

# PARTIAL CAUCHY DATA FOR GENERAL SECOND-ORDER ELLIPTIC OPERATORS IN TWO DIMENSIONS

OLEG YU. IMANUVILOV, GUNTHER UHLMANN, AND MASAHIRO YAMAMOTO

ABSTRACT. We consider the inverse problem of determining the coefficients of a general second-order elliptic operator in two dimensions by measuring the corresponding Cauchy data on an arbitrary open subset of the boundary. We show that one can determine the coefficients of the operator up to natural obstructions such as conformal invariance, gauge transformations and diffeomorphism invariance. We use the main result to prove that the *curl* of the magnetic field and the electric potential are uniquely determined by measuring partial Cauchy data associated to the magnetic Schrödinger equation measured on an arbitrary open subset of the boundary. We also show that any two of the three coefficients of a second order elliptic operator whose principal part is the Laplacian, are uniquely determined by their partial Cauchy data.

## 1. Introduction

Let  $\Omega \subset \mathbf{R}^2$  be a bounded domain with smooth boundary  $\partial\Omega = \cup_{k=1}^{\mathcal{N}} \gamma_k$ , where  $\gamma_k$ ,  $1 \leq k \leq \mathcal{N}$ , are smooth closed contours, and  $\gamma_{\mathcal{N}}$  is the external contour.

Let  $\tilde{\Gamma} \subset \partial\Omega$  be an arbitrarily fixed non-empty relatively open subset of  $\partial\Omega$  and let  $\Gamma_0 = \partial\Omega \setminus \tilde{\Gamma}$ . Let  $\nu$  be the unit outward normal vector to  $\partial\Omega$  and let  $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$ .

Henceforth we set  $i = \sqrt{-1}$ ,  $x_1, x_2 \in \mathbf{R}$ ,  $z = x_1 + ix_2$ ,  $\bar{z}$  denotes the complex conjugate of  $z \in \mathbf{C}$ , and we identify  $x = (x_1, x_2) \in \mathbf{R}^2$  with  $z = x_1 + ix_2 \in \mathbf{C}$ . We also denote  $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2})$ ,  $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2})$ .

Let  $u \in H^1(\Omega)$  be a solution to the following boundary value problem

$$(1.1) \quad L(x, D)u = \Delta_g u + 2A \frac{\partial u}{\partial z} + 2B \frac{\partial u}{\partial \bar{z}} + qu = 0, \quad u|_{\Gamma_0} = 0, \quad u|_{\tilde{\Gamma}} = f.$$

Here  $\Delta_g$  denotes the Laplace-Beltrami operator associated to the Riemannian metric  $g$ . We assume that  $g$  is a positive definite symmetric matrix in  $\Omega$  and

$$(1.2) \quad \Delta_g = \frac{1}{\sqrt{\det g}} \sum_{j,k=1}^2 \frac{\partial}{\partial x_k} (\sqrt{\det g} g^{jk} \frac{\partial}{\partial x_j}),$$

where  $\{g^{jk}\}$  denotes the inverse of  $g = \{g_{jk}\}$ . From now on we assume that  $g \in C^{7+\alpha}(\bar{\Omega})$ ,  $(A, B, q), (A_j, B_j, q_j) \in C^{5+\alpha}(\bar{\Omega}) \times C^{5+\alpha}(\bar{\Omega}) \times C^{4+\alpha}(\bar{\Omega})$ ,  $j = 1, 2$  for some  $\alpha \in (0, 1)$  are complex-valued functions. Henceforth  $\alpha$  denotes a constant such that  $0 < \alpha < 1$ .

We set

$$L_j(x, D) = \Delta_{g_j} + 2A_j \frac{\partial}{\partial z} + 2B_j \frac{\partial}{\partial \bar{z}} + q_j, \quad j = 1, 2.$$

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We define the partial Cauchy data by

$$(1.3) \quad \mathcal{C}_{g,A,B,q} = \left\{ \left( u|_{\bar{\Gamma}}, \frac{\partial u}{\partial \nu_g} \Big|_{\bar{\Gamma}} \right) \mid (\Delta_g + 2A \frac{\partial}{\partial z} + 2B \frac{\partial}{\partial \bar{z}} + q)u = 0 \text{ in } \Omega, \quad u \in H^1(\Omega), u|_{\Gamma_0} = 0 \right\},$$

where  $\frac{\partial}{\partial \nu_g} = \sqrt{\det g} \sum_{j,k=1}^2 g^{jk} \nu_k \frac{\partial}{\partial x_j}$ .

The goal of this paper is to determine the coefficients of the operator  $L$ . In the general case this is impossible. As for the invariance of the Cauchy data, there are of the following three types.

(i) The partial Cauchy data for the operators  $e^{-\eta} L(x, D) e^{\eta}$  and  $L(x, D)$  are the same provided that  $\eta \in C^{6+\alpha}(\bar{\Omega})$  is a complex-valued function and  $\eta|_{\bar{\Gamma}} = \frac{\partial \eta}{\partial \nu} \Big|_{\bar{\Gamma}} = 0$ .

(ii) Let  $\beta \in C^{7+\alpha}(\bar{\Omega})$  be a positive function on  $\bar{\Omega}$ . The partial Cauchy data for the operators  $L(x, D)$  and  $\frac{1}{\beta} L(x, D) = \Delta_{\beta g} + \frac{1}{\beta} (2A \frac{\partial}{\partial z} + 2B \frac{\partial}{\partial \bar{z}} + q)$  are exactly the same.

(iii) Let  $F \in C^{8+\alpha}(\bar{\Omega}) : \bar{\Omega} \rightarrow \bar{\Omega}$  be a diffeomorphism such that  $F|_{\bar{\Gamma}} = Id$ . For any metric  $g$  and complex valued functions  $A, B, q$ , we introduce a metric  $F^*g$  and functions  $A_F, B_F, q_F$  by

$$(1.4) \quad \begin{aligned} F^*g &= ((DF) \circ g \circ (DF)^T) \circ F^{-1}, \\ A_F &= \{(A+B) \left( \frac{\partial F_1}{\partial x_1} - i \frac{\partial F_2}{\partial x_1} \right) + i(B-A) \left( \frac{\partial F_1}{\partial x_2} - i \frac{\partial F_2}{\partial x_2} \right)\} \circ F^{-1} | \det DF^{-1}|, \\ B_F &= \{(A+B) \left( \frac{\partial F_1}{\partial x_1} + i \frac{\partial F_2}{\partial x_1} \right) + i(B-A) \left( \frac{\partial F_1}{\partial x_2} + i \frac{\partial F_2}{\partial x_2} \right)\} \circ F^{-1} | \det DF^{-1}|, \\ q &= | \det DF^{-1}| (q \circ F^{-1}), \end{aligned}$$

where  $DF$  denotes the differential of  $F$ ,  $(DF)^T$  its transpose and  $\circ$  denotes matrix composition.

Then the operator

$$K(x, D) = \Delta_{F^*g_1} + 2A_F \frac{\partial}{\partial z} + 2B_F \frac{\partial}{\partial \bar{z}} + q_F$$

and the operator  $L(x, D)$  have the same partial Cauchy data.

We show the converse and state our main result below.

Assume that for some  $\alpha \in (0, 1)$  and  $\alpha' > 0$

$$g_{jk} \in C^{7+\alpha}(\bar{\Omega}), \quad g_{jk} = g_{kj} \quad \forall k, j \in \{1, 2\}, \quad \sum_{j,k=1}^2 g_{jk} \xi_k \xi_j \geq \alpha' |\xi|^2, \quad \xi \in \mathbf{R}^2.$$

Consider the following set of functions

$$(1.5) \quad \eta \in C^{6+\alpha}(\bar{\Omega}), \quad \frac{\partial \eta}{\partial \nu} \Big|_{\bar{\Gamma}} = 0, \quad \eta|_{\bar{\Gamma}} = 0.$$

We have

**Theorem 1.1.** *Suppose that for some  $\alpha \in (0, 1)$ , there exists a positive function  $\tilde{\beta} \in C^{7+\alpha}(\bar{\Omega})$  such that  $(g_1 - \tilde{\beta}g_2)|_{\tilde{\Gamma}} = \frac{\partial(g_1 - \tilde{\beta}g_2)}{\partial\nu}|_{\tilde{\Gamma}} = 0$ . Then  $\mathcal{C}_{g_1, A_1, B_1, q_1} = \mathcal{C}_{g_2, A_2, B_2, q_2}$  if and only if there exist a diffeomorphism  $F \in C^{8+\alpha}(\bar{\Omega})$ ,  $F : \bar{\Omega} \rightarrow \bar{\Omega}$  satisfying  $F|_{\tilde{\Gamma}} = Id$ , a positive function  $\beta \in C^{7+\alpha}(\bar{\Omega})$  and a complex valued function  $\eta$  satisfying (1.5) such that*

$$L_2(x, D) = e^{-\eta}K(x, D)e^{\eta},$$

where

$$K(x, D) = \Delta_{\beta F^*g_1} + \frac{2}{\beta}A_{1,F}\frac{\partial}{\partial z} + \frac{2}{\beta}B_{1,F}\frac{\partial}{\partial \bar{z}} + \frac{1}{\beta}q_{1,F}.$$

and the functions  $F^*g_1, A_{1,F}, B_{1,F}, q_{1,F}$  are defined for  $g_1, A_1, B_1, q_1$  by (1.4).

We point out that we can prove that that we can assume  $(g_1 - \tilde{\beta}g_2)|_{\tilde{\Gamma}} = 0$ . However we can not determine the normal derivatives as pointed out in [23] for the case of the operator  $\Delta_g$ . Next we discuss the case of an anisotropic conductivity problem which is an independent interest. In this case the conductivity depends on direction and is represented by a positive definite symmetric matrix  $\sigma^{-1} = \{\sigma^{jk}\}$ . The conductivity equation with voltage potential  $f$  on  $\partial\Omega$  is given by

$$\sum_{j,k=1}^2 \frac{\partial}{\partial x_j} \left( \sigma^{jk} \frac{\partial u}{\partial x_k} \right) = 0 \quad \text{in } \Omega,$$

$$u|_{\partial\Omega} = f.$$

We define the partial Cauchy data by

$$(1.6) \quad \mathcal{V}_\sigma = \left\{ \left( f|_{\tilde{\Gamma}}, \sum_{j,k=1}^2 \sigma^{jk} \nu_j \frac{\partial u}{\partial x_k} \Big|_{\tilde{\Gamma}} \right) \mid \sum_{j,k=1}^2 \frac{\partial}{\partial x_j} \left( \sigma^{jk} \frac{\partial u}{\partial x_k} \right) = 0 \right. \\ \left. \text{in } \Omega, \quad u \in H^1(\Omega), \quad u|_{\partial\Omega} = f, \quad \text{supp } f \subset \tilde{\Gamma} \right\}.$$

It has been known for a long time that even in the case of  $\tilde{\Gamma} = \partial\Omega$ , that the full Cauchy data  $\mathcal{V}_\sigma$  does not determine  $\sigma$  uniquely in the anisotropic case [20]. Let  $F : \bar{\Omega} \rightarrow \bar{\Omega}$  be a diffeomorphism such that  $F(x) = x$  for  $x$  on  $\tilde{\Gamma}$ . Then

$$\mathcal{V}_{|\det DF^{-1}|F^*\sigma} = \mathcal{V}_\sigma.$$

In the case of full Cauchy data (i.e.,  $\tilde{\Gamma} = \partial\Omega$ ), the question whether one can determine the conductivity up to the above obstruction has been solved in two dimensions for  $C^2$  conductivities in [24], Lipschitz conductivities in [30] and merely  $L^\infty$  conductivities in [3]. The method of proof in all these papers is the reduction to the isotropic case using isothermal coordinates [1]. We have

**Theorem 1.2.** *Let  $\sigma_1, \sigma_2 \in C^{7+\alpha}(\bar{\Omega})$  with some  $\alpha \in (0, 1)$  be positive definite symmetric matrices on  $\bar{\Omega}$ . If  $\mathcal{V}_{\sigma_1} = \mathcal{V}_{\sigma_2}$ , then there exists a diffeomorphism  $F : \bar{\Omega} \rightarrow \bar{\Omega}$  satisfying  $F|_{\tilde{\Gamma}} = Id$  and  $F \in C^{8+\alpha}(\bar{\Omega})$  such that*

$$|\det DF^{-1}|F^*\sigma_1 = \sigma_2.$$

For the isotropic case, the corresponding result is proved in [16]. The proof of Theorem 1.2 is given in section 6.

Now we take the matrix  $g$  to be the identity matrix. We consider the problem of determining a complex-valued potential  $q$  and complex-valued coefficients  $A$  and  $B$  in a bounded two dimensional domain from the Cauchy data measured on an arbitrary open subset of the boundary for the associated second-order elliptic operator  $\Delta + 2A \frac{\partial}{\partial z} + 2B \frac{\partial}{\partial \bar{z}} + q$ . Specific cases of interest are the magnetic Schrödinger operator and the Laplacian with convection terms. We remark that general second order elliptic operators can be reduced to this form by using isothermal coordinates (e.g., [1]). The case of the conductivity equation and the Schrödinger have been considered in [16]. For global uniqueness results in the two dimensional case for the conductivity equation with full data measurements under different regularity assumptions see [2], [6], [24]. Such a problem originates in [9].

Next we will consider the case where the principal part of  $L_j$  is the Laplacian (i.e.,  $g = I$ ; the identity matrix). Then our next result is the following:

**Theorem 1.3.** *Assume that  $\mathcal{C}_{I,A_1,B_1,q_1} = \mathcal{C}_{I,A_2,B_2,q_2}$ . Then*

$$(1.7) \quad A_1 = A_2, \quad B_1 = B_2 \quad \text{on} \quad \tilde{\Gamma},$$

$$(1.8) \quad -2 \frac{\partial}{\partial z} (A_1 - A_2) - (B_1 - B_2) A_1 - (A_1 - A_2) B_2 + (q_1 - q_2) = 0 \quad \text{in } \Omega,$$

$$(1.9) \quad -2 \frac{\partial}{\partial \bar{z}} (B_1 - B_2) - (A_1 - A_2) B_1 - (B_1 - B_2) A_2 + (q_1 - q_2) = 0 \quad \text{in } \Omega.$$

**Remark.** In the case that  $A_1 = A_2$  and  $B_1 = B_2$  in  $\Omega$ , Theorem 1.3 yields that  $q_1 = q_2$ , which is the main result in [16]. The latter result was extended to Riemann surfaces in [13]. The case of full data in two dimensions was settled in [7]. This case is closely related to the inverse conductivity problem, or Calderón's problem. See the articles [24], [6], [2] in two dimensions.

Theorem 1.3 yields

**Corollary 1.1.** *The relation  $\mathcal{C}_{I,A_1,B_1,q_1} = \mathcal{C}_{I,A_2,B_2,q_2}$  holds true if and only if there exists a function  $\eta \in C^{6+\alpha}(\bar{\Omega})$ ,  $\eta|_{\tilde{\Gamma}} = \frac{\partial \eta}{\partial \nu}|_{\tilde{\Gamma}} = 0$  such that*

$$(1.10) \quad L_1(x, D) = e^{-\eta} L_2(x, D) e^{\eta}.$$

*Proof of Corollary 1.1.* We only prove the necessity since the sufficiency of the condition is easy to check. By (1.8) and (1.9), we have  $\frac{\partial}{\partial z} (A_1 - A_2) = \frac{\partial}{\partial \bar{z}} (B_1 - B_2)$ . This equality is equivalent to

$$\frac{\partial(\hat{A} - \hat{B})}{\partial x_1} = i \frac{\partial(\hat{B} + \hat{A})}{\partial x_2} \quad \text{where} \quad (\hat{A}, \hat{B}) = (A_1 - A_2, B_1 - B_2).$$

Applying Lemma 1.1 (p.313) of [31], we obtain that there exists a function  $\tilde{\eta}$  with domain  $\Omega^0$  which satisfies

$$(1.11) \quad \tilde{\eta} = \eta_0 + h, \quad \nabla \tilde{\eta} \in C^{5+\alpha}(\bar{\Omega}), \quad \Delta h = 0 \quad \text{in } \Omega^0,$$

$$[h]|_{\Sigma_k} \text{ are constants, } \left[ \frac{\partial h}{\partial \nu_k} \right] |_{\Sigma_k} = \frac{\partial h}{\partial \nu} |_{\gamma_{\mathcal{N}}} = 0 \quad \forall k \in \{1, \dots, \mathcal{N}\}$$

and

$$(i(\widehat{B} + \widehat{A}), (\widehat{A} - \widehat{B})) = \nabla \tilde{\eta}.$$

Here  $\Omega^0 = \Omega \setminus \Sigma$  is simply connected where  $\Sigma = \cup_{k=1}^{\mathcal{N}-1} \Sigma_k$ ,  $\Sigma_j \cap \Sigma_k = \emptyset$  for  $j \neq k$ ,  $\Sigma_k$  are smooth curves which do not self-intersect and are orthogonal to  $\partial\Omega$ . We choose a normal vector  $\nu_k = \nu_k(x)$ ,  $1 \leq k \leq \mathcal{N} - 1$  to  $\Sigma_k$  at  $x$  contained in the interior  $\Sigma_k^0$  of the closed curve  $\Sigma_k$ . Then, for  $x \in \Sigma_k^0$ , we set  $[h](x) = \lim_{y \rightarrow x, (\tilde{x}y, \nu_k) > 0} h(y) - \lim_{y \rightarrow x, (\tilde{x}y, \nu_k) < 0} h(y)$  where  $(\cdot, \cdot)$  denotes the scalar product in  $\mathbf{R}^2$ . Setting  $2\eta = -i\tilde{\eta}$ , we have

$$((\widehat{B} + \widehat{A}), i(\widehat{B} - \widehat{A})) = 2\nabla\eta.$$

Therefore by (1.8)

$$(1.12) \quad q_1 = q_2 + \Delta\eta + 4\frac{\partial\eta}{\partial z}\frac{\partial\eta}{\partial \bar{z}} + 2\frac{\partial\eta}{\partial z}A_2 + 2\frac{\partial\eta}{\partial \bar{z}}B_2.$$

The operator  $L_1(x, D)$  given by (1.10) has the Laplace operator as the principal part, the coefficients of  $\frac{\partial}{\partial x_1}$  is  $A_2 + B_2 + 2\frac{\partial\eta}{\partial x_2}$ , the coefficient of  $\frac{\partial}{\partial x_2}$  is  $i(B_2 - A_2) + 2\frac{\partial\eta}{\partial x_1}$ , and the coefficient of the zero order term is given by the right-hand side of (1.12). By (1.7) we have that  $\frac{\partial\eta}{\partial\nu}|_{\tilde{\Gamma}} = 0$  and  $\eta|_{\tilde{\Gamma}} = \mathcal{C}$  where the function  $\mathcal{C}(x)$  is equal to constant on each connected component of  $\tilde{\Gamma}$ .

Let us show that the function  $\eta$  is continuous. Our proof is by contradiction. Suppose that  $\eta$  is discontinuous say along the curve  $\Sigma_j$ . Let the function  $u_2 \in H^1(\Omega)$  be a solution to the following boundary value problem

$$(1.13) \quad L_2(x, D)u_2 = 0 \quad \text{in } \Omega, \quad u_2|_{\Gamma_0} = 0.$$

Assume in addition that  $u_2$  is not identically equal to zero on  $\Sigma_j$ . Let  $\tilde{\Gamma}_1$  be one connected component of the set  $\tilde{\Gamma}$  and  $\mathcal{C}|_{\tilde{\Gamma}_1} = \widehat{C}$ . Without loss of generality we may assume that  $\widehat{C} = 0$ . Indeed if  $\widehat{C} \neq 0$ , then we replace  $\eta$  by the function  $\eta - \widehat{C}$ . Since the partial Cauchy data of the operators  $L_1(x, D)$  and  $L_2(x, D)$  are the same, there exists a solution  $u_1$  to the following boundary value problem

$$(1.14) \quad L_1(x, D)u_1 = 0 \quad \text{in } \Omega, \quad u_1 = u_2 \quad \text{on } \partial\Omega, \quad \frac{\partial u_1}{\partial\nu} = \frac{\partial u_2}{\partial\nu} \quad \text{on } \tilde{\Gamma}.$$

Then the function  $v = e^{-\eta}u_2$  verifies

$$L_1(x, D)v = 0 \quad \text{in } \Omega^0, \quad v|_{\Gamma_0} = 0.$$

Since on  $\eta = \frac{\partial\eta}{\partial\nu} = 0$  on  $\tilde{\Gamma}_1$  we have that  $v \equiv u_1$ . However  $u_1 \in H^1(\Omega)$  and  $v$  is discontinuous along one part of  $\Sigma_j$ . Thus we arrive at a contradiction.

Let us show that  $\mathcal{C} \equiv 0$ . Suppose that there exists another connected component of  $\tilde{\Gamma}$  of the set  $\tilde{\Gamma}$  such that  $\mathcal{C}|_{\tilde{\Gamma}_2} \neq 0$ . Suppose that the functions  $u_1, u_2$  satisfy (1.13) and (1.14) and  $u_1|_{\tilde{\Gamma}_2}$  not identically zero.

Then the function  $v = e^{-\eta}u_2$  verifies

$$L_1(x, D)v = 0 \quad \text{in } \Omega, \quad v|_{\Gamma_0} = 0.$$

Moreover, since on  $\eta = \frac{\partial \eta}{\partial \nu} = 0$  on  $\tilde{\Gamma}_1$ , we have that

$$v = u_1, \quad \frac{\partial v}{\partial \nu} = \frac{\partial u_1}{\partial \nu} \quad \text{on } \tilde{\Gamma}_1.$$

By the uniqueness of the Cauchy problem for second-order elliptic equations, we have  $v \equiv u_1$ . In particular  $v = u_1$  on  $\tilde{\Gamma}_2$ . Since  $u_1 = u_2$  on  $\partial\Omega$ , this implies that  $e^{-\eta}|_{\tilde{\Gamma}_2} = 1$ . We arrived at a contradiction. The proof of the corollary is completed.  $\square$

We now apply our result to the case of the magnetic Schrödinger operator. Denote  $\tilde{A} = (\tilde{A}_1, \tilde{A}_2)$ , where  $\tilde{A}_j$  are real-valued,  $\tilde{\mathcal{A}} = \tilde{A}_1 - i\tilde{A}_2$ ,  $\text{rot } \tilde{A} = \frac{\partial \tilde{A}_2}{\partial x_1} - \frac{\partial \tilde{A}_1}{\partial x_2}$ ,  $D_k = \frac{1}{i} \frac{\partial}{\partial x_k}$ . Consider the magnetic Schrödinger operator

$$(1.15) \quad \mathcal{L}_{\tilde{A}, \tilde{q}}(x, D) = \sum_{k=1}^2 (D_k + \tilde{A}_k)^2 + \tilde{q}.$$

Let us define the following set of partial Cauchy data

$$\tilde{\mathcal{C}}_{\tilde{A}, \tilde{q}} = \left\{ \left( u|_{\tilde{\Gamma}}, \frac{\partial u}{\partial \nu} |_{\tilde{\Gamma}} \right) \mid \mathcal{L}_{\tilde{A}, \tilde{q}}(x, D)u = 0 \text{ in } \Omega, u|_{\Gamma_0} = 0, u \in H^1(\Omega) \right\}.$$

For the case of full data in two dimensions, it is known that there is a gauge invariance in this problem and we can recover at best the *curl* of the magnetic field [29]. The same is valid for the three dimensional case with partial Cauchy data [12]. We prove here that the converse holds in two dimensions.

**Corollary 1.2.** *Let real-valued vector fields  $\tilde{A}^{(1)}, \tilde{A}^{(2)} \in C^{5+\alpha}(\bar{\Omega})$  and complex-valued potentials  $\tilde{q}^{(1)}, \tilde{q}^{(2)} \in C^{4+\alpha}(\bar{\Omega})$  with some  $\alpha \in (0, 1)$  satisfy  $\tilde{\mathcal{C}}_{\tilde{A}^{(1)}, \tilde{q}^{(1)}} = \tilde{\mathcal{C}}_{\tilde{A}^{(2)}, \tilde{q}^{(2)}}$ . Then  $\tilde{q}^{(1)} = \tilde{q}^{(2)}$ ,  $\text{rot } \tilde{A}^{(1)} = \text{rot } \tilde{A}^{(2)}$  in  $\Omega$  and  $\tilde{A}^{(1)} = \tilde{A}^{(2)}$  on  $\tilde{\Gamma}$ .*

*Proof.* A straightforward calculation gives

$$(1.16) \quad \begin{aligned} \mathcal{L}_{\tilde{A}, \tilde{q}}(x, D) &= -\Delta + \frac{2}{i} \tilde{A}_1 \frac{\partial}{\partial x_1} + \frac{2}{i} \tilde{A}_2 \frac{\partial}{\partial x_2} + |\tilde{A}|^2 + \frac{1}{i} \frac{\partial \tilde{A}_1}{\partial x_1} + \frac{1}{i} \frac{\partial \tilde{A}_2}{\partial x_2} + \tilde{q} \\ &= -\Delta + \frac{2}{i} \overline{\tilde{\mathcal{A}}} \frac{\partial}{\partial z} + \frac{2}{i} \tilde{\mathcal{A}} \frac{\partial}{\partial \bar{z}} + \frac{2}{i} \frac{\partial \tilde{\mathcal{A}}}{\partial z} - \text{rot } \tilde{A} + |\tilde{A}|^2 + \tilde{q}. \end{aligned}$$

Then the operator  $\mathcal{L}_{\tilde{A}, \tilde{q}}(x, D)$  is a particular case of (1.1) with the metric  $g = \{\delta_{ij}\}$   $A = -\frac{1}{i} \overline{\tilde{\mathcal{A}}}$ ,  $B = -\frac{1}{i} \tilde{\mathcal{A}}$ ,  $q = -(\frac{2}{i} \frac{\partial \tilde{\mathcal{A}}}{\partial z} - \text{rot } \tilde{A} + |\tilde{A}|^2 + \tilde{q})$ . Suppose that Schrödinger operators with the vector fields  $\tilde{A}^{(1)}, \tilde{A}^{(2)}$  and the potentials  $\tilde{q}^{(1)}, \tilde{q}^{(2)}$  have the same partial Cauchy data. Then (1.8) gives

$$\text{rot } \tilde{A}^{(1)} - \text{rot } \tilde{A}^{(2)} + \tilde{q}^{(2)} - \tilde{q}^{(1)} \equiv 0$$

and (1.9) gives

$$(1.17) \quad \frac{2}{i} \frac{\partial \tilde{\mathcal{A}}^{(1)}}{\partial \bar{z}} - \frac{2}{i} \frac{\partial \tilde{\mathcal{A}}^{(2)}}{\partial \bar{z}} - \frac{2}{i} \frac{\partial \overline{\tilde{\mathcal{A}}^{(1)}}}{\partial z} + \frac{2}{i} \frac{\partial \overline{\tilde{\mathcal{A}}^{(2)}}}{\partial z} + \text{rot } \tilde{A}^{(1)} - \text{rot } \tilde{A}^{(2)} + \tilde{q}^{(2)} - \tilde{q}^{(1)} \equiv 0.$$

Using the identity  $\frac{2}{i} \frac{\partial \tilde{\mathcal{A}}}{\partial \bar{z}} - \frac{2}{i} \frac{\partial \overline{\tilde{\mathcal{A}}}}{\partial z} = -2 \text{rot } \tilde{A}$ , we transform (1.17) to the form

$$-(\text{rot } \tilde{A}^{(1)} - \text{rot } \tilde{A}^{(2)}) + \tilde{q}^{(2)} - \tilde{q}^{(1)} \equiv 0.$$

The proof of the corollary is completed.  $\square$

There is another way to define partial Cauchy data for the Schrödinger operator.

$$\widehat{C}_{\widetilde{A}, \widetilde{q}} = \left\{ \left( u|_{\widetilde{\Gamma}}, \left( \frac{\partial u}{\partial \nu} + i(\widetilde{A}, \nu)u \right) |_{\widetilde{\Gamma}} \right) \mid \mathcal{L}_{\widetilde{A}, \widetilde{q}}(x, D)u = 0 \text{ in } \Omega, u|_{\Gamma_0} = 0, u \in H^1(\Omega) \right\}.$$

**Corollary 1.3.** *Let real-valued vector fields  $\widetilde{A}^{(1)}, \widetilde{A}^{(2)} \in C^{5+\alpha}(\overline{\Omega})$  and complex-valued potentials  $\widetilde{q}^{(1)}, \widetilde{q}^{(2)} \in C^{4+\alpha}(\overline{\Omega})$  with some  $\alpha \in (0, 1)$  satisfy  $\widehat{C}_{\widetilde{A}^{(1)}, \widetilde{q}^{(1)}} = \widehat{C}_{\widetilde{A}^{(2)}, \widetilde{q}^{(2)}}$ . Then  $\widetilde{q}^{(1)} = \widetilde{q}^{(2)}$ , and  $\text{rot } \widetilde{A}^{(1)} = \text{rot } \widetilde{A}^{(2)}$  in  $\Omega$ .*

*Proof.* Suppose that there exist two vector fields and potentials  $(\widetilde{A}^{(j)}, \widetilde{q}^{(j)})$  such that  $\widehat{C}_{\widetilde{A}^{(1)}, \widetilde{q}^{(1)}} = \widehat{C}_{\widetilde{A}^{(2)}, \widetilde{q}^{(2)}}$ . Consider a complex valued function  $\eta \in C^{6+\alpha}(\overline{\Omega})$ ,  $\eta|_{\widetilde{\Gamma}} = 0$  such that  $i(\nu, \widetilde{A}^{(1)} - \widetilde{A}^{(2)}) = -\frac{1}{i} \frac{\partial \eta}{\partial \nu}$  on  $\widetilde{\Gamma}$ . Then  $\widetilde{C}_{\widetilde{A}^{(1)}, \widetilde{q}^{(1)}} = \widetilde{C}_{\widetilde{A}^{(2)} + i\nabla \eta, \widetilde{q}^{(2)}}$ . Applying Corollary 1.2, we finish the proof.  $\square$

Corollaries 1.2 and 1.3 were new in Nov. 2009 when the second author posed a preliminary version of this manuscript with the proof of Theorem 1.3, i.e. the metric  $g$  is the Euclidean metric. Since then proofs for the magnetic Schrödinger equation and full data in the Euclidean setting was given in [?] and in the Riemann surface case in [?]. In two dimensions, Sun proved in [29] that for measurements on the whole boundary, the uniqueness holds assuming that both the magnetic potential and the electric potential are small. Kang and Uhlmann proved global uniqueness for the case of measurements on the whole boundary for a special case of the magnetic Schrödinger equation, namely the Pauli Hamiltonian [18]. In dimensions  $n \geq 3$ , global uniqueness was shown in [25] for the case of full data. The regularity assumptions in the result were improved in [26] and [27]. The case of partial data was considered in [12], based on the methods of [19] and [8], with an improvement on the regularity of the coefficients in [21].

Our main theorem implies that the partial Cauchy data can uniquely determine any two of  $(A, B, q)$ . First we can prove that  $A$  and  $B$  are uniquely determined if  $q$  is known. Consider the operator

$$(1.18) \quad L(x, D)u = \Delta u + a(x) \frac{\partial u}{\partial x_1} + b(x) \frac{\partial u}{\partial x_2} + q(x)u.$$

Here  $a, b, q$  are complex-valued functions. Let us define the following set of partial Cauchy data

$$\widetilde{C}_{a,b} = \left\{ \left( u|_{\widetilde{\Gamma}}, \frac{\partial u}{\partial \nu} |_{\widetilde{\Gamma}} \right) \mid \Delta u + a(x) \frac{\partial u}{\partial x_1} + b(x) \frac{\partial u}{\partial x_2} + q(x)u = 0 \text{ in } \Omega, u|_{\Gamma_0} = 0, u \in H^1(\Omega) \right\}.$$

We have

**Corollary 1.4.** *Let  $\alpha \in (0, 1)$  and two pairs of complex-valued coefficients  $(a^{(1)}, b^{(1)}) \in C^{5+\alpha}(\overline{\Omega}) \times C^{5+\alpha}(\overline{\Omega})$  and  $(a^{(2)}, b^{(2)}) \in C^{5+\alpha}(\overline{\Omega}) \times C^{5+\alpha}(\overline{\Omega})$  satisfy  $\widetilde{C}_{a^{(1)}, b^{(1)}} = \widetilde{C}_{a^{(2)}, b^{(2)}}$ . Then  $(a^{(1)}, b^{(1)}) \equiv (a^{(2)}, b^{(2)})$ .*

*Proof.* Taking into account that  $\frac{\partial}{\partial x_1} = \left( \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \right)$  and  $\frac{\partial}{\partial x_2} = i \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right)$ , we can rewrite the operator (1.18) in the form

$$L(x, D)u = \Delta u + (a(x) + ib(x)) \frac{\partial u}{\partial z} + (a(x) - ib(x)) \frac{\partial u}{\partial \bar{z}} + q(x)u.$$

For the pairs  $(a^{(1)}, b^{(1)})$  and  $(a^{(2)}, b^{(2)})$ , let the corresponding operators defined by (1.18) have the same partial Cauchy data. Denote  $2A_k(x) = a^{(k)}(x) + ib^{(k)}(x)$  and  $2B_k(x) = a^{(k)}(x) - ib^{(k)}(x)$ . By (1.8), we have

$$(1.19) \quad -2\frac{\partial}{\partial z}(A_1 - A_2) - (B_1 - B_2)A_1 - (A_1 - A_2)B_2 = 0 \quad \text{in } \Omega,$$

$$(1.20) \quad -2\frac{\partial}{\partial \bar{z}}(B_1 - B_2) - (A_1 - A_2)B_1 - (B_1 - B_2)A_2 = 0 \quad \text{in } \Omega.$$

Applying to equation (1.19) the operator  $2\frac{\partial}{\partial \bar{z}}$  and to equation (1.20) the operator  $2\frac{\partial}{\partial z}$ , we have

$$(1.21) \quad -\Delta(A_1 - A_2) - 2\frac{\partial}{\partial \bar{z}}((B_1 - B_2)A_1 + (A_1 - A_2)B_2) = 0 \quad \text{in } \Omega,$$

$$(1.22) \quad -\Delta(B_1 - B_2) - 2\frac{\partial}{\partial z}((A_1 - A_2)B_1 + (B_1 - B_2)A_2) = 0 \quad \text{in } \Omega.$$

By (1.7)

$$(A_1 - A_2)|_{\tilde{\Gamma}} = (B_1 - B_2)|_{\tilde{\Gamma}} = 0.$$

Using these identities and equations (1.19) and (1.20), we obtain

$$\frac{\partial(A_1 - A_2)}{\partial \nu}|_{\tilde{\Gamma}} = \frac{\partial(B_1 - B_2)}{\partial \nu}|_{\tilde{\Gamma}} = 0.$$

The uniqueness of the Cauchy problem for the system (1.21)-(1.22) can be proved in the standard way by using a Carleman estimate (e.g., [15]). Therefore we have  $A_1 = A_2$  and  $B_1 = B_2$  in  $\Omega$ .  $\square$

We remark that Corollary 1.4 generalizes the result of [11] uniqueness is obtained assuming that the measurements are made on the whole boundary. In dimensions  $n \geq 3$ , global uniqueness was shown in [10] for the case of full data.

Similarly to Corollary 1.4, we can prove that the partial Cauchy data can uniquely determine a potential  $q$  and one of the coefficients  $A$  and  $B$  in (1.1).

**Corollary 1.5.** *For  $j = 1, 2$ , let  $(A_j, B_j, q_j) \in C^{5+\alpha}(\bar{\Omega}) \times C^{5+\alpha}(\bar{\Omega}) \times C^{4+\alpha}(\bar{\Omega})$  for some  $\alpha \in (0, 1)$  and be complex-valued. We assume either  $A_1 = A_2$  or  $B_1 = B_2$  in  $\Omega$ . Then  $\mathcal{C}_{I, A_1, B_1, q_1} = \mathcal{C}_{I, A_2, B_2, q_2}$  implies  $(A_1, B_1, q_1) = (A_2, B_2, q_2)$ .*

Corollaries 1.4 and 1.5 mean that the partial Cauchy data on  $\tilde{\Gamma}$  uniquely determine any two coefficients of the three coefficients of a second-order elliptic operator whose principal part is the Laplacian.

The proof of Theorem 1.1 uses isothermal coordinates, the Carleman estimate obtained in section 2, and Theorem 1.3. In this case we need to prove a new Carleman estimate with degenerate harmonic weights to construct appropriate complex geometrical optics solutions. These solutions are different than the case of zero magnetic potential. The new form of these solutions considerably complicate the arguments, especially the asymptotic expansions needed to analyze the behavior of the solutions. In Section 2 we prove the Carleman estimate which we need. In Section 3 we state the estimates and asymptotics which we will use in the construction of the complex geometrical optics solutions. This construction is done in Section 4. The proof of Theorem 1.3 is completed in Section 5. In section 7 and 8 we discuss some technical lemmas needed in the previous sections.



