Algebra 2, Semester 1 2015
Jarod Alper
Homework 11
Due: Friday, May 29

Problem 11.1. Suppose K is a finite field extension of $\mathbb{Q}$. Let $K \subseteq L$ be a Galois field extension and $K \subseteq K'$ be a finite field extension. Show that $K' \subseteq K'L$ is a Galois field extension and

$$\mathrm{Gal}(K'L/K') \cong \mathrm{Gal}(L/L \cap K')$$

Remark: This abstract result was used in the proof in class of the theorem stating that a polynomial is solvable by radicals if and only if the Galois group of the splitting field is solvable.

Problem 11.2. Determine the Galois group of the splitting fields over $\mathbb{Q}$ of the following polynomials:
(a) $f(x) = x^3 + 2x + 1$.
(b) $f(x) = x^4 + 3x - 3$.
(c) $f(x) = x^4 + 5x^2 - 5$.

Hint: You may want to use the follwing formulae:
- the discriminant of $f(x) = x^3 + ax + b$ is $-4a^3 - 27b^2$.
- the discriminant of $f(x) = x^4 + ax + b$ is $-27a^4 + 256b^3$
- the resolvent cubic of $f(x) = x^4 + ax + b$ is $x^3 - 4bx + a^2$.

Problem 11.3. Show that for any prime $p \neq 3, 5$, the polynomial

$$x^4 + px + p \in \mathbb{Q}[x]$$

is irreducible and the Galois group of the splitting field is S_4.
Extra credit: What happens if $p = 3$ or 5?