Algebra 2, Semester 1 2015
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Homework 8
Due: Wednesday, April 29

Problem 8.1. Let $K \subseteq L$ be a field extension.

(a) Show that
$$\mathrm{Gal}(L/K) := \{K\text{-automorphisms } \sigma \colon L \to L\}$$
is a group under composition.

(b) Let $f(x) \in K[x]$ be a polynomial and $\alpha \in L$ be a root of $f(x)$. Show that if $\sigma \colon L \to L$ is a K-automorphism, then $\sigma(\alpha)$ is also a root of $f(x)$.

Problem 8.2. Consider the field $\mathbb{C}(t)$. As usual, we denote $\mathrm{Gal}(\mathbb{C}(t)/\mathbb{C})$ as the group of all $\mathbb{C}$-automorphisms $\sigma \colon \mathbb{C}(t) \to \mathbb{C}(t)$.

(a) For each n, let $\rho_n = e^{2\pi i/n}$. Show that there is a well-defined field automorphism $\sigma_n \colon \mathbb{C}(t) \to \mathbb{C}(t)$ defined by $\sigma_n(t) = \rho_n t$.

(b) Conclude that $\mathrm{Gal}(\mathbb{C}(t)/\mathbb{C})$ is not finite.

(c) Let $H_n \subseteq \mathrm{Gal}(\mathbb{C}(t)/\mathbb{C})$ be the subgroup generated by σ_n. Show that $\mathbb{C}(t)^{H_n} \cong \mathbb{C}(t^n)$.

Problem 8.3. Let L be the splitting field of $f(x) = x^4 - 2$ over $\mathbb{Q}$.

(a) Determine $\mathrm{Gal}(L/\mathbb{Q})$.

(b) Give the complete correspondence between intermediate field extensions $\mathbb{Q} \subseteq L' \subseteq L$ and subgroups $H \subseteq \mathrm{Gal}(L/\mathbb{Q})$.

Problem 8.4. Let L be the splitting field of $f(x) = x^3 - 2$ over $\mathbb{Q}$. Let $\rho = e^{2\pi i/3}$ be a primitive 3rd root of unity. We saw in the HW Problem 7.2 and in Tutorial 6 that $\mathrm{Gal}(L/\mathbb{Q}) \cong S_3$ is generated by automorphisms σ, τ, where
$$\begin{aligned} \sigma \colon L &\to L & \qquad \tau \colon L &\to L \\ \sqrt[3]{2} &\mapsto \rho\sqrt[3]{2} & \sqrt[3]{2} &\mapsto \sqrt[3]{2} \\ \rho &\mapsto \rho & \rho &\mapsto \rho^2 \end{aligned}$$

(a) Let $H = \{1, \sigma, \sigma^2\}$. Compute the orbit $\{\alpha_1, \alpha_2, \alpha_3\}$ of the element $\sqrt[3]{2} + \rho$ under the action of H.

(b) Show directly that the coefficients of the polynomial $f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$ are in $\mathbb{Q}(\rho)$.

(c) Conclude that $f(x)$ is the minimal polynomial of $\sqrt[3]{2} + \rho$ over $\mathbb{Q}(\rho)$.

Problem 8.5. Let $K \subseteq L$ be a Galois field extension with $\mathrm{Gal}(L/K) \cong \mathbb{Z}/4\mathbb{Z}$. Show that L is the splitting field of a polynomial

$$f(x) = (x^2 - a)^2 - b$$

for elements $a, b \in K$ such that $a \neq 0$, $\sqrt{b} \notin K$ and $\sqrt{a^2 - b} \in K$.