## 2. CARLEMAN ESTIMATE

**Notations.** We use throughout the paper the following notations.  $i = \sqrt{-1}$ ,  $x_1, x_2, \xi_1, \xi_2 \in \mathbf{R}$ ,  $z = x_1 + ix_2$ ,  $\zeta = \xi_1 + i\xi_2$ ,  $\bar{z}$  denotes the complex conjugate of  $z \in \mathbf{C}$ ,  $D_k = \frac{1}{i} \frac{\partial}{\partial x_k}$ ,  $\beta = (\beta_1, \beta_2)$  where  $\beta_j \in \mathbb{N}_+$ . We identify  $x = (x_1, x_2) \in \mathbf{R}^2$  with  $z = x_1 + ix_2 \in \mathbf{C}$ . We set  $\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2})$ ,  $\partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2})$ ,  $\mathcal{O}_\epsilon = \{x \in \Omega | \text{dist}(x, \partial\Omega) \leq \epsilon\}$ . Throughout the paper we use both notations  $\partial_z$  and  $\frac{\partial}{\partial z}$ , etc. and for example we denote  $\partial_z^2 = \frac{\partial^2}{\partial z^2}$ . We say that a function  $a(x)$  is antiholomorphic in  $\Omega$  if  $\partial_z a(x)|_\Omega \equiv 0$ . The tangential derivative on the boundary is given by  $\frac{\partial}{\partial \bar{\tau}} = \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2}$ , with  $\nu = (\nu_1, \nu_2)$  the unit outer normal to  $\partial\Omega$ . The ball of radius  $\delta$  centered at  $\hat{x}$  is denoted by  $B(\hat{x}, \delta) = \{x \in \mathbf{R}^2 | |x - \hat{x}| < \delta\}$ . The corresponding sphere is denoted by  $S(\hat{x}, \delta) = \{x \in \mathbf{R}^2 | |x - \hat{x}| = \delta\}$ . If  $f : \mathbf{R}^2 \rightarrow \mathbf{R}^1$  is a function, then  $f''$  is the Hessian matrix with entries  $\frac{\partial^2 f}{\partial x_i \partial x_j}$ . Let  $\|\cdot\|_{H^{k,\tau}(\Omega)} = \|\cdot\|_{H^k(\Omega)} + |\tau|^{2k} \|\cdot\|_{L^2(\Omega)}$  be the norm in the standard semiclassical Sobolev space with inner product given by  $(\cdot, \cdot)_{H^{k,\tau}(\Omega)} = (\cdot, \cdot)_{H^k(\Omega)} + |\tau|^{2k} (\cdot, \cdot)_{L^2(\Omega)}$ . For any positive function  $d$  we introduce the space  $L_d^2(\Omega) = \{v(x) | \|v\|_{L_d^2(\Omega)} = (\int_\Omega d|v|^2 dx)^{\frac{1}{2}} < \infty\}$ .  $\mathcal{L}(X, Y)$  denotes the Banach space of all bounded linear operators from a Banach space  $X$  to another Banach space  $Y$ . By  $o_X(\frac{1}{\tau^\kappa})$  we denote a function  $f(\tau, \cdot)$  such that  $\|f(\tau, \cdot)\|_X = o(\frac{1}{\tau^\kappa})$  as  $|\tau| \rightarrow +\infty$ . Finally for any  $\tilde{x} \in \partial\Omega$  we introduce the left and right tangential derivatives as follows:

$$\mathbf{D}_+(\tilde{x})f = \lim_{s \rightarrow +0} \frac{f(\ell(s)) - f(\tilde{x})}{s}$$

where  $\ell(0) = \tilde{x}$ ,  $\ell(s)$  is a parametrization of  $\partial\Omega$  near  $\tilde{x}$ ,  $s$  is the length of the curve, and we are moving clockwise as  $s$  increases;

$$\mathbf{D}_-(\tilde{x})f = \lim_{s \rightarrow -0} \frac{f(\tilde{\ell}(s)) - f(\tilde{x})}{s}$$

where  $\tilde{\ell}(0) = \tilde{x}$ ,  $\tilde{\ell}(s)$  is the parametrization of  $\partial\Omega$  near  $\tilde{x}$ ,  $s$  is the length of the curve, and we are moving counterclockwise as  $s$  increases.

For some  $\alpha \in (0, 1)$  we consider a function  $\Phi(z) = \varphi(x_1, x_2) + i\psi(x_1, x_2) \in C^{6+\alpha}(\bar{\Omega})$  with real-valued  $\varphi$  and  $\psi$  such that

$$(2.1) \quad \frac{\partial \Phi}{\partial \bar{z}}(z) = 0 \quad \text{in } \Omega, \quad \text{Im } \Phi|_{\Gamma_0^*} = 0$$

where  $\Gamma_0^*$  is an open set on  $\partial\Omega$  such that  $\Gamma_0 \subset\subset \Gamma_0^*$ . Denote by  $\mathcal{H}$  the set of all the critical points of the function  $\Phi$

$$\mathcal{H} = \{z \in \bar{\Omega} | \frac{\partial \Phi}{\partial z}(z) = 0\}.$$

Assume that  $\Phi$  has no critical points on  $\bar{\Gamma}$ , and that all critical points are nondegenerate:

$$(2.2) \quad \mathcal{H} \cap \partial\Omega \subset \Gamma_0, \quad \frac{\partial^2 \Phi}{\partial z^2}(z) \neq 0, \quad \forall z \in \mathcal{H}.$$

Then  $\Phi$  has only a finite number of critical points and we can set:

$$(2.3) \quad \mathcal{H} \setminus \Gamma_0 = \{\tilde{x}_1, \dots, \tilde{x}_\ell\}, \quad \mathcal{H} \cap \Gamma_0 = \{\tilde{x}_{\ell+1}, \dots, \tilde{x}_{\ell+\ell'}\}.$$

The following proposition was proved in [16].

**Proposition 2.1.** *Let  $\tilde{x}$  be an arbitrary point in  $\Omega$ . There exists a sequence of functions  $\{\Phi_\epsilon\}_{\epsilon \in (0,1)}$  satisfying (2.1) such that all the critical points of  $\Phi_\epsilon$  are nondegenerate and there exists a sequence  $\{\tilde{x}_\epsilon\}, \epsilon \in (0,1)$  such that*

$$\tilde{x}_\epsilon \in \mathcal{H}_\epsilon = \{z \in \bar{\Omega} \mid \frac{\partial \Phi_\epsilon}{\partial z}(z) = 0\}, \quad \tilde{x}_\epsilon \rightarrow \tilde{x} \text{ as } \epsilon \rightarrow +0.$$

Moreover for any  $j$  from  $\{1, \dots, \mathcal{N}\}$  we have

$$\mathcal{H}_\epsilon \cap \gamma_j = \emptyset \quad \text{if } \gamma_j \cap \tilde{\Gamma} \neq \emptyset,$$

$$\mathcal{H}_\epsilon \cap \gamma_j \subset \Gamma_0 \quad \text{if } \gamma_j \cap \tilde{\Gamma} = \emptyset,$$

$$\text{Im } \Phi_\epsilon(\tilde{x}_\epsilon) \notin \{\text{Im } \Phi_\epsilon(x) \mid x \in \mathcal{H}_\epsilon \setminus \{\tilde{x}_\epsilon\}\} \text{ and } \text{Im } \Phi_\epsilon(\tilde{x}_\epsilon) \neq 0.$$

In order to prove (1.7) we need the following proposition.

**Proposition 2.2.** *Let  $\hat{\Gamma}_* \subset \subset \tilde{\Gamma}$  be an arc with left endpoint  $x_-$  and right endpoint  $x_+$  oriented clockwise. For any  $\hat{x} \in \text{Int } \Gamma_*$  there exists a function  $\Phi(z)$  which satisfies (2.1), (2.2),  $\text{Im } \Phi|_{\partial\Omega \setminus \Gamma_*} = 0$  and*

$$(2.4) \quad \hat{x} \in \mathcal{G} = \{x \in \hat{\Gamma}_* \mid \frac{\partial \text{Im } \Phi}{\partial \bar{\tau}}(x) = 0\}, \quad \text{card } \mathcal{G} < \infty,$$

$$(2.5) \quad \left(\frac{\partial}{\partial \bar{\tau}}\right)^2 \text{Im } \Phi(x) \neq 0 \quad \forall x \in \mathcal{G} \setminus \{x_-, x_+\},$$

Moreover

$$(2.6) \quad \text{Im } \Phi(\hat{x}) \neq \text{Im } \Phi(x) \quad \forall x \in \mathcal{G} \setminus \{\hat{x}\} \text{ and } \quad \text{Im } \Phi(\hat{x}) \neq 0.$$

$$(2.7) \quad \mathbf{D}_+(x_-) \left(\frac{\partial}{\partial \bar{\tau}}\right)^6 \text{Im } \Phi \neq 0, \quad \mathbf{D}_-(x_+) \left(\frac{\partial}{\partial \bar{\tau}}\right)^6 \text{Im } \Phi \neq 0.$$

*Proof.* Denote  $\hat{\Gamma}_0^* = \partial\Omega \setminus \hat{\Gamma}_*$ . Let  $\hat{x}_-, \hat{x}_+ \in \partial\Omega$  be points such that the arc  $[\hat{x}_-, \hat{x}_+] \subset (x_-, x_+)$  and  $\hat{x} \in (\hat{x}_-, \hat{x}_+)$  be an arbitrary point and  $x_0 \in (x_-, x_+)$  be another fixed point from the interval  $(\hat{x}, \hat{x}_+)$ . We claim that there exists a pair  $(\varphi, \psi) \in C^6(\bar{\Omega}) \times C^6(\bar{\Omega})$  which solves the system of Cauchy-Riemann equations in  $\Omega$  such that

$$\text{A) } \psi|_{\hat{\Gamma}_0^*} = 0, \quad \left|\frac{\partial \varphi}{\partial \bar{\tau}}\right|_{\gamma_j \setminus \hat{\Gamma}_*} > 0 \text{ if } \gamma_j \cap \hat{\Gamma}_* \neq \emptyset, \quad \frac{\partial \psi}{\partial \bar{\tau}}(\hat{x}) = 0, \quad \left(\frac{\partial}{\partial \bar{\tau}}\right)^2 \psi(\hat{x}) \neq 0,$$

$$\text{A') } \mathbf{D}_+(x_-) \left(\frac{\partial}{\partial \bar{\tau}}\right)^6 \psi(x) \neq 0, \quad \mathbf{D}_-(x_+) \left(\frac{\partial}{\partial \bar{\tau}}\right)^6 \psi(x) \neq 0,$$

B) The restriction of the function  $\psi$  to the arc  $[\hat{x}_-, \hat{x}_+]$  is a Morse function,

$$\text{C) } \frac{\partial \psi}{\partial \bar{\tau}} > 0 \text{ on } (x_-, \hat{x}_-], \quad \frac{\partial \psi}{\partial \bar{\tau}} < 0 \text{ on } [\hat{x}_+, x_+),$$

$$\text{D) } \psi(\hat{x}) \notin \{\psi(x) \mid x \in \partial\Omega \setminus \{\hat{x}\}, \frac{\partial \psi}{\partial \bar{\tau}}(x) = 0\},$$

E) if  $\gamma_j \cap \hat{\Gamma}_* = \emptyset$ , then the restriction of the function  $\varphi$  on  $\gamma_j$  has only two nondegenerate critical points.

Such a pair of functions may be constructed in the following way. Let  $\gamma_1 \cap \hat{\Gamma}_* \neq \emptyset$  and  $\gamma_j \cap \hat{\Gamma}_* = \emptyset$  for all  $j \in \{2, \dots, \mathcal{N}\}$ . First, by Corollary 7.1 in the Appendix, for some  $\alpha \in (0, 1)$ , there exists a solution  $(\tilde{\varphi}, \tilde{\psi}) \in C^{6+\alpha}(\bar{\Omega}) \times C^{6+\alpha}(\bar{\Omega})$  to the Cauchy-Riemann equations with the following boundary data

$$\tilde{\psi}|_{\partial\Omega \setminus [x_0, \hat{x}_+]} = \psi_*, \quad \frac{\partial \tilde{\varphi}}{\partial \bar{\tau}}|_{\gamma_0 \setminus [x_0, \hat{x}_+]} < \beta < 0$$

and such that if  $\gamma_j \cap \hat{\Gamma}_* = \emptyset$  the function  $\tilde{\varphi}$  has only two nondegenerate critical points located on the contour  $\gamma_j$ . The function  $\psi_*$  has the following properties:  $\psi_*|_{\hat{\Gamma}_0^*} = 0$ ,  $\frac{\partial \psi_*}{\partial \bar{\tau}} > 0$  on  $(x_-, \hat{x}_-]$ ,  $\frac{\partial \psi_*}{\partial \bar{\tau}} < 0$  on  $[\hat{x}_+, x_+)$ . The function  $\psi_*$  on the set  $[\hat{x}_-, x_0]$  has only one critical point  $\hat{x}$  and  $\psi_*(\hat{x}) \neq 0$ . On the set  $(x_0, \hat{x}_+)$  the Cauchy data is not fixed. The restriction of the function  $\tilde{\psi}$  on  $[x_0, \hat{x}_+]$  can be approximated in the space  $C^{6+\alpha}([x_0, \hat{x}_+])$  by a sequence of Morse functions  $\{g_\epsilon\}_{\epsilon \in (0,1)}$  such that

$$\left(\frac{\partial}{\partial \bar{\tau}}\right)^k \tilde{\psi}(x) = \left(\frac{\partial}{\partial \bar{\tau}}\right)^k g_\epsilon(x) \quad x \in \{\hat{x}_+, x_0\}, \quad k \in \{0, 1, \dots, 6\},$$

and

$$\psi_*(\hat{x}) \notin \{g_\epsilon(x) | \frac{\partial g_\epsilon(x)}{\partial \bar{\tau}} = 0\}.$$

Let us consider some arc  $\mathcal{J} \subset \subset (x_-, \hat{x}_-)$ . On this arc we have  $\frac{\partial \tilde{\psi}}{\partial \bar{\tau}} > 0$ , say,

$$(2.8) \quad \frac{\partial \tilde{\psi}}{\partial \bar{\tau}} > \beta' > 0 \quad \text{on } \mathcal{J} \quad \text{for some positive } \beta'.$$

Let  $(\varphi_\epsilon, \psi_\epsilon) \in C^{6+\alpha}(\bar{\Omega}) \times C^{6+\alpha}(\bar{\Omega})$  be a solution to the Cauchy-Riemann equations with boundary data  $\psi_\epsilon = 0$  on  $\partial\Omega \setminus (\mathcal{J} \cup [x_0, \hat{x}_+])$  and  $\psi_\epsilon = g_\epsilon - \tilde{\psi}$  on  $[x_0, \hat{x}_+]$  and on  $\mathcal{J}$  the Cauchy data is chosen in such a way that

$$(2.9) \quad \|\psi_\epsilon\|_{C^{6+\alpha}(\partial\Omega)} + \|\varphi_\epsilon\|_{C^{6+\alpha}(\partial\Omega)} \rightarrow 0 \quad \text{as} \quad \|g_\epsilon - \tilde{\psi}\|_{C^{6+\alpha}([x_0, \hat{x}_+])} \rightarrow 0.$$

By (2.8), (2.9) for all small positive  $\epsilon$ , the restriction of the function  $\tilde{\psi} + \psi_\epsilon$  to  $\partial\Omega$  satisfies

$$\begin{aligned} (\tilde{\psi} + \psi_\epsilon)|_{\Gamma_0^*} = 0, \quad \frac{\partial(\tilde{\psi} + \psi_\epsilon)}{\partial \nu}|_{\gamma_0 \setminus [x_0, \hat{x}_+]} < 0, \quad \frac{\partial(\tilde{\psi} + \psi_\epsilon)}{\partial \bar{\tau}} > 0 \quad \text{on } [x_-, \hat{x}_-], \\ \frac{\partial(\tilde{\psi} + \psi_\epsilon)}{\partial \bar{\tau}} < 0 \quad \text{on } [\hat{x}_+, x_+], \quad (\tilde{\psi} + \psi_\epsilon)|_{[x_0, \hat{x}_+]} = g_\epsilon, \quad (\tilde{\psi} + \psi_\epsilon)|_{[\hat{x}_-, x_0]} = \psi_*. \end{aligned}$$

If  $j \geq 2$  then the restriction of the function  $\varphi_\epsilon + \tilde{\varphi}$  on  $\gamma_j$  has only two critical points located on the contour  $\gamma_j \subset \hat{\Gamma}_0^*$ . These critical points are nondegenerate if  $\epsilon$  is sufficiently small.

Therefore the restriction of the function  $(\tilde{\psi} + \psi_\epsilon)$  on  $\hat{\Gamma}_*$  has a finite number of a critical points. Some of these points may be the critical points of  $(\tilde{\psi} + \psi_\epsilon)$  considered as a function on  $\bar{\Omega}$ . We change slightly the function  $(\tilde{\psi} + \psi_\epsilon)$  such that all of its critical points are in  $\bar{\Omega}$ . Suppose that function  $\tilde{\psi} + \psi_\epsilon$  has critical points on  $\hat{\Gamma}_*$ . Then these critical points should be among the set of critical points of the function  $g_\epsilon$ , otherwise it would be the point  $\hat{x}$ . We

denote these points by  $\hat{x}_1, \dots, \hat{x}_m$ . Let  $(\hat{\varphi}, \hat{\psi}) \in C^{6+\alpha}(\bar{\Omega}) \times C^{6+\alpha}(\bar{\Omega})$  be a solution to the Cauchy-Riemann problem (7.1) with the following boundary data

$$\hat{\psi}|_{\Gamma_0^*} = 0, \quad \hat{\psi}(\hat{x}) = 1, \quad \hat{\psi}|_{\mathcal{G} \setminus \{\hat{x}\}} = 0, \quad \left| \frac{\partial \hat{\psi}}{\partial \nu} \right|_{\overline{\gamma_0 \setminus \mathcal{J}}} > 0.$$

For all small positive  $\epsilon_1$  the function  $\tilde{\psi} + \psi_\epsilon + \epsilon_1 \hat{\psi}$  does not have a critical point on  $\partial\Omega$  and the restriction of this function on  $\tilde{\Gamma}$  has a finite number of nondegenerate critical points. Therefore we take  $(\tilde{\varphi} + \varphi_\epsilon + \epsilon_1 \hat{\varphi}, \tilde{\psi} + \psi_\epsilon + \epsilon_1 \hat{\psi})$  as the pairs of functions satisfying A) - E).

The function  $\varphi + i\psi$  with pair  $(\varphi, \psi)$  satisfying conditions A)-E) satisfies all the hypotheses of Proposition 2.2 except that some of its critical points might possibly be degenerate. In order to fix this problem we consider a perturbation of the function  $\varphi + i\psi$  which is constructed in the following way. By Proposition 7.2, there exists a holomorphic function  $w$  in  $\Omega$  such that

$$(2.10) \quad \operatorname{Im} w|_{\Gamma_0^*} = 0, \quad w|_{\mathcal{H}_0} = \frac{\partial w}{\partial z}|_{\mathcal{H}_0} = 0, \quad \frac{\partial^2 w}{\partial z^2}|_{\mathcal{H}_0} \neq 0.$$

Denote  $\Phi_\delta = \varphi + i\psi + \delta w$ . For all sufficiently small positive  $\delta$ , we have

$$\mathcal{H}_0 \subset \mathcal{H}_\delta \equiv \{x \in \Omega \mid \frac{\partial}{\partial z} \Phi_\delta(x) = 0\}.$$

We now show that for all sufficiently small positive  $\delta$ , all critical points of the function  $\Phi_\delta$  are nondegenerate. Let  $\tilde{x}$  be a critical point of the function  $\varphi + i\psi$ . If  $\tilde{x}$  is a nondegenerate critical point, by the implicit function theorem, there exists a ball  $B(\tilde{x}, \delta_1)$  such that the function  $\Phi_\delta$  in this ball has only one nondegenerate critical point for all small  $\delta$ . Let  $\tilde{x}$  be a degenerate critical point of  $\varphi + i\psi$ . Without loss of generality we may assume that  $\tilde{x} = 0$ . In some neighborhood of 0, we have  $\frac{\partial \Phi_\delta}{\partial z} = \sum_{k=1}^{\infty} c_k z^{k+\hat{k}} - \delta \sum_{k=1}^{\infty} b_k z^k$  for some natural number  $\hat{k}$  and some  $c_1 \neq 0$ . Moreover (2.10) implies  $b_1 \neq 0$ . Let  $(x_{1,\delta}, x_{2,\delta}) \in \mathcal{H}_\delta$  and  $z_\delta = x_{1,\delta} + ix_{2,\delta} \rightarrow 0$ . Then either

$$(2.11) \quad z_\delta = 0 \quad \text{or} \quad z_\delta^{\hat{k}} = \delta b_1 / c_1 + o(\delta) \quad \text{as } \delta \rightarrow 0.$$

Therefore  $\frac{\partial^2 \Phi_\delta}{\partial z^2}(z_\delta) \neq 0$  for all sufficiently small  $\delta$ .  $\square$

Let  $\alpha \in (0, 1)$  and  $\mathcal{A}, \mathcal{B} \in C^{6+\alpha}(\bar{\Omega})$  be two complex-valued solutions to the boundary value problem

$$(2.12) \quad 2 \frac{\partial \mathcal{A}}{\partial \bar{z}} = -A \quad \text{in } \Omega, \quad \operatorname{Im} \mathcal{A}|_{\Gamma_0} = 0, \quad 2 \frac{\partial \mathcal{B}}{\partial z} = -B \quad \text{in } \Omega, \quad \operatorname{Im} \mathcal{B}|_{\Gamma_0} = 0.$$

Consider the following boundary value problem

$$(2.13) \quad \frac{\partial a}{\partial \bar{z}} = 0 \quad \text{in } \Omega, \quad \frac{\partial d}{\partial z} = 0 \quad \text{in } \Omega, \quad (ae^{\mathcal{A}} + de^{\mathcal{B}})|_{\Gamma_0} = \beta.$$

The existence of such functions  $a(z)$  and  $d(\bar{z})$  is given by the following proposition.

**Proposition 2.3.** *Let  $\alpha \in (0, 1)$ ,  $\mathcal{A}$  and  $\mathcal{B}$  be as in (2.12). If  $\beta \in C^{5+\alpha}(\bar{\Gamma}_0)$  (2.13) has at least one solution  $(a, d) \in C^{5+\alpha}(\bar{\Omega}) \times C^{5+\alpha}(\bar{\Omega})$  such that*

$$(2.14) \quad \|(a, d)\|_{C^{5+\alpha}(\bar{\Omega}) \times C^{5+\alpha}(\bar{\Omega})} \leq C_1 \|\beta\|_{C^{5+\alpha}(\bar{\Gamma}_0)}.$$

If  $\beta \in H^{\frac{1}{2}}(\Gamma_0)$ , then (2.13) has at least one solution  $(a, d) \in H^1(\Omega) \times H^1(\Omega)$  such that

$$(2.15) \quad \|(a, d)\|_{H^1(\Omega) \times H^1(\Omega)} \leq C_2 \|\beta\|_{H^{\frac{1}{2}}(\Gamma_0)}.$$

*Proof.* Let  $\tilde{\Omega}$  be a domain in  $\mathbb{R}^2$  with smooth boundary such that  $\Omega \subset \tilde{\Omega}$  and there exists an open subdomain  $\tilde{\Gamma}_0 \subset \partial\tilde{\Omega}$  satisfying  $\bar{\Gamma}_0 \subset \tilde{\Gamma}_0$ . Denote  $\Gamma^* = \partial\tilde{\Omega} \setminus \tilde{\Gamma}_0$ . We extend  $\mathcal{A}, \mathcal{B}$  to  $\tilde{\Gamma}_0$  keeping the regularity and we extend  $\beta$  to  $\tilde{\Gamma}_0$  in such a way that  $\|\beta\|_{H^{\frac{1}{2}}(\tilde{\Gamma}_0)} \leq C_3 \|\beta\|_{H^{\frac{1}{2}}(\Gamma_0)}$  or  $\|\beta\|_{C^{5+\alpha}(\tilde{\Gamma}_0)} \leq C_3 \|\beta\|_{C^{5+\alpha}(\Gamma_0)}$  where the constant  $C_3$  is independent of  $\beta$ . By the trace theorem there exist a constant  $C_4$  independent of  $\beta$ , and a pair  $(r, \tilde{r})$  such that  $(re^{\mathcal{A}} + \tilde{r}e^{\mathcal{B}})|_{\tilde{\Gamma}_0} = \beta$  and if  $\beta \in H^{\frac{1}{2}}(\tilde{\Gamma}_0)$  then  $(r, \tilde{r}) \in H^1(\Omega) \times H^1(\Omega)$  and

$$\|(r, \tilde{r})\|_{H^1(\Omega) \times H^1(\Omega)} \leq C_4 \|\beta\|_{H^{\frac{1}{2}}(\Gamma_0)}.$$

Similarly if  $\beta \in C^{5+\alpha}(\tilde{\Gamma}_0)$  then  $(r, \tilde{r}) \in C^{5+\alpha}(\bar{\Omega}) \times C^{5+\alpha}(\bar{\Omega})$  and

$$\|(r, \tilde{r})\|_{C^{5+\alpha}(\bar{\Omega}) \times C^{5+\alpha}(\bar{\Omega})} \leq C_5 \|\beta\|_{C^{5+\alpha}(\Gamma_0)}.$$

Let  $f = \frac{\partial r}{\partial z}$  and  $\tilde{f} = \frac{\partial \tilde{r}}{\partial \bar{z}}$ . For any  $\epsilon$  from  $(0, 1)$  consider the extremal problem

$$J_\epsilon(p, \tilde{p}) = \|(p, \tilde{p})\|_{L^2(\tilde{\Omega})}^2 + \frac{1}{\epsilon} \left\| \frac{\partial p}{\partial z} - f \right\|_{L^2(\tilde{\Omega})}^2 + \frac{1}{\epsilon} \left\| \frac{\partial \tilde{p}}{\partial \bar{z}} - \tilde{f} \right\|_{L^2(\tilde{\Omega})}^2 \rightarrow \inf, \quad (p, \tilde{p}) \in \mathcal{K},$$

where  $\mathcal{K} = \{(h_1, h_2) \in L^2(\tilde{\Omega}) \times L^2(\tilde{\Omega}) \mid (h_1 e^{\mathcal{A}} + h_2 e^{\mathcal{B}})|_{\tilde{\Gamma}_0} = 0\}$ . Denote the solution to this extremal problem as  $(p_\epsilon, \tilde{p}_\epsilon)$ . Then

$$J'_\epsilon(p_\epsilon, \tilde{p}_\epsilon)(\delta, \tilde{\delta}) = 0 \quad \forall (\delta, \tilde{\delta}) \in \mathcal{K}.$$

Hence

$$(2.16) \quad ((p_\epsilon, \tilde{p}_\epsilon), (\delta, \tilde{\delta}))_{L^2(\tilde{\Omega})} + \frac{1}{\epsilon} \left( \frac{\partial p_\epsilon}{\partial z} - f, \frac{\partial \delta}{\partial z} \right)_{L^2(\tilde{\Omega})} + \frac{1}{\epsilon} \left( \frac{\partial \tilde{p}_\epsilon}{\partial \bar{z}} - \tilde{f}, \frac{\partial \tilde{\delta}}{\partial \bar{z}} \right)_{L^2(\tilde{\Omega})} = 0 \quad \forall (\delta, \tilde{\delta}) \in \mathcal{K}.$$

Denote  $P_\epsilon = -\frac{1}{\epsilon}(\frac{\partial p_\epsilon}{\partial z} - f)$ ,  $\tilde{P}_\epsilon = -\frac{1}{\epsilon}(\frac{\partial \tilde{p}_\epsilon}{\partial \bar{z}} - \tilde{f})$ . From (2.16) we obtain

$$(2.17) \quad \frac{\partial P_\epsilon}{\partial \bar{z}} = p_\epsilon, \quad \frac{\partial \tilde{P}_\epsilon}{\partial z} = \tilde{p}_\epsilon, \quad P_\epsilon|_{\Gamma^*} = \tilde{P}_\epsilon|_{\Gamma^*} = 0, \quad ((\nu_1 + i\nu_2)P_\epsilon e^{\mathcal{B}} - (\nu_1 - i\nu_2)\tilde{P}_\epsilon e^{\mathcal{A}})|_{\tilde{\Gamma}_0} = 0.$$

We claim that there exists a constant  $C_6$  independent of  $\epsilon$  such that

$$(2.18) \quad \|(P_\epsilon, \tilde{P}_\epsilon)\|_{H^1(\tilde{\Omega})} \leq C_6 (\|(p_\epsilon, \tilde{p}_\epsilon)\|_{L^2(\tilde{\Omega})} + \|(P_\epsilon, \tilde{P}_\epsilon)\|_{L^2(\tilde{\Omega})}).$$

It clearly suffices to prove the estimate (2.18) locally assuming that  $\text{supp}(p_\epsilon, \tilde{p}_\epsilon)$  is in a small neighborhood of zero and the vector  $(0, 1)$  is orthogonal to  $\partial\Omega$  on the intersection of this neighborhood with the boundary. Using a conformal transformation we may assume that  $\partial\Omega \cap \text{supp} P_\epsilon, \partial\Omega \cap \text{supp} \tilde{P}_\epsilon \subset \{x_1 = 0\}$ . In order to prove (2.18) we consider the system of equations

$$(2.19) \quad \frac{\partial \mathbf{u}}{\partial x_2} + \hat{B} \frac{\partial \mathbf{u}}{\partial x_1} = \mathbf{F}, \quad \text{supp } \mathbf{u} \subset B(0, \delta) \cap \{x|x_2 \geq 0\}.$$

Here  $\mathbf{u} = (u_1, u_2, u_3, u_4) = (\operatorname{Re} P_\epsilon, \operatorname{Im} P_\epsilon, \operatorname{Re} \tilde{P}_\epsilon, \operatorname{Im} \tilde{P}_\epsilon)$ ,  $\mathbf{F} = 2(\operatorname{Re} p_\epsilon, \operatorname{Im} p_\epsilon, \operatorname{Re} \tilde{p}_\epsilon, \operatorname{Im} \tilde{p}_\epsilon)$ ,  $\hat{B} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ . The matrix  $\hat{B}$  has two eigenvalues  $\pm i$  and four linearly independent eigenvectors:

$$\mathbf{q}_3 = (0, 0, 1, i), \mathbf{q}_4 = (1, -i, 0, 0) \quad \text{corresponding to the eigenvalue } -i,$$

$$\mathbf{q}_1 = (1, i, 0, 0), \mathbf{q}_2 = (0, 0, 1, -i) \quad \text{corresponding to the eigenvalue } i.$$

We set  $\mathbf{r}_1 = (\nu_1 e^{\mathcal{B}}, -\nu_2 e^{\mathcal{B}}, -\nu_1 e^{\mathcal{A}}, -\nu_2 e^{\mathcal{A}})$ ,  $\mathbf{r}_2 = (\nu_2 e^{\mathcal{B}}, \nu_1 e^{\mathcal{B}}, \nu_2 e^{\mathcal{A}}, -\nu_1 e^{\mathcal{A}})$ . Consider the matrix  $D = \{d_{j\ell}\}$  where  $d_{j\ell} = r_j \cdot q_\ell$ . We have

$$D = \begin{pmatrix} (\nu_1 - i\nu_2)e^{\mathcal{B}} & -(\nu_1 - i\nu_2)e^{\mathcal{A}} \\ (\nu_2 + i\nu_1)e^{\mathcal{B}} & (\nu_2 + i\nu_1)e^{\mathcal{A}} \end{pmatrix}.$$

Since the Lopatinski determinant  $\det D \neq 0$  we obtain (2.18) (see e.g. [33]).

Next we need to get rid of the second term in the right hand side of (2.18). Suppose that for any  $\tilde{C}$  one can find  $\epsilon$  such that the estimate

$$\|(P_\epsilon, \tilde{P}_\epsilon)\|_{H^1(\tilde{\Omega})} \leq \tilde{C} \|(p_\epsilon, \tilde{p}_\epsilon)\|_{L^2(\tilde{\Omega})}$$

fails. That is, there exist a sequence  $\epsilon_k \rightarrow +\infty$  and a sequence  $\{C_{\epsilon_k}\}$  such that  $\lim_{\epsilon_k \rightarrow +\infty} C_{\epsilon_k} = +\infty$  and

$$\left\| \frac{(p_{\epsilon_k}, \tilde{p}_{\epsilon_k})}{\|(P_{\epsilon_k}, \tilde{P}_{\epsilon_k})\|_{H^1(\tilde{\Omega})}} \right\|_{L^2(\tilde{\Omega})} < \frac{1}{C_{\epsilon_k}}.$$

We set  $(Q_{\epsilon_k}, \tilde{Q}_{\epsilon_k}) = (P_{\epsilon_k}, \tilde{P}_{\epsilon_k}) / \|(P_{\epsilon_k}, \tilde{P}_{\epsilon_k})\|_{H^1(\tilde{\Omega})}$  and  $(q_{\epsilon_k}, \tilde{q}_{\epsilon_k}) = (p_{\epsilon_k}, \tilde{p}_{\epsilon_k}) / \|(P_{\epsilon_k}, \tilde{P}_{\epsilon_k})\|_{H^1(\tilde{\Omega})}$ . Then  $\|(q_{\epsilon_k}, \tilde{q}_{\epsilon_k})\|_{L^2(\tilde{\Omega})} \rightarrow 0$  as  $\epsilon_k \rightarrow 0$ . Passing to the limit in (2.17) we have

$$\frac{\partial Q}{\partial \bar{z}} = 0 \quad \text{in } \tilde{\Omega}, \quad \frac{\partial \tilde{Q}}{\partial z} = 0 \quad \text{in } \tilde{\Omega}, \quad Q|_{\Gamma^*} = \tilde{Q}|_{\Gamma^*} = 0.$$

By the uniqueness for the Cauchy problem for the operator  $\partial_z$  we conclude  $Q \equiv \tilde{Q} \equiv 0$ . On the other hand, since  $\|(Q_{\epsilon_k}, \tilde{Q}_{\epsilon_k})\|_{H^1(\tilde{\Omega})} = 1$ , we can extract a subsequence, denoted the same, which is convergent in  $L^2(\tilde{\Omega})$ . Therefore the sequence  $\{(Q_{\epsilon_k}, \tilde{Q}_{\epsilon_k})\}$  converges to zero in  $L^2(\tilde{\Omega}) \times L^2(\tilde{\Omega})$ . By (2.18), we have  $1/C_7 \leq \|(q_{\epsilon_k}, \tilde{q}_{\epsilon_k})\|_{L^2(\tilde{\Omega})} + \|(Q_{\epsilon_k}, \tilde{Q}_{\epsilon_k})\|_{L^2(\tilde{\Omega})}$ . Therefore  $\liminf_{\epsilon_k \rightarrow 0} \|(Q_{\epsilon_k}, \tilde{Q}_{\epsilon_k})\|_{L^2(\tilde{\Omega})} \neq 0$ , and this is a contradiction. Hence

$$\|(P_{\epsilon_k}, \tilde{P}_{\epsilon_k})\|_{H^1(\tilde{\Omega})} \leq C_8 \|(p_{\epsilon_k}, \tilde{p}_{\epsilon_k})\|_{L^2(\tilde{\Omega})}, \quad \forall \epsilon > 0.$$

Let us plug in (2.16) the function  $(p_{\epsilon_k}, \tilde{p}_{\epsilon_k})$  instead of  $(\delta, \tilde{\delta})$ . Then, by the above inequality, in view of the definitions of  $P_{\epsilon_k}$  and  $\tilde{P}_{\epsilon_k}$ , we have

$$\begin{aligned} \|(p_{\epsilon_k}, \tilde{p}_{\epsilon_k})\|_{L^2(\tilde{\Omega})}^2 &\leq C_9 \|(f, \tilde{f}), (P_{\epsilon_k}, \tilde{P}_{\epsilon_k})\|_{L^2(\tilde{\Omega})} \leq C_{10} \|(f, \tilde{f})\|_{L^2(\tilde{\Omega})} \|(P_{\epsilon_k}, \tilde{P}_{\epsilon_k})\|_{L^2(\tilde{\Omega})} \\ &\leq C_{11} \|(f, \tilde{f})\|_{L^2(\tilde{\Omega})} \|(p_{\epsilon_k}, \tilde{p}_{\epsilon_k})\|_{L^2(\tilde{\Omega})}. \end{aligned}$$

This inequality implies that the sequence  $(p_{\epsilon_k}, \tilde{p}_{\epsilon_k})$  is bounded in  $L^2(\tilde{\Omega})$  and

$$\left(\frac{\partial p_{\epsilon_k}}{\partial z}, \frac{\partial \tilde{p}_{\epsilon_k}}{\partial \bar{z}}\right) \rightarrow (f, \tilde{f}) \quad \text{in } L^2(\tilde{\Omega}) \times L^2(\tilde{\Omega}).$$

Then we construct a solution to (2.13) such that

$$(2.20) \quad \|(p, \tilde{p})\|_{L^2(\tilde{\Omega})} \leq C_{12} \|(f, \tilde{f})\|_{L^2(\tilde{\Omega})}.$$

Observe that we can write the boundary value problem

$$\frac{\partial p}{\partial z} = f \quad \text{in } \Omega, \quad \frac{\partial \tilde{p}}{\partial \bar{z}} = \tilde{f} \quad \text{in } \Omega, \quad (pe^A + \tilde{p}e^B)|_{\tilde{\Gamma}_0} = 0$$

in the form of system (2.19) with  $\mathbf{u} = (\operatorname{Re} p, \operatorname{Im} p, \operatorname{Re} \tilde{p}, \operatorname{Im} \tilde{p})$ ,  $\mathbf{F} = 2(\operatorname{Re} f, \operatorname{Im} f, \operatorname{Re} \tilde{f}, \operatorname{Im} \tilde{f})$ . We set  $\mathbf{r}_1 = (e^A, -e^A, -e^B, -e^B)$ ,  $\mathbf{r}_2 = (e^A, e^A, e^B, -e^B)$ . Consider the matrix  $D = \{d_{j\ell}\}$  where  $d_{j\ell} = r_j \cdot q_\ell$ . We have

$$D = \begin{pmatrix} e^B & -e^A \\ e^B & e^A \end{pmatrix}.$$

Since the Lopatinski determinant  $\det D \neq 0$  the estimate (2.20) imply (2.14) and (2.15) (see e.g., [33] Theorem 4.1.2). This completes the proof of the proposition.  $\square$

The following proposition was proven in [16]:

**Proposition 2.4.** *Let  $\Phi$  satisfy (2.1), (2.2) and the function  $C = C_1 + iC_2$  belongs to  $C^1(\bar{\Omega})$ . Let  $\tilde{f} \in L^2(\Omega)$ , and  $\tilde{v} \in H^1(\Omega)$  be a solution to*

$$(2.21) \quad 2\frac{\partial}{\partial z}\tilde{v} - \tau\frac{\partial\Phi}{\partial z}\tilde{v} + C\tilde{v} = \tilde{f} \quad \text{in } \Omega$$

or  $\tilde{v}$  be a solution to

$$(2.22) \quad 2\frac{\partial}{\partial \bar{z}}\tilde{v} - \tau\frac{\partial\bar{\Phi}}{\partial \bar{z}}\tilde{v} + C\tilde{v} = \tilde{f} \quad \text{in } \Omega.$$

In the case (2.21) we have

$$(2.23) \quad \begin{aligned} & \left\| \frac{\partial \tilde{v}}{\partial x_1} - i \operatorname{Im} \left( \tau \frac{\partial \Phi}{\partial z} - C \right) \tilde{v} \right\|_{L^2(\Omega)}^2 - \int_{\partial\Omega} (\tau(\nabla\varphi, \nu) - (\nu_1 C_1 + \nu_2 C_2)) |\tilde{v}|^2 d\sigma - \int_{\Omega} \left( \frac{\partial C_1}{\partial x_1} + \frac{\partial C_2}{\partial x_2} \right) |\tilde{v}|^2 dx \\ & \quad + \operatorname{Re} \int_{\partial\Omega} i \left( \left( \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2} \right) \tilde{v} \right) \bar{\tilde{v}} d\sigma \\ & \quad + \left\| -\frac{1}{i} \frac{\partial \tilde{v}}{\partial x_2} - \operatorname{Re} \left( \tau \frac{\partial \Phi}{\partial z} - C \right) \tilde{v} \right\|_{L^2(\Omega)}^2 = \|\tilde{f}\|_{L^2(\Omega)}^2. \end{aligned}$$

In the case (2.22) we have

$$(2.24) \quad \begin{aligned} & \left\| \frac{\partial \tilde{v}}{\partial x_1} - i \operatorname{Im} \left( \tau \frac{\partial \bar{\Phi}}{\partial \bar{z}} - C \right) \tilde{v} \right\|_{L^2(\Omega)}^2 - \int_{\partial\Omega} (\tau(\nabla\varphi, \nu) - (\nu_1 C_1 - \nu_2 C_2)) |\tilde{v}|^2 d\sigma - \int_{\Omega} \left( \frac{\partial C_1}{\partial x_1} - \frac{\partial C_2}{\partial x_2} \right) |\tilde{v}|^2 dx \\ & \quad + \operatorname{Re} \int_{\partial\Omega} i \left( \left( -\nu_2 \frac{\partial}{\partial x_1} + \nu_1 \frac{\partial}{\partial x_2} \right) \tilde{v} \right) \bar{\tilde{v}} d\sigma \\ & \quad + \left\| \frac{1}{i} \frac{\partial \tilde{v}}{\partial x_2} - \operatorname{Re} \left( \tau \frac{\partial \bar{\Phi}}{\partial \bar{z}} - C \right) \tilde{v} \right\|_{L^2(\Omega)}^2 = \|\tilde{f}\|_{L^2(\Omega)}^2. \end{aligned}$$

Consider the boundary value problem

$$\begin{cases} \mathcal{K}(x, D)u = (4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} + 2A\frac{\partial}{\partial z} + 2B\frac{\partial}{\partial \bar{z}})u = f & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

For this problem we have the following Carleman estimate with boundary terms.

