what times t within the first 10 seconds is the ant's direction of travel parallel to the line $x + y = 2$? (*Please round your answer to three digits after the decimal.*)

Solution: We compute that $x'(t) = 3t^3 + 45$ and $y'(t) = -24t$. Thus

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{-24t}{3t^2 + 45}.$$

The line $x + y = 2$ has slope -1 so we simply need to find all values t for which

$$\frac{dy}{dx} = \frac{-24t}{3t^2 + 45} = -1.$$

In other words, we need to solve $-24t = -3t^2 - 45$ or $t^2 - 8t + 15 = 0$. This latter equation factors as $(t - 5)(t - 3) = 0$ and we obtain the times $t = 3, 5$.

(b) At what time within the first 10 seconds does the ant attain its maximal speed? (*Please round your answer to three digits after the decimal.*) *Solution:*

The maximal speed is attained at the same time at which the *square* of the speed is maximized. Let $f(t)$ be the square of the speed, namely,

$$f(t) = x'(t)^2 + y'(t)^2 = (3t^2 + 45)^2 + (24)^2 t^2 = 9t^4 + 846t^2 + 2025$$

and its derivative is given by

$$f'(t) = 36t^3 + 1692t = 36t(t^2 + 47).$$

The solutions to $f'(t) = 0$ are $t = 0$. The only critical number of $f(x)$ in $[0, 10]$ is 0. So we only need to evaluate $f(t)$ at 0 and 10. We compute $f(0) = 2025$ and $f(10) = 176625$. The maximal speed is attained at $t = 10$.

Problem 5. Consider the function
$$f(x) = x^4 - 6x^2 + 4.$$

(a) Using interval notation, determine where $f(x)$ is increasing and decreasing.

Solution: Since $f'(x) = 4x^3 - 12x = 4x(x^2 - 3)$, we see that $f(x)$ is increasing on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$ and decreasing on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$.

(b) Determine the critical numbers of $f(x)$.

Solution: The derivative $f'(x) = 4x(x^2 - 3)$ exists everywhere and is zero precisely at $0, \pm\sqrt{3}$.

(c) Using interval notation, determine where $f(x)$ is concave up and down.

Solution: Since $f''(x) = 12x^2 - 12 = 12(x^2 - 1)$, we see that $f(x)$ is concave up on $(-\infty, -1) \cup (1, \infty)$ and concave down on $(-1, 1)$.

(d) Determine the inflection points of $f(x)$.

Solution: The concavity switches at ± 1.

(e) For each critical number, determine whether it is a local minimum, local maximum or neither.

Solution: We use the second derivative test. Since $f''(\pm\sqrt{3}) > 0$ and $f''(0) < 0$, we see that $f(x)$ attains a local minima at both $\pm\sqrt{3}$ and a local maximum at 0.