Algebra 2, Semester 1 2015
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Tutorials 7 and 8
Friday, May 1 and Wednesday, May 6

Tutorial 7.1. Let G be a finite group and $H \subseteq G$ be a subgroup. Show that if the index $|G : H| = 2$, then H is normal in G.

Tutorial 7.2. Find the Galois group of the splitting field of $f(x) = x^6 - 2x^3 - 1$ over $\mathbb{Q}$.

Tutorial 7.3. Can you write down a Galois field extension $\mathbb{Q} \subseteq L$ with $\mathrm{Gal}(L/\mathbb{Q}) \cong S_4$?

Tutorial 7.4. Let K be a field and consider the field $K(x_1, \ldots, x_n)$ consisting of quotients of polynomials $f(x_1, \ldots, x_n)/g(x_1, \ldots, x_n)$. Recall the symmetric functions

$$\begin{aligned} s_1 &= x_1 + x_2 + \cdots + x_n \\ s_2 &= x_1x_2 + x_1x_3 + \cdots x_{n-1}x_n \\ &\vdots \\ s_n &= x_1x_2 \cdots x_n \end{aligned}$$

(a) Show that S_n is naturally a subgroup of $\mathrm{Aut}(K(x_1, \ldots, x_n))$ where permutations act on elements of $L = K(x_1, \ldots, x_n)$ by permuting the indices of the variables.
(b) Show that
$$K(x_1, \ldots, x_n)^{S_n} \cong K(s_1, \ldots, s_n).$$
(c) Conclude that $\mathrm{Gal}(K(x_1, \ldots, x_n)/K(s_1, \ldots, s_n)) \cong S_n$
(d) Now let H be any finite group. Use part (c) and the fundamental theorem of Galois theory to construct a Galois extension of fields $L' \subseteq L$ such that $\mathrm{Gal}(L/L') = H$.

Extra credit: Can you do the same with $L' = \mathbb{Q}$? That is, given a finite group H, can you find a field extension $\mathbb{Q} \subseteq L$ such that $\mathrm{Gal}(L/\mathbb{Q})$? If you can, I'll not only give you an "A+" in the class but I can guarantee that you'll be admitted to Stanford for a math PhD.

Tutorial 7.5. Recall that a permutation $\sigma \in S_n$ is said to be *even* if it can be written as a product $\sigma = \tau_1 \cdots \tau_{2k}$ of an even number of transpositions τ_i. A transposition τ is a permutation of the form (ij) for integers $1 \leq i < j \leq n$.

(a) Show that a permutation σ is even if and only if the number of pairs (i, j) satisfying $1 \leq i < j \leq n$ and $\sigma(i) > \sigma(j)$ is even.
(b) Show that the subset $A_n \subseteq S_n$ consisting of even permutations is a normal subgroup of index 2.