**Proposition 2.5.** *Suppose that  $\Phi$  satisfies (2.1), (2.2),  $u \in H_0^1(\Omega)$  and  $\|A\|_{L^\infty(\bar{\Omega})} + \|B\|_{L^\infty(\bar{\Omega})} \leq K$ . Then there exist  $\tau_0 = \tau_0(K, \Phi)$  and  $C_{13} = C_{13}(K, \Phi)$  independent of  $u$  and  $\tau$  such that for all  $|\tau| > \tau_0$*

$$(2.25) \quad \begin{aligned} & |\tau| \|ue^{\tau\varphi}\|_{L^2(\Omega)}^2 + \|ue^{\tau\varphi}\|_{H^1(\Omega)}^2 + \|\frac{\partial u}{\partial \nu} e^{\tau\varphi}\|_{L^2(\Gamma_0)}^2 + \tau^2 \|\frac{\partial \Phi}{\partial z} |ue^{\tau\varphi}|\|_{L^2(\Omega)}^2 \\ & \leq C_{13} (\|(\mathcal{K}(x, D)u)e^{\tau\varphi}\|_{L^2(\Omega)}^2 + |\tau| \int_{\bar{\Gamma}^*} |\frac{\partial u}{\partial \nu}|^2 e^{2\tau\varphi} d\sigma). \end{aligned}$$

*Proof.* Denote  $\tilde{v} = ue^{\tau\varphi}$ ,  $\mathcal{K}(x, D)u = f$ . Observe that  $\varphi(x_1, x_2) = \frac{1}{2}(\Phi(z) + \overline{\Phi(z)})$ . Therefore

$$(2.26) \quad \begin{aligned} e^{\tau\varphi} \Delta(e^{-\tau\varphi} \tilde{v}) &= (2\frac{\partial}{\partial z} - \tau\frac{\partial \Phi}{\partial z})(2\frac{\partial}{\partial \bar{z}} - \tau\frac{\partial \bar{\Phi}}{\partial \bar{z}}) \tilde{v} = \\ & (2\frac{\partial}{\partial \bar{z}} - \tau\frac{\partial \bar{\Phi}}{\partial \bar{z}})(2\frac{\partial}{\partial z} - \tau\frac{\partial \Phi}{\partial z}) \tilde{v} = \tilde{f} = (f - 2B\frac{\partial u}{\partial \bar{z}} - 2A\frac{\partial u}{\partial z})e^{\tau\varphi}. \end{aligned}$$

Assume now that  $u$  is a real valued function. Denote  $\tilde{w} = (2\frac{\partial}{\partial \bar{z}} - \tau\frac{\partial \bar{\Phi}}{\partial \bar{z}}) \tilde{v}$ .

Thanks to the zero Dirichlet boundary condition for  $u$  we have

$$\tilde{w}|_{\partial\Omega} = 2\frac{\partial \tilde{v}}{\partial \bar{z}}|_{\partial\Omega} = (\nu_1 + i\nu_2) \frac{\partial \tilde{v}}{\partial \nu}|_{\partial\Omega}.$$

Let  $\mathcal{C}$  be some smooth function in  $\Omega$  such that

$$2\frac{\partial \mathcal{C}}{\partial z} = C(x) = C_1(x) + iC_2(x) \quad \text{in } \Omega, \quad \text{Im } \mathcal{C} = 0 \quad \text{on } \Gamma_0,$$

where  $\vec{C} = (C_1, C_2)$  is the smooth function in  $\Omega$  such that

$$\text{div } \vec{C} = 1 \quad \text{in } \Omega, \quad (\nu, \vec{C}) = -1 \quad \text{on } \Gamma_0.$$

By Proposition 2.4 we have the following integral equalities:

$$(2.27) \quad \begin{aligned} & \|\frac{\partial(\tilde{w}e^{N\mathcal{C}})}{\partial x_1} - i\text{Im}(\tau\frac{\partial \bar{\Phi}}{\partial \bar{z}} + N\mathcal{C})(\tilde{w}e^{N\mathcal{C}})\|_{L^2(\Omega)} - \int_{\partial\Omega} (\tau(\nabla\varphi, \nu) + N(\nu_1 C_1 + \nu_2 C_2)) \frac{\partial \tilde{v}}{\partial \nu} e^{N\mathcal{C}}|^2 d\sigma \\ & + N \int_{\Omega} |\tilde{w}e^{N\mathcal{C}}|^2 dx + \text{Re} \int_{\partial\Omega} i((\nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2})(\tilde{w}e^{N\mathcal{C}})) \overline{\tilde{w}e^{N\mathcal{C}}} d\sigma + \\ & + \|\frac{1}{i} \frac{\partial(\tilde{w}e^{N\mathcal{C}})}{\partial x_2} - \text{Re}(\tau\frac{\partial \Phi}{\partial z} + N\mathcal{C})(\tilde{w}e^{N\mathcal{C}})\|_{L^2(\Omega)}^2 = \|\tilde{f}e^{\tau\varphi + N\mathcal{C}}\|_{L^2(\Omega)}^2. \end{aligned}$$



We now simplify the integral  $\operatorname{Re} i \int_{\partial\Omega} ((\nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2})(\tilde{w}e^{Nc})) \overline{\tilde{w}e^{Nc}} d\sigma$ . We recall that  $\tilde{v} = ue^{\tau\varphi}$  and  $\tilde{w} = (\nu_1 + i\nu_2) \frac{\partial \tilde{v}}{\partial \nu} = (\nu_1 + i\nu_2) \frac{\partial u}{\partial \nu} e^{\tau\varphi}$ . Denote  $R + iP = (\nu_1 + i\nu_2)e^{N\operatorname{Im}c}$ . Therefore

$$(2.28) \quad \begin{aligned} & \operatorname{Re} \int_{\partial\Omega} i((\nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2})(\tilde{w}e^{Nc})) \overline{\tilde{w}e^{Nc}} d\sigma = \\ & \operatorname{Re} \int_{\partial\Omega} i((\nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2})[(R + iP) \frac{\partial u}{\partial \nu} e^{\tau\varphi + N\operatorname{Re}c}]) (R - iP) \frac{\partial u}{\partial \nu} e^{\tau\varphi + N\operatorname{Re}c} d\sigma = \\ & \operatorname{Re} \int_{\partial\Omega} i[(\nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2})(R + iP)] \left| \frac{\partial(\tilde{v}e^{Nc})}{\partial \nu} \right|^2 (R - iP) d\sigma. \end{aligned}$$

Using the above formula we obtain

$$(2.29) \quad \begin{aligned} & \left\| \frac{\partial(\tilde{w}e^{Nc})}{\partial x_1} - i\operatorname{Im}(\tau \frac{\partial \bar{\Phi}}{\partial \bar{z}} + NC)(\tilde{w}e^{Nc}) \right\|_{L^2(\Omega)} - \int_{\partial\Omega} (\tau(\nabla\varphi, \nu) + N(\nu_1 C_1 + \nu_2 C_2)) \left| \frac{\partial \tilde{v}}{\partial \nu} e^{Nc} \right|^2 d\sigma \\ & + N \int_{\Omega} |\tilde{w}e^{Nc}|^2 dx + \operatorname{Re} \int_{\partial\Omega} i[(\nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2})(R + iP)] \left| \frac{\partial(\tilde{v}e^{N\operatorname{Re}c})}{\partial \nu} \right|^2 (R - iP) d\sigma \\ & + \left\| -\frac{1}{i} \frac{\partial(\tilde{w}e^{Nc})}{\partial x_2} - \operatorname{Re}(\tau \frac{\partial \Phi}{\partial z} + NC)(\tilde{w}e^{Nc}) \right\|_{L^2(\Omega)}^2 = \|\tilde{f}e^{\tau\varphi + Nc}\|_{L^2(\Omega)}^2 \end{aligned}$$

Taking the parameter  $N$  sufficiently large positive and taking into account that the function  $R + iP$  is independent of  $N$  on  $\Gamma_0$  we conclude from (2.29)

$$(2.30) \quad \begin{aligned} & - \int_{\partial\Omega} (\tau(\nabla\varphi, \nu) + \frac{N}{2}(\nu_1 C_1 + \nu_2 C_2)) \left| \frac{\partial \tilde{v}}{\partial \nu} e^{Nc} \right|^2 d\sigma + N \int_{\Omega} |\tilde{w}e^{Nc}|^2 dx \\ & \leq \|\tilde{f}e^{\tau\varphi} e^{Nc}\|_{L^2(\Omega)}^2 + C(N) \int_{\Gamma} \left| \frac{\partial \tilde{v}}{\partial \nu} e^{Nc} \right|^2 d\sigma \end{aligned}$$

A simple computation gives

$$(2.31) \quad \begin{aligned} & 4 \left\| \frac{\partial(\tilde{v}e^{N\operatorname{Re}c})}{\partial \bar{z}} \right\|_{L^2(\Omega)}^2 + \tau^2 \left\| \frac{\partial \bar{\Phi}}{\partial \bar{z}} (\tilde{v}e^{N\operatorname{Re}c}) \right\|_{L^2(\Omega)}^2 = \left\| 2 \frac{\partial(\tilde{v}e^{N\operatorname{Re}c})}{\partial \bar{z}} - \tau \frac{\partial \bar{\Phi}}{\partial \bar{z}} (\tilde{v}e^{N\operatorname{Re}c}) \right\|_{L^2(\Omega)}^2 = \\ & \left\| e^{N\operatorname{Re}c} (2 \frac{\partial \tilde{v}}{\partial \bar{z}} - (\tau \frac{\partial \bar{\Phi}}{\partial \bar{z}} + 2 \frac{\partial N\operatorname{Re}c}{\partial \bar{z}}) \tilde{v}) \right\|_{L^2(\Omega)}^2 \\ & \leq 2 \|\tilde{w}e^{Nc}\|_{L^2(\Omega)}^2 + C(N) \|ue^{\tau\varphi}\|_{L^2(\Omega)}^2. \end{aligned}$$

Since by assumption (2.2), the function  $\Phi$  has zeros of at most second order, there exists a constant  $C_{14} > 0$  independent of  $\tau$  such that

$$(2.32) \quad \tau \|\tilde{v}e^{N\operatorname{Re}c}\|_{L^2(\Omega)}^2 \leq C_{14} (\|\tilde{v}e^{N\operatorname{Re}c}\|_{H^1(\Omega)}^2 + \tau^2 \left\| \frac{\partial \bar{\Phi}}{\partial \bar{z}} \tilde{v}e^{N\operatorname{Re}c} \right\|_{L^2(\Omega)}^2).$$

Therefore by (2.30)-(2.32) there exists  $N_0 > 0$  such that for any  $N > N_0$  there exists  $\tau_0(N)$  that

$$(2.33) \quad \begin{aligned} & - \int_{\partial\Omega} (\tau(\nabla\varphi, \nu) + \frac{N}{2}(\nu_1 C_1 + \nu_2 C_2)) \left| \frac{\partial \tilde{v}}{\partial \nu} e^{Nc} \right|^2 d\sigma + \frac{N}{2} \int_{\Omega} |\tilde{w}e^{Nc}|^2 dx \\ & \tau \|\tilde{v}e^{N\operatorname{Re}c}\|_{L^2(\Omega)}^2 + \|\tilde{v}e^{N\operatorname{Re}c}\|_{H^1(\Omega)}^2 + \tau^2 \left\| \frac{\partial \bar{\Phi}}{\partial \bar{z}} \tilde{v}e^{N\operatorname{Re}c} \right\|_{L^2(\Omega)}^2 \\ & \leq \|\tilde{f}e^{\tau\varphi} e^{Nc}\|_{L^2(\Omega)}^2 + C_{15}(N) \int_{\Gamma^*} \left| \frac{\partial \tilde{v}}{\partial \nu} e^{Nc} \right|^2 d\sigma \end{aligned}$$

In order to drop the assumption that  $u$  is the real valued function we obtain the (2.33) separately for the real and imaginary parts of  $u$  and combine them. This concludes the proof of the proposition.  $\square$

As a corollary we derive a Carleman inequality for the function  $u$  which satisfies the integral equality

$$(2.34) \quad (u, \mathcal{K}(x, D)^*w)_{L^2(\Omega)} + (f, w)_{H^{1,\tau}(\Omega)} + (ge^{\tau\varphi}, e^{-\tau\varphi}w)_{H^{\frac{1}{2},\tau}(\tilde{\Gamma})} = 0$$

for all  $w \in \mathcal{X} = \{w \in H^1(\Omega) | w|_{\Gamma_0} = 0, \mathcal{K}(x, D)^*w \in L^2(\Omega)\}$ . We have

**Corollary 2.1.** *Suppose that  $\Phi$  satisfies (2.1), (2.2),  $f \in H^1(\Omega)$ ,  $g \in H^{\frac{1}{2}}(\tilde{\Gamma})$ ,  $u \in L^2(\Omega)$  and the coefficients  $A, B \in \{C \in C^1(\bar{\Omega}) | \|C\|_{C^1(\bar{\Omega})} \leq K\}$ . Then there exist  $\tau_0 = \tau_0(K, \Phi)$  and  $C_{16} = C_{16}(K, \Phi)$ , independent of  $u$  and  $\tau$ , such that for solutions of (2.34):*

$$(2.35) \quad \|ue^{\tau\varphi}\|_{L^2(\Omega)}^2 \leq C_{16}|\tau|(\|fe^{\tau\varphi}\|_{H^{1,\tau}(\Omega)}^2 + \|ge^{\tau\varphi}\|_{H^{\frac{1}{2},\tau}(\tilde{\Gamma})}^2) \quad \forall |\tau| \geq \tau_0.$$

*Proof.* Let  $\epsilon$  be some positive number and  $d(x)$  be a smooth positive function of  $\tilde{\Gamma}$  which blow up like  $\frac{1}{|x-y|^8}$  for any  $y \in \partial\tilde{\Gamma}$ . Consider the extremal problem

$$(2.36) \quad J_\epsilon(w) = \frac{1}{2}\|we^{-\tau\varphi}\|_{L^2(\Omega)}^2 + \frac{1}{2\epsilon}\|\mathcal{K}(x, D)^*w - ue^{2\tau\varphi}\|_{L^2(\Omega)}^2 + \frac{1}{2|\tau|}\|we^{-\tau\varphi}\|_{L_a^2(\tilde{\Gamma})}^2 \rightarrow \inf,$$

$$(2.37) \quad w \in \hat{\mathcal{X}} = \{w \in H^{\frac{1}{2}}(\Omega) | \mathcal{K}(x, D)^*w \in L^2(\Omega), w|_{\Gamma_0} = 0\}.$$

There exists a unique solution to (2.36), (2.37) which we denote by  $\hat{w}_\epsilon$ . By Fermat's theorem

$$J'_\epsilon(\hat{w}_\epsilon)[\delta] = 0 \quad \forall \delta \in \hat{\mathcal{X}}.$$

Using the notation  $p_\epsilon = \frac{1}{\epsilon}(\mathcal{K}(x, D)^*\hat{w}_\epsilon - ue^{2\tau\varphi})$  this implies

$$(2.38) \quad \mathcal{K}(x, D)p_\epsilon + \hat{w}_\epsilon e^{-2\tau\varphi} = 0 \quad \text{in } \Omega, \quad p_\epsilon|_{\partial\Omega} = 0, \quad \frac{\partial p_\epsilon}{\partial\nu}|_{\tilde{\Gamma}} = d\frac{\hat{w}_\epsilon}{|\tau|}e^{-2\tau\varphi}.$$

By Proposition 2.5 we have

$$(2.39) \quad \begin{aligned} & |\tau| \|p_\epsilon e^{\tau\varphi}\|_{L^2(\Omega)}^2 + \|p_\epsilon e^{\tau\varphi}\|_{H^1(\Omega)}^2 + \left\| \frac{\partial p_\epsilon}{\partial\nu} e^{\tau\varphi} \right\|_{L^2(\Gamma_0)}^2 + \tau^2 \left\| \frac{\partial\Phi}{\partial z} |p_\epsilon e^{\tau\varphi}| \right\|_{L^2(\Omega)}^2 \\ & \leq C_{11} (\|\hat{w}_\epsilon e^{-\tau\varphi}\|_{L^2(\Omega)}^2 + \frac{1}{|\tau|} \int_{\tilde{\Gamma}^*} |\hat{w}_\epsilon|^2 e^{-2\tau\varphi} d\sigma) \leq 2C_{17} J_\epsilon(\hat{w}_\epsilon). \end{aligned}$$

Taking the scalar product of equation (2.38) with  $\hat{w}_\epsilon$  we obtain

$$2J_\epsilon(\hat{w}_\epsilon) + (ue^{2\tau\varphi}, p_\epsilon)_{L^2(\Omega)} = 0.$$

Applying to the second term of the above equality estimate (2.39) we have

$$|\tau| J_\epsilon(\hat{w}_\epsilon) \leq C_{18} \|ue^{\tau\varphi}\|_{L^2(\Omega)}^2.$$

Using this estimate we pass to the limit in (2.38) as  $\epsilon$  goes to zero. We obtain

$$(2.40) \quad \mathcal{K}(x, D)p + \hat{w}e^{-2\tau\varphi} = 0 \quad \text{in } \Omega, \quad p|_{\partial\Omega} = 0, \quad \frac{\partial p}{\partial\nu}|_{\tilde{\Gamma}} = d\frac{\hat{w}}{|\tau|}e^{-2\tau\varphi},$$

$$(2.41) \quad \mathcal{K}(x, D)^*\hat{w} - ue^{2\tau\varphi} = 0 \quad \text{in } \Omega, \quad \hat{w}|_{\Gamma_0} = 0,$$

and

$$(2.42) \quad |\tau| \|\widehat{w}e^{-\tau\varphi}\|_{L^2(\Omega)}^2 + \|\widehat{w}e^{-\tau\varphi}\|_{L^2(\tilde{\Gamma})}^2 \leq C_{19} \|ue^{\tau\varphi}\|_{L^2(\Omega)}^2.$$

Since  $\widehat{w} \in L^2(\Omega)$  we have  $p \in H^2(\Omega)$  and by the trace theorem  $\frac{\partial p}{\partial \nu} \in H^{\frac{1}{2}}(\partial\Omega)$ . The relation (2.40) implies  $\widehat{w} \in H^{\frac{1}{2}}(\tilde{\Gamma})$ . But since  $\widehat{w} \in L_d^2(\tilde{\Gamma})$  and  $\widehat{w}|_{\Gamma_0} = 0$  we have  $\widehat{w} \in H^{\frac{1}{2}}(\partial\Omega)$ . By (2.39)-(2.42) we get

$$(2.43) \quad \|\widehat{w}e^{-\tau\varphi}\|_{H^{\frac{1}{2},\tau}(\partial\Omega)} \leq C_{20} |\tau|^{\frac{1}{2}} \|ue^{\tau\varphi}\|_{L^2(\Omega)}.$$

Taking the scalar product of (2.41) with  $\widehat{w}e^{-2\tau\varphi}$  and using the estimates (2.43), (2.42) we get

$$(2.44) \quad \frac{1}{|\tau|} \|\nabla \widehat{w}e^{-\tau\varphi}\|_{L^2(\Omega)}^2 + |\tau| \|\widehat{w}e^{-\tau\varphi}\|_{L^2(\Omega)}^2 + \frac{1}{|\tau|} \|\widehat{w}e^{-\tau\varphi}\|_{H^{\frac{1}{2},\tau}(\tilde{\Gamma})}^2 \leq C_{21} \|ue^{\tau\varphi}\|_{L^2(\Omega)}^2.$$

From this estimate and a standard duality argument, the statement of Corollary 2.1 follows immediately.  $\square$

Consider the following problem

$$(2.45) \quad L(x, D)u = fe^{\tau\varphi} \quad \text{in } \Omega, \quad u|_{\Gamma_0} = ge^{\tau\varphi}.$$

We have

**Proposition 2.6.** *Let  $A, B \in C^{5+\alpha}(\overline{\Omega})$ ,  $q \in L^\infty(\Omega)$  and  $\epsilon, \alpha$  be a small positive numbers. There exists  $\tau_0 > 0$  such that for all  $|\tau| > \tau_0$  there exists a solution to the boundary value problem (2.45) such that*

$$(2.46) \quad \frac{1}{\sqrt{|\tau|}} \|\nabla ue^{-\tau\varphi}\|_{L^2(\Omega)} + \sqrt{|\tau|} \|ue^{-\tau\varphi}\|_{L^2(\Omega)} \leq C_{22} (\|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2},\tau}(\Gamma_0)}).$$

*Let  $\epsilon$  be a sufficiently small positive number. If  $\text{supp } f \subset G_\epsilon = \{x \in \Omega | \text{dist}(x, \mathcal{H}) > \epsilon\}$  and  $g = 0$  then there exists  $\tau_0 > 0$  such that for all  $|\tau| > \tau_0$  there exists a solution to the boundary value problem (2.45) such that*

$$(2.47) \quad \|\nabla ue^{-\tau\varphi}\|_{L^2(\Omega)} + |\tau| \|ue^{-\tau\varphi}\|_{L^2(\Omega)} \leq C_{23}(\epsilon) \|f\|_{L^2(\Omega)}.$$

*Proof.* First we reduce the problem (2.45) to the case  $g = 0$ . Let  $r(z)$  be a holomorphic function and  $\tilde{r}(\bar{z})$  be an antiholomorphic function such that  $(e^{\mathcal{A}}r + e^{\mathcal{B}}\tilde{r})|_{\Gamma_0} = g$  where  $\mathcal{A}, \mathcal{B} \in C^{6+\alpha}(\overline{\Omega})$  are defined as in (2.12). The existence of such functions  $r, \tilde{r}$  follows from Proposition 2.3, and these functions can be chosen in such a way that

$$\|r\|_{H^1(\Omega)} + \|\tilde{r}\|_{H^1(\Omega)} \leq C_{24} \|g\|_{H^{\frac{1}{2}}(\Gamma_0)}.$$

We look for a solution  $u$  in the form

$$u = (e^{\mathcal{A}+\tau\Phi}r + e^{\mathcal{B}+\tau\bar{\Phi}}\tilde{r}) + \tilde{u},$$

where

$$(2.48) \quad L(x, D)\tilde{u} = \tilde{f}e^{\tau\varphi} \quad \text{in } \Omega, \quad \tilde{u}|_{\Gamma_0} = 0$$

and  $\tilde{f} = f - (q - 2\frac{\partial A}{\partial z} - AB)e^{\mathcal{A}}re^{i\tau\psi} - (q - 2\frac{\partial B}{\partial \bar{z}} - AB)e^{\mathcal{B}}\tilde{r}e^{-i\tau\psi}$ .

In order to prove (2.46) we consider the following extremal problem:

$$(2.49) \quad \tilde{I}_\epsilon(u) = \frac{1}{2} \|ue^{-\tau\varphi}\|_{H^{1,\tau}(\Omega)}^2 + \frac{1}{2\epsilon} \|L(x, D)u - \tilde{f}e^{\tau\varphi}\|_{L^2(\Omega)}^2 + \frac{1}{2} \|ue^{-\tau\varphi}\|_{H^{\frac{1}{2},\tau}(\tilde{\Gamma})}^2 \rightarrow \inf,$$

$$(2.50) \quad u \in \mathcal{Y} = \{w \in H^1(\Omega) | w|_{\Gamma_0} = 0, L(x, D)w \in L^2(\Omega)\}.$$

There exists a unique solution to problem (2.49), (2.50) which we denote as  $\hat{u}_\epsilon$ . By Fermat's theorem

$$(2.51) \quad \tilde{I}'_\epsilon(\hat{u}_\epsilon)[\delta] = 0 \quad \forall \delta \in \mathcal{Y}.$$

Let  $p_\epsilon = \frac{1}{\epsilon}(L(x, D)\hat{u}_\epsilon - \tilde{f}e^{\tau\varphi})$ . Applying Corollary 2.1 we obtain from (2.51)

$$(2.52) \quad \frac{1}{|\tau|} \|p_\epsilon e^{\tau\varphi}\|_{L^2(\Omega)}^2 \leq C_{25} (\|\hat{u}_\epsilon e^{-\tau\varphi}\|_{H^{1,\tau}(\Omega)}^2 + \|\hat{u}_\epsilon e^{-\tau\varphi}\|_{H^{\frac{1}{2},\tau}(\tilde{\Gamma})}^2) \leq 2C_{25} \tilde{I}_\epsilon(\hat{u}_\epsilon).$$

Substituting in (2.51) with  $\delta = \hat{u}_\epsilon$  we get

$$2\tilde{I}_\epsilon(\hat{u}_\epsilon) + (\tilde{f}e^{\tau\varphi}, p_\epsilon)_{L^2(\Omega)} = 0.$$

Applying to this equality estimate (2.52) we have

$$\tilde{I}_\epsilon(\hat{u}_\epsilon) \leq C_{26} |\tau| \|\tilde{f}\|_{L^2(\Omega)}^2.$$

Using this estimate we pass to the limit as  $\epsilon \rightarrow +0$ . We obtain

$$(2.53) \quad L(x, D)u - \tilde{f}e^{\tau\varphi} = 0 \quad \text{in } \Omega, \quad u|_{\Gamma_0} = 0,$$

and

$$(2.54) \quad \|ue^{-\tau\varphi}\|_{H^{1,\tau}(\Omega)}^2 + \|ue^{-\tau\varphi}\|_{L^2(\tilde{\Gamma})}^2 \leq C_{27} |\tau| \|\tilde{f}\|_{L^2(\Omega)}^2.$$

Since  $\|\tilde{f}\|_{L^2(\Omega)} \leq C_{28} (\|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\Gamma_0)})$ , inequality (2.54) implies (2.46).

In order to prove (2.47) we consider the following extremal problem

$$(2.55) \quad \tilde{J}_\epsilon(u) = \frac{1}{2} \|ue^{-\tau\varphi}\|_{L^2(\Omega)}^2 + \frac{1}{2\epsilon} \|L(x, D)u - fe^{\tau\varphi}\|_{L^2(\Omega)}^2 + \frac{1}{2|\tau|} \|ue^{-\tau\varphi}\|_{L^2_d(\tilde{\Gamma})}^2 \rightarrow \inf,$$

$$(2.56) \quad u \in \tilde{\mathcal{X}} = \{w \in H^{\frac{1}{2}}(\Omega) | L(x, D)w \in L^2(\Omega), w|_{\Gamma_0} = 0\}.$$

(Here  $d(x)$  is a smooth positive function of  $\tilde{\Gamma}$  which blow up like  $\frac{1}{|x-y|^8}$  for any  $y \in \partial\tilde{\Gamma}$ .) There exists a unique solution to problem (2.55), (2.56) which we denote as  $\hat{u}_\epsilon$ . By Fermat's theorem

$$\tilde{J}'_\epsilon(\hat{u}_\epsilon)[\delta] = 0 \quad \forall \delta \in \tilde{\mathcal{X}}.$$

This equality implies

$$(2.57) \quad L(x, D)^* p_\epsilon + \hat{u}_\epsilon e^{-2\tau\varphi} = 0 \quad \text{in } \Omega, \quad \hat{p}_\epsilon|_{\partial\Omega} = 0, \quad \frac{\partial p_\epsilon}{\partial \nu}|_{\tilde{\Gamma}} = d \frac{\hat{u}_\epsilon}{|\tau|} e^{-2\tau\varphi}.$$

By Proposition 2.5

$$\frac{1}{|\tau|} \|p_\epsilon e^{\tau\varphi}\|_{H^{1,\tau}(\Omega)}^2 + \left\| \frac{\partial p_\epsilon}{\partial \nu} e^{\tau\varphi} \right\|_{L^2(\Gamma_0)}^2 + \tau^2 \left\| \frac{\partial \Phi}{\partial z} |p_\epsilon e^{\tau\varphi}| \right\|_{L^2(\Omega)}^2$$

$$(2.58) \quad \leq C_{29}(\|\widehat{u}_\epsilon e^{-\tau\varphi}\|_{L^2(\Omega)}^2 + \frac{1}{|\tau|} \int_{\widetilde{\Gamma}^*} |\widehat{u}_\epsilon|^2 e^{-2\tau\varphi} d\sigma) \leq C_{30} \widetilde{J}_\epsilon(\widehat{u}_\epsilon).$$

Taking the scalar product of equation (2.57) with  $\widehat{u}_\epsilon$  we obtain

$$2\widetilde{J}_\epsilon(\widehat{u}_\epsilon) + (fe^{\tau\varphi}, p_\epsilon)_{L^2(\Omega)} = 0.$$

Applying to this equality estimate (2.58) we have

$$(2.59) \quad |\tau|^2 \widetilde{J}_\epsilon(\widehat{u}_\epsilon) \leq C_{31} \|f\|_{L^2(\Omega)}^2.$$

Using this estimate we pass to the limit in (2.57). We conclude that

$$(2.60) \quad L(x, D)^* p + ue^{-2\tau\varphi} = 0 \quad \text{in } \Omega, \quad p|_{\partial\Omega} = 0, \quad \frac{\partial p}{\partial\nu}|_{\widetilde{\Gamma}} = \frac{u}{|\tau|} e^{-2\tau\varphi},$$

$$(2.61) \quad L(x, D)u - fe^{\tau\varphi} = 0 \quad \text{in } \Omega, \quad u|_{\Gamma_0} = 0.$$

Moreover (2.59) implies

$$(2.62) \quad |\tau|^2 \|ue^{-\tau\varphi}\|_{L^2(\Omega)}^2 + |\tau| \|ue^{-\tau\varphi}\|_{L^2(\widetilde{\Gamma})}^2 \leq C_{32} \|f\|_{L^2(\Omega)}^2.$$

This finishes the proof of the proposition.  $\square$

### 3. ESTIMATES AND ASYMPTOTICS

In this section we prove some estimates and obtain asymptotic expansions needed in the construction of the complex geometrical optics solutions in Section 4.

Consider the operator

$$(3.1) \quad \begin{aligned} L_1(x, D) &= 4\frac{\partial}{\partial z}\frac{\partial}{\partial\bar{z}} + 2A_1\frac{\partial}{\partial z} + 2B_1\frac{\partial}{\partial\bar{z}} + q_1 = \\ &= (2\frac{\partial}{\partial z} + B_1)(2\frac{\partial}{\partial\bar{z}} + A_1) + q_1 - 2\frac{\partial A_1}{\partial z} - A_1B_1 = \\ &= (2\frac{\partial}{\partial\bar{z}} + A_1)(2\frac{\partial}{\partial z} + B_1) + q_1 - 2\frac{\partial B_1}{\partial\bar{z}} - A_1B_1. \end{aligned}$$

Let  $\mathcal{A}_1, \mathcal{B}_1 \in C^{6+\alpha}(\overline{\Omega})$ , with some  $\alpha \in (0, 1)$ , satisfy

$$(3.2) \quad 2\frac{\partial \mathcal{A}_1}{\partial\bar{z}} = -A_1 \quad \text{in } \Omega, \quad \text{Im } \mathcal{A}_1|_{\Gamma_0} = 0, \quad 2\frac{\partial \mathcal{B}_1}{\partial z} = -B_1 \quad \text{in } \Omega, \quad \text{Im } \mathcal{B}_1|_{\Gamma_0} = 0$$

and let  $\mathcal{A}_2, \mathcal{B}_2 \in C^{6+\alpha}(\overline{\Omega})$  be defined similarly. Observe that

$$(2\frac{\partial}{\partial\bar{z}} + A_1)e^{\mathcal{A}_1} = 0 \quad \text{in } \Omega, \quad (2\frac{\partial}{\partial z} + B_1)e^{\mathcal{B}_1} = 0 \quad \text{in } \Omega.$$

Therefore if  $a(z), \Phi(z)$  are holomorphic functions and  $b(\bar{z})$  is an antiholomorphic function, we have

$$\begin{aligned} L_1(x, D)(e^{\mathcal{A}_1} a e^{\tau\Phi}) &= (q_1 - 2\frac{\partial A_1}{\partial z} - A_1B_1)e^{\mathcal{A}_1} a e^{\tau\Phi}, \\ L_1(x, D)(e^{\mathcal{B}_1} b e^{\tau\bar{\Phi}}) &= (q_1 - 2\frac{\partial B_1}{\partial\bar{z}} - A_1B_1)e^{\mathcal{B}_1} b e^{\tau\bar{\Phi}}. \end{aligned}$$

Let us introduce the operators:

$$\partial_{\bar{z}}^{-1} g = \frac{1}{2\pi i} \int_{\Omega} \frac{g(\xi_1, \xi_2)}{\zeta - z} d\zeta \wedge d\bar{\zeta} = -\frac{1}{\pi} \int_{\Omega} \frac{g(\xi_1, \xi_2)}{\zeta - z} d\xi_1 d\xi_2,$$

$$\partial_z^{-1}g = -\frac{1}{2\pi i} \int_{\Omega} \frac{\overline{g(\xi_1, \xi_2)}}{\zeta - z} d\zeta \wedge d\bar{\zeta} = -\frac{1}{\pi} \int_{\Omega} \frac{g(\xi_1, \xi_2)}{\bar{\zeta} - \bar{z}} d\xi_1 d\xi_2.$$

We have (e.g., p.47, 56, 72 in [32]):

**Proposition 3.1. A)** *Let  $m \geq 0$  be an integer number and  $\alpha \in (0, 1)$ . Then  $\partial_{\bar{z}}^{-1}, \partial_z^{-1} \in \mathcal{L}(C^{m+\alpha}(\bar{\Omega}), C^{m+\alpha+1}(\bar{\Omega}))$ .*

**B)** *Let  $1 \leq p \leq 2$  and  $1 < \gamma < \frac{2p}{2-p}$ . Then  $\partial_{\bar{z}}^{-1}, \partial_z^{-1} \in \mathcal{L}(L^p(\Omega), L^\gamma(\Omega))$ .*

**C)** *Let  $1 < p < \infty$ . Then  $\partial_{\bar{z}}^{-1}, \partial_z^{-1} \in \mathcal{L}(L^p(\Omega), W_p^1(\Omega))$ .*

Assume that  $\mathcal{A}, \mathcal{B}, A, B$  satisfy (2.12). Setting  $T_B g = e^{\mathcal{B}} \partial_z^{-1}(e^{-\mathcal{B}} g)$  and  $P_A g = e^{\mathcal{A}} \partial_{\bar{z}}^{-1}(e^{-\mathcal{A}} g)$ , for any  $g \in C^\alpha(\bar{\Omega})$  we have

$$\left(2\frac{\partial}{\partial z} + B\right)T_B g = g \quad \text{in } \Omega, \quad \left(2\frac{\partial}{\partial \bar{z}} + A\right)P_A g = g \quad \text{in } \Omega.$$

We define two other operators:

$$(3.3) \quad \mathcal{R}_{\tau, A} g = \frac{1}{2} e^{\mathcal{A}} e^{\tau(\bar{\Phi} - \Phi)} \partial_{\bar{z}}^{-1}(g e^{-\mathcal{A}} e^{\tau(\Phi - \bar{\Phi})}), \quad \tilde{\mathcal{R}}_{\tau, B} g = \frac{1}{2} e^{\mathcal{B}} e^{\tau(\bar{\Phi} - \Phi)} \partial_z^{-1}(g e^{-\mathcal{B}} e^{\tau(\Phi - \bar{\Phi})})$$

for  $\mathcal{A}, \mathcal{B}, A, B$  satisfying (2.12).

