Algebra 2, Semester 1 2015
Jarod Alper
Homework 3
Due: Tuesday, March 10

Problem 3.1. (a) Show that there are ideals in $\mathbb{Z}[x]$ which are not generated by a single element.

(b) Show that the element 6 in the ring $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ does not factor uniquely. That is, write 6 has a product $6 = ab$ and $6 = a'b'$ in two distinct ways such that a, b, a', b' are irreducible, and both a and b are not units times a' or b'. Conclude that there is an irreducible element of $\mathbb{Z}[\sqrt{-5}]$ which does *not* generate a prime ideal.

Problem 3.2. Show $\mathbb{Q}[x]/(x^2 + x + 1) \cong \{a + b\omega \mid a, b \in \mathbb{Q}\}$ where $\omega = e^{2\pi i/3}$. Describe explicitly additional, multiplication and division on the right hand side.

Problem 3.3. Prove that $x^5 - x^2 + 1 \in \mathbb{Q}[x]$ is irreducible. (Hint: consider $\mathbb{F}_2$)

Problem 3.4. *Eisenstein's criterion with a twist.*

(a) Let a be any integer. Prove that a polynomial $f(x) \in \mathbb{Z}[x]$ is irreducible iff $f(x + a) \in \mathbb{Z}[x]$ is irreducible.

(b) Use this trick to prove that $x^3 - 3x^2 + 9x - 5$ is irreducible.

(c) Use this trick to prove that, for any prime p, the polynomial $x^{p-1} + x^{p-2} + \ldots + x + 1$ is irreducible.

Problem 3.5. Consider the field extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$.

(a) What is the degree, $[\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}) : \mathbb{Q}]$, of this field extension?

(b) Prove that this is a primitive field extension; that is, find an element α such that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$.