Algebra 2, Semester 1 2015
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Tutorial 10
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Tutorial 10.1.

(a) Show that a primitive 14th root of unity satisfies the sextic equation
$$x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 = 0.$$

(b) Solve the equation from part (1) by radicals.
Hint: Substitute $y = x + 1/x$.

Tutorial 10.2.

(a) Can you give an example of an element $\alpha \in \mathbb{C}$ which can be expressed by radicals but such that $\mathbb{Q} \subseteq \mathbb{Q}(\alpha)$ is not a radical field extension (i.e. $\mathbb{Q}(\alpha)$ is not obtained from $\mathbb{Q}$ by adjoining an nth root of an element $a \in \mathbb{Q}$).

(b) Can you give an example of an element $\alpha \in \mathbb{C}$ which can be expressed by radicals but such that there does not exist a sequence of field extensions
$$\mathbb{Q} = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_r = \mathbb{Q}(\alpha)$$
where each $K_i \subseteq K_{i+1}$ is a radical field extension.

Tutorial 10.3. Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{-3}, \sqrt[3]{5})$.

(a) Prove that $\mathbb{Q} \subseteq L$ is a Galois extension with Galois group
$$\mathrm{Gal}(L/\mathbb{Q}) \cong \mathbb{Z}/2 \times S_3.$$

(b) Give the complete correspondence between intermediate field extensions $\mathbb{Q} \subseteq L' \subseteq L$ and subgroups $H \subseteq \mathrm{Gal}(L/\mathbb{Q})$.

Tutorial 10.4. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible quartic.

(a) Show that the discriminant of f is equal to the discriminant of the resolvent cubic of f.

(b) Show that if $\mathrm{Gal}(f) = \mathbb{Z}/4\mathbb{Z}$ then the discriminant of f is positive.

Tutorial 10.5. Determine the Galois group of the splitting fields over $\mathbb{Q}$ of the following polynomials:

(a) $f(x) = x^3 + 3x + 1$.
(b) $f(x) = x^3 - 3x + 1$.
(c) $f(x) = x^4 + 4x^2 - 2$
(d) $f(x) = x^4 + x^2 + 1$

(e) $f(x) = x^4 + 36x + 63$

Hint: You may want to use the follwing formulae:

- the discriminant of $f(x) = x^3 + ax + b$ is $-4a^3 - 27b^2$.
- the discriminant of $f(x) = x^4 + ax + b$ is $-27a^4 + 256b^3$.
- the resolvent cubic of $f(x) = x^4 + ax + b$ is $x^3 - 4bx + a^2$.
- the discriminant of $f(x) = x^4 + ax^2 + b$ is $16b(a^2 - 4b)^2$.