The following proposition follows from straightforward calculations.

**Proposition 3.2.** *Let  $g \in C^\alpha(\bar{\Omega})$  for some positive  $\alpha$ . The function  $\mathcal{R}_{\tau, A} g$  is a solution to*

$$(3.4) \quad 2\frac{\partial}{\partial \bar{z}} \mathcal{R}_{\tau, A} g - 2\tau \frac{\partial \bar{\Phi}}{\partial \bar{z}} \mathcal{R}_{\tau, A} g + A \mathcal{R}_{\tau, A} g = g \quad \text{in } \Omega.$$

The function  $\tilde{\mathcal{R}}_{\tau, B} g$  solves

$$(3.5) \quad 2\frac{\partial}{\partial z} \tilde{\mathcal{R}}_{\tau, B} g + 2\tau \frac{\partial \Phi}{\partial z} \tilde{\mathcal{R}}_{\tau, B} g + B \tilde{\mathcal{R}}_{\tau, B} g = g \quad \text{in } \Omega.$$

Using the stationary phase argument (e.g., Bleistein and Handelsman [4]), we will show

**Proposition 3.3.** *Let  $g \in L^1(\Omega)$  and a function  $\Phi$  satisfy (2.1), (2.2). Then*

$$\lim_{|\tau| \rightarrow +\infty} \int_{\Omega} g e^{\tau(\Phi(z) - \bar{\Phi}(\bar{z}))} dx = 0.$$

*Proof.* Let  $\{g_k\}_{k=1}^\infty \in C_0^\infty(\Omega)$  be a sequence of functions such that  $g_k \rightarrow g$  in  $L^1(\Omega)$ . Let  $\epsilon > 0$  be arbitrary. Suppose that  $\hat{j}$  is large enough such that  $\|g - g_{\hat{j}}\|_{L^1(\Omega)} \leq \frac{\epsilon}{2}$ . Then

$$\left| \int_{\Omega} g e^{\tau(\Phi(z) - \bar{\Phi}(\bar{z}))} dx \right| \leq \left| \int_{\Omega} (g - g_{\hat{j}}) e^{\tau(\Phi(z) - \bar{\Phi}(\bar{z}))} dx \right| + \left| \int_{\Omega} g_{\hat{j}} e^{\tau(\Phi(z) - \bar{\Phi}(\bar{z}))} dx \right|.$$

The first term on the right-hand side of this inequality is less than  $\epsilon/2$  and the second goes to zero as  $|\tau|$  approaches to infinity by the stationary phase argument (see e.g. [4]).  $\square$

We have

**Proposition 3.4.** *Let  $g \in C^2(\Omega)$ ,  $g|_{\mathcal{O}_\epsilon} = 0$  and  $g|_{\mathcal{H}} = 0$ . Then for any  $1 \leq p < \infty$*

$$(3.6) \quad \left\| \mathcal{R}_{\tau, A} g + \frac{g}{2\tau \partial_z \bar{\Phi}} \right\|_{L^p(\Omega)} + \left\| \tilde{\mathcal{R}}_{\tau, B} g - \frac{g}{2\tau \partial_z \Phi} \right\|_{L^p(\Omega)} = o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty.$$

*Proof.* We give a proof of the asymptotic formula for  $\widetilde{\mathcal{R}}_{\tau,B}g$ . The proof for  $\mathcal{R}_{\tau,A}g$  is similar. Let  $\widetilde{g}(\zeta, \bar{\zeta}) = ge^{-\mathcal{B}}$ . Then

$$\begin{aligned} 2e^{-\mathcal{B}}\mathcal{R}_{\tau,B}g &= -\frac{e^{\tau(\overline{\Phi}-\Phi)}}{\pi} \int_{\Omega} \frac{\widetilde{g}(\zeta, \bar{\zeta})}{\bar{\zeta} - \bar{z}} e^{\tau(\Phi(\zeta) - \overline{\Phi(\bar{\zeta})})} d\xi_1 d\xi_2 \\ &= -\frac{e^{\tau(\overline{\Phi}-\Phi)}}{\pi} \lim_{\delta \rightarrow +0} \int_{\Omega \setminus B(z, \delta)} \frac{\widetilde{g}(\zeta, \bar{\zeta})}{\bar{\zeta} - \bar{z}} e^{\tau(\Phi(\zeta) - \overline{\Phi(\bar{\zeta})})} d\xi_1 d\xi_2. \end{aligned}$$

Let  $z = x_1 + ix_2$  and  $x = (x_1, x_2)$  be not a critical point of the function  $\Phi$ . Then

$$\begin{aligned} (3.7) \quad 2e^{-\mathcal{B}}\mathcal{R}_{\tau,B}g &= -\frac{e^{\tau(\overline{\Phi}-\Phi)}}{\pi\tau} \lim_{\delta \rightarrow +0} \int_{\Omega \setminus B(x, \delta)} \frac{\widetilde{g}(\zeta, \bar{\zeta})}{\bar{\zeta} - \bar{z}} \frac{\partial_{\zeta} e^{\tau(\Phi(\zeta) - \overline{\Phi(\bar{\zeta})})}}{\partial_{\zeta} \Phi(\zeta)} d\xi_1 d\xi_2 \\ &= \frac{e^{\tau(\overline{\Phi}-\Phi)}}{\pi\tau} \lim_{\delta \rightarrow +0} \int_{\Omega \setminus B(x, \delta)} \frac{1}{\bar{\zeta} - \bar{z}} \frac{\partial}{\partial \zeta} \left( \frac{\widetilde{g}(\zeta, \bar{\zeta})}{\partial_{\zeta} \Phi(\zeta)} \right) e^{\tau(\Phi(\zeta) - \overline{\Phi(\bar{\zeta})})} d\xi_1 d\xi_2 \\ &\quad - \frac{e^{\tau(\overline{\Phi}-\Phi)}}{\pi\tau} \lim_{\delta \rightarrow +0} \int_{S(x, \delta)} \frac{\widetilde{g}(\zeta, \bar{\zeta})}{\bar{\zeta} - \bar{z}} \frac{(\tilde{\nu}_1 - i\tilde{\nu}_2)}{2\partial_{\zeta} \Phi(\zeta)} e^{\tau(\Phi(\zeta) - \overline{\Phi(\bar{\zeta})})} d\xi_1 d\xi_2. \end{aligned}$$

Since  $\widetilde{g}|_{\mathcal{H}} = 0$ , we have

$$(3.8) \quad \left| \frac{\partial}{\partial \zeta} \left( \frac{\widetilde{g}(\zeta, \bar{\zeta})}{\partial_{\zeta} \Phi(\zeta)} \right) \right| \leq C \sum_{k=1}^{\ell} \frac{\|\widetilde{g}\|_{C^1(\overline{\Omega})}}{|\zeta - \tilde{x}_k|} \in L^p(\Omega) \quad \forall p \in (1, 2).$$

Hence, passing to the limit in (3.7) we get

$$2e^{-\mathcal{B}}\mathcal{R}_{\tau,B}g = \frac{e^{\tau(\overline{\Phi}-\Phi)}}{\pi\tau} \int_{\Omega} \frac{1}{\bar{\zeta} - \bar{z}} \frac{\partial}{\partial \zeta} \left( \frac{\widetilde{g}(\zeta, \bar{\zeta})}{\partial_{\zeta} \Phi(\zeta)} \right) e^{\tau(\Phi(\zeta) - \overline{\Phi(\bar{\zeta})})} d\xi_1 d\xi_2 - \frac{\widetilde{g}(z, \bar{z})}{\tau \partial_z \Phi(z)}.$$

Denote  $G_{\tau}(x) = \int_{\Omega} \frac{1}{\bar{\zeta} - \bar{z}} \frac{\partial}{\partial \zeta} \left( \frac{\widetilde{g}(\zeta, \bar{\zeta})}{\partial_{\zeta} \Phi(\zeta)} \right) e^{\tau(\Phi(\zeta) - \overline{\Phi(\bar{\zeta})})} d\xi_1 d\xi_2$ . By Proposition 3.3, we see that

$$(3.9) \quad G_{\tau}(x) \longrightarrow 0 \quad \text{as } |\tau| \rightarrow +\infty \quad \forall x \in \overline{\Omega}.$$

Denote

$$T(\xi_1, \xi_2) = \left| \frac{\partial_{\bar{\zeta}} \widetilde{g}(\zeta, \bar{\zeta})}{\partial_{\zeta} \Phi(\zeta)} \right| \chi_{\Omega},$$

where  $\chi_{\Omega}$  is the characteristic function of  $\Omega$ .

Clearly

$$(3.10) \quad |G_{\tau}(x)| \leq \int_{\Omega} \frac{|T(\xi_1, \xi_2)|}{|z - \zeta|} d\xi_1 d\xi_2 \quad a.e. \text{ in } \Omega \quad \forall \tau.$$

By (3.8)  $T$  belongs to  $L^p(\mathbb{R}^2)$  for any  $p \in (1, 2)$ . For any  $f \in L^p(\mathbb{R}^2)$ , we set

$$I_r f(z) = \int_{\mathbb{R}^2} |z - \zeta|^{-\frac{2}{r}} f(\zeta, \bar{\zeta}) d\xi_1 d\xi_2.$$

Then, by the Hardy-Littlewood-Sobolev inequality, if  $r > 1$  and  $\frac{1}{r} = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)$  for  $1 < p < q < \infty$ , then

$$\|I_r f\|_{L^q(\mathbb{R}^2)} \leq C_{p,q} \|f\|_{L^p(\mathbb{R}^2)}.$$

Set  $r = 2$ . Then we have to choose  $\frac{1}{p} - \frac{1}{q} = \frac{1}{2}$ , that is, we can arbitrarily choose  $p > 2$  close to 2, so that  $q$  is arbitrarily large. Hence  $\int_{\Omega} \frac{T}{|z-\zeta|} d\xi_1 d\xi_2$  belongs to  $L^q(\Omega)$  with positive  $q$ . By (3.9), (3.10) and the dominated convergence theorem

$$G_{\tau} \rightarrow 0 \quad \text{in } L^q(\Omega) \quad \forall q \in (1, \infty).$$

The proof of the proposition is finished.  $\square$

We now consider the contribution from the critical points.

**Proposition 3.5.** *Let  $\Phi$  satisfy (2.1) and (2.2). Let  $g \in C^{4+\alpha}(\overline{\Omega})$  for some  $\alpha > 0$ ,  $g|_{\mathcal{O}_{\epsilon}} = 0$  and  $g|_{\mathcal{H}} = 0$ . Then there exist constants  $p_k$  such that*

$$(3.11) \quad \int_{\Omega} g e^{\tau(\Phi(z) - \overline{\Phi}(\overline{z}))} dx = \frac{1}{\tau^2} \sum_{k=1}^{\ell} p_k e^{2i\tau\psi(\tilde{x}_k)} + o\left(\frac{1}{\tau^2}\right) \quad \text{as } |\tau| \rightarrow +\infty.$$

*Proof.* Let  $\delta > 0$  be a sufficiently small number and  $\tilde{e}_k \in C_0^{\infty}(B(\tilde{x}_k, \delta))$ ,  $\tilde{e}_k|_{B(\tilde{x}_k, \delta/2)} \equiv 1$ . By the stationary phase argument

$$\begin{aligned} I(\tau) &= \int_{\Omega} g e^{\tau(\Phi - \overline{\Phi})} dx = \sum_{k=1}^{\ell} \int_{B(\tilde{x}_k, \delta)} \tilde{e}_k g e^{\tau(\Phi - \overline{\Phi})} dx + o\left(\frac{1}{\tau^2}\right) = \\ &= \sum_{k=1}^{\ell} e^{2i\tau\psi(\tilde{x}_k)} \int_{B(\tilde{x}_k, \delta)} \tilde{e}_k g e^{\tau(\Phi - \overline{\Phi}) - 2i\tau\psi(\tilde{x}_k)} dx + o\left(\frac{1}{\tau^2}\right) \quad \text{as } |\tau| \rightarrow +\infty. \end{aligned}$$

Since all the critical points of  $\Phi$  are nondegenerate, in some neighborhood of  $\tilde{x}_k$  one can take local coordinates such that  $\Phi - \overline{\Phi} - 2i\tau\psi(\tilde{x}_k) = z^2 - \overline{z}^2$ . Therefore

$$I(\tau) = \sum_{k=1}^{\ell} e^{2i\tau\psi(\tilde{x}_k)} \int_{B(0, \delta')} q_k e^{\tau(z^2 - \overline{z}^2)} dx + o\left(\frac{1}{\tau^2}\right) \quad \text{as } |\tau| \rightarrow +\infty,$$

where  $q_k \in C_0^4(B(0, \delta'))$  and  $q_k(0) = 0$ . Hence there exist functions  $r_{1,k}, r_{2,k} \in C_0^3(B(0, \delta'))$  such that  $q_k = 2zr_{1,k} + 2\overline{z}r_{2,k}$ . Integrating by parts, one can decompose  $I(\tau)$  as

$$\begin{aligned} I(\tau) &= -\frac{1}{\tau} \sum_{k=1}^{\ell} e^{2i\tau\psi(\tilde{x}_k)} \int_{B(0, \delta')} \left( \frac{\partial r_{1,k}}{\partial z} - \frac{\partial r_{2,k}}{\partial \overline{z}} \right) e^{\tau(z^2 - \overline{z}^2)} dx + o\left(\frac{1}{\tau^2}\right) = \\ &= -\frac{1}{\tau} \sum_{k=1}^{\ell} e^{2i\tau\psi(\tilde{x}_k)} \int_{B(0, \delta')} \left( \frac{\partial r_{1,k}}{\partial z} - \frac{\partial r_{2,k}}{\partial \overline{z}} \right)(0) \chi(x) e^{\tau(z^2 - \overline{z}^2)} dx \\ &= -\frac{1}{\tau} \sum_{k=1}^{\ell} e^{2i\tau\psi(\tilde{x}_k)} \int_{B(0, \delta')} \tilde{q}_k e^{\tau(z^2 - \overline{z}^2)} dx + o\left(\frac{1}{\tau^2}\right) \quad \text{as } |\tau| \rightarrow +\infty, \end{aligned}$$

where  $\chi, \tilde{q}_k \in C_0^2(B(0, \delta'))$ ,  $\chi|_{B(0, \delta'/2)} \equiv 1$  and  $\tilde{q}_k(0) = 0$ . Hence there exist functions  $\tilde{r}_{1,k}, \tilde{r}_{2,k} \in C_0^1(B(0, \delta'))$  such that  $\tilde{q}_k = 2z\tilde{r}_{1,k} + 2\overline{z}\tilde{r}_{2,k}$ . Integrating by parts and applying Proposition 3.4 we obtain

$$\lim_{|\tau| \rightarrow +\infty} \tau \int_{B(0, \delta')} \tilde{q}_k e^{\tau(z^2 - \overline{z}^2)} dx = - \sum_{k=1}^{\ell} e^{2i\tau\psi(\tilde{x}_k)} \lim_{|\tau| \rightarrow +\infty} \int_{B(0, \delta')} \left( \frac{\partial \tilde{r}_{1,k}}{\partial z} - \frac{\partial \tilde{r}_{2,k}}{\partial \overline{z}} \right) e^{\tau(z^2 - \overline{z}^2)} dx = 0.$$



Therefore (3.11) follows from a standard application of stationary phase. The proof of the proposition is completed.  $\square$

**Proposition 3.6.** *Let  $0 < \epsilon' < \epsilon$ , a function  $\Phi$  satisfy (2.1), (2.2) and  $\overline{\mathcal{O}}_\epsilon \cap (\mathcal{H} \setminus \Gamma_0) = \emptyset$ . Suppose that  $g \in C^\alpha(\overline{\Omega}) \cap H^1(\Omega)$  for some  $\alpha \in (0, 1)$ ,  $g|_{\mathcal{O}_\epsilon} = 0$  and  $g|_{\mathcal{H}} = 0$ . Then*

$$(3.12) \quad |\tau| \|\tilde{\mathcal{R}}_{\tau, B} g\|_{L^\infty(\mathcal{O}_{\epsilon'})} + \|\nabla \tilde{\mathcal{R}}_{\tau, B} g\|_{L^\infty(\mathcal{O}_{\epsilon'})} \leq C_1(\epsilon', \alpha) \|g\|_{C^\alpha(\overline{\Omega}) \cap H^1(\Omega)}.$$

Moreover

$$(3.13) \quad \|\nabla \tilde{\mathcal{R}}_{\tau, B} g\|_{L^2(\Omega)} + |\tau|^{\frac{1}{2}} \|\tilde{\mathcal{R}}_{\tau, B} g\|_{L^2(\Omega)} + |\tau| \left\| \frac{\partial \Phi}{\partial z} \tilde{\mathcal{R}}_{\tau, B} g \right\|_{L^2(\Omega)} \leq C_2(\epsilon', \alpha) \|g\|_{C^\alpha(\overline{\Omega}) \cap H^1(\Omega)}.$$

*Proof.* Denote  $\tilde{g} = g e^{-B}$ . Let  $x = (x_1, x_2)$  be an arbitrary point from  $\mathcal{O}_{\epsilon'}$  and  $z = x_1 + i x_2$ . Then

$$-\pi \partial_z^{-1}(e^{\tau(\Phi - \bar{\Phi})} \tilde{g}) = \int_{\Omega} \frac{\tilde{g} e^{\tau(\Phi - \bar{\Phi})}}{\bar{\zeta} - \bar{z}} d\xi_1 d\xi_2 = \lim_{\delta \rightarrow +0} \sum_{k=1}^{\ell} \int_{\Omega \setminus B(\tilde{x}_k, \delta)} \frac{\tilde{g} e^{\tau(\Phi - \bar{\Phi})}}{\bar{\zeta} - \bar{z}} d\xi_1 d\xi_2.$$

Integrating by parts and taking  $\delta$  sufficiently small we have

$$(3.14) \quad \begin{aligned} -\pi \partial_z^{-1}(e^{\tau(\Phi - \bar{\Phi})} \tilde{g}) &= -\frac{1}{\tau} \lim_{\delta \rightarrow +0} \int_{\Omega \setminus \cup_{k=1}^{\ell} B(\tilde{x}_k, \delta)} \frac{\frac{\partial \tilde{g}}{\partial \bar{\zeta}}}{(\bar{\zeta} - \bar{z}) \frac{\partial \Phi}{\partial \bar{\zeta}}} e^{\tau(\Phi - \bar{\Phi})} d\xi_1 d\xi_2 \\ &\quad + \frac{1}{\tau} \lim_{\delta \rightarrow +0} \int_{\Omega \setminus \cup_{k=1}^{\ell} B(\tilde{x}_k, \delta)} \frac{\tilde{g} \frac{\partial^2 \Phi}{\partial \bar{\zeta}^2}}{(\bar{\zeta} - \bar{z}) \left(\frac{\partial \Phi}{\partial \bar{\zeta}}\right)^2} e^{\tau(\Phi - \bar{\Phi})} d\xi_1 d\xi_2 \\ &\quad + \frac{1}{2\tau} \lim_{\delta \rightarrow +0} \int_{\cup_{k=1}^{\ell} S(\tilde{x}_k, \delta)} (\tilde{\nu}_1 - i \tilde{\nu}_2) \frac{\tilde{g}}{(\bar{\zeta} - \bar{z}) \frac{\partial \Phi}{\partial \bar{\zeta}}} e^{\tau(\Phi - \bar{\Phi})} d\sigma. \end{aligned}$$

Since  $g|_{\mathcal{H}} = 0$  we have that  $\|g\|_{C^0(S(\tilde{x}_k, \delta))} \leq \delta^\alpha \|g\|_{C^\alpha(\overline{\Omega})}$ . Using this inequality and the fact that all the critical points of  $\Phi$  are nondegenerate we obtain

$$\frac{1}{2\tau} \lim_{\delta \rightarrow +0} \int_{\cup_{k=1}^{\ell} S(\tilde{x}_k, \delta)} (\tilde{\nu}_1 - i \tilde{\nu}_2) \frac{\tilde{g}}{(\bar{\zeta} - \bar{z}) \frac{\partial \Phi}{\partial \bar{\zeta}}} e^{\tau(\Phi - \bar{\Phi})} d\sigma = 0.$$

Since  $\left| \frac{\tilde{g} \frac{\partial^2 \Phi}{\partial \bar{\zeta}^2}}{\left(\frac{\partial \Phi}{\partial \bar{\zeta}}\right)^2}(\zeta, \bar{\zeta}) \right| \leq C_3 \|\tilde{g}\|_{C^\alpha(\overline{\Omega})} \sum_{k=1}^{\ell} \frac{1}{|\zeta - \tilde{x}_k|^{2-\alpha}}$  we see that  $\frac{\tilde{g} \frac{\partial^2 \Phi}{\partial \bar{\zeta}^2}}{\left(\frac{\partial \Phi}{\partial \bar{\zeta}}\right)^2}(\zeta, \bar{\zeta}) \in L^1(\Omega)$  and

$$(3.15) \quad \begin{aligned} -\pi \partial_z^{-1}(e^{\tau(\Phi - \bar{\Phi})} \tilde{g}) &= -\frac{1}{\tau} \int_{\Omega} \frac{\frac{\partial \tilde{g}}{\partial \bar{\zeta}}}{(\bar{\zeta} - \bar{z}) \frac{\partial \Phi}{\partial \bar{\zeta}}} e^{\tau(\Phi - \bar{\Phi})} d\xi_1 d\xi_2 \\ &\quad + \frac{1}{\tau} \int_{\Omega} \frac{\tilde{g} \frac{\partial^2 \Phi}{\partial \bar{\zeta}^2}}{(\bar{\zeta} - \bar{z}) \left(\frac{\partial \Phi}{\partial \bar{\zeta}}\right)^2} e^{\tau(\Phi - \bar{\Phi})} d\xi_1 d\xi_2. \end{aligned}$$

From this equality, Proposition 3.3 and definition (3.3) of the operator  $\tilde{\mathcal{R}}_{\tau, B}$ , the estimate (3.12) follows immediately. To prove (3.13) we observe

$$\frac{\partial \tilde{\mathcal{R}}_{\tau, B} g}{\partial \bar{z}} = \frac{\partial \mathcal{B}}{\partial \bar{z}} \tilde{\mathcal{R}}_{\tau, B} g + \tilde{\mathcal{R}}_{\tau, B} \left\{ \frac{\partial g}{\partial \bar{z}} - \frac{\partial \mathcal{B}}{\partial \bar{z}} g \right\} + \frac{\tau}{2\pi} e^{\tau(\Phi - \bar{\Phi}) + \mathcal{B}} \int_{\Omega} \frac{\frac{\partial \bar{\Phi}(\zeta)}{\partial \bar{\zeta}} - \frac{\partial \bar{\Phi}(z)}{\partial \bar{z}}}{\bar{\zeta} - \bar{z}} \tilde{g} e^{\tau(\Phi - \bar{\Phi})} d\xi_1 d\xi_2.$$

By Proposition 3.1

$$\left\| \frac{\partial \mathcal{B}}{\partial \bar{z}} \tilde{\mathcal{R}}_{\tau, B} g + \tilde{\mathcal{R}}_{\tau, B} \left\{ \frac{\partial g}{\partial \bar{z}} - \frac{\partial \mathcal{B}}{\partial \bar{z}} g \right\} \right\|_{L^2(\Omega)} \leq C_4 \|g\|_{H^1(\Omega)}.$$

Using arguments similar to (3.14), (3.15) we obtain

$$\left\| \frac{\tau}{2\pi} \int_{\Omega} \frac{\frac{\partial \bar{\Phi}(\zeta)}{\partial \bar{\zeta}} - \frac{\partial \bar{\Phi}(z)}{\partial \bar{z}}}{\bar{\zeta} - \bar{z}} \tilde{g} e^{\tau(\Phi - \bar{\Phi})} d\xi_1 d\xi_2 \right\|_{L^2(\Omega)} \leq C_5 \|g\|_{C^\alpha(\bar{\Omega}) \cap H^1(\Omega)}.$$

Hence

$$\left\| \frac{\partial \tilde{\mathcal{R}}_{\tau, B} g}{\partial \bar{z}} \right\|_{L^2(\Omega)} \leq C_6 \|g\|_{C^\alpha(\bar{\Omega}) \cap H^1(\Omega)}.$$

Combining this estimate with (3.12) we conclude

$$\|\nabla \tilde{\mathcal{R}}_{\tau, B} g\|_{L^2(\Omega)} \leq C_7 \|g\|_{C^\alpha(\bar{\Omega}) \cap H^1(\Omega)}.$$

Using this estimate and equation (3.4) we have

$$|\tau| \left\| \frac{\partial \Phi}{\partial z} \tilde{\mathcal{R}}_{\tau, B} g \right\|_{L^2(\Omega)} \leq C_8 \|g\|_{C^\alpha(\bar{\Omega}) \cap H^1(\Omega)},$$

finishing the proof of the proposition.  $\square$

Let  $e_1, e_2 \in C^\infty(\bar{\Omega})$  be functions such that

$$(3.16) \quad e_1 + e_2 = 1 \quad \text{in } \Omega,$$

$e_2$  vanishes in some neighborhood of  $\mathcal{H} \setminus \Gamma_0$  and  $e_1$  vanishes in a neighborhood of  $\partial\Omega$ .

**Proposition 3.7.** *Let for some  $\alpha \in (0, 1)$   $A, B \in C^{5+\alpha}(\bar{\Omega})$ , and the functions  $\mathcal{A}, \mathcal{B} \in C^{6+\alpha}(\bar{\Omega})$  satisfy (2.12). Let  $e_1, e_2$  be defined as in (3.16). Let  $g \in L^p(\Omega)$  for some  $p > 2$ ,  $\text{supp } g \subset \subset \text{supp } e_1$  and  $\text{dist}(\Gamma_0, \text{supp } g) > 0$ . We define  $u$  by*

$$u = \tilde{\mathcal{R}}_{\tau, B}(e_1(P_A g - \tilde{M}e^A)) + \frac{e_2(P_A g - \tilde{M}e^A)}{2\tau \partial_z \Phi},$$

where  $\tilde{M} = \tilde{M}(z)$  is a polynomial such that  $\frac{\partial^k}{\partial z^k}(P_A g - \tilde{M}e^A)|_{\mathcal{H}} = 0$  for any  $k$  from  $\{0, \dots, 6\}$ . Then we have

$$(3.17) \quad \mathcal{P}(x, D)(ue^{\tau\Phi}) \triangleq \left(2\frac{\partial}{\partial \bar{z}} + A\right)\left(2\frac{\partial}{\partial z} + B\right)(ue^{\tau\Phi}) = ge^{\tau\Phi} + \frac{e^{\tau\varphi}}{|\tau|} h_\tau \quad \text{as } |\tau| \rightarrow +\infty,$$

where

$$\|h_\tau\|_{L^\infty(\Omega)} \leq C_9(p) \|g\|_{L^p(\Omega)}$$

and for some sufficiently small positive  $\epsilon'$  we have:

$$(3.18) \quad \frac{1}{|\tau|^{\frac{1}{2}}} \|\nabla u\|_{L^2(\Omega)} + |\tau|^{\frac{1}{2}} \|u\|_{L^2(\Omega)} + \|u\|_{H^{1,\tau}(\mathcal{O}_{\epsilon'})} \leq C_{10} \|g\|_{L^p(\Omega)}.$$

*Proof.* By Proposition 3.1  $P_A g$  belongs to  $W_p^1(\Omega)$ . Since  $p > 2$ , by the Sobolev embedding theorem there exists  $\alpha > 0$  such that  $P_A g \in C^\alpha(\bar{\Omega})$ . By properties of elliptic operators and the fact that  $\text{supp } e_2 \cap \text{supp } g = \{\emptyset\}$  we have that  $P_A g \in C^5(\text{supp } e_2)$ . The estimate (3.18) follows from Proposition 3.6. Short calculations give

$$(3.19) \quad \mathcal{P}(x, D)(ue^{\tau\Phi}) = ge^{\tau\Phi} + \frac{e^{\tau\Phi}}{\tau} \mathcal{P}(x, D) \left( \frac{e_2(P_A g - \widetilde{M}e^A)}{2\partial_z \Phi} \right).$$

This formula implies (3.17) with  $h_\tau = e^{i\tau\psi} \mathcal{P}(x, D) \left( \frac{e_2(P_A g - \widetilde{M}e^A)}{2\partial_z \Phi} \right) / \text{sign } \tau$ .  $\square$

The following proposition will play a critically important role in the construction of the complex geometric optic solutions.

**Proposition 3.8.** *Let  $f \in L^p(\Omega)$  for some  $p > 2$ ,  $\text{dist}(\bar{\Gamma}_0, \text{supp } f) > 0$ ,  $q \in H^{\frac{1}{2}}(\Gamma_0)$ ,  $\epsilon'$  be a small positive number such that  $O_{\epsilon'} \cap (\mathcal{H} \setminus \Gamma_0) = \emptyset$ . Then there exists  $\tau_0$  such that for all  $|\tau| > \tau_0$  there exists a solution to the boundary value problem*

$$(3.20) \quad L(x, D)w = fe^{\tau\Phi} \quad \text{in } \Omega, \quad w|_{\Gamma_0} = qe^{\tau\varphi}/\tau$$

such that

$$\sqrt{|\tau|} \|we^{-\tau\varphi}\|_{L^2(\Omega)} + \frac{1}{\sqrt{|\tau|}} \|(\nabla w)e^{-\tau\varphi}\|_{L^2(\Omega)} + \|we^{-\tau\varphi}\|_{H^{1,\tau}(O_{\epsilon'})} \leq C_{11}(\|f\|_{L^p(\Omega)} + \|q\|_{H^{\frac{1}{2}}(\Gamma_0)}).$$

*Proof.* Let  $\chi \in C_0^\infty(\Omega)$  be equal to one in some neighborhood of the set  $\mathcal{H} \setminus \Gamma_0$ . By Proposition 2.6 there exists a solution to the problem (3.20) with inhomogeneous term  $(1 - \chi)f$  and boundary data  $q/\tau$  such that

$$(3.21) \quad \|w_1 e^{-\tau\varphi}\|_{H^{1,\tau}(\Omega)} \leq C_{12}(\|f\|_{L^2(\Omega)} + \|q\|_{H^{\frac{1}{2}}(\Gamma_0)}).$$

Denote  $w_2 = \widetilde{\mathcal{R}}_{-\tau, B}(e_1(P_A(\chi f) - \widetilde{M}e^A)) + \frac{e_2(P_A(\chi f) - \widetilde{M}e^A)}{2\tau\partial_z \Phi}$  where  $\widetilde{M} = \widetilde{M}(z)$  is a polynomial such that  $\frac{\partial^k}{\partial z^k}(P_A(\chi f) - \widetilde{M}e^A)|_{\mathcal{H}} = 0$  for any  $k$  from  $\{0, \dots, 6\}$ . Let  $q_\tau$  be the restriction of  $w_2$  to  $\Gamma_0$ . By (3.12) there exists a constant  $C_{13}$  independent of  $\tau$  such that

$$(3.22) \quad |\tau|^{\frac{1}{2}} \|q_\tau\|_{H^{\frac{1}{2}}(\Gamma_0)} \leq C_{13} \|f\|_{L^p(\Omega)}.$$

By Proposition 3.7 there exists a constant  $C_{14}$  independent of  $\tau$  such that

$$(3.23) \quad \sqrt{|\tau|} \|w_2 e^{-\tau\varphi}\|_{L^2(\Omega)} + \frac{1}{\sqrt{|\tau|}} \|\nabla w_2 e^{-\tau\varphi}\|_{L^2(\Omega)} + \|w_2 e^{-\tau\varphi}\|_{H^{1,\tau}(O_{\epsilon'})} \leq C_{14} \|f\|_{L^p(\Omega)}.$$

Let  $\tilde{a}_\tau, \tilde{b}_\tau \in H^1(\Omega)$  be holomorphic and antiholomorphic functions, respectively, such that  $(\tilde{a}_\tau e^A + \tilde{b}_\tau e^B)|_{\Gamma_0} = -q_\tau$ . By (3.22) and Proposition 2.3 there exist constants  $C_{15}, C_{16}$  independent of  $\tau$  such that

$$(3.24) \quad \|\tilde{a}_\tau\|_{H^1(\Omega)} + \|\tilde{b}_\tau\|_{H^1(\Omega)} \leq C_{15} \|q_\tau\|_{H^{\frac{1}{2}}(\Gamma_0)} \leq C_{16} \frac{\|f\|_{L^p(\Omega)}}{\sqrt{|\tau|}}.$$

The function  $W = (w_2 + \tilde{a}_\tau e^A)e^{\tau\Phi} + \tilde{b}_\tau e^{B+\tau\bar{\Phi}}$  satisfies

$$L(x, D)W = \chi f e^{\tau\Phi} + e^{\tau\varphi} \frac{\tilde{h}_\tau}{\sqrt{|\tau|}} \quad \text{in } \Omega, \quad W|_{\Gamma_0} = 0,$$

where

$$(3.25) \quad \|\tilde{h}_\tau\|_{L^2(\Omega)} \leq C_{17}\|f\|_{L^2(\Omega)}$$

with some constant  $C_{17}$  independent of  $\tau$ . By (3.23), (3.24)

$$(3.26) \quad \sqrt{|\tau|}\|We^{-\tau\varphi}\|_{L^2(\Omega)} + \frac{1}{\sqrt{|\tau|}}\|\nabla We^{-\tau\varphi}\|_{L^2(\Omega)} + \|We^{-\tau\varphi}\|_{H^{1,\tau}(\mathcal{O}_{e'})} \leq C_{18}\|f\|_{L^p(\Omega)}.$$

Let  $\tilde{W}$  be a solution to problem (2.45) with inhomogeneous term and boundary data  $f = -\frac{\tilde{h}_\tau}{\sqrt{|\tau|}}$ ,  $g \equiv 0$  respectively given by Proposition 2.6. The estimate (2.46) has the form

$$(3.27) \quad \|\tilde{W}e^{-\tau\varphi}\|_{H^{1,\tau}(\Omega)} \leq C_{19}\|\tilde{h}_\tau\|_{L^2(\Omega)} \leq C_{20}\|f\|_{L^2(\Omega)}.$$

Then the function  $w_1 + W + \tilde{W}$  solves (3.20). The estimate (6.11) follows from (3.21), (3.26) and (3.27). The proof of the proposition is completed.  $\square$

#### 4. COMPLEX GEOMETRICAL OPTICS SOLUTIONS

For a complex-valued vector field  $(A_1, B_1)$  and complex-valued potential  $q_1$  we will construct solutions to the boundary value problem

$$(4.1) \quad L_1(x, D)u_1 = 0 \quad \text{in } \Omega, \quad u_1|_{\Gamma_0} = 0$$

of the form

$$(4.2) \quad u_1(x) = a_\tau(z)e^{A_1+\tau\Phi} + d_\tau(\bar{z})e^{B_1+\tau\bar{\Phi}} + u_{11}e^{\tau\varphi} + u_{12}e^{\tau\psi}.$$

Here  $\mathcal{A}_1$  and  $\mathcal{B}_1$  are defined by (3.2) respectively for  $A_1$  and  $B_1$ ,  $a_\tau(z) = a(z) + \frac{a_1(z)}{\tau} + \frac{a_{2,\tau}(z)}{\tau^2}$ ,  $d_\tau(\bar{z}) = d(\bar{z}) + \frac{d_1(\bar{z})}{\tau} + \frac{d_{2,\tau}(\bar{z})}{\tau^2}$ ,

$$(4.3) \quad a, d \in C^{5+\alpha}(\bar{\Omega}), \quad \frac{\partial a}{\partial \bar{z}} = 0 \text{ in } \Omega, \quad \frac{\partial d}{\partial z} = 0 \text{ in } \Omega,$$

$$(4.4) \quad (ae^{A_1} + de^{B_1})|_{\Gamma_0} = 0.$$

Let  $\tilde{x}$  be some fixed point from  $\mathcal{H} \setminus \partial\Omega$ . Suppose in addition that

$$(4.5) \quad \frac{\partial^k a}{\partial z^k}|_{\mathcal{H} \cap \partial\Omega} = \frac{\partial^k d}{\partial \bar{z}^k}|_{\mathcal{H} \cap \partial\Omega} = 0 \quad \forall k \in \{0, \dots, 5\}, \quad a|_{\mathcal{H} \setminus \{\tilde{x}\}} = d|_{\mathcal{H} \setminus \{\tilde{x}\}} = 0, \quad a(\tilde{x}) \neq 0, d(\tilde{x}) \neq 0.$$

Such functions exist by Proposition 7.2.

Denote

$$g_1 = T_{B_1}((q_1 - 2\frac{\partial B_1}{\partial \bar{z}} - A_1 B_1)de^{B_1}) - M_2(\bar{z})e^{B_1}, \quad g_2 = P_{A_1}((q_1 - 2\frac{\partial A_1}{\partial z} - A_1 B_1)ae^{A_1}) - M_1(z)e^{A_1},$$

where  $M_1(z)$  and  $M_2(\bar{z})$  are polynomials such that

$$(4.6) \quad \frac{\partial^k g_1}{\partial \bar{z}^k}|_{\mathcal{H}} = \frac{\partial^k g_2}{\partial z^k}|_{\mathcal{H}} = 0 \quad \forall k \in \{0, \dots, 6\}, \quad \frac{\partial g_1}{\partial z} = \frac{\partial g_2}{\partial \bar{z}} = 0 \text{ on } \mathcal{H} \setminus \{\tilde{x}\}.$$

Thanks to our assumptions on the regularity of  $A_1, B_1$  and  $q$ ,  $g_1, g_2$  belong to  $C^{6+\alpha}(\bar{\Omega})$ .

Note that by (4.6), (4.5)

$$(4.7) \quad \frac{\partial^{k+j} g_1}{\partial z^k \partial \bar{z}^j} |_{\mathcal{H} \cap \partial \Omega} = \frac{\partial^{k+j} g_2}{\partial z^k \partial \bar{z}^j} |_{\mathcal{H} \cap \partial \Omega} = 0 \quad \text{if } k+j \leq 6.$$

The function  $a_1(z)$  is holomorphic in  $\Omega$  and  $d_1(\bar{z})$  is antiholomorphic in  $\Omega$  and

$$a_1 e^{\mathcal{A}_1} + d_1 e^{\mathcal{B}_1} = \frac{g_1}{2\partial_z \Phi} + \frac{g_2}{2\partial_z \Phi} \quad \text{on } \Gamma_0.$$

The existence of such functions is given again by Proposition 2.3. Observe that by (4.7) the functions  $\frac{e_2 g_1}{\partial_z \Phi}, \frac{e_2 g_2}{\partial_z \Phi} \in C^4(\bar{\Omega})$ . Let

$$\widehat{g}_1 = T_{B_1} \left( (q_1 - 2 \frac{\partial B_1}{\partial \bar{z}} - A_1 B_1) d_1 e^{\mathcal{B}_1} \right) - \widehat{M}_2(\bar{z}) e^{\mathcal{B}_1}, \quad \widehat{g}_2 = P_{A_1} \left( (q_1 - 2 \frac{\partial A_1}{\partial z} - A_1 B_1) a_1 e^{\mathcal{A}_1} \right) - \widehat{M}_1(z) e^{\mathcal{A}_1},$$

where  $\widehat{M}_1(z)$  and  $\widehat{M}_2(\bar{z})$  are polynomials such that

$$(4.8) \quad \frac{\partial^k \widehat{g}_1}{\partial \bar{z}^k} |_{\mathcal{H}} = \frac{\partial^k \widehat{g}_2}{\partial z^k} |_{\mathcal{H}} = 0 \quad \forall k \in \{0, \dots, 3\}.$$

Henceforth we recall (3.3). The function  $u_{11}$  is given by

$$(4.9) \quad \begin{aligned} u_{11} = & -e^{-i\tau\psi} \mathcal{R}_{-\tau, A_1} \{e_1(g_1 + \widehat{g}_1/\tau)\} - e^{-i\tau\psi} \frac{e_2(g_1 + \frac{\widehat{g}_1}{\tau})}{2\tau \partial_z \Phi} + \frac{e^{-i\tau\psi}}{4\tau^2 \partial_z \Phi} L_1(x, D) \left( \frac{e_2 g_1}{\partial_z \Phi} \right) \\ & - e^{i\tau\psi} \widetilde{\mathcal{R}}_{\tau, B_1} \{e_1(g_2 + \widehat{g}_2/\tau)\} - e^{i\tau\psi} \frac{e_2(g_2 + \frac{\widehat{g}_2}{\tau})}{2\tau \partial_z \Phi} + \frac{e^{i\tau\psi}}{4\tau^2 \partial_z \Phi} L_1(x, D) \left( \frac{e_2 g_2}{\partial_z \Phi} \right). \end{aligned}$$

Now let us determine the functions  $u_{12}$ ,  $a_2(z)$  and  $d_2(\bar{z})$ .

First we can obtain the following asymptotic formulae for any point on the boundary of  $\Omega$ :

$$(4.10) \quad \mathcal{R}_{-\tau, A_1} \{e_1 g_1\} |_{\partial \Omega} = \frac{e^{\mathcal{A}_1 + 2i\tau\psi}}{2\tau^2 |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \left( \frac{e^{-2i\tau\psi(\tilde{x})} \sigma_1(\tilde{x})}{(z - \tilde{z})^2} + \frac{e^{-2i\tau\psi(\tilde{x})} m_1(\tilde{x})}{(\tilde{z} - z)} \right) + \mathcal{W}_{\tau, 1},$$

$$(4.11) \quad \widetilde{\mathcal{R}}_{\tau, B_1} \{e_1 g_2\} |_{\partial \Omega} = \frac{e^{\mathcal{B}_1 - 2i\tau\psi}}{2\tau^2 |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \left( \frac{e^{2i\tau\psi(\tilde{x})} \tilde{\sigma}_1(\tilde{x})}{(\bar{z} - \tilde{z})^2} + \frac{e^{2i\tau\psi(\tilde{x})} \tilde{m}_1(\tilde{x})}{(\tilde{z} - \bar{z})} \right) + \mathcal{W}_{\tau, 2},$$

where

$$(4.12) \quad \sigma_1(\tilde{x}) = \frac{\partial_z \tilde{g}_1(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})}, \quad m_1(\tilde{x}) = \frac{1}{2} \left( \frac{\partial_z \tilde{g}_1(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} \frac{\partial_z^3 \Phi(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} + \frac{\partial_z^2 \tilde{g}_1(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} - \frac{\partial_z^2 \tilde{g}_1(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} \right),$$

$$(4.13) \quad \tilde{\sigma}_1(\tilde{x}) = \frac{\partial_{\bar{z}} \tilde{g}_2(\tilde{x})}{\partial_{\bar{z}}^2 \Phi(\tilde{x})}, \quad \tilde{m}_1(\tilde{x}) = \frac{1}{2} \left( \frac{\partial_{\bar{z}} \tilde{g}_2(\tilde{x})}{\partial_{\bar{z}}^2 \Phi(\tilde{x})} \frac{\partial_{\bar{z}}^3 \Phi(\tilde{x})}{\partial_{\bar{z}}^2 \Phi(\tilde{x})} - \frac{\partial_{\bar{z}}^2 \tilde{g}_2(\tilde{x})}{\partial_{\bar{z}}^2 \Phi(\tilde{x})} + \frac{\partial_{\bar{z}}^2 \tilde{g}_2(\tilde{x})}{\partial_{\bar{z}}^2 \Phi(\tilde{x})} \right),$$

$\tilde{g}_1 = e^{-\mathcal{A}_1} g_1$ ,  $\tilde{g}_2 = e^{-\mathcal{B}_1} g_2$  and  $\mathcal{W}_{\tau, 1}, \mathcal{W}_{\tau, 2} \in H^{\frac{1}{2}}(\Gamma_0)$  satisfy

$$(4.14) \quad \|\mathcal{W}_{\tau, 1}\|_{H^{\frac{1}{2}}(\Gamma_0)} + \|\mathcal{W}_{\tau, 2}\|_{H^{\frac{1}{2}}(\Gamma_0)} = o\left(\frac{1}{\tau^2}\right) \quad \text{as } |\tau| \rightarrow +\infty.$$

The proof of (4.10) and (4.11) is given in Section 8.

Denote

$$p_+(x) = e^{\mathcal{A}_1(x)} \left( \frac{\sigma_1(\tilde{x})}{(z - \tilde{z})^2} + \frac{m_1(\tilde{x})}{(\tilde{z} - z)} \right), \quad p_-(x) = e^{\mathcal{B}_1(x)} \left( \frac{\tilde{\sigma}_1(\tilde{x})}{(\bar{z} - \tilde{z})^2} + \frac{\tilde{m}_1(\tilde{x})}{(\tilde{z} - \bar{z})} \right).$$

Thanks to Proposition 2.3 we can define functions  $a_{2,\pm}(z) \in C^2(\bar{\Omega})$  and  $d_{2,\pm}(\bar{z}) \in C^2(\bar{\Omega})$  satisfying

$$(4.15) \quad a_{2,\pm} e^{\mathcal{A}_1} + d_{2,\pm} e^{\mathcal{B}_1} = p_{\pm} \quad \text{on } \Gamma_0.$$

Straightforward computations give

$$(4.16) \quad \begin{aligned} & L_1(x, D) \left( (a + \frac{a_1}{\tau}) e^{\mathcal{A}_1 + \tau\Phi} + (d + \frac{d_1}{\tau}) e^{\mathcal{B}_1 + \tau\bar{\Phi}} + e^{\tau\varphi} u_{11} \right) \\ &= (q_1 - 2\frac{\partial A_1}{\partial z} - A_1 B_1) e^{\tau\Phi} \left( -\tilde{\mathcal{R}}_{\tau, B_1} \{e_1(g_2 + \hat{g}_2/\tau)\} - \frac{e_2(g_2 + \hat{g}_2/\tau)}{2\tau\partial_z\Phi} \right) \\ &+ (q_1 - 2\frac{\partial B_1}{\partial \bar{z}} - A_1 B_1) e^{\tau\bar{\Phi}} \left( -\mathcal{R}_{-\tau, A_1} \{e_1(g_1 + \hat{g}_1/\tau)\} - \frac{e_2(g_1 + \hat{g}_1/\tau)}{2\tau\partial_z\Phi} \right) \\ &+ \frac{e^{\tau\Phi}}{\tau^2} L_1(x, D) \left( \frac{1}{4\partial_z\Phi} L_1(x, D) \left( \frac{e_2 g_2}{\partial_z\Phi} \right) \right) + \frac{e^{\tau\bar{\Phi}}}{\tau^2} L_1(x, D) \left( \frac{1}{4\partial_z\Phi} L_1(x, D) \left( \frac{e_2 g_1}{\partial_z\Phi} \right) \right). \end{aligned}$$

Using Proposition 3.4 we transform the right-hand side of (4.16) as follows.

$$(4.17) \quad \begin{aligned} & L_1(x, D) \left( (a + \frac{a_1}{\tau}) e^{\mathcal{A}_1 + \tau\Phi} + (d + \frac{d_1}{\tau}) e^{\mathcal{B}_1 + \tau\bar{\Phi}} + u_{11} e^{\tau\varphi} \right) \\ &= -(q_1 - 2\frac{\partial A_1}{\partial z} - A_1 B_1) e^{\tau\Phi} \frac{g_1}{2\tau\partial_z\Phi} \\ &- (q_1 - 2\frac{\partial B_1}{\partial \bar{z}} - A_1 B_1) e^{\tau\bar{\Phi}} \frac{g_2}{2\tau\partial_z\Phi} + \chi_{\mathcal{O}_\epsilon} O_{L^4(\Omega)} \left( \frac{1}{\tau^2} \right) + \chi_{\Omega \setminus \mathcal{O}_\epsilon} O_{L^4(\Omega)} \left( \frac{1}{\tau^2} \right) \quad \text{as } |\tau| \rightarrow +\infty. \end{aligned}$$

We are looking for  $u_{12}$  in the form  $u_{12} = u_0 + u_{-1}$ . The function  $u_{-1}$  is given by

$$(4.18) \quad u_{-1} = \frac{e^{i\tau\psi}}{\tau} \tilde{\mathcal{R}}_{\tau, B_1} \{e_1 g_5\} + \frac{e^{-i\tau\psi}}{\tau} \mathcal{R}_{-\tau, A_1} \{e_1 g_6\} + \frac{e_2 g_5 e^{i\tau\psi}}{2\tau^2 \partial_z\Phi} + \frac{e_2 g_6 e^{-i\tau\psi}}{2\tau^2 \partial_z\Phi},$$

where

$$(4.19) \quad g_5 = \frac{P_{A_1} \left( (q_1 - 2\frac{\partial A_1}{\partial z} - A_1 B_1) g_1 \right) - M_5(z) e^{\mathcal{A}_1}}{2\partial_z\Phi}, \quad g_6 = \frac{T_{B_1} \left( (q_1 - 2\frac{\partial B_1}{\partial \bar{z}} - A_1 B_1) g_2 \right) - M_6(\bar{z}) e^{\mathcal{B}_1}}{2\partial_z\Phi}.$$

Here  $M_5(z), M_6(\bar{z})$  are polynomials such that

$$g_5|_{\mathcal{H}} = g_6|_{\mathcal{H}} = \nabla g_5|_{\mathcal{H}} = \nabla g_6|_{\mathcal{H}} = 0.$$

Using Proposition 2.3 we introduce functions  $a_{2,0}, d_{2,0} \in C^2(\bar{\Omega})$  (holomorphic and antiholomorphic respectively) such that

$$(4.20) \quad a_{2,0} e^{\mathcal{A}_1} + d_{2,0} e^{\mathcal{B}_1} = \frac{g_5}{2\partial_z\Phi} + \frac{g_6}{2\partial_z\Phi} \quad \text{on } \Gamma_0.$$

Next we claim that

$$(4.21) \quad \mathcal{R}_{-\tau, A_1} \{e_1 g_6\}|_{\Gamma_0} = o\left(\frac{1}{\tau}\right) \text{ as } |\tau| \rightarrow +\infty, \quad \tilde{\mathcal{R}}_{\tau, B_1} \{e_1 g_5\}|_{\Gamma_0} = o\left(\frac{1}{\tau}\right) \text{ as } |\tau| \rightarrow +\infty.$$

To see this, let us introduce the function  $\mathcal{F}$  with domain  $\Gamma_0$ .

$$\mathcal{F} = 2e^{-\mathcal{A}_1} e^{\tau(\Phi - \bar{\Phi})} \mathcal{R}_{-\tau, \mathcal{A}_1} \{e_1 g_6\} = \partial_{\bar{z}}^{-1} (e_1 e^{-\mathcal{A}_1 + \tau(\Phi - \bar{\Phi})} \frac{T_{B_1}((q_1 - 2\frac{\partial B_1}{\partial \bar{z}} - A_1 B_1)g_2) - M_6 e^{\mathcal{B}_1}}{2\partial_z \bar{\Phi}}).$$

Denoting  $r(x) = e^{\mathcal{A}_1} \frac{T_{B_1}((q_1 - 2\frac{\partial B_1}{\partial \bar{z}} - A_1 B_1)g_2) - M_6(\bar{z})e^{\mathcal{B}_1}}{2\partial_z \bar{\Phi}}$  we have

$$\mathcal{F}(x) = -\frac{1}{\pi} \int_{\Omega} \frac{e_1(x)r(x)e^{2i\tau\psi}}{\bar{\zeta} - \bar{z}} d\xi_1 d\xi_2 = \frac{1}{2i\pi\tau} \int_{\Omega} \sum_{k=1}^2 \frac{\partial}{\partial x_k} \left( \frac{\partial\psi}{\partial x_k} \frac{e_1(x)r(x)}{|\nabla\psi|^2} \frac{r(x)}{\bar{\zeta} - \bar{z}} \right) e^{2i\tau\psi} d\xi_1 d\xi_2.$$

Since  $\sum_{k=1}^2 \frac{\partial}{\partial x_k} \left( \frac{\partial\psi}{|\nabla\psi|^2} \frac{e_1(x)r(x)}{\bar{\zeta} - \bar{z}} \right) \in L^1(\Omega)$ , we have  $\mathcal{F} = o(\frac{1}{\tau})$ . This proves (4.21).

Now we finish the construction of functions  $a_{2,\tau}(z)$  and  $d_{2,\tau}(\bar{z})$  by setting

$$d_{2,\tau}(\bar{z}) = d_{2,0}(\bar{z}) + \frac{d_{2,+}(\bar{z})e^{2i\tau\psi(\bar{x})} + d_{2,-}(\bar{z})e^{-2i\tau\psi(\bar{x})}}{2|\det \psi''(\bar{x})|^{\frac{1}{2}}},$$

$$a_{2,\tau}(z) = a_{2,0}(z) + \frac{a_{2,+}(z)e^{2i\tau\psi(\bar{x})} + a_{2,-}(z)e^{-2i\tau\psi(\bar{x})}}{2|\det \psi''(\bar{x})|^{\frac{1}{2}}},$$

where  $a_{2,\pm}, d_{2,\pm}$  satisfy (4.15). To complete the construction of a solution to (4.1) we define  $u_0$  as the solution to the inhomogeneous problem

$$(4.22) \quad L_1(x, D)(u_0 e^{\tau\varphi}) = h_1 e^{\tau\varphi} \quad \text{in } \Omega,$$

$$(4.23) \quad u_0 e^{\tau\varphi} = e^{\tau\varphi} \mathbf{m}_1 \quad \text{on } \Gamma_0,$$

where

$$h_1(\tau) = -e^{-\tau\varphi} L_1(x, D)(a_{\tau} e^{\mathcal{A}_1 + \tau\Phi} + d_{\tau} e^{\mathcal{B}_1 + \tau\bar{\Phi}} + u_{11} e^{\tau\varphi} + u_{-1} e^{\tau\varphi}),$$

$$\mathbf{m}_1 = -e^{-\tau\varphi} (a_{\tau} e^{\mathcal{A}_1 + \tau\Phi} + d_{\tau} e^{\mathcal{B}_1 + \tau\bar{\Phi}} + u_{11} e^{\tau\varphi} + u_{-1} e^{\tau\varphi})|_{\Gamma_0}.$$

Observe that by (4.17) - (4.19)  $h_1(\tau)$  can be represented in the form  $h_1(\tau) = h_{11} + h_{12}$  where

$$(4.24) \quad \|h_{11}\|_{L^4(\Omega)} = O\left(\frac{1}{\tau^2}\right), \quad \|h_{12}\|_{L^4(\Omega)} = o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty$$

and for some positive  $\epsilon$

$$\text{supp } h_{11} \subset \mathcal{O}_{\epsilon}, \quad \text{dist}(\text{supp } h_{12}, \partial\Omega) > 0$$

and by (4.8), (4.14), (4.15), (4.20)

$$(4.25) \quad \|u_0\|_{H^{\frac{1}{2}}(\Gamma_0)} = o\left(\frac{1}{\tau^2}\right) \quad \text{as } |\tau| \rightarrow +\infty.$$

By Proposition 2.6 and Proposition 3.8 there exists a solution to (4.22), (4.24) such that

$$(4.26) \quad \frac{1}{\sqrt{|\tau|}} \|u_0\|_{H^1(\Omega)} + \sqrt{|\tau|} \|u_0\|_{L^2(\Omega)} + \|u_0\|_{H^{1,\tau}(\mathcal{O}_{\epsilon})} = o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty.$$

**4.1. Complex geometrical optics solutions for the adjoint operator.** We now construct complex geometrical optics solutions for the adjoint operator. This parallels the previous construction since the adjoint has a similar form.

Consider the operator  $L_2(x, D) = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} + 2A_2\frac{\partial}{\partial z} + 2B_2\frac{\partial}{\partial \bar{z}} + q_2$ . Its adjoint has the form

$$\begin{aligned} L_2(x, D)^* &= 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} - 2\overline{A_2}\frac{\partial}{\partial \bar{z}} - 2\overline{B_2}\frac{\partial}{\partial z} + \overline{q_2} - 2\frac{\partial \overline{A_2}}{\partial \bar{z}} - \frac{\partial \overline{B_2}}{\partial z} \\ &= (2\frac{\partial}{\partial z} - \overline{A_2})(2\frac{\partial}{\partial \bar{z}} - \overline{B_2}) + q_2 - 2\frac{\partial \overline{A_2}}{\partial \bar{z}} - \overline{A_2 B_2} \\ &= (2\frac{\partial}{\partial \bar{z}} - \overline{B_2})(2\frac{\partial}{\partial z} - \overline{A_2}) + q_2 - 2\frac{\partial \overline{B_2}}{\partial z} - \overline{A_2 B_2}. \end{aligned}$$

Next we construct solution to the following boundary value problem:

$$(4.27) \quad L_2(x, D)^*v = 0 \quad \text{in } \Omega, \quad v|_{\Gamma_0} = 0.$$

We construct solutions to (4.27) of the form

$$(4.28) \quad v(x) = b_\tau(z)e^{\mathcal{B}_2 - \tau\Phi} + c_\tau(\bar{z})e^{\mathcal{A}_2 - \tau\bar{\Phi}} + v_{11}e^{-\tau\varphi} + v_{12}e^{-\tau\bar{\varphi}}, \quad v|_{\Gamma_0} = 0.$$

Here  $\mathcal{A}_2, \mathcal{B}_2 \in C^{6+\alpha}(\overline{\Omega})$  satisfy

$$(4.29) \quad 2\frac{\partial \mathcal{A}_2}{\partial z} = \overline{A_2} \quad \text{in } \Omega, \quad \text{Im } \mathcal{A}_2|_{\Gamma_0} = 0, \quad 2\frac{\partial \mathcal{B}_2}{\partial \bar{z}} = \overline{B_2} \quad \text{in } \Omega, \quad \text{Im } \mathcal{B}_2|_{\Gamma_0} = 0,$$

and  $b_\tau(z) = b(z) + \frac{b_1(z)}{\tau} + \frac{b_{2,\tau}(z)}{\tau^2}$ ,  $c_\tau(\bar{z}) = c(\bar{z}) + \frac{c_1(\bar{z})}{\tau} + \frac{c_{2,\tau}(\bar{z})}{\tau^2}$  and

$$(4.30) \quad b, c \in C^{5+\alpha}(\overline{\Omega}), \quad \frac{\partial b}{\partial \bar{z}} = 0 \text{ in } \Omega, \quad \frac{\partial c}{\partial z} = 0 \text{ in } \Omega,$$

$$(4.31) \quad (be^{\mathcal{B}_2} + ce^{\mathcal{A}_2})|_{\Gamma_0} = 0,$$

$$(4.32) \quad \frac{\partial^k b}{\partial z^k}|_{\mathcal{H} \cap \partial\Omega} = \frac{\partial^k c}{\partial \bar{z}^k}|_{\mathcal{H} \cap \partial\Omega} = 0 \quad \forall k \in \{0, \dots, 5\}, \quad b|_{\mathcal{H} \setminus \{\tilde{x}\}} = c|_{\mathcal{H} \setminus \{\tilde{x}\}} = 0, \quad b(\tilde{x}) \neq 0, \quad c(\tilde{x}) \neq 0.$$

The existence of the functions  $b$  and  $c$  is given by Proposition 7.2 . Denote

$$g_3 = P_{-\overline{B_2}}((\overline{q_2} - 2\frac{\partial \overline{A_2}}{\partial \bar{z}} - \overline{A_2 B_2})be^{\mathcal{B}_2}) - M_3(z)e^{\mathcal{B}_2}, \quad g_4 = T_{-\overline{A_2}}((\overline{q_2} - 2\frac{\partial \overline{B_2}}{\partial z} - \overline{A_2 B_2})ce^{\mathcal{A}_2}) - M_4(\bar{z})e^{\mathcal{A}_2},$$

where the polynomials  $M_3(z), M_4(\bar{z})$  are chosen such that

$$(4.33) \quad \frac{\partial^k g_3}{\partial z^k}|_{\mathcal{H}} = \frac{\partial^k g_4}{\partial \bar{z}^k}|_{\mathcal{H}} = 0 \quad \forall k \in \{0, \dots, 6\}.$$

By (4.33), (4.32)

$$(4.34) \quad \frac{\partial^{k+j} g_3}{\partial z^k \partial \bar{z}^j}|_{\mathcal{H} \cap \partial\Omega} = \frac{\partial^{k+j} g_4}{\partial z^k \partial \bar{z}^j}|_{\mathcal{H} \cap \partial\Omega} = 0 \quad \forall k+j \leq 6, \quad \frac{\partial g_4}{\partial z} = \frac{\partial g_3}{\partial \bar{z}} = 0 \text{ on } \mathcal{H} \setminus \{\tilde{x}\}.$$

Observe that by (4.34)  $\frac{g_3}{\partial z^k}, \frac{g_4}{\partial \bar{z}^k} \in C^{4+\alpha}(\overline{\Omega})$ . Using Proposition 2.3 we introduce a holomorphic function  $b_1(z) \in C^2(\overline{\Omega})$  and an antiholomorphic function  $c_1(\bar{z}) \in C^2(\overline{\Omega})$  such that

$$(4.35) \quad b_1 e^{\mathcal{B}_2} + c_1 e^{\mathcal{A}_2} = \frac{e_2 g_3}{2\partial_z \Phi} + \frac{e_2 g_4}{2\partial_{\bar{z}} \bar{\Phi}} \quad \text{on } \Gamma_0.$$



Let

$$\widehat{g}_3 = P_{-\overline{B_2}}((\overline{q_2} - 2 \frac{\partial \overline{A_2}}{\partial \overline{z}} - \overline{A_2 B_2}) b_1 e^{\mathcal{B}_2}) - \widehat{M}_3(z) e^{\mathcal{B}_2}, \quad \widehat{g}_4 = T_{-\overline{A_2}}((\overline{q_2} - 2 \frac{\partial \overline{B_2}}{\partial z} - \overline{A_2 B_2}) c_1 e^{\mathcal{A}_2}) - \widehat{M}_4(\overline{z}) e^{\mathcal{A}_2},$$

where the polynomials  $\widehat{M}_3(z), \widehat{M}_4(\overline{z})$  are chosen such that

$$(4.36) \quad \frac{\partial^k \widehat{g}_3}{\partial z^k} |_{\mathcal{H}} = \frac{\partial^k \widehat{g}_4}{\partial \overline{z}^k} |_{\mathcal{H}} = 0 \quad \forall k \in \{0, \dots, 3\}.$$

The function  $v_{11}$  is defined by

$$(4.37) \quad v_{11} = -e^{-i\tau\psi} \widetilde{\mathcal{R}}_{-\tau, -\overline{A_2}} \{e_1(g_3 + \widehat{g}_3/\tau)\} + \frac{e^{-i\tau\psi} e_2(g_3 + \widehat{g}_3/\tau)}{2\tau \partial_z \Phi} - e^{i\tau\psi} \mathcal{R}_{\tau, -\overline{B_2}} \{e_1(g_4 + \widehat{g}_4/\tau)\} + \frac{e^{i\tau\psi} e_2(g_4 + \widehat{g}_4/\tau)}{2\tau \partial_z \Phi} \\ - \frac{e^{-i\tau\psi}}{4\tau^2 \partial_z \Phi} L_2(x, D)^* \left( \frac{e_2 g_3}{\partial_z \Phi} \right) - \frac{e^{i\tau\psi}}{4\tau^2 \partial_z \Phi} L_2(x, D)^* \left( \frac{e_2 g_4}{\partial_z \Phi} \right).$$

Here we set

$$\mathcal{R}_{\tau, -\overline{B_2}} \{g\} = \frac{1}{2} e^{\mathcal{B}_2} e^{\tau(\overline{\Phi} - \Phi)} \partial_{\overline{z}}^{-1} (g e^{-\mathcal{B}_2} e^{\tau(\Phi - \overline{\Phi})}) \\ \widetilde{\mathcal{R}}_{-\tau, -\overline{A_2}} \{g\} = \frac{1}{2} e^{\mathcal{A}_2} e^{\tau(\Phi - \overline{\Phi})} \partial_z^{-1} (g e^{-\mathcal{A}_2} e^{\tau(\overline{\Phi} - \Phi)})$$

provided that  $A_2, B_2, \mathcal{A}_2, \mathcal{B}_2$  satisfy (4.29). By Proposition 8.1 the following asymptotic formulae hold:

$$(4.38) \quad \widetilde{\mathcal{R}}_{-\tau, -\overline{A_2}} \{e_1 g_3\} |_{\partial\Omega} = \frac{1}{2\tau^2} \frac{e^{\mathcal{A}_2 + 2\tau i\psi}}{|\det \psi''(\tilde{x})|^{\frac{1}{2}}} \left( \frac{e^{-2i\tau\psi(\tilde{x})} r_1(\tilde{x})}{(\overline{z} - \tilde{z})^2} + \frac{e^{-2i\tau\psi(\tilde{x})} t_1(\tilde{x})}{(\tilde{z} - \overline{z})} \right) + \widetilde{\mathcal{W}}_{2,\tau}$$

and

$$(4.39) \quad \mathcal{R}_{\tau, -\overline{B_2}} \{e_1 g_4\} |_{\partial\Omega} = \frac{1}{2\tau^2} \frac{e^{\mathcal{B}_2 - 2\tau i\psi}}{|\det \psi''(\tilde{x})|^{\frac{1}{2}}} \left( \frac{e^{2i\tau\psi(\tilde{x})} \tilde{r}_1(\tilde{x})}{(z - \tilde{z})^2} + \frac{e^{2i\tau\psi(\tilde{x})} \tilde{t}_1(\tilde{x})}{(\tilde{z} - z)} \right) + \widetilde{\mathcal{W}}_{1,\tau},$$

where

$$(4.40) \quad r_1(\tilde{x}) = \frac{\partial_{\overline{z}} \tilde{g}_3(\tilde{x})}{\partial_{\overline{z}}^2 \Phi(\tilde{x})}, \quad t_1(\tilde{x}) = \frac{1}{2} \left( \frac{\partial_{\overline{z}} \tilde{g}_3(\tilde{x})}{\partial_{\overline{z}}^2 \Phi(\tilde{x})} \frac{\partial_{\overline{z}}^3 \overline{\Phi}(\tilde{x})}{\partial_{\overline{z}}^2 \Phi(\tilde{x})} + \frac{\partial_z^2 \tilde{g}_3(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} - \frac{\partial_{\overline{z}}^2 \tilde{g}_3(\tilde{x})}{\partial_{\overline{z}}^2 \Phi(\tilde{x})} \right),$$

$$(4.41) \quad \tilde{r}_1(\tilde{x}) = \frac{\partial_z \tilde{g}_4(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})}, \quad \tilde{t}_1(\tilde{x}) = \frac{1}{2} \left( \frac{\partial_z \tilde{g}_4(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} \frac{\partial_z^3 \Phi(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} - \frac{\partial_z^2 \tilde{g}_4(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} + \frac{\partial_{\overline{z}}^2 \tilde{g}_4(\tilde{x})}{\partial_{\overline{z}}^2 \Phi(\tilde{x})} \right),$$

$\tilde{g}_3 = e^{-\mathcal{A}_2} g_3, \tilde{g}_4 = e^{-\mathcal{B}_2} g_4$ . Here the functions  $\widetilde{\mathcal{W}}_{\tau,1}, \widetilde{\mathcal{W}}_{\tau,2} \in H^{\frac{1}{2}}(\Gamma_0)$  satisfy

$$(4.42) \quad \|\widetilde{\mathcal{W}}_{\tau,1}\|_{H^{\frac{1}{2}}(\Gamma_0)} + \|\widetilde{\mathcal{W}}_{\tau,2}\|_{H^{\frac{1}{2}}(\Gamma_0)} = o\left(\frac{1}{\tau^2}\right) \quad \text{as } |\tau| \rightarrow +\infty.$$

Using Proposition 2.3 we define the holomorphic functions  $b_{2,\pm}(z) \in C^2(\overline{\Omega})$  and antiholomorphic  $c_{2,\pm}(\overline{z}) \in C^2(\overline{\Omega})$  such that

$$(4.43) \quad b_{2,\pm} e^{\mathcal{B}_2} + c_{2,\pm} e^{\mathcal{A}_2} = \tilde{p}_{\pm} \quad \text{on } \Gamma_0,$$

where  $\tilde{p}_k$  is defined as

$$\tilde{p}_+(x) = e^{\mathcal{B}_2} \left( \frac{\tilde{r}_1(\tilde{x})}{(z - \tilde{z})^2} + \frac{\tilde{t}_1(\tilde{x})}{(\tilde{z} - z)} \right), \quad \tilde{p}_-(x) = e^{\mathcal{A}_2} \left( \frac{r_1(\tilde{x})}{(\overline{z} - \tilde{z})^2} + \frac{t_1(\tilde{x})}{(\tilde{z} - \overline{z})} \right).$$

Similarly to (4.17), there exist two positive numbers  $\epsilon$  and  $\epsilon'$  such that

$$(4.44) \quad \begin{aligned} & L_2(x, D)^* \left( (b + \frac{b_1}{\tau}) e^{\mathcal{B}_2 - \tau\Phi} + (\bar{b} + \frac{c_1}{\tau}) e^{\mathcal{A}_2 - \tau\bar{\Phi}} + v_{11} e^{-\tau\varphi} \right) \\ &= \frac{g_3 e^{-\tau\Phi}}{2\tau\partial_z\Phi} (\bar{q}_2 - 2\frac{\partial\bar{\mathcal{A}}_2}{\partial\bar{z}} - \overline{\mathcal{A}_2\mathcal{B}_2}) - \frac{g_4 e^{-\tau\bar{\Phi}}}{2\tau\partial_z\bar{\Phi}} (\bar{q}_2 - 2\frac{\partial\bar{\mathcal{B}}_2}{\partial z} - \overline{\mathcal{A}_2\mathcal{B}_2}) \end{aligned}$$

$$(4.45) \quad + \chi_{\Omega \setminus \mathcal{O}_\epsilon} o_{L^4}(\Omega) + \chi_{\mathcal{O}_\epsilon} o_{L^4}(\Omega) \left( \frac{1}{\tau} \right).$$

We are looking for  $v_{12}$  in the form  $v_{12} = v_0 + v_{-1}$ . The function  $v_{-1}$  is given by

$$(4.46) \quad v_{-1} = -\frac{e^{\tau i\psi}}{\tau} \mathcal{R}_{\tau, -\bar{\mathcal{B}}_2} \{e_1 g_7\} - \frac{e^{-\tau i\psi}}{\tau} \tilde{\mathcal{R}}_{-\tau, -\bar{\mathcal{A}}_2} \{e_1 g_8\} + \frac{e_2 g_7}{2\tau^2 \partial_z \Phi} + \frac{e_2 g_8}{2\tau^2 \partial_z \bar{\Phi}},$$

where

$$(4.47) \quad g_7 = \frac{P_{-\bar{\mathcal{B}}_2}((q_2 - 2\frac{\partial\bar{\mathcal{B}}_2}{\partial z} - \overline{\mathcal{A}_2\mathcal{B}_2})g_3) - M_7(z)e^{\mathcal{B}_2}}{2\partial_z\Phi}, \quad g_8 = \frac{T_{-\bar{\mathcal{A}}_2}((q_2 - 2\frac{\partial\bar{\mathcal{A}}_2}{\partial\bar{z}} - \overline{\mathcal{A}_2\mathcal{B}_2})g_4) - M_8(\bar{z})e^{\mathcal{A}_2}}{2\partial_z\bar{\Phi}},$$

and  $M_7(z), M_8(\bar{z})$  are polynomials such that

$$(4.48) \quad g_7|_{\mathcal{H}} = g_8|_{\mathcal{H}} = \nabla g_7|_{\mathcal{H}} = \nabla g_8|_{\mathcal{H}} = 0.$$

Using Proposition 2.3 we introduce functions  $b_{2,0}, c_{2,0} \in C^2(\bar{\Omega})$  such that

$$(4.49) \quad b_{2,0} e^{\mathcal{B}_2} + c_{2,0} e^{\mathcal{A}_2} = \frac{g_7}{2\partial_z\Phi} + \frac{g_8}{2\partial_z\bar{\Phi}} \quad \text{on } \Gamma_0.$$

Similarly to (4.21) we have

$$\left( \frac{1}{\tau} \mathcal{R}_{\tau, -\bar{\mathcal{B}}_2} \{e_1 g_8\} + \frac{1}{\tau} \tilde{\mathcal{R}}_{-\tau, -\bar{\mathcal{A}}_2} \{e_1 g_7\} \right)|_{\Gamma_0} = o\left(\frac{1}{\tau^2}\right) \quad \text{as } |\tau| \rightarrow +\infty.$$

Now we finish the construction of  $b_{2,\tau}(z)$  and  $c_{2,\tau}(\bar{z})$  by setting

$$(4.50) \quad b_{2,\tau}(\bar{z}) = b_{2,0}(\bar{z}) + \frac{b_{2,+}(\bar{z})e^{2i\tau\psi(\bar{x})} + b_{2,-}(\bar{z})e^{-2i\tau\psi(\bar{x})}}{2|\det\psi''(\bar{x})|^{\frac{1}{2}}}$$

and

$$(4.51) \quad c_{2,\tau}(z) = c_{2,0}(z) + \frac{c_{2,+}(z)e^{2i\tau\psi(\bar{x})} + c_{2,-}(z)e^{-2i\tau\psi(\bar{x})}}{2|\det\psi''(\bar{x})|^{\frac{1}{2}}},$$

where  $b_{2,+}, c_{2,-}$  are defined in (4.43).

Consider the following boundary value problem

$$(4.52) \quad L_2(x, D)^*(e^{-\tau\varphi} v_0) = h_2 e^{-\tau\varphi} \quad \text{in } \Omega,$$

$$(4.53) \quad e^{-\tau\varphi} v_0|_{\Gamma_0} = \mathbf{m}_2 e^{-\tau\varphi},$$

where

$$h_2 = -e^{\tau\varphi} L_2(x, D)^*(b_\tau e^{\mathcal{B}_2 - \tau\Phi} + c_\tau e^{\mathcal{A}_2 - \tau\bar{\Phi}} + v_{11} e^{-\tau\varphi} + v_{-1} e^{-\tau\varphi})$$

and

$$\mathbf{m}_2 = -e^{\tau\varphi} (b_\tau e^{\mathcal{A}_2 - \tau\Phi} + c_\tau e^{\mathcal{B}_2 - \tau\bar{\Phi}} + v_{11} e^{-\tau\varphi} + v_{-1} e^{-\tau\varphi}).$$

By (4.44)-(4.47) represent the function  $h_2$  in the form  $h_2 = h_{21} + h_{22}$  where for some positive  $\epsilon$

$$\text{supp } h_{21} \subset \mathcal{O}_\epsilon, \quad \text{dist}(\text{supp } h_{22}, \partial\Omega) > 0.$$

The norms of the functions  $h_{2j}$  are estimated as

$$(4.54) \quad \|h_{21}\|_{L^4(\Omega)} = O\left(\frac{1}{\tau^2}\right), \quad \|h_{22}\|_{L^4(\Omega)} = o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty.$$

By (4.50), (4.51), (4.43), (4.42), (4.30) we have

$$(4.55) \quad \|v_0\|_{H^{\frac{1}{2}}(\Gamma_0)} = o\left(\frac{1}{\tau^2}\right) \quad \text{as } |\tau| \rightarrow +\infty.$$

Thanks to (4.54), (4.55), by Proposition 2.6 and Proposition 3.8 for sufficiently small positive  $\epsilon$  there exists a solution to problem (4.52), (4.53) such that

$$(4.56) \quad \frac{1}{\sqrt{|\tau|}} \|v_0\|_{H^1(\Omega)} + \sqrt{|\tau|} \|v_0\|_{L^2(\Omega)} + \|v_0\|_{H^{1,\tau}(\mathcal{O}_\epsilon)} = o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty.$$

## 5. PROOF OF THEOREM 1.3.

Let  $u_1$  be a complex geometrical optics solution as in (4.2). Let  $u_2$  be a solution to the following boundary value problem

$$(5.1) \quad L_2(x, D)u_2 = 0 \quad \text{in } \Omega, \quad u_2|_{\partial\Omega} = u_1|_{\partial\Omega}, \quad \frac{\partial u_2}{\partial\nu}|_{\tilde{\Gamma}} = \frac{\partial u_1}{\partial\nu}|_{\tilde{\Gamma}}.$$

Setting  $u = u_1 - u_2$ ,  $q = q_1 - q_2$  we have

$$(5.2) \quad L_2(x, D)u + 2(A_1 - A_2)\frac{\partial u_1}{\partial z} + 2(B_1 - B_2)\frac{\partial u_1}{\partial \bar{z}} + qu_1 = 0 \quad \text{in } \Omega,$$

$$(5.3) \quad u|_{\partial\Omega} = 0, \quad \frac{\partial u}{\partial\nu}|_{\tilde{\Gamma}} = 0.$$

Let  $v$  be a solution to (4.27) in the form (4.28). Taking the scalar product of (5.2) with  $\bar{v}$  in  $L^2(\Omega)$  we obtain

$$(5.4) \quad 0 = \int_{\Omega} (2(A_1 - A_2)\frac{\partial u_1}{\partial z} + 2(B_1 - B_2)\frac{\partial u_1}{\partial \bar{z}} + qu_1)\bar{v} dx.$$

Our goal is to obtain the asymptotic formula for the right hand side of (5.4). We have

**Proposition 5.1.** *The following asymptotic formula is valid as  $|\tau| \rightarrow +\infty$ :*

$$\begin{aligned}
(5.5) \quad I_0 &= (qu_1, v)_{L^2(\Omega)} = \int_{\Omega} (qa\bar{c}e^{(\mathcal{A}_1+\bar{\mathcal{A}}_2)} + qd\bar{b}e^{(\mathcal{B}_1+\bar{\mathcal{B}}_2)}) dx \\
&+ \int_{\Omega} \left( \frac{q}{\tau} (a_1b + a\bar{c}_1) e^{(\mathcal{A}_1-\bar{\mathcal{B}}_2)} + \frac{q}{\tau} (\bar{a}b_1 + \bar{b}d_1) e^{(\mathcal{A}_2-\bar{\mathcal{B}}_1)} \right) dx \\
&+ \frac{1}{\tau} \int_{\Omega} \left( \frac{a\bar{g}_4 e^{\mathcal{A}_1}}{2\partial_z\Phi} - \frac{\bar{c}g_2 e^{\bar{\mathcal{A}}_2}}{2\partial_z\Phi} - \frac{\bar{b}g_1 e^{\bar{\mathcal{B}}_2}}{2\partial_z\bar{\Phi}} + \frac{d\bar{g}_3 e^{\mathcal{B}_1}}{2\partial_z\bar{\Phi}} \right) dx \\
&+ 2\pi \frac{(qab)(\tilde{x})e^{(\mathcal{A}_1+\bar{\mathcal{B}}_2+2\tau i\psi)(\tilde{x})} + (qd\bar{c})(\tilde{x})e^{(\mathcal{B}_1+\bar{\mathcal{A}}_2-2i\tau\psi)(\tilde{x})}}{\tau |\det\psi(\tilde{x})|^{\frac{1}{2}}} \\
&+ \frac{1}{2\tau i} \int_{\partial\Omega} qab\bar{e}^{\mathcal{A}_1+\bar{\mathcal{B}}_2+2\tau i\psi} \frac{(\nu, \nabla\psi)}{|\nabla\psi|^2} d\sigma - \frac{1}{2\tau i} \int_{\partial\Omega} qd\bar{c}e^{\mathcal{B}_1+\bar{\mathcal{A}}_2-2\tau i\psi} \frac{(\nu, \nabla\psi)}{|\nabla\psi|^2} d\sigma + o\left(\frac{1}{\tau}\right).
\end{aligned}$$

*Proof.* By (4.2), (4.9), (4.26) and Proposition 3.4 we have

$$(5.6) \quad u_1(x) = \left( a(z) + \frac{a_1(z)}{\tau} \right) e^{\mathcal{A}_1+\tau\Phi} + \left( d(\bar{z}) + \frac{d_1(\bar{z})}{\tau} \right) e^{\mathcal{B}_1+\tau\bar{\Phi}} - \frac{g_1 e^{\tau\bar{\Phi}}}{2\tau\partial_z\bar{\Phi}} - \frac{g_2 e^{\tau\Phi}}{2\tau\partial_z\Phi} + o_{L^2(\Omega)}\left(\frac{1}{\tau}\right).$$

Using (4.28), (4.37), (4.56) and Proposition 3.4 we get

$$(5.7) \quad v(x) = \left( b(z) + \frac{b_1(z)}{\tau} \right) e^{\mathcal{B}_2-\tau\Phi} + \left( c(\bar{z}) + \frac{c_1(\bar{z})}{\tau} \right) e^{\mathcal{A}_2-\tau\bar{\Phi}} + \frac{g_4 e^{-\tau\bar{\Phi}}}{2\tau\partial_z\bar{\Phi}} + \frac{g_3 e^{-\tau\Phi}}{2\tau\partial_z\Phi} + o_{L^2(\Omega)}\left(\frac{1}{\tau}\right).$$

By (5.6), (5.7) we obtain

$$\begin{aligned}
(qu_1, v)_{L^2(\Omega)} &= \left( q \left( a + \frac{a_1}{\tau} \right) e^{\mathcal{A}_1+\tau\Phi} + \left( d + \frac{d_1}{\tau} \right) e^{\mathcal{B}_1+\tau\bar{\Phi}} - \frac{g_1 e^{\tau\bar{\Phi}}}{2\tau\partial_z\bar{\Phi}} - \frac{g_2 e^{\tau\Phi}}{2\tau\partial_z\Phi} + o_{L^2(\Omega)}\left(\frac{1}{\tau}\right) \right), \\
&\left( b + \frac{b_1}{\tau} \right) e^{\mathcal{B}_2-\tau\Phi} + \left( c + \frac{c_1}{\tau} \right) e^{\mathcal{A}_2-\tau\bar{\Phi}} + \frac{g_4 e^{-\tau\bar{\Phi}}}{2\tau\partial_z\bar{\Phi}} + \frac{g_3 e^{-\tau\Phi}}{2\tau\partial_z\Phi} + o_{L^2(\Omega)}\left(\frac{1}{\tau}\right) \Big)_{L^2(\Omega)} = \\
&\int_{\Omega} \left( q(d\bar{b} + \frac{1}{\tau}(d_1\bar{b} + d\bar{b}_1)) e^{\mathcal{B}_1+\bar{\mathcal{B}}_2} + q(a\bar{c} + \frac{1}{\tau}(a\bar{c}_1 + a_1\bar{c})) e^{\mathcal{A}_1+\bar{\mathcal{A}}_2} \right) dx \\
&+ \frac{1}{\tau} \int_{\Omega} \left( \frac{a\bar{g}_4 e^{\mathcal{A}_1}}{2\partial_z\Phi} - \frac{\bar{c}g_2 e^{\bar{\mathcal{A}}_2}}{2\partial_z\Phi} - \frac{\bar{b}g_1 e^{\bar{\mathcal{B}}_2}}{2\partial_z\bar{\Phi}} + \frac{d\bar{g}_3 e^{\mathcal{B}_1}}{2\partial_z\bar{\Phi}} \right) dx \\
&+ \int_{\Omega} q \left( a\bar{b} e^{\mathcal{A}_1+\bar{\mathcal{B}}_2+\tau(\Phi-\bar{\Phi})} + d\bar{c} e^{\mathcal{B}_1+\bar{\mathcal{A}}_2+\tau(\bar{\Phi}-\Phi)} \right) dx + o\left(\frac{1}{\tau}\right).
\end{aligned}$$

Applying the stationary phase argument to the last integral on the right hand side of this formula we finish the proof of Proposition 5.1.  $\square$

We set

$$(5.8) \quad \mathcal{U}(x) = a_{\tau}(z) e^{\mathcal{A}_1(x)+\tau\Phi(z)} + d_{\tau}(\bar{z}) e^{\mathcal{B}_1(x)+\tau\bar{\Phi}(\bar{z})}, \quad \mathcal{V}(x) = b_{\tau}(z) e^{\mathcal{B}_2(x)-\tau\Phi(z)} + c_{\tau}(\bar{z}) e^{\mathcal{A}_2(x)-\tau\bar{\Phi}(\bar{z})}.$$

Short calculations give:

$$\begin{aligned}
& I_1 \equiv 2((A_1 - A_2) \frac{\partial \mathcal{U}}{\partial z}, \mathcal{V})_{L^2(\Omega)} \\
& = (2(A_1 - A_2)) \left( \left( \frac{\partial \mathcal{A}_1}{\partial z} + \tau \frac{\partial \Phi}{\partial z} \right) a_\tau + \frac{\partial a_\tau}{\partial z} \right) e^{A_1 + \tau \Phi} + d_\tau \frac{\partial \mathcal{B}_1}{\partial z} e^{\mathcal{B}_1 + \tau \bar{\Phi}}, \\
& \quad \left( b_\tau e^{\mathcal{B}_2 - \tau \Phi} + c_\tau e^{A_2 - \tau \bar{\Phi}} \right)_{L^2(\Omega)} \\
& = \sum_{k=1}^3 \tau^{2-k} \kappa_k - \int_{\Omega} (A_1 - A_2) B_1 d_\tau \bar{c}_\tau e^{\mathcal{B}_1 + \bar{A}_2 - 2i\tau\psi} dx \\
& \quad - \left( 2 \frac{\partial}{\partial z} (A_1 - A_2) a_\tau e^{A_1 + \tau \Phi}, b_\tau e^{\mathcal{B}_2 - \tau \Phi} \right)_{L^2(\Omega)} + \frac{1}{\tau} \mathcal{I}_1(\partial\Omega) \\
& \quad - (2(A_1 - A_2) a_\tau e^{A_1 + \tau \Phi}, \frac{\partial \mathcal{B}_2}{\partial \bar{z}} b_\tau e^{\mathcal{B}_2 - \tau \Phi})_{L^2(\Omega)} \\
& \quad + \int_{\partial\Omega} (A_1 - A_2) (\nu_1 - i\nu_2) a_\tau \bar{b}_\tau e^{A_1 + \bar{B}_2 + 2i\tau\psi} d\sigma + o\left(\frac{1}{\tau}\right) \\
& = \sum_{k=1}^3 \tau^{2-k} \kappa_k + \int_{\Omega} \left\{ -(A_1 - A_2) B_1 d_\tau \bar{c}_\tau e^{\mathcal{B}_1 + \bar{A}_2 - 2i\tau\psi} \right. \\
& \quad \left. - (A_1 - A_2) B_2 a_\tau \bar{b}_\tau e^{A_1 + \bar{B}_2 + 2i\tau\psi} - 2 \frac{\partial}{\partial z} (A_1 - A_2) a_\tau \bar{b}_\tau e^{A_1 + \bar{B}_2 + 2i\tau\psi} \right\} dx \\
(5.9) \quad & + \int_{\partial\Omega} (A_1 - A_2) (\nu_1 - i\nu_2) a_\tau \bar{b}_\tau e^{A_1 + \bar{B}_2 + 2i\tau\psi} d\sigma + \frac{1}{\tau} \mathcal{I}_1(\partial\Omega) + o\left(\frac{1}{\tau}\right)
\end{aligned}$$

and

$$\begin{aligned}
& I_2 \equiv ((B_1 - B_2) \frac{\partial \mathcal{U}}{\partial \bar{z}}, \mathcal{V})_{L^2(\Omega)} \\
& = (2(B_1 - B_2)) \left( a_\tau e^{A_1 + \tau \Phi} \frac{\partial \mathcal{A}_1}{\partial \bar{z}} + \frac{\partial}{\partial \bar{z}} \left( d_\tau e^{\mathcal{B}_1 + \tau \bar{\Phi}} \right) \right), b_\tau e^{\mathcal{B}_2 - \tau \Phi} + c_\tau e^{A_2 - \tau \bar{\Phi}})_{L^2(\Omega)} \\
& = \sum_{k=1}^3 \tau^{2-k} \tilde{\kappa}_k + \int_{\Omega} 2(B_1 - B_2) \frac{\partial \mathcal{A}_1}{\partial \bar{z}} a_\tau \bar{b}_\tau e^{A_1 + \bar{B}_2 + 2i\tau\psi} dx - \left( 2 \frac{\partial}{\partial \bar{z}} (B_1 - B_2) d_\tau e^{\mathcal{B}_1 + \tau \bar{\Phi}}, \right. \\
& \quad \left. c_\tau e^{A_2 - \tau \bar{\Phi}} \right)_{L^2(\Omega)} - \left( 2(B_1 - B_2) d_\tau e^{\mathcal{B}_1 + \tau \bar{\Phi}}, \frac{\partial \mathcal{A}_2}{\partial z} c_\tau e^{A_2 - \tau \bar{\Phi}} \right)_{L^2(\Omega)} \\
& \quad + \int_{\partial\Omega} (B_1 - B_2) (\nu_1 + i\nu_2) d_\tau \bar{c}_\tau e^{\mathcal{B}_1 + \bar{A}_2 - 2i\tau\psi} d\sigma + \frac{1}{\tau} \mathcal{I}_2(\partial\Omega) + o\left(\frac{1}{\tau}\right) \\
& = \sum_{k=1}^3 \tau^{2-k} \tilde{\kappa}_k + \int_{\Omega} \left\{ -(B_1 - B_2) A_1 a_\tau \bar{b}_\tau e^{A_1 + \bar{B}_2 + 2i\tau\psi} \right. \\
& \quad \left. - 2 \frac{\partial}{\partial \bar{z}} (B_1 - B_2) d_\tau \bar{c}_\tau e^{\mathcal{B}_1 + \bar{A}_2 - 2i\tau\psi} - (B_1 - B_2) d_\tau \bar{c}_\tau A_2 e^{\mathcal{B}_1 + \bar{A}_2 - 2i\tau\psi} \right\} dx \\
(5.10) \quad & + \int_{\partial\Omega} (B_1 - B_2) (\nu_1 + i\nu_2) d_\tau \bar{c}_\tau e^{\mathcal{B}_1 + \bar{A}_2 - 2i\tau\psi} d\sigma + \frac{1}{\tau} \mathcal{I}_2(\partial\Omega) + o\left(\frac{1}{\tau}\right).
\end{aligned}$$

Here  $\kappa_k, \tilde{\kappa}_k$  are some constants independent of  $\tau$  but may depend on  $A_j, B_j, \Phi$ . The terms  $\mathcal{I}_1(\partial\Omega), \mathcal{I}_2(\partial\Omega)$  are given by

$$\begin{aligned}
\mathcal{I}_1(\partial\Omega) &= \int_{\Omega} (A_1 - A_2) e^{\mathcal{A}_1 + \overline{\mathcal{A}}_2} \frac{\partial\Phi}{\partial z} \bar{c} \left( \frac{a_{2,+} e^{2\tau i\psi(\tilde{x})} + a_{2,-} e^{-2\tau i\psi(\tilde{x})}}{|\det \psi''(\tilde{x})|^{\frac{1}{2}}} \right) dx \\
&+ \int_{\Omega} (A_1 - A_2) e^{\mathcal{A}_1 + \overline{\mathcal{A}}_2} \frac{\partial\Phi}{\partial z} a \left( \frac{c_{2,+} e^{2\tau i\psi(\tilde{x})} + c_{2,-} e^{-2\tau i\psi(\tilde{x})}}{|\det \psi''(\tilde{x})|^{\frac{1}{2}}} \right) dx = \\
&- 2 \int_{\Omega} \frac{\partial}{\partial \bar{z}} e^{\mathcal{A}_1 + \overline{\mathcal{A}}_2} \frac{\partial\Phi}{\partial z} \bar{c} \left( \frac{a_{2,+} e^{2\tau i\psi(\tilde{x})} + a_{2,-} e^{-2\tau i\psi(\tilde{x})}}{|\det \psi''(\tilde{x})|^{\frac{1}{2}}} \right) dx \\
&- 2 \int_{\Omega} \frac{\partial}{\partial \bar{z}} e^{\mathcal{A}_1 + \overline{\mathcal{A}}_2} \frac{\partial\Phi}{\partial z} a \left( \frac{c_{2,+} e^{2\tau i\psi(\tilde{x})} + c_{2,-} e^{-2\tau i\psi(\tilde{x})}}{|\det \psi''(\tilde{x})|^{\frac{1}{2}}} \right) dx = \\
&- \int_{\partial\Omega} (\nu_1 + i\nu_2) e^{\mathcal{A}_1 + \overline{\mathcal{A}}_2} \frac{\partial\Phi}{\partial z} \bar{c} \left( \frac{a_{2,+} e^{2\tau i\psi(\tilde{x})} + a_{2,-} e^{-2\tau i\psi(\tilde{x})}}{|\det \psi''(\tilde{x})|^{\frac{1}{2}}} \right) d\sigma \\
(5.11) \quad &- \int_{\partial\Omega} (\nu_1 + i\nu_2) e^{\mathcal{A}_1 + \overline{\mathcal{A}}_2} \frac{\partial\Phi}{\partial z} a \left( \frac{c_{2,+} e^{2\tau i\psi(\tilde{x})} + c_{2,-} e^{-2\tau i\psi(\tilde{x})}}{|\det \psi''(\tilde{x})|^{\frac{1}{2}}} \right) d\sigma
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}_2(\partial\Omega) &= \int_{\Omega} (B_1 - B_2) e^{\mathcal{B}_1 + \overline{\mathcal{B}}_2} \frac{\partial\bar{\Phi}}{\partial \bar{z}} \bar{b} \left( \frac{d_{2,+} e^{2\tau i\psi(\tilde{x})} + d_{2,-} e^{-2\tau i\psi(\tilde{x})}}{|\det \psi''(\tilde{x})|^{\frac{1}{2}}} \right) dx \\
&+ \int_{\Omega} (B_1 - B_2) e^{\mathcal{B}_1 + \overline{\mathcal{B}}_2} \frac{\partial\bar{\Phi}}{\partial \bar{z}} d \left( \frac{b_{2,+} e^{2\tau i\psi(\tilde{x})} + b_{2,-} e^{-2\tau i\psi(\tilde{x})}}{|\det \psi''(\tilde{x})|^{\frac{1}{2}}} \right) dx = \\
&- 2 \int_{\Omega} \frac{\partial}{\partial z} e^{\mathcal{B}_1 + \overline{\mathcal{B}}_2} \frac{\partial\bar{\Phi}}{\partial \bar{z}} \bar{b} \left( \frac{d_{2,+} e^{2\tau i\psi(\tilde{x})} + d_{2,-} e^{-2\tau i\psi(\tilde{x})}}{|\det \psi''(\tilde{x})|^{\frac{1}{2}}} \right) dx \\
&- 2 \int_{\Omega} \frac{\partial}{\partial z} e^{\mathcal{B}_1 + \overline{\mathcal{B}}_2} \frac{\partial\bar{\Phi}}{\partial \bar{z}} d \left( \frac{b_{2,+} e^{2\tau i\psi(\tilde{x})} + b_{2,-} e^{-2\tau i\psi(\tilde{x})}}{|\det \psi''(\tilde{x})|^{\frac{1}{2}}} \right) dx = \\
&- \int_{\partial\Omega} (\nu_1 - i\nu_2) e^{\mathcal{B}_1 + \overline{\mathcal{B}}_2} \frac{\partial\bar{\Phi}}{\partial \bar{z}} \bar{b} \left( \frac{d_{2,+} e^{2\tau i\psi(\tilde{x})} + d_{2,-} e^{-2\tau i\psi(\tilde{x})}}{|\det \psi''(\tilde{x})|^{\frac{1}{2}}} \right) d\sigma \\
(5.12) \quad &- \int_{\partial\Omega} (\nu_1 - i\nu_2) e^{\mathcal{B}_1 + \overline{\mathcal{B}}_2} \frac{\partial\bar{\Phi}}{\partial \bar{z}} d \left( \frac{b_{2,+} e^{2\tau i\psi(\tilde{x})} + b_{2,-} e^{-2\tau i\psi(\tilde{x})}}{|\det \psi''(\tilde{x})|^{\frac{1}{2}}} \right) d\sigma.
\end{aligned}$$

Denote

$$U_1 = -e^{\tau\bar{\Phi}} \mathcal{R}_{-\tau, A_1} \{e_1 g_1\}, \quad U_2 = -e^{\tau\Phi} \tilde{\mathcal{R}}_{\tau, B_1} \{e_1 g_2\}.$$

A short calculation gives

$$(5.13) \quad 2 \frac{\partial U_1}{\partial \bar{z}} = (-e_1 g_1 + A_1 \mathcal{R}_{-\tau, A_1} \{e_1 g_1\}) e^{\tau\bar{\Phi}}$$

and

$$(5.14) \quad 2\frac{\partial U_2}{\partial z} = (-e_1 g_2 + B_1 \tilde{\mathcal{R}}_{\tau, B_1} \{e_1 g_2\}) e^{\tau \Phi}.$$

We have

$$(5.15) \quad \begin{aligned} \frac{\partial}{\partial z} \mathcal{R}_{-\tau, A_1} \{e_1 g_1\} &= \frac{\partial \mathcal{A}_1}{\partial z} \mathcal{R}_{-\tau, A_1} \{e_1 g_1\} + \tau \frac{\partial \Phi}{\partial z} \mathcal{R}_{-\tau, A_1} \{e_1 g_1\} + \mathcal{R}_{-\tau, A_1} \left\{ \frac{\partial(e_1 g_1)}{\partial z} \right\} \\ &= -\mathcal{R}_{-\tau, A_1} \left\{ e_1 g_1 \frac{\partial \mathcal{A}_1}{\partial z} \right\} - \tau \mathcal{R}_{-\tau, A_1} \left\{ \frac{\partial \Phi}{\partial z} e_1 g_1 \right\} + \mathcal{R}_{-\tau, A_1} \left\{ \frac{\partial(e_1 g_1)}{\partial z} \right\} \\ &\quad + \tau \frac{e^{A_1}}{2\pi} e^{-\tau(\bar{\Phi}-\Phi)} \int_{\Omega} \frac{\frac{\partial \Phi}{\partial \zeta}(\zeta) - \frac{\partial \Phi}{\partial z}(z)}{\zeta - z} (e_1 g_1 e^{-A_1})(\xi_1, \xi_2) e^{\tau(\bar{\Phi}(\zeta)-\Phi(\zeta))} d\xi_1 d\xi_2 \\ &\quad + \frac{e^{A_1}}{2\pi} e^{-\tau(\bar{\Phi}-\Phi)} \int_{\Omega} \frac{\frac{\partial \mathcal{A}_1}{\partial \zeta}(\zeta, \bar{\zeta}) - \frac{\partial \mathcal{A}_1}{\partial z}(z, \bar{z})}{\zeta - z} (e_1 g_1 e^{-A_1})(\xi_1, \xi_2) e^{\tau(\bar{\Phi}(\zeta)-\Phi(\zeta))} d\xi_1 d\xi_2. \end{aligned}$$

Let

$$\mathfrak{G}(x, g, \mathcal{A}, \tau) = -\frac{1}{2\pi} \int_{\Omega} \frac{(\tau \frac{\partial \Phi(\zeta)}{\partial \zeta} + \frac{\partial \mathcal{A}(\zeta, \bar{\zeta})}{\partial \zeta}) - (\tau \frac{\partial \Phi(z)}{\partial z} + \frac{\partial \mathcal{A}(z, \bar{z})}{\partial z})}{\zeta - z} e_1 g e^{-\mathcal{A}} e^{\tau(\bar{\Phi}-\Phi)} d\xi_1 d\xi_2.$$

We set

$$\begin{aligned} \mathfrak{G}_1(x, \tau) &= \mathfrak{G}(x, g_1, \mathcal{A}_1, \tau), \quad \mathfrak{G}_2(x, \tau) = \overline{\mathfrak{G}(x, \bar{g}_2, \bar{\mathcal{B}}_1, \tau)}, \\ \mathfrak{G}_3(x, \tau) &= \overline{\mathfrak{G}(x, \bar{g}_3, \bar{\mathcal{A}}_2, -\tau)}, \quad \mathfrak{G}_4(x, \tau) = \mathfrak{G}(x, g_4, \mathcal{B}_2, -\tau). \end{aligned}$$

By (3.16), (4.6), (5.15) and Proposition 3.4, we obtain

$$(5.16) \quad \frac{\partial}{\partial z} \mathcal{R}_{-\tau, A_1} \{e_1 g_1\} = \mathcal{R}_{-\tau, A_1} \left\{ \frac{\partial(e_1 g_1)}{\partial z} \right\} - e^{A_1} e^{-\tau(\bar{\Phi}-\Phi)} \mathfrak{G}_1(\cdot, \tau) + o_{L^2(\Omega)}\left(\frac{1}{\tau}\right).$$

Simple computations provide the formula

$$(5.17) \quad \begin{aligned} \frac{\partial}{\partial \bar{z}} \tilde{\mathcal{R}}_{\tau, B_1} \{e_1 g_2\} &= \frac{\partial \mathcal{B}_1}{\partial \bar{z}} \tilde{\mathcal{R}}_{\tau, B_1} \{e_1 g_2\} + \tau \frac{\partial \bar{\Phi}}{\partial \bar{z}} \tilde{\mathcal{R}}_{\tau, B_1} \{e_1 g_2\} + \tilde{\mathcal{R}}_{\tau, B_1} \left\{ \frac{\partial(e_1 g_2)}{\partial \bar{z}} \right\} \\ &\quad - \tau \tilde{\mathcal{R}}_{\tau, B_1} \left\{ \frac{\partial \bar{\Phi}}{\partial \bar{z}} e_1 g_2 \right\} - \tilde{\mathcal{R}}_{\tau, B_1} \left\{ \frac{\partial \mathcal{B}_1}{\partial \bar{z}} e_1 g_2 \right\} = \tilde{\mathcal{R}}_{\tau, B_1} \left\{ \frac{\partial(e_1 g_2)}{\partial \bar{z}} \right\} \\ &\quad + \tau \frac{e^{B_1}}{2\pi} e^{\tau(\bar{\Phi}-\Phi)} \int_{\Omega} \frac{\frac{\partial \bar{\Phi}}{\partial \zeta}(\bar{\zeta}) - \frac{\partial \bar{\Phi}}{\partial \bar{z}}(\bar{z})}{\bar{\zeta} - \bar{z}} (e_1 g_2 e^{-B_1})(\xi_1, \xi_2) e^{\tau(\Phi(\zeta)-\bar{\Phi}(\bar{\zeta}))} d\xi_1 d\xi_2 \\ &\quad + \frac{e^{B_1}}{2\pi} e^{\tau(\bar{\Phi}-\Phi)} \int_{\Omega} \frac{\frac{\partial \mathcal{B}_1}{\partial \zeta}(\zeta, \bar{\zeta}) - \frac{\partial \mathcal{B}_1}{\partial \bar{z}}(z, \bar{z})}{\bar{\zeta} - \bar{z}} (e_1 g_2 e^{-B_1})(\xi_1, \xi_2) e^{\tau(\Phi(\zeta)-\bar{\Phi}(\bar{\zeta}))} d\xi_1 d\xi_2. \end{aligned}$$

By (3.16), (4.6), (5.17), Proposition 3.4 we have

$$(5.18) \quad \frac{\partial}{\partial \bar{z}} \tilde{\mathcal{R}}_{\tau, B_1} \{e_1 g_2\} = \tilde{\mathcal{R}}_{\tau, B_1} \left\{ \frac{\partial(e_1 g_2)}{\partial \bar{z}} \right\} - e^{B_1} e^{\tau(\bar{\Phi}-\Phi)} \mathfrak{G}_2(\cdot, \tau) + o_{L^2(\Omega)}\left(\frac{1}{\tau}\right).$$

Denote

$$V_1 = -e^{-\tau \Phi} \tilde{\mathcal{R}}_{-\tau, -\bar{A}_2} \{e_1 g_3\}, \quad V_2 = -e^{-\tau \bar{\Phi}} \mathcal{R}_{\tau, -\bar{B}_2} \{e_1 g_4\}, \quad \mathcal{P}(x, D) = 2(A_1 - A_2) \frac{\partial}{\partial z} + 2(B_1 - B_2) \frac{\partial}{\partial \bar{z}},$$

$$\begin{aligned}\mathcal{Q}_+ &= -(B_1 - B_2)A_1 - (A_1 - A_2)B_2 - 2\frac{\partial}{\partial z}(A_1 - A_2) + (q_1 - q_2), \\ \mathcal{Q}_- &= -(A_1 - A_2)B_1 - (B_1 - B_2)A_2 - 2\frac{\partial}{\partial \bar{z}}(B_1 - B_2) + (q_1 - q_2).\end{aligned}$$

The following proposition is proved in Section 8.

**Proposition 5.2.** *There exist two numbers  $\kappa, \kappa_0$  independent of  $\tau$  such that the following asymptotic formula holds true:*

$$\begin{aligned}(\mathcal{P}(x, D)(U_1 + U_2), b_\tau e^{\mathcal{B}_2 - \tau\Phi} + c_\tau e^{\mathcal{A}_2 - \tau\bar{\Phi}})_{L^2(\Omega)} + (\mathcal{P}(x, D)(a_\tau e^{\mathcal{A}_1 + \tau\Phi} + d_\tau e^{\mathcal{B}_1 + \tau\bar{\Phi}}), V_1 + V_2)_{L^2(\Omega)} = \\ \kappa + \frac{\kappa_0}{\tau} - 2 \int_{\partial\Omega} (\nu_1 + i\nu_2) e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2} \overline{c_\tau(\bar{z})} \mathfrak{G}_1(x, \tau) d\sigma + 2 \int_{\partial\Omega} (\nu_1 - i\nu_2) e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} d_\tau(\bar{z}) \overline{\mathfrak{G}_4(x, \tau)} d\sigma \\ + 2 \int_{\partial\Omega} a_\tau(z) \overline{\mathfrak{G}_3(x, \tau)} (\nu_1 + i\nu_2) e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2} d\sigma - 2 \int_{\partial\Omega} (\nu_1 - i\nu_2) e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} \overline{b_\tau(z)} \mathfrak{G}_2(x, \tau) d\sigma \\ + \frac{e^{-2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \left\{ \frac{\partial g_4(\tilde{x})}{\partial z} \right\} e^{-\bar{\mathcal{B}}_2(\tilde{x})} \int_{\partial\Omega} \frac{(\nu_1 - i\nu_2) d e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2}}{\tilde{z} - \bar{z}} d\sigma \\ - \frac{e^{-2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_1(\tilde{x})}{\partial z} e^{-\mathcal{A}_1(\tilde{x})} \int_{\partial\Omega} \frac{(\nu_1 + i\nu_2) \bar{c} e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2}}{\tilde{z} - z} d\sigma \\ + \frac{e^{2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_3(\tilde{x})}{\partial \bar{z}} e^{-\bar{\mathcal{A}}_2(\tilde{x})} \int_{\partial\Omega} \frac{(\nu_1 + i\nu_2) a e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2}}{\tilde{z} - z} d\sigma \\ - \frac{e^{2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_2(\tilde{x})}{\partial \bar{z}} e^{-\mathcal{B}_1(\tilde{x})} \int_{\partial\Omega} \frac{(\nu_1 - i\nu_2) \bar{b} e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2}}{\tilde{z} - \bar{z}} d\sigma \\ (5.19) \quad - \frac{2\pi(\mathcal{Q}_+ \bar{a} \bar{b} e^{\mathcal{A}_1 + \bar{\mathcal{B}}_2 + 2i\tau\psi} + \mathcal{Q}_- \bar{c} d e^{\mathcal{B}_1 + \bar{\mathcal{A}}_2 - 2i\tau\psi})(\tilde{x})}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}}.\end{aligned}$$

By (5.13), (5.14), (5.16), (5.18) and Proposition 3.3 there exists a constant  $\mathcal{C}_0$  independent of  $\tau$  such that

$$\begin{aligned}(\mathcal{P}(x, D)(U_1 + U_2), V_1 + V_2)_{L^2(\Omega)} &= ((A_1 - A_2)(-2 \left( \mathcal{R}_{-\tau, A_1} \left\{ \frac{\partial(e_1 g_1)}{\partial z} \right\} \right. \\ &- \left. e^{\mathcal{A}_1} e^{-\tau(\bar{\Phi} - \Phi)} \mathfrak{G}_1 + o_{L^2(\Omega)}\left(\frac{1}{\tau}\right) \right) e^{\tau\bar{\Phi}} \\ &+ (-e_1 g_2 + \frac{B_1 e_1 g_2}{2\tau \partial_z \Phi} + o_{L^2(\Omega)}\left(\frac{1}{\tau}\right)) e^{\tau\Phi}, V_1 + V_2)_{L^2(\Omega)} \\ &+ ((B_1 - B_2)(-e_1 g_1 + A_1 \frac{e_1 g_1}{2\tau \partial_z \Phi} + o_{L^2(\Omega)}\left(\frac{1}{\tau}\right)) e^{\tau\bar{\Phi}}, V_1 + V_2)_{L^2(\Omega)} \\ &- (2(B_1 - B_2)(\tilde{\mathcal{R}}_{\tau, B_1} \left\{ \frac{\partial(e_1 g_2)}{\partial \bar{z}} \right\} - e^{\mathcal{B}_1} e^{\tau(\bar{\Phi} - \Phi)} \mathfrak{G}_2 + o_{L^2(\Omega)}\left(\frac{1}{\tau}\right)) e^{\tau\Phi}, V_1 + V_2)_{L^2(\Omega)} \\ &= \frac{\mathcal{C}_0}{\tau} + o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty.\end{aligned}$$



Next we claim that

$$(5.20) \quad (\mathcal{P}(x, D)(u_0 e^{\tau\varphi}), v)_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty,$$

and

$$(5.21) \quad (\mathcal{P}(x, D)u, v_0 e^{-\tau\varphi})_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty.$$

Let us first prove (5.21). By (4.26), (4.56), (5.16), (5.18), Proposition 3.2 and Proposition 3.4 we have

$$(5.22) \quad (\mathcal{P}(x, D)u, v_0 e^{-\tau\varphi})_{L^2(\Omega)} = (\mathcal{P}(x, D)\mathcal{U}, v_0 e^{-\tau\varphi})_{L^2(\Omega)} + o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty.$$

We remind that the function  $\mathcal{U}$  and  $\mathcal{V}$  are defined by (5.8). By (4.56) we obtain from (5.22)

$$(5.23) \quad (\mathcal{P}(x, D)u, v_0 e^{-\tau\varphi})_{L^2(\Omega)} = \tau \int_{\Omega} 2\chi \left( \frac{\partial\Phi}{\partial z} (A_1 - A_2) a e^{A_1 + i\tau\psi} + \frac{\partial\bar{\Phi}}{\partial\bar{z}} (B_1 - B_2) b e^{B_1 - i\tau\psi} \right) \bar{v}_0 dx + o\left(\frac{1}{\tau}\right)$$

as  $|\tau| \rightarrow +\infty$ . Here  $\chi \in C_0^\infty(\bar{\Omega})$  is a function such that  $\chi \equiv 1$  in some neighborhood of  $\text{supp } e_2$  and  $\mathcal{H} \setminus \partial\Omega \subset \text{supp } e_2$ .

By (4.22) and (4.52) the functions  $v_{0,+} = e^{-i\tau\psi} \bar{v}_0$  and  $v_{0,-} = e^{i\tau\psi} \bar{v}_0$  satisfy  $e^{\tau\bar{\Phi}} L_2(x, D)^*(e^{-\tau\bar{\Phi}} v_{0,+}) = \bar{h}_2 e^{i\tau\psi}$  and  $e^{\tau\bar{\Phi}} L_2(x, D)^*(e^{-\tau\bar{\Phi}} v_{0,-}) = \bar{h}_2 e^{-i\tau\psi}$ . More explicitly, there exist two first-order operators  $\mathcal{P}_k(x, D)$  such that

$$e^{\tau\bar{\Phi}} L_2(x, D)^*(e^{-\tau\bar{\Phi}} v_{0,+}) = \Delta \bar{v}_{0,+} - 2\tau \frac{\partial\Phi}{\partial z} \left( 2 \frac{\partial \bar{v}_{0,+}}{\partial \bar{z}} - A_2 \bar{v}_{0,+} \right) + \mathcal{P}_1(x, D) \bar{v}_{0,+} = o_{L^2(\Omega)}\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty$$

and

$$e^{\tau\bar{\Phi}} L_2(x, D)^*(e^{-\tau\bar{\Phi}} v_{0,-}) = \Delta \bar{v}_{0,-} - 2\tau \frac{\partial\bar{\Phi}}{\partial\bar{z}} \left( 2 \frac{\partial \bar{v}_{0,-}}{\partial z} - B_2 \bar{v}_{0,-} \right) + \mathcal{P}_2(x, D) \bar{v}_{0,-} = o_{L^2(\Omega)}\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty.$$

In the above equalities we used (4.24) and (4.54).

Let  $\chi_1 \in C_0^\infty(\bar{\Omega})$  be a function such that  $\chi_1 \equiv 1$  on  $\text{supp } \chi$  and  $g \in C^2(\bar{\Omega})$ . Taking the scalar product of the first equation with  $\chi_1 g$  we obtain

$$\int_{\Omega} 2\tau \frac{\partial\Phi}{\partial z} \bar{v}_{0,+} \chi_1 \left( 2 \frac{\partial}{\partial \bar{z}} + A_2 \right) g dx = o\left(\frac{1}{\tau}\right) - \int_{\Omega} \left( \bar{v}_{0,+} (\Delta + \mathcal{P}_1(x, D)^*)(\chi_1 g) + 2\tau \bar{v}_{0,+} \frac{\partial\Phi}{\partial z} g \left( \frac{\partial}{\partial \bar{z}} + A_2 \right) \chi_1 \right) dx.$$

By (4.56) we have

$$(5.24) \quad \int_{\Omega} \tau \frac{\partial\Phi}{\partial z} \bar{v}_{0,+} \chi_1 \left( 2 \frac{\partial}{\partial \bar{z}} + A_2 \right) g dx = o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty.$$

Taking the scalar product of the second equation with  $\chi_1 g$  where  $g \in C^2(\bar{\Omega})$  we have

$$\int_{\Omega} 2\tau \frac{\partial\bar{\Phi}}{\partial\bar{z}} \bar{v}_{0,-} \chi_1 \left( 2 \frac{\partial}{\partial z} + B_2 \right) g dx = o\left(\frac{1}{\tau}\right) - \int_{\Omega} \left( \bar{v}_{0,-} (\Delta + \mathcal{P}_2(x, D)^*)(\chi_1 g) + 2\tau \bar{v}_{0,-} g \frac{\partial\bar{\Phi}}{\partial\bar{z}} \left( 2 \frac{\partial}{\partial z} + B_2 \right) \chi_1 \right) dx.$$

By (4.56) we get

$$(5.25) \quad \int_{\Omega} 2\tau \frac{\partial\bar{\Phi}}{\partial\bar{z}} \bar{v}_{0,-} \chi_1 \left( 2 \frac{\partial}{\partial z} + B_2 \right) g dx = o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty.$$

Taking  $g$  such that  $(2 \frac{\partial}{\partial \bar{z}} + A_2)g = (A_1 - A_2)e^{A_1}a(z)$  in (5.24) and  $g$  such that  $(2 \frac{\partial}{\partial z} + B_2)g = (B_1 - B_2)b(\bar{z})e^{B_1}$  in (5.25) from (5.22) we obtain (5.21).

In order to prove (5.20) we observe that

$$(5.26) \quad \begin{aligned} (\mathcal{P}(x, D)(u_0 e^{\tau\varphi}), v)_{L^2(\Omega)} &= (\mathcal{P}(x, D)(u_0 e^{\tau\varphi}), \mathcal{V})_{L^2(\Omega)} + o\left(\frac{1}{\tau}\right) = (\mathcal{P}(x, D)(u_0 e^{\tau\varphi}), \chi\mathcal{V})_{L^2(\Omega)} + o\left(\frac{1}{\tau}\right) \\ &= (u_0 e^{\tau\varphi}, \mathcal{P}(x, D)^*(\chi\mathcal{V}))_{L^2(\Omega)} + o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty. \end{aligned}$$

Then we can finish the proof of (5.20) using arguments similar to (5.23)-(5.24).

$$(5.27) \quad \begin{aligned} \text{Denote } \mathcal{M}_1 &= \frac{1}{4\partial_{\bar{z}}\Phi} L_1(x, D)\left(\frac{e_2 g_1}{\partial_z \Phi}\right), \mathcal{M}_2 = \frac{1}{4\partial_z \Phi} L_1(x, D)\left(\frac{e_2 g_2}{\partial_z \Phi}\right), \mathcal{M}_3 = -\frac{1}{4\partial_z \Phi} L_2(x, D)^* \left(\frac{e_2 g_3}{\partial_z \Phi}\right), \\ \mathcal{M}_4 &= -\frac{1}{4\partial_{\bar{z}}\Phi} L_2(x, D)^* \left(\frac{e_2 g_4}{\partial_{\bar{z}}\Phi}\right). \text{ Then there exists a constant } \mathcal{C} \text{ independent of } \tau \text{ such that} \\ (\mathcal{P}(x, D)(e^{\tau\Phi} \frac{\mathcal{M}_1}{\tau^2} + e^{\tau\bar{\Phi}} \frac{\mathcal{M}_2}{\tau^2}), v)_{L^2(\Omega)} &+ (\mathcal{P}(x, D)u, e^{-\tau\Phi} \frac{\mathcal{M}_3}{\tau^2} + e^{-\tau\bar{\Phi}} \frac{\mathcal{M}_4}{\tau^2})_{L^2(\Omega)} = \frac{\mathcal{C}}{\tau} + o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty. \end{aligned}$$

Denote  $\mathcal{X}_1 = -\frac{e_2 g_1}{2\partial_{\bar{z}}\Phi}$ ,  $\mathcal{X}_2 = -\frac{e_2 g_2}{2\partial_z \Phi}$ ,  $\mathcal{X}_3 = \frac{e_2 g_3}{2\partial_z \Phi}$ ,  $\mathcal{X}_4 = \frac{e_2 g_4}{2\partial_{\bar{z}}\Phi}$ . Then, using the stationary phase argument we conclude

$$(5.28) \quad \begin{aligned} (\mathcal{P}(x, D)(e^{\tau\Phi} \frac{\mathcal{X}_2}{\tau} + e^{\tau\bar{\Phi}} \frac{\mathcal{X}_1}{\tau}), v)_{L^2(\Omega)} &+ (\mathcal{P}(x, D)u, e^{-\tau\Phi} \frac{\mathcal{X}_3}{\tau} + e^{-\tau\bar{\Phi}} \frac{\mathcal{X}_4}{\tau})_{L^2(\Omega)} = \\ \mathcal{C}_0 + \frac{\mathcal{C}_{-1}}{\tau} + \frac{1}{\tau} \int_{\tilde{\Gamma}} &((A_1 - A_2) \frac{\partial\Phi}{\partial z} \mathcal{X}_2 \bar{b} e^{\bar{B}_2} e^{2\tau i\psi} - (B_1 - B_2) \frac{\partial\bar{\Phi}}{\partial \bar{z}} \mathcal{X}_1 b e^{\bar{A}_2} e^{-2\tau i\psi}) \frac{(\nabla\psi, \nu)}{2i|\nabla\psi|^2} d\sigma \\ + \frac{1}{\tau} \int_{\tilde{\Gamma}} &((A_1 - A_2) \frac{\partial\Phi}{\partial z} \mathcal{X}_3 a e^{A_1} e^{2\tau i\psi} - (B_1 - B_2) \frac{\partial\bar{\Phi}}{\partial \bar{z}} \mathcal{X}_4 \bar{a} e^{B_1} e^{-2\tau i\psi}) \frac{(\nabla\psi, \nu)}{2i|\nabla\psi|^2} d\sigma + o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty. \end{aligned}$$

Next we show that

**Proposition 5.3.** *Under the conditions of Theorem 1.3*

$$(5.29) \quad A_1 = A_2, \quad B_1 = B_2 \quad \text{on } \tilde{\Gamma}$$

and for any function  $\Phi$  satisfying (2.1), (2.2) and for any functions  $a, b, c, d$  satisfying (4.3), (4.4), (4.30), (4.31) we have

$$(5.30) \quad \mathfrak{I}(\Phi, a, b, c, d) = \int_{\tilde{\Gamma}} \left\{ (\nu_1 + i\nu_2) \frac{\partial\Phi}{\partial z} a(z) \overline{c(\bar{z})} e^{A_1 + \bar{A}_2} + (\nu_1 - i\nu_2) \frac{\partial\bar{\Phi}}{\partial \bar{z}} d(\bar{z}) \overline{b(z)} e^{B_1 + \bar{B}_2} \right\} d\sigma = 0.$$

*Proof.* Let  $\hat{x}$  be an arbitrary point from  $\text{Int } \tilde{\Gamma}$  and  $\Gamma_*$  be an arc containing  $\hat{x}$  such that  $\Gamma_* \subset \subset \tilde{\Gamma}$ . By Proposition 2.2 there exists a weight function  $\Phi$  satisfying (2.4) and (2.6).

Then the boundary integrals in (5.9), (5.10) have the following asymptotic:

$$\begin{aligned}
& \int_{\bar{\Gamma}} (B_1 - B_2) d_\tau(\bar{z}) \overline{c_\tau(\bar{z})} e^{\mathcal{B}_1 + \overline{\mathcal{A}_2} - 2i\tau\psi} d\sigma + \int_{\bar{\Gamma}} (A_1 - A_2)(\nu_1 - i\nu_2) a_\tau(z) \overline{b_\tau(z)} e^{\mathcal{A}_1 + \overline{\mathcal{B}_2} + 2i\tau\psi} d\sigma \\
&= \sum_{x \in \mathcal{G} \setminus \{x_-, x_+\}} \left\{ \left( \frac{2\pi}{i \frac{\partial^2 \psi}{\partial \bar{\tau}^2}(x)} \right)^{\frac{1}{2}} (\bar{c}d(B_1 - B_2))(x) \frac{e^{(\mathcal{B}_1 + \overline{\mathcal{A}_2} - 2i\tau\psi)(x)}}{\sqrt{\tau}} \right. \\
&+ \left. \left( \frac{2\pi}{-i \frac{\partial^2 \psi}{\partial \bar{\tau}^2}(x)} \right)^{\frac{1}{2}} (a\bar{b}(A_1 - A_2))(x) \frac{e^{(\mathcal{A}_1 + \overline{\mathcal{B}_2} + 2i\tau\psi)(x)}}{\sqrt{\tau}} \right\} \\
(5.31) \quad & + O\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty.
\end{aligned}$$

We remind that the set  $\mathcal{G}$  is introduced in (2.4). Moreover, in order to avoid the contribution from the points  $x_\pm$  functions  $a, b$  are chosen in such a way that

$$(5.32) \quad \frac{\partial^{|\beta|} a}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}}(x_\pm) = \frac{\partial^{|\beta|} b}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}}(x_\pm) = \frac{\partial^{|\beta|} c}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}}(x_\pm) = \frac{\partial^{|\beta|} d}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}}(x_\pm) = 0 \quad \forall |\beta| \in \{0, \dots, 5\}.$$

Let  $\tilde{\chi}_1 \in C^\infty(\partial\Omega)$  be a function such that it is equal 1 near points  $x_\pm$  and has support located in a small neighborhood of these points. Then

$$\begin{aligned}
& \int_{\Gamma_*} \tilde{\chi}_1(B_1 - B_2) d_\tau \bar{c}_\tau e^{\mathcal{B}_1 + \overline{\mathcal{A}_2} - 2i\tau\psi} d\sigma + \int_{\Gamma_*} \tilde{\chi}_1(A_1 - A_2)(\nu_1 - i\nu_2) a_\tau \bar{b}_\tau e^{\mathcal{A}_1 + \overline{\mathcal{B}_2} + 2i\tau\psi} d\sigma = \\
& \int_{\Gamma_*} \frac{\tilde{\chi}_1(B_1 - B_2) d_\tau \bar{c}_\tau e^{\mathcal{B}_1 + \overline{\mathcal{A}_2}}}{-2i\tau \frac{\partial \psi}{\partial \bar{\tau}}} \frac{\partial e^{-2i\tau\psi}}{\partial \bar{\tau}} d\sigma + \int_{\Gamma_*} \frac{\tilde{\chi}_1(A_1 - A_2)(\nu_1 - i\nu_2) a_\tau \bar{b}_\tau e^{\mathcal{A}_1 + \overline{\mathcal{B}_2}}}{2i\tau \frac{\partial \psi}{\partial \bar{\tau}}} \frac{\partial e^{2i\tau\psi}}{\partial \bar{\tau}} d\sigma = \\
& \int_{\Gamma_*} \frac{\partial}{\partial \bar{\tau}} \left( \frac{\tilde{\chi}_1(B_1 - B_2) d_\tau \bar{c}_\tau e^{\mathcal{B}_1 + \overline{\mathcal{A}_2}}}{2i\tau \frac{\partial \psi}{\partial \bar{\tau}}} \right) e^{-2i\tau\psi} d\sigma \\
& - \int_{\Gamma_*} \frac{\partial}{\partial \bar{\tau}} \left( \frac{\tilde{\chi}_1(A_1 - A_2)(\nu_1 - i\nu_2) a_\tau \bar{b}_\tau e^{\mathcal{A}_1 + \overline{\mathcal{B}_2}}}{2i\tau \frac{\partial \psi}{\partial \bar{\tau}}} \right) e^{2i\tau\psi} d\sigma = O\left(\frac{1}{\tau}\right).
\end{aligned}$$

In order to obtain the last equality we used that by (5.32) and (2.7) the functions

$$\frac{\partial}{\partial \bar{\tau}} \left( \frac{\tilde{\chi}_1(B_1 - B_2) d_\tau \bar{c}_\tau e^{\mathcal{B}_1 + \overline{\mathcal{A}_2}}}{2i \frac{\partial \psi}{\partial \bar{\tau}}} \right), \quad \frac{\partial}{\partial \bar{\tau}} \left( \frac{\tilde{\chi}_1(A_1 - A_2)(\nu_1 - i\nu_2) a_\tau \bar{b}_\tau e^{\mathcal{A}_1 + \overline{\mathcal{B}_2}}}{2i \frac{\partial \psi}{\partial \bar{\tau}}} \right)$$

are bounded. By (5.5), (5.9)-(5.12), (5.19)-(5.21), (5.27) - (5.28) and (5.31), we can represent the right-hand side of (5.4) as

$$\begin{aligned}
O\left(\frac{1}{\tau}\right) &= \tau F_1 + F_0 + \sum_{x \in \mathcal{G} \setminus \{x_-, x_+\}} \left( \left( \frac{2\pi}{i \frac{\partial^2 \psi}{\partial \bar{\tau}^2}(x)} \right)^{\frac{1}{2}} (\bar{c}d(B_1 - B_2))(x) \frac{e^{(\mathcal{B}_1 + \overline{\mathcal{A}_2} - 2i\tau\psi)(x)}}{\sqrt{\tau}} \right. \\
&+ \left. \left( \frac{2\pi}{-i \frac{\partial^2 \psi}{\partial \bar{\tau}^2}(x)} \right)^{\frac{1}{2}} (a\bar{b}(A_1 - A_2))(x) \frac{e^{(\mathcal{B}_1 + \overline{\mathcal{A}_2} + 2i\tau\psi)(x)}}{\sqrt{\tau}} \right).
\end{aligned}$$

Taking into account that  $F_1$  is equal to the left-hand side of (5.30) we obtain the equality (5.30). Using (2.6) and applying Bohr's theorem (e.g., [5], p.393), we obtain (5.29).  $\square$

Thanks to (5.5), (5.9)-(5.12), (5.19)-(5.21), (5.27)-(5.29), (5.31) we can write down the right-hand side of (5.4) as

$$\begin{aligned}
(5.33) \quad I_0 + I_1 + I_2 &= \sum_{k=1}^3 \tau^{2-k} (\kappa_k + \tilde{\kappa}_k) + \kappa \\
&+ \int_{\Gamma_0} (A_1 - A_2) (\nu_1 - i\nu_2) a_\tau \bar{b}_\tau e^{\mathcal{A}_1 + \bar{\mathcal{B}}_2} d\sigma + \int_{\Gamma_0} (B_1 - B_2) (\nu_1 + i\nu_2) d_\tau \bar{c}_\tau e^{\mathcal{B}_1 + \bar{\mathcal{A}}_2} d\sigma \\
&- \frac{1}{2\tau i} \int_{\Gamma_0} \mathcal{Q}_+ \bar{a} b e^{\mathcal{B}_1 + \bar{\mathcal{A}}_2} \frac{(\nabla\psi, \nu)}{|\nabla\psi|^2} d\sigma - \frac{1}{2\tau i} \int_{\Gamma_0} \mathcal{Q}_- \bar{a} b e^{\mathcal{A}_1 + \bar{\mathcal{B}}_2} \frac{(\nabla\psi, \nu)}{|\nabla\psi|^2} d\sigma \\
&- \frac{(\mathcal{Q}_+ \bar{a} b)(\tilde{x}) e^{(\mathcal{A}_1 + \bar{\mathcal{B}}_2 + 2i\tau\psi)(\tilde{x})} + (\mathcal{Q}_- \bar{d} \bar{c})(\tilde{x}) e^{(\mathcal{B}_1 + \bar{\mathcal{A}}_2 - 2i\tau\psi)(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \\
&+ \frac{1}{\tau} (\mathcal{I}_1(\partial\Omega) + \mathcal{I}_2(\partial\Omega)) - 2 \int_{\partial\Omega} (\nu_1 + i\nu_2) e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2} \bar{c}_\tau(\bar{z}) \mathfrak{G}_1(x, \tau) d\sigma \\
&+ 2 \int_{\partial\Omega} (\nu_1 - i\nu_2) e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} d_\tau(\bar{z}) \overline{\mathfrak{G}_4(x, \tau)} d\sigma + 2 \int_{\partial\Omega} (\nu_1 + i\nu_2) e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2} a_\tau(z) \overline{\mathfrak{G}_3(x, \tau)} d\sigma \\
&- 2 \int_{\partial\Omega} (\nu_1 - i\nu_2) e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} \bar{b}_\tau(z) \mathfrak{G}_2(x, \tau) d\sigma \\
&+ \frac{e^{-2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \left\{ \frac{\partial g_4(\tilde{x})}{\partial z} \right\} e^{-\bar{\mathcal{B}}_2(\tilde{x})} \int_{\partial\Omega} \frac{(\nu_1 - i\nu_2) d e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2}}{\tilde{z} - \bar{z}} d\sigma \\
&- \frac{e^{-2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_1(\tilde{x})}{\partial z} e^{-\mathcal{A}_1(\tilde{x})} \int_{\partial\Omega} \frac{(\nu_1 + i\nu_2) \bar{c} e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2}}{\tilde{z} - z} d\sigma \\
&+ \frac{e^{2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_3(\tilde{x})}{\partial \bar{z}} e^{-\bar{\mathcal{A}}_2(\tilde{x})} \int_{\partial\Omega} \frac{(\nu_1 + i\nu_2) a e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2}}{\tilde{z} - z} d\sigma \\
&- \frac{e^{2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_2(\tilde{x})}{\partial \bar{z}} e^{-\mathcal{B}_1(\tilde{x})} \int_{\partial\Omega} \frac{(\nu_1 - i\nu_2) \bar{b} e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2}}{\tilde{z} - \bar{z}} d\sigma + o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty.
\end{aligned}$$

We note that  $\kappa_k$  and  $\tilde{\kappa}_k$  denote generic constants which are independent of  $\tau$ . In order to transform some terms in the above equality, we need the following proposition:

**Proposition 5.4.** *There exist a holomorphic function  $\Theta \in H^{\frac{1}{2}}(\Omega)$  and an antiholomorphic function  $\tilde{\Theta} \in H^{\frac{1}{2}}(\Omega)$  such that*

$$(5.34) \quad \Theta|_{\tilde{\Gamma}} = e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2}, \quad \tilde{\Theta}|_{\tilde{\Gamma}} = e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2}$$

and

$$(5.35) \quad e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} \Theta - e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2} \tilde{\Theta} = 0 \quad \text{on } \Gamma_0.$$

*Proof.* Consider the extremal problem:

$$(5.36) \quad J(\Psi, \tilde{\Psi}) = \|e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2} \frac{\partial \Phi}{\partial z} a \bar{c} - \Psi\|_{L^2(\tilde{\Gamma})}^2 + \|e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} \frac{\partial \bar{\Phi}}{\partial \bar{z}} \bar{b} d - \tilde{\Psi}\|_{L^2(\tilde{\Gamma})}^2 \rightarrow \inf,$$

$$(5.37) \quad \frac{\partial \Psi}{\partial \bar{z}} = 0 \quad \text{in } \Omega, \quad \frac{\partial \tilde{\Psi}}{\partial z} = 0 \quad \text{in } \Omega, \quad ((\nu_1 + i\nu_2)\Psi + (\nu_1 - i\nu_2)\tilde{\Psi})|_{\Gamma_0} = 0.$$

Here the functions  $a, b, c, d$  satisfy (4.3), (4.4), (4.30) and (4.31). Denote the unique solution to this extremal problem (5.36), (5.37) as  $(\widehat{\Psi}, \widehat{\tilde{\Psi}})$ . Applying Lagrange's principle we obtain

$$(5.38) \quad \operatorname{Re}(e^{\mathcal{A}_1 + \overline{\mathcal{A}_2}} \frac{\partial \Phi}{\partial z} a \bar{c} - \widehat{\Psi}, \delta)_{L^2(\tilde{\Gamma})} + \operatorname{Re}(e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}} \frac{\partial \bar{\Phi}}{\partial \bar{z}} \bar{b} d - \widehat{\tilde{\Psi}}, \tilde{\delta})_{L^2(\tilde{\Gamma})} = 0$$

for any  $\delta, \tilde{\delta}$  from  $H^{\frac{1}{2}}(\Omega)$  such that

$$\frac{\partial \delta}{\partial \bar{z}} = 0 \quad \text{in } \Omega, \quad \frac{\partial \tilde{\delta}}{\partial z} = 0 \quad \text{in } \Omega, \quad (\nu_1 + i\nu_2)\delta|_{\Gamma_0} = -(\nu_1 - i\nu_2)\tilde{\delta}|_{\Gamma_0}$$

and there exist two functions  $P, \tilde{P} \in H^{\frac{1}{2}}(\Omega)$  such that

$$(5.39) \quad \frac{\partial P}{\partial \bar{z}} = 0 \quad \text{in } \Omega, \quad \frac{\partial \tilde{P}}{\partial z} = 0 \quad \text{in } \Omega,$$

$$(5.40) \quad (\nu_1 + i\nu_2)P = \overline{e^{\mathcal{A}_1 + \overline{\mathcal{A}_2}} \frac{\partial \Phi}{\partial z} a \bar{c} - \widehat{\Psi}} \quad \text{on } \tilde{\Gamma}, \quad (\nu_1 - i\nu_2)\tilde{P} = \overline{e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}} \frac{\partial \bar{\Phi}}{\partial \bar{z}} \bar{b} d - \widehat{\tilde{\Psi}}} \quad \text{on } \tilde{\Gamma},$$

$$(5.41) \quad (P - \tilde{P})|_{\Gamma_0} = 0.$$

Denote  $\Psi_0(z) = \frac{1}{2i}(P(z) - \overline{\tilde{P}(\bar{z})})$ ,  $\Phi_0(z) = \frac{1}{2}(P(z) + \overline{\tilde{P}(\bar{z})})$ . By (5.41)

$$(5.42) \quad \operatorname{Im} \Psi_0|_{\Gamma_0} = \operatorname{Im} \Phi_0|_{\Gamma_0} = 0.$$

Hence

$$(5.43) \quad P = (\Phi_0 + i\Psi_0), \quad \overline{\tilde{P}} = (\Phi_0 - i\Psi_0).$$

From (5.38) taking  $\delta = \widehat{\Psi}$  and  $\tilde{\delta} = \widehat{\tilde{\Psi}}$  we have

$$(5.44) \quad \operatorname{Re}(e^{\mathcal{A}_1 + \overline{\mathcal{A}_2}} \frac{\partial \Phi}{\partial z} a \bar{c} - \widehat{\Psi}, \widehat{\Psi})_{L^2(\tilde{\Gamma})} + \operatorname{Re}(e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}} \frac{\partial \bar{\Phi}}{\partial \bar{z}} \bar{b} d - \widehat{\tilde{\Psi}}, \widehat{\tilde{\Psi}})_{L^2(\tilde{\Gamma})} = 0.$$

By (5.39), (5.40) and (5.43), we have

$$\begin{aligned} H_1 &= \operatorname{Re}(e^{\mathcal{A}_1 + \overline{\mathcal{A}_2}} \frac{\partial \Phi}{\partial z} a \bar{c} - \widehat{\Psi}, e^{\mathcal{A}_1 + \overline{\mathcal{A}_2}} \frac{\partial \Phi}{\partial z} a \bar{c})_{L^2(\tilde{\Gamma})} + \operatorname{Re}(e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}} \frac{\partial \bar{\Phi}}{\partial \bar{z}} \bar{b} d - \widehat{\tilde{\Psi}}, e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}} \frac{\partial \bar{\Phi}}{\partial \bar{z}} \bar{b} d)_{L^2(\tilde{\Gamma})} \\ &= \operatorname{Re}((\nu_1 - i\nu_2)\overline{\tilde{P}}, e^{\mathcal{A}_1 + \overline{\mathcal{A}_2}} \frac{\partial \Phi}{\partial z} a \bar{c})_{L^2(\tilde{\Gamma})} + \operatorname{Re}((\nu_1 + i\nu_2)\overline{\tilde{P}}, e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}} \frac{\partial \bar{\Phi}}{\partial \bar{z}} \bar{b} d)_{L^2(\tilde{\Gamma})} = \\ &2\operatorname{Re}((\nu_1 - i\nu_2)\overline{(\Phi_0 + i\Psi_0)}, e^{\mathcal{A}_1 + \overline{\mathcal{A}_2}} \frac{\partial \Phi}{\partial z} a \bar{c})_{L^2(\tilde{\Gamma})} + 2\operatorname{Re}((\nu_1 + i\nu_2)(\Phi_0 - i\Psi_0), e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}} \frac{\partial \bar{\Phi}}{\partial \bar{z}} \bar{b} d)_{L^2(\tilde{\Gamma})}. \end{aligned}$$

We can rewrite

$$(5.45) \quad 2\operatorname{Re}((\nu_1 - i\nu_2)\overline{\Phi_0}, e^{\mathcal{A}_1 + \overline{\mathcal{A}_2}} \frac{\partial \Phi}{\partial z} a \bar{c})_{L^2(\tilde{\Gamma})} + 2\operatorname{Re}((\nu_1 + i\nu_2)\Phi_0, e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}} \frac{\partial \bar{\Phi}}{\partial \bar{z}} \bar{b} d)_{L^2(\tilde{\Gamma})} = \mathfrak{I}(\Phi, \Phi_0 a, b, c, \Phi_0 d) + \overline{\mathfrak{I}(\Phi, \Phi_0 a, b, c, \Phi_0 d)}$$

and

$$\begin{aligned}
& 2\operatorname{Re}((\nu_1 - i\nu_2)\overline{(i\Psi_0)}, e^{\mathcal{A}_1 + \overline{\mathcal{A}_2}} \frac{\partial \Phi}{\partial z} a \bar{c})_{L^2(\tilde{\Gamma})} + 2\operatorname{Re}((\nu_1 + i\nu_2)(-i\Psi_0), e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}} \frac{\partial \overline{\Phi}}{\partial \bar{z}} b d)_{L^2(\tilde{\Gamma})} = \\
& -2\operatorname{Im}((\nu_1 - i\nu_2)\overline{a c \Psi_0}, e^{\mathcal{A}_1 + \overline{\mathcal{A}_2}} \frac{\partial \Phi}{\partial z})_{L^2(\tilde{\Gamma})} - 2\operatorname{Im}((\nu_1 + i\nu_2)\overline{b d \Psi_0}, e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}} \frac{\partial \overline{\Phi}}{\partial \bar{z}})_{L^2(\tilde{\Gamma})} = \\
& -\frac{1}{i} \int_{\tilde{\Gamma}} ((\nu_1 - i\nu_2)\overline{a c \Psi_0} \frac{\partial \overline{\Phi}}{\partial \bar{z}} e^{\overline{\mathcal{A}_1 - \mathcal{A}_2}} - (\nu_1 + i\nu_2)a \bar{c} \Psi_0 \frac{\partial \Phi}{\partial z} e^{\mathcal{A}_1 + \overline{\mathcal{A}_2}}) d\sigma \\
& -\frac{1}{i} \int_{\tilde{\Gamma}} ((\nu_1 + i\nu_2)\overline{b d \Psi_0} \frac{\partial \Phi}{\partial z} e^{\overline{\mathcal{B}_1 + \mathcal{B}_2}} - (\nu_1 - i\nu_2)\overline{b d \Psi_0} \frac{\partial \overline{\Phi}}{\partial \bar{z}} e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}}) d\sigma = \\
(5.46) \quad & \frac{1}{i} (\mathfrak{I}(\Phi, a\Psi_0, b, c, d\Psi_0) - \overline{\mathfrak{I}(\Phi, a\Psi_0, b, c, d\Psi_0)}).
\end{aligned}$$

Then by (5.42), (5.45), (5.46) and Proposition 5.3,  $H_1 = 0$ . Taking into account (5.44) we obtain that  $J(\widehat{\Psi}, \widehat{\Psi}) = 0$ . Consequently, setting  $\Theta = \widehat{\Psi}/(\partial_z \Phi a \bar{c})$  and  $\widetilde{\Theta} = \widehat{\Psi}/(\partial_z \Phi d \bar{b})$  we obtain (5.34).

Observe that

$$(5.47) \quad (\nu_1 + i\nu_2) \frac{\partial \Phi}{\partial z} = -(\nu_1 - i\nu_2) \frac{\partial \overline{\Phi}}{\partial \bar{z}} \quad \text{on } \Gamma_0.$$

In order to see this we argue as follows. We have that  $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2})(\varphi + i\psi) = \frac{1}{2}(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \psi}{\partial x_2}) + \frac{i}{2}(\frac{\partial \varphi}{\partial x_2} + \frac{\partial \psi}{\partial x_1})$ . Hence  $\frac{\partial \varphi}{\partial x_1} = \frac{\partial \psi}{\partial x_2}$ ,  $\frac{\partial \varphi}{\partial x_2} = -\frac{\partial \psi}{\partial x_1}$ ,  $\frac{\partial \varphi}{\partial \nu} = -\frac{\partial \psi}{\partial \bar{\tau}}$  and  $\frac{\partial \psi}{\partial \nu} = \frac{\partial \varphi}{\partial \bar{\tau}}$ . Observe that  $(\nu_1 + i\nu_2) \frac{\partial}{\partial z} = \frac{1}{2}(\nu_1 \frac{\partial}{\partial x_1} + \nu_2 \frac{\partial}{\partial x_2}) + \frac{i}{2}(\nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2}) = \frac{1}{2}(\frac{\partial}{\partial \nu} + i\frac{\partial}{\partial \bar{\tau}})$  and  $(\nu_1 - i\nu_2) \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial \nu} - i\frac{\partial}{\partial \bar{\tau}})$ . Hence

$$(\nu_1 + i\nu_2) \frac{\partial \Phi}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial \nu} + i\frac{\partial}{\partial \bar{\tau}})(\varphi + i\psi) = \frac{1}{2}(\frac{\partial \varphi}{\partial \nu} - \frac{\partial \psi}{\partial \bar{\tau}}) + \frac{i}{2}(\frac{\partial \varphi}{\partial \bar{\tau}} + \frac{\partial \psi}{\partial \nu}) = -\frac{\partial \psi}{\partial \bar{\tau}} + i\frac{\partial \varphi}{\partial \bar{\tau}}.$$

Therefore

$$(\nu_1 - i\nu_2) \frac{\partial \overline{\Phi}}{\partial \bar{z}} = \overline{\frac{\partial \psi}{\partial \bar{\tau}} + i\frac{\partial \varphi}{\partial \bar{\tau}}} = -\frac{\partial \psi}{\partial \bar{\tau}} - i\frac{\partial \varphi}{\partial \bar{\tau}}.$$

Taking into account that  $\psi|_{\Gamma_0} = 0$  we obtain (5.47).

From (5.47), (4.3), (4.4), (4.30), (4.31) and (5.37) we obtain (5.35). The proof of the proposition is completed. In general  $\Phi, a, b, c, d$  may have a finite number of zeros in  $\Omega$ . At these zeros  $\Theta, \widetilde{\Theta}$  may have poles. On the other hand observe that  $\Theta, \widetilde{\Theta}$  are independent of a particular choice of the functions  $\Phi, a, b, c, d$ . Making small perturbations of these functions we can shift the position of the zeros. Therefore we may assume that there are no poles for  $\Theta, \widetilde{\Theta}$ .  $\square$

Thanks to Proposition 5.4, we can rewrite (5.33) as

$$\begin{aligned}
(5.48) \quad & o\left(\frac{1}{\tau}\right) = \sum_{k=1}^3 \tau^{2-k} \tilde{F}_k \\
& - \frac{\pi}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \left\{ (\mathcal{Q}_+ a \bar{b})(\tilde{x}) e^{(\mathcal{A}_1 + \bar{\mathcal{B}}_2 + 2\tau i \psi)(\tilde{x})} + (\mathcal{Q}_- d \bar{c})(\tilde{x}) e^{(\mathcal{B}_1 + \bar{\mathcal{A}}_2 - 2i\tau \psi)(\tilde{x})} \right\} \\
& - \frac{1}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \int_{\Gamma_0} (\nu_1 + i\nu_2) (e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2} - \Theta) \frac{\partial \Phi}{\partial z} a(z) \overline{(c_{2,+} e^{2\tau i \psi(\tilde{x})} + c_{2,-} e^{-2\tau i \psi(\tilde{x})})} d\sigma \\
& - \frac{1}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \int_{\Gamma_0} (\nu_1 - i\nu_2) (e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} - \tilde{\Theta}) \frac{\partial \bar{\Phi}}{\partial \bar{z}} \bar{b}(z) (d_{2,+} e^{2\tau i \psi(\tilde{x})} + d_{2,-} e^{-2\tau i \psi(\tilde{x})}) d\sigma \\
& - \frac{1}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \int_{\Gamma_0} (\nu_1 + i\nu_2) (e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2} - \Theta) \frac{\partial \Phi}{\partial z} \overline{c(\bar{z})} (a_{2,+} e^{2\tau i \psi(\tilde{x})} + a_{2,-} e^{-2\tau i \psi(\tilde{x})}) d\sigma \\
& - \frac{1}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \int_{\Gamma_0} (\nu_1 - i\nu_2) (e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} - \tilde{\Theta}) \frac{\partial \bar{\Phi}}{\partial \bar{z}} d(\bar{z}) \overline{(b_{2,+} e^{2\tau i \psi(\tilde{x})} + b_{2,-} e^{-2\tau i \psi(\tilde{x})})} d\sigma \\
& - 2 \int_{\Gamma_0} (\nu_1 + i\nu_2) (e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2} - \Theta) \overline{c_\tau(\bar{z})} \mathfrak{G}_1(x, \tau) d\sigma + 2 \int_{\Gamma_0} (\nu_1 - i\nu_2) (e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} - \tilde{\Theta}) d_\tau(\bar{z}) \overline{\mathfrak{G}_4(x, \tau)} d\sigma \\
& + 2 \int_{\Gamma_0} (\nu_1 + i\nu_2) (e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2} - \Theta) a_\tau(z) \overline{\mathfrak{G}_3(x, \tau)} d\sigma - 2 \int_{\Gamma_0} (\nu_1 - i\nu_2) (e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} - \tilde{\Theta}) \overline{b_\tau(z)} \mathfrak{G}_2(x, \tau) d\sigma \\
& + \frac{e^{-2i\tau \psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_4(\tilde{x})}{\partial z} e^{-\bar{\mathcal{B}}_2(\tilde{x})} \left( \int_{\Gamma_0} \frac{(\nu_1 - i\nu_2) d(e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} - \tilde{\Theta})}{\bar{z} - \bar{z}} d\sigma - 2\pi(d\tilde{\Theta})(\tilde{x}) \right) \\
& - \frac{e^{-2i\tau \psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_1(\tilde{x})}{\partial z} e^{-\mathcal{A}_1(\tilde{x})} \left( \int_{\Gamma_0} \frac{(\nu_1 + i\nu_2) \bar{c}(e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2} - \Theta)}{\bar{z} - z} d\sigma - 2\pi(\bar{c}\Theta)(\tilde{x}) \right) \\
& + \frac{e^{2i\tau \psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_3(\tilde{x})}{\partial \bar{z}} e^{-\bar{\mathcal{A}}_2(\tilde{x})} \left( \int_{\Gamma_0} \frac{(\nu_1 + i\nu_2) a(e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2} - \Theta)}{\bar{z} - z} d\sigma - 2\pi(a\Theta)(\tilde{x}) \right) \\
& - \frac{e^{2i\tau \psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_2(\tilde{x})}{\partial \bar{z}} e^{-\mathcal{B}_1(\tilde{x})} \left( \int_{\Gamma_0} \frac{(\nu_1 - i\nu_2) \bar{b}(e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} - \tilde{\Theta})}{\bar{z} - \bar{z}} d\sigma - 2\pi(\bar{b}\Theta)(\tilde{x}) \right).
\end{aligned}$$

Here  $\tilde{F}_k$  are some constants independent of  $\tau$ .

Then, using (4.15), (5.35), (5.47), on  $\Gamma_0$  we have

$$\begin{aligned}
(5.49) \quad & -(\nu_1 + i\nu_2) \frac{\partial \Phi}{\partial z} (e^{\mathcal{A}_1 + \overline{\mathcal{A}}_2} - \Theta) \bar{c} (a_{2,+} e^{2\tau i \psi(\tilde{x})} + a_{2,-} e^{-2\tau i \psi(\tilde{x})}) \\
& -(\nu_1 - i\nu_2) \frac{\partial \bar{\Phi}}{\partial \bar{z}} (e^{\mathcal{B}_1 + \overline{\mathcal{B}}_2} - \tilde{\Theta}) \bar{b} (d_{2,+} e^{2\tau i \psi(\tilde{x})} + d_{2,-} e^{-2\tau i \psi(\tilde{x})}) = \\
& -(\nu_1 + i\nu_2) \frac{\partial \Phi}{\partial z} \bar{c} \left( (e^{\mathcal{A}_1 + \overline{\mathcal{A}}_2} - \Theta) a_{2,+} e^{2\tau i \psi(\tilde{x})} + (e^{\mathcal{B}_1 + \overline{\mathcal{A}}_2} - \tilde{\Theta} e^{\overline{\mathcal{A}}_2 - \overline{\mathcal{B}}_2}) d_{2,+} e^{2\tau i \psi(\tilde{x})} \right) \\
& -(\nu_1 - i\nu_2) \frac{\partial \bar{\Phi}}{\partial \bar{z}} \bar{b} \left( (e^{\mathcal{A}_1 + \overline{\mathcal{B}}_2} - \Theta e^{\overline{\mathcal{B}}_2 - \overline{\mathcal{A}}_2}) a_{2,-} e^{-2\tau i \psi(\tilde{x})} + (e^{\mathcal{B}_1 + \overline{\mathcal{B}}_2} - \tilde{\Theta}) d_{2,-} e^{-2\tau i \psi(\tilde{x})} \right) = \\
& -(\nu_1 + i\nu_2) \frac{\partial \Phi}{\partial z} \bar{c} \left( (e^{\mathcal{A}_1 + \overline{\mathcal{A}}_2} - \Theta) a_{2,+} e^{2\tau i \psi(\tilde{x})} + (e^{\mathcal{B}_1 + \overline{\mathcal{A}}_2} - \Theta e^{-\mathcal{A}_1 + \mathcal{B}_1}) d_{2,+} e^{2\tau i \psi(\tilde{x})} \right) \\
& -(\nu_1 - i\nu_2) \frac{\partial \bar{\Phi}}{\partial \bar{z}} \bar{b} \left( (e^{\mathcal{A}_1 + \overline{\mathcal{B}}_2} - \tilde{\Theta} e^{\mathcal{A}_1 - \mathcal{B}_1}) a_{2,-} e^{-2\tau i \psi(\tilde{x})} + (e^{\mathcal{B}_1 + \overline{\mathcal{B}}_2} - \tilde{\Theta}) d_{2,-} e^{-2\tau i \psi(\tilde{x})} \right) = \\
& -(\nu_1 + i\nu_2) \frac{\partial \Phi}{\partial z} (e^{\overline{\mathcal{A}}_2} - \Theta e^{-\mathcal{A}_1}) \bar{c} p_+ e^{2\tau i \psi(\tilde{x})} - (\nu_1 - i\nu_2) \frac{\partial \bar{\Phi}}{\partial \bar{z}} (e^{\overline{\mathcal{B}}_2} - \tilde{\Theta} e^{-\mathcal{B}_1}) \bar{b} p_- e^{-2\tau i \psi(\tilde{x})}
\end{aligned}$$

and

$$\begin{aligned}
(5.50) \quad & -(\nu_1 + i\nu_2) (e^{\mathcal{A}_1 + \overline{\mathcal{A}}_2} - \Theta) \frac{\partial \Phi}{\partial z} a \left( \overline{c_{2,+} e^{2\tau i \psi(\tilde{x})} + c_{2,-} e^{-2\tau i \psi(\tilde{x})}} \right) \\
& -(\nu_1 - i\nu_2) (e^{\mathcal{B}_1 + \overline{\mathcal{B}}_2} - \tilde{\Theta}) \frac{\partial \bar{\Phi}}{\partial \bar{z}} d \left( \overline{b_{2,+} e^{2\tau i \psi(\tilde{x})} + b_{2,-} e^{-2\tau i \psi(\tilde{x})}} \right) = \\
& -(\nu_1 + i\nu_2) \frac{\partial \Phi}{\partial z} a \left( (e^{\mathcal{A}_1 + \overline{\mathcal{A}}_2} - \Theta) \overline{c_{2,-} e^{-2\tau i \psi(\tilde{x})}} + (e^{\mathcal{A}_1 + \overline{\mathcal{B}}_2} - e^{-\mathcal{B}_1 + \mathcal{A}_1} \tilde{\Theta}) \overline{b_{2,-} e^{-2\tau i \psi(\tilde{x})}} \right) \\
& -(\nu_1 - i\nu_2) \frac{\partial \bar{\Phi}}{\partial \bar{z}} d \left( (e^{\mathcal{B}_1 + \overline{\mathcal{A}}_2} - \Theta e^{-\mathcal{A}_1 + \mathcal{B}_1}) \overline{c_{2,+} e^{-2\tau i \psi(\tilde{x})}} + (e^{\mathcal{B}_1 + \overline{\mathcal{B}}_2} - \tilde{\Theta}) \overline{b_{2,+} e^{2\tau i \psi(\tilde{x})}} \right) = \\
& -(\nu_1 + i\nu_2) \frac{\partial \Phi}{\partial z} a \left( (e^{\mathcal{A}_1 + \overline{\mathcal{A}}_2} - \Theta) \overline{c_{2,-} e^{-2\tau i \psi(\tilde{x})}} + (e^{\mathcal{A}_1 + \overline{\mathcal{B}}_2} - e^{\overline{\mathcal{B}}_2 - \overline{\mathcal{A}}_2} \Theta) \overline{b_{2,-} e^{-2\tau i \psi(\tilde{x})}} \right) \\
& -(\nu_1 - i\nu_2) \frac{\partial \bar{\Phi}}{\partial \bar{z}} d \left( (e^{\mathcal{B}_1 + \overline{\mathcal{A}}_2} - \Theta e^{\overline{\mathcal{A}}_2 - \overline{\mathcal{B}}_2}) \overline{c_{2,+} e^{2\tau i \psi(\tilde{x})}} + (e^{\mathcal{B}_1 + \overline{\mathcal{B}}_2} - \tilde{\Theta}) \overline{b_{2,+} e^{2\tau i \psi(\tilde{x})}} \right) = \\
& -(\nu_1 + i\nu_2) \frac{\partial \Phi}{\partial z} (e^{\mathcal{A}_1} - \Theta e^{-\overline{\mathcal{A}}_2}) \overline{a \tilde{p}_- e^{-2\tau i \psi(\tilde{x})}} - (\nu_1 - i\nu_2) \frac{\partial \bar{\Phi}}{\partial \bar{z}} (e^{\mathcal{B}_1} - \tilde{\Theta} e^{-\overline{\mathcal{B}}_2}) \overline{d \tilde{p}_+ e^{2\tau i \psi(\tilde{x})}}.
\end{aligned}$$



Using (5.49), (5.50) and Proposition 8.2 we rewrite (5.48) as

$$\begin{aligned}
(5.51) \quad o\left(\frac{1}{\tau}\right) &= \sum_{k=1}^3 \tau^{2-k} \tilde{F}_k \\
&- \frac{\pi}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \left\{ (\mathcal{Q}_+ a \bar{b})(\tilde{x}) e^{(\mathcal{A}_1 + \bar{\mathcal{B}}_2 + 2i\tau\psi)(\tilde{x})} + (\mathcal{Q}_- d \bar{c})(\tilde{x}) e^{(\mathcal{B}_1 + \bar{\mathcal{A}}_2 - 2i\tau\psi)(\tilde{x})} \right\} \\
&- \frac{e^{-2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \int_{\Gamma_0} (\nu_1 + i\nu_2) (e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2} - \Theta) a \frac{\overline{(\partial_z g_3 e^{-\mathcal{A}_2})(\tilde{x})}}{\tilde{z} - z} d\sigma \\
&+ \frac{e^{-2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \int_{\Gamma_0} (\nu_1 + i\nu_2) (e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2} - \Theta) \overline{c(\tilde{z})} \frac{(\partial_z g_1 e^{-\mathcal{A}_1})(\tilde{x})}{\tilde{z} - z} d\sigma \\
&- \frac{e^{-2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \int_{\Gamma_0} (\nu_1 - i\nu_2) (e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} - \tilde{\Theta}) d \frac{\overline{(\partial_z g_4 e^{-\mathcal{B}_2})(\tilde{x})}}{\tilde{z} - \bar{z}} d\sigma \\
&+ \frac{e^{2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \int_{\Gamma_0} (\nu_1 - i\nu_2) (e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} - \tilde{\Theta}) \overline{b(z)} \frac{(\partial_z g_2 e^{-\mathcal{B}_1})(\tilde{x})}{\tilde{z} - \bar{z}} d\sigma \\
&+ \frac{e^{-2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_4(\tilde{x})}{\partial z} e^{-\bar{\mathcal{B}}_2(\tilde{x})} \left( \int_{\Gamma_0} \frac{(\nu_1 - i\nu_2) d(e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} - \tilde{\Theta})}{\tilde{z} - \bar{z}} d\sigma - 2\pi(d\tilde{\Theta})(\tilde{x}) \right) \\
&- \frac{e^{-2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_1(\tilde{x})}{\partial z} e^{-\mathcal{A}_1(\tilde{x})} \left( \int_{\Gamma_0} \frac{(\nu_1 + i\nu_2) \bar{c}(e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2} - \Theta)}{\tilde{z} - z} d\sigma - 2\pi(\bar{c}\Theta)(\tilde{x}) \right) \\
&+ \frac{e^{2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_3(\tilde{x})}{\partial \bar{z}} e^{-\bar{\mathcal{A}}_2(\tilde{x})} \left( \int_{\Gamma_0} \frac{(\nu_1 + i\nu_2) a(e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2} - \Theta)}{\tilde{z} - z} d\sigma - 2\pi(a\Theta)(\tilde{x}) \right) \\
&- \frac{e^{2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_2(\tilde{x})}{\partial \bar{z}} e^{-\mathcal{B}_1(\tilde{x})} \left( \int_{\Gamma_0} \frac{(\nu_1 - i\nu_2) \bar{b}(e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} - \tilde{\Theta})}{\tilde{z} - \bar{z}} d\sigma - 2\pi(\bar{b}\tilde{\Theta})(\tilde{x}) \right).
\end{aligned}$$

Let  $\eta$  be a smooth function such that  $\eta$  is zero in some neighborhood of  $\partial\Omega$  and  $\eta(\tilde{x}) \neq 0$ . Observe that the partial Cauchy data of the operators  $L_2(x, D)$  and the operator  $e^{-s\eta} L_1(x, D) e^{s\eta}$  are exactly the same. Therefore we have the analog of (5.51) for these two operators with  $\mathcal{A}_1$  and  $\mathcal{B}_1$  replaced by  $\mathcal{A}_1 - s\eta$  and  $\mathcal{B}_1 - s\eta$ . The coefficients  $A_1, B_1$  should be replaced by  $A_1 + 2s \frac{\partial \eta}{\partial \bar{z}}, B_1 + 2s \frac{\partial \eta}{\partial z}$ . The functions  $\mathcal{Q}_\pm$  will not change. The function  $q_1$  should be replaced by  $q_1 + s\Delta\eta + s^2|\nabla\eta|^2 + 2sA_1 \frac{\partial \eta}{\partial z} + 2sB_1 \frac{\partial \eta}{\partial \bar{z}}$ . This immediately implies that  $(\mathcal{Q}_+ a \bar{b})(\tilde{x}) = (\mathcal{Q}_- d \bar{c})(\tilde{x}) = 0$ . The proof of the theorem is completed.  $\square$

## 6. PROOF OF THEOREM 1.1

Suppose that we have two operators

$$L_1(x, D) = \Delta_{g_1} + 2A_1 \frac{\partial}{\partial z} + 2B_1 \frac{\partial}{\partial \bar{z}} + q_1$$

and

$$L_2(x, D) = \Delta_{g_2} + 2A_2 \frac{\partial}{\partial z} + 2B_2 \frac{\partial}{\partial \bar{z}} + q_2$$

with the same partial Cauchy data. Multiplying the metric  $g_2$ , if necessary, by some positive smooth function  $\tilde{\beta}$ , we may assume that

$$(6.1) \quad \frac{\partial^\ell}{\partial \nu^\ell} (g_1^{jk} - g_2^{jk})|_{\tilde{\Gamma}} = 0, \quad \ell \in \{0, 1\}.$$

We note that  $\{g_1^{jk}\}$  denotes the inverse matrix to  $g_1 = \{g_{1,jk}\}$ . Without loss of generality, we may assume that there exists a smooth positive function  $\mu_2$  such that  $g_2 = \mu_2 I$ . Indeed, using isothermal coordinates we make a change of variables in the operator  $L_2(x, D)$  such that  $g_2 = \mu_2 I$ . Then we make the same changes of variables in the operator  $L_1(x, D)$ . The partial Cauchy data of both operators obtained by this change of variables are the same.

Let  $\omega$  be a subdomain in  $\mathbb{R}^2$  such that  $\Omega \cap \omega = \emptyset$ ,  $\partial\omega \cap \partial\Omega = \tilde{\Gamma}$  and the boundary of the domain  $\tilde{\Omega} = \text{Int}(\Omega \cup \omega)$  is smooth. We extend  $\mu_2$  in  $\tilde{\Omega}$  as a smooth positive function and set  $g_1^{jk} = \frac{1}{\mu_2} I$  in  $\omega$ . By (6.1)  $g_1 \in C^1(\tilde{\Omega})$ . There exists an isothermal mapping  $\chi_1 = (\chi_{1,1}, \chi_{1,2})$  such that the operator  $L_1(x, D)$  is transformed to

$$(6.2) \quad Q_1(y, D) = \frac{1}{\mu_1} \Delta + 2C_1 \frac{\partial}{\partial z} + 2D_1 \frac{\partial}{\partial \bar{z}} + r_1 \quad y \in \chi_1(\tilde{\Omega}),$$

where  $\mu_1$  is a smooth positive function in  $\chi_1(\tilde{\Omega})$  and  $C_1, D_1, r_1$  are some smooth complex valued functions. Consider a solution to the boundary value problem

$$Q_1(y, D)w = 0 \quad \text{in } \chi_1(\tilde{\Omega}), \quad w|_{\chi_1(\Gamma_0)} = 0$$

of the form (4.2) with a holomorphic weight function  $\Phi_1$ . Then the function  $u_1(x) = w(\chi_1(x))$  is solution to

$$L_1(x, D)u_1 = 0 \quad \text{in } \tilde{\Omega}, \quad u_1|_{\Gamma_0} = 0.$$

Since the partial Cauchy data for the operators  $L_1(x, D)$  and  $L_2(x, D)$  are the same, there exists a function  $u_2$  such that

$$(6.3) \quad L_2(x, D)u_2 = 0 \quad \text{in } \Omega, \quad u_2|_{\Gamma_0} = 0, \quad \left( \frac{\partial u_1}{\partial \nu_{g_1}} - \frac{\partial u_2}{\partial \nu_{g_2}} \right) |_{\tilde{\Gamma}} = 0.$$

Using (6.1) and (6.3) we extend  $u_2 \in H^1(\tilde{\Omega})$  in  $\tilde{\Omega}$  such that

$$(6.4) \quad u_1|_\omega = u_2|_\omega.$$

Let  $\varphi_2$  be a harmonic function in  $\tilde{\Omega}$  such that

$$\frac{\partial \varphi_2}{\partial \nu} |_{\Gamma_0} = 0, \quad \varphi_2 = \text{Re } \Phi_1 \circ \chi_1 \quad \text{on } \partial\tilde{\Omega} \setminus \Gamma_0.$$

We claim that

$$(6.5) \quad \varphi_2 = \text{Re } \Phi_1 \circ \chi_1 \quad \text{in } \omega.$$

A difficulty comes from the fact that the function  $\varphi_2$  is continuous on  $\bar{\Omega}$  (see e.g [28]) but the derivatives of  $\varphi_2$  may be discontinuous at some points of  $\partial\Gamma_0$ . First we observe that it suffices to prove (6.5) for the functions such that  $\text{Im } \Phi_1 = 0$  on some open set  $\mathcal{O}_{\Phi_1} \subset \partial\chi_1(\tilde{\Omega})$  such that  $\overline{\chi_1(\Gamma_0)} \subset \mathcal{O}_{\Phi_1}$ . Indeed, without loss of generality, assume that  $\partial\tilde{\Omega} \setminus \Gamma_0$  is an arc

with two endpoints  $x_{\pm}$ . Let the sequences  $\{x_{\epsilon,-}\}, \{x_{\epsilon,+}\} \subset \partial\tilde{\Omega} \setminus \Gamma_0$  be such that  $x_{\epsilon,\pm} \rightarrow x_{\pm}$  as  $\epsilon \rightarrow 0$ . Consider a sequence of holomorphic functions  $\{\Phi_{1,\epsilon}\}_{\epsilon \in (0,1)}$  such that

$$\frac{\partial\Phi_{1,\epsilon}}{\partial\bar{z}} = 0 \quad \text{in } \chi_1(\tilde{\Omega}), \quad \text{Im } \Phi_{1,\epsilon}|_{\chi_1(\Gamma_{0,\epsilon})} = 0,$$

$$\Phi_{1,\epsilon} \rightarrow \Phi_1 \quad \text{in } C^1(\overline{\tilde{\Gamma}_{\epsilon}}),$$

where  $\overline{\tilde{\Gamma}_{\epsilon}} \subset \tilde{\Gamma}$  is the arc between points  $x_{\epsilon,-}, x_{\epsilon,+}$  and  $\Gamma_{0,\epsilon} = \partial\tilde{\Omega} \setminus \tilde{\Gamma}_{\epsilon}$ . We define  $\varphi_{2,\epsilon}$  as

$$\frac{\partial\varphi_{2,\epsilon}}{\partial\nu}|_{\Gamma_{0,\epsilon}} = 0, \quad \varphi_{2,\epsilon} = \text{Re } \Phi_{1,\epsilon} \circ \chi_1 \quad \text{on } \partial\tilde{\Omega} \setminus \Gamma_{0,\epsilon}.$$

First we assume that

$$(6.6) \quad \varphi_{2,\epsilon} = \text{Re } \Phi_{1,\epsilon} \circ \chi_1 \quad \text{on } \omega.$$

Passing to the limit in the above equality we obtain (6.5).

Now we concentrate on the proof of (6.6). Let  $\Phi_{1,\epsilon}$  be one of the functions in the sequence  $\{\Phi_{1,\epsilon}\}_{\epsilon \in (0,1)}$ . Consider a sequence of domains  $\tilde{\Omega}_{\epsilon}$  such that  $\tilde{\Omega}_{\epsilon} \subset \tilde{\Omega}$ ,  $\partial\tilde{\Omega}_{\epsilon} \cap \partial\tilde{\Omega} = \Gamma_0$  and  $\text{dist}(\partial\tilde{\Omega}_{\epsilon} \setminus \Gamma_0, \tilde{\Gamma}) \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Then the function  $\varphi_{2,\epsilon}$  is smooth on  $\tilde{\Omega}_{\epsilon}$ . Let us take as  $u_1$  the CGO solution constructed in the previous sections. Thanks to the Carleman estimate (2.25) there exists  $\tau_0 = \tau_0(\epsilon)$  such that

$$(6.7) \quad \|e^{-\tau\varphi_{2,\epsilon}} u_2\|_{L^2(\tilde{\Omega}_{\epsilon})} \leq C_0 |\tau e^{\delta_{\epsilon}|\tau}| \quad \forall |\tau| \geq \tau_0,$$

where  $C_0 = C_0(\epsilon)$  is independent of  $\tau$  and  $\delta_{\epsilon} \rightarrow 0$  as  $\epsilon \rightarrow 0$ . On the other hand  $u_1 = e^{\tau \text{Re } \Phi_{1,\epsilon} \circ \chi_1} ((a_{\tau} e^{c_1 + i\tau \text{Im } \Phi_{1,\epsilon}} + b_{\tau} e^{\mathcal{D}_1 - i\tau \text{Im } \Phi_{1,\epsilon}}) \circ \chi_1 + O(\frac{1}{\tau}))$ . Here we note that  $\mathcal{C}_1, \mathcal{D}_1 \in C^{6+\alpha}(\overline{\tilde{\Omega}_{\epsilon}})$  are defined similarly to (3.2):

$$2\frac{\partial\mathcal{C}_1}{\partial\bar{z}} = -C_1 \quad \text{in } \tilde{\Omega}_{\epsilon}, \quad \text{Im } \mathcal{C}_1|_{\Gamma_0} = 0, \quad 2\frac{\partial\mathcal{D}_1}{\partial z} = -D_1 \quad \text{in } \tilde{\Omega}_{\epsilon}, \quad \text{Im } \mathcal{D}_1|_{\Gamma_0} = 0.$$

Then by (6.4) the following holds true:

$$(6.8) \quad e^{\tau\varphi_{2,\epsilon}} (e^{-\tau\varphi_{2,\epsilon}} u_2) = e^{\tau \text{Re } \Phi_{1,\epsilon} \circ \chi_1} ((a_{\tau} e^{c_1 + i\tau \text{Im } \Phi_{1,\epsilon}} + b_{\tau} e^{\mathcal{D}_1 - i\tau \text{Im } \Phi_{1,\epsilon}}) \circ \chi_1 + O(\frac{1}{\tau})) \quad \forall x \in \omega.$$

This equality implies (6.5) immediately. Indeed, let for some point  $\hat{x}$  from  $\omega$

$$(6.9) \quad \varphi_{2,\epsilon}(\hat{x}) \neq \text{Re } \Phi_{1,\epsilon} \circ \chi_1(\hat{x}).$$

Then there exists a ball  $B(\hat{x}, \delta') \subset \omega$  such that

$$(6.10) \quad |\varphi_{2,\epsilon}(x) - \text{Re } \Phi_{1,\epsilon} \circ \chi_1(x)| > \alpha' > 0 \quad \forall x \in \overline{B(\hat{x}, \delta')}.$$

Let us fix positive  $\varepsilon_1$  such that  $\overline{B(\hat{x}, \delta')} \subset \Omega_{\varepsilon_1}$  and  $2\delta_{\varepsilon_1} < \alpha'$ . Form (6.8) by (6.7) and (6.10) we have

$$C' e^{|\tau|\alpha'} \text{Vol}(B(\hat{x}, \delta'))^{\frac{1}{2}} \leq \|e^{\tau(\text{Re } \Phi_{1,\epsilon} \circ \chi_1 - \varphi_{2,\epsilon})} (((a_{\tau} e^{c_1 + i\tau \text{Im } \Phi_{1,\epsilon}} + b_{\tau} e^{\mathcal{D}_1 - i\tau \text{Im } \Phi_{1,\epsilon}}) \circ \chi_1 + O(\frac{1}{\tau}))\|_{L^2(B(\hat{x}, \delta'))}$$

$$= \|e^{-\tau\varphi_{2,\epsilon}} u_2\|_{L^2(B(\hat{x}, \delta'))} \leq C_0 |\tau| e^{\delta_{\epsilon}|\tau|},$$

where  $\tau > \tau_0$  if  $\varphi_{2,\epsilon}(\hat{x}) < \text{Re } \Phi_{1,\epsilon} \circ \chi_1(\hat{x})$  and  $\tau < -\tau_0$  if  $\varphi_{2,\epsilon}(\hat{x}) > \text{Re } \Phi_{1,\epsilon} \circ \chi_1(\hat{x})$ . The above inequality contradicts (6.9).

Let  $\Xi = \chi_{1,1} + i\chi_{1,2}$ . Using the Cauchy-Riemann equations, we construct the multivalued function  $\psi_2$  such that  $\Phi_2 = \varphi_2 + i\psi_2$  is holomorphic on the Riemann surface associated with  $\tilde{\Omega}$ . Moreover we take the function  $\Phi_1$  which can be holomorphically extended in some domain  $\mathcal{O}$  such that  $\chi_1(\tilde{\Omega}) \subset \mathcal{O}$ . Observe that

$$\Phi_2 = \Phi_1 \circ \Xi \quad \text{on } \omega.$$

Then  $\Xi = \Phi_1^{-1} \circ \Phi_2$  in  $\omega$ .

We claim that the function  $\Xi$  can be extended up to a single valued holomorphic function  $\tilde{\Xi}$  on  $\tilde{\Omega}$  such that  $\tilde{\Xi} : \tilde{\Omega} \rightarrow \chi_1(\tilde{\Omega})$ ,  $\tilde{\Xi}(\tilde{\Omega}) = \chi_1(\tilde{\Omega})$  and  $\partial_z \tilde{\Xi} \neq 0$ . First we show that the function  $\Xi$  can be extended along any curve connecting two points in  $\Omega$ . Our proof is by contradiction. Let  $\gamma$  be such a continuous curve connecting a point  $z_1$  in  $\omega$  and a point  $z_2$  in  $\Omega$  such that the function  $\Xi$  can not be extended along  $\gamma$ . Consider the parametrization of the curve  $\gamma$  such that we are moving from the point  $z_1 = \gamma(0)$  to the point  $z_2 = \gamma(1)$ . Let  $\hat{z} = \gamma(\kappa)$  be the first point on  $\gamma$  around which the holomorphic continuation of the function  $\Xi$  is impossible. Consider the function  $\Phi_1$  such that  $\{z | \partial_z \Phi_1 = 0\} \cap \{z | z = \tilde{\Xi}(\gamma(s)) \quad s \in [0, \kappa]\} = \emptyset$ . Observe that

$$\Phi_2(\gamma(s)) = \Phi_1 \circ \Xi(\gamma(s)) \quad \forall s \in [0, \kappa].$$

Indeed let  $\hat{s} = \sup_{s \in X} s$  where  $X = \{s | \text{there exists } \delta(s) > 0 \text{ such that } \Phi_2(z) = \Phi_1 \circ \Xi(z) \quad \forall z \in B(\gamma(s), \delta)\}$ . Let  $\hat{s} < \kappa$ . Since  $\partial_z \Phi_1(\gamma(\hat{s})) \neq 0$   $\Phi_1^{-1} \circ \Phi_2$  is holomorphic with a domain which contains the ball centered at  $\gamma(\hat{s})$ . Since  $\tilde{\Xi} = \Phi_1^{-1} \circ \Phi_2$  on some open set this equality holds true on this ball which is a contradiction with the definition of  $\hat{s}$ .

Now we consider the situation at the point  $\hat{z}$ . Since we can not extend  $\Xi$  around this point we would have  $\Pi = \{\hat{z} | \Phi_1(\hat{z}) = \Phi_2(\hat{z})\} \subset \chi_1(\partial\tilde{\Omega})$ . Since  $\partial_z \Phi_1(\hat{z}) \neq 0$  we can extend  $\Xi$  on some ball centered at  $\hat{z}$ . (Of course such extension might not be the one we are looking for since  $\tilde{\Xi} : \tilde{\Omega} \rightarrow \chi_1(\tilde{\Omega})$  might not be valid.) Consider a perturbation of the function  $\Phi_1$ :  $\Phi_1 + \epsilon\Psi_1$ , where  $\Psi_1$  is a smooth holomorphic function in  $\mathcal{O}$  such that  $Im\Psi_1|_{\chi_1(\Gamma_0) \cup \Pi} = 0$ . This perturbation generates a perturbation of the function  $\Phi_2$ :  $\Phi_2 + \epsilon\Psi_2$ , where

$$(6.11) \quad \Delta Re \Psi_2 = 0 \quad \text{in } \tilde{\Omega}, \quad \frac{\partial Re \Psi_2}{\partial \nu}|_{\Gamma_0} = 0, \quad Re \Psi_2 = Re \Psi_1 \circ \chi_1 \quad \text{on } \partial\tilde{\Omega} \setminus \Gamma_0.$$

For these new functions we still have

$$\Phi_2 + \epsilon\Psi_2 = (\Phi_1 + \epsilon\Psi_1) \circ \Xi \quad \text{on } \omega.$$

For all sufficiently small  $\epsilon$ , the function  $(\Phi_1 + \epsilon\Psi_1)$  does not have a critical point on  $\tilde{\Omega}$ . Therefore the function  $(\Phi_1 + \epsilon\Psi_1)^{-1} \circ (\Phi_2 + \epsilon\Psi_2)$  can be holomorphically continued along of  $\gamma$  up to the point  $\hat{z}$ . Denote this extension on some ball centered at  $\hat{z}$  as  $\tilde{\Xi}_\epsilon$ . Obviously  $\tilde{\Xi}_\epsilon = \tilde{\Xi}$ . Making a choice of  $\Psi_1$  in such a way that  $Im\Psi_2(\hat{z}) \neq Im\Psi_1(\tilde{\Xi}(\hat{z}))$  we obtain that this equality is impossible.

Let us show that the function  $\tilde{\Xi}$  does not have a critical points in  $\tilde{\Omega}$ . Our proof is again by contradiction. Suppose that  $\hat{z}$  is a critical point of  $\tilde{\Xi}$ . If such critical points exist these points are critical points of the function  $\Phi_2$ . Consider the perturbation of the function  $\Phi_1$ :  $\Phi_1 + \epsilon\Psi_1$ , where  $\Psi_1$  is a smooth holomorphic function in  $\chi_1(\tilde{\Omega})$  such that  $Im\Psi_1|_{\chi_1(\Gamma_0)} = 0$  and such that for the function  $\Phi_2$  given by (6.11) we have  $\partial_z \Psi_2(\hat{z}) \neq 0$ . The mapping  $\tilde{\Xi}$  is

still the same but a position of the critical point for the function  $\Phi_2 + \epsilon\Psi_2$  changes which is a contradiction.

Let us show that  $\tilde{\Xi}$  is the single valued function. Our proof is by contradiction. Let  $\tilde{\Xi}$  be the multivalued function around of some point  $\tilde{z}$ . Then there exists holomorphic  $\Phi_1$  such that  $\partial_z\Phi_1(\tilde{z}) = 0$  and  $\Phi_2 = \Phi_1 \circ \tilde{\Xi}$ . Obviously

$$(6.12) \quad \{z|z = \tilde{\Xi}(\hat{z})\} \subset \{z|\partial_z\Phi_1 = 0\}.$$

Let  $\Psi_1$  is a smooth holomorphic function in  $\mathcal{O}$  such that  $Im\Psi_1|_{\chi_1(\Gamma_0)\cup\Pi} = 0$  and  $\partial\Psi_1 \neq 0$  for all  $z \in \{z|\partial_z\Phi_1 = 0\}$ . Then for the function  $\Phi_1 + \epsilon\Psi_1$  we again should have  $\tilde{\Xi}(\hat{z}) \subset \{z|\partial_z(\Phi_1 + \epsilon\Psi_1) = 0\}$ . This contradicts to (6.12).

If  $\tilde{\Xi}(\tilde{\Omega}) \neq \chi_1(\tilde{\Omega})$  we still have that  $\tilde{\Xi}(\tilde{\Omega}) \subset \chi_1(\tilde{\Omega})$ . On the other hand, on the boundary of  $\chi_1(\tilde{\Omega}) \setminus \tilde{\Xi}(\tilde{\Omega})$  the imaginary part of the function  $\Phi_1$  is zero. This is impossible.

In the domain  $\Omega$  consider the new infinitesimal coordinates for the operator  $P_1$  given by the mapping  $\tilde{\Xi}^{-1} \circ \Xi(x)$ . In these coordinates, the operator  $P_1(x, D)$  has the form

$$(6.13) \quad \tilde{Q}(x, D) = \frac{1}{\tilde{\mu}_1}\Delta + 2\tilde{A}_1\frac{\partial}{\partial z} + 2\tilde{B}_1\frac{\partial}{\partial \bar{z}} + \tilde{q}_1.$$

Since  $\tilde{\Xi}^{-1} \circ \Xi|_{\tilde{\Gamma}} = Id$ , the Cauchy data for the operators  $L_2$  and  $\tilde{Q}$  are exactly the same. The operators  $L_2$  and  $\tilde{Q}$  are particular cases of the operator (1.1). Since  $(\mu_2 - \tilde{\mu}_1)|_{\tilde{\Gamma}} = 0$  the partial Cauchy data  $\mathcal{C}_{\mu_2 I, A_2, B_2, q_2}$  and  $\mathcal{C}_{\tilde{\mu}_1 I, \tilde{A}_1, \tilde{B}_1, \tilde{q}_1}$  are equal. We multiply the operator  $Q$  by the function  $\tilde{\mu}_1/\mu_2$  and denote the resulting operator as  $\hat{Q}(x, D) = \frac{\tilde{\mu}_1}{\mu_2}\tilde{Q}(x, D)$ . Therefore by Corollary 1.1 there exists a function  $\eta$  which satisfies (1.5) such that  $L_2(x, D) = e^{-\eta}\hat{Q}(x, D)e^{\eta}$ . The proof of the theorem is completed.  $\square$

**Proof of Theorem 1.2.** First we observe that in order to prove the statement of this theorem it suffices to prove it in the case when instead of the whole  $\tilde{\Gamma}$  the input and output both are measured on an arbitrary small neighborhood of the point  $\hat{x}$ .

Now one can consider only the case when  $\tilde{\Gamma} \subset \{x_1 = 0\}$  is a small neighborhood of the point 0. We observe that if

$$(6.14) \quad (\sigma_1^{22} - \sigma_2^{22})|_{\tilde{\Gamma}} = \frac{\partial}{\partial x_2}(\sigma_1^{22} - \sigma_2^{22})|_{\tilde{\Gamma}} = 0$$

then repeating the proof of [21] we obtain

$$(6.15) \quad \sigma_1 = \sigma_2 \quad \text{on } \tilde{\Gamma}; \quad \frac{\partial\sigma_1}{\partial x_2} = \frac{\partial\sigma_2}{\partial x_2} \quad \text{on } \tilde{\Gamma}.$$

Let us show that there exists a diffeomorphism

$$(6.16) \quad F : \Omega \rightarrow \Omega, \quad F(x)|_{\tilde{\Gamma}} = x$$

such that for the metric  $\tilde{\sigma}_1 = |det DF^{-1}|F^*\sigma_1$  we have

$$\tilde{\sigma}_1^{22} = \sigma_2^{22} \quad \text{on } \tilde{\Gamma}.$$

First assume that we are already have

$$(6.17) \quad \sigma_1^{22}(0) = \sigma_2^{22}(0) \quad \text{and} \quad \frac{\partial}{\partial x_2}(\sigma_1^{22} - \sigma_2^{22})(0) = 0.$$

Let  $y = y(x)$  be some diffeomorphism of  $\Omega$  into itself. By  $x = x(y)$  we denote the inverse mapping. Then

$$\tilde{\sigma}_1^{22} = \sigma_1^{11} \left( \frac{\partial x_1}{\partial y_2} \right)^2 / \frac{\partial x_2}{\partial y_2} + 2\sigma_1^{12} \frac{\partial x_1}{\partial y_2} + \sigma_1^{22} \frac{\partial x_2}{\partial y_2}.$$

This equality and (6.17) immediately imply that as the perturbation of the identity mapping one can construct the diffeomorphism of  $\Omega$  into itself which satisfy (6.16) such that (6.14) hold true .

Let us construct the diffeomorphism which satisfies (6.14) such that (6.17) holds true. Let  $\rho$  be a smooth function such that  $\rho|_{\partial\Omega} = 0$  and  $\rho$  is strictly positive in  $\Omega$  and  $\frac{\partial \rho}{\partial \nu}|_{\partial\Omega} < 0$ . Consider the system of ODE

$$\frac{dy_1}{dt} = \rho(y)f_1(y), \quad \frac{dy_2}{dt} = \rho(y)f_2(y).$$

The corresponding phase flow  $g^s$  is a diffeomorphism of  $\Omega$  into itself such that  $g^s(x) = x$  on  $\partial\Omega$ . Let  $f_1(0) \neq 0$  and  $f_2(0) \neq 0$ . With appropriate choice of  $f_1, f_2$  one can arrange that  $\frac{\partial x_1(0)}{\partial y_2} = 0$  and  $\frac{\partial x_2(0)}{\partial y_2} = \sigma_2^{22}(0)/\sigma_1^{22}(0)$ . Then the first equality in (6.17) holds true. Adjusting the second derivatives of  $f_1, f_2$  we can arrange that  $\frac{\partial^2 x_1(0)}{\partial y_2^2} = 0$  and  $\frac{\partial^2 x_2(0)}{\partial y_2^2} = \frac{1}{\sigma_1^{22}(0)} \left( \frac{\partial \sigma_2^{22}(0)}{\partial y_2} - \frac{\partial \sigma_1^{22}(0)}{\partial y_2} \frac{\partial x_2(0)}{\partial y_2} \right)$ . Then the second equality in (6.17) holds true.

Now (6.15) is established. The rest of the proof of Theorem 1.2 as in the proof of Theorem 1.1.  $\square$

## 7. APPENDIX I

Consider the following problem for the Cauchy-Riemann equations

$$(7.1) \quad L(\phi, \psi) = \left( \frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_2}, \frac{\partial \phi}{\partial x_2} + \frac{\partial \psi}{\partial x_1} \right) = 0 \quad \text{in } \Omega, \quad (\phi, \psi)|_{\Gamma_0} = (b_1(x), b_2(x)),$$

$$\frac{\partial^l}{\partial z^l} (\phi + i\psi)(\hat{x}_j) = c_{0,j}, \quad \forall j \in \{1, \dots, N\} \quad \text{and } \forall l \in \{0, \dots, 5\}.$$

Here  $\hat{x}_1, \dots, \hat{x}_N$  be an arbitrary fixed points in  $\Omega$ . We consider the pair  $b_1, b_2$  and complex numbers  $\vec{C} = (c_{0,1}, c_{1,1}, c_{2,1}, c_{3,1}, c_{4,1}, c_{5,1}, \dots, c_{0,N}, c_{1,N}, c_{2,N}, c_{3,N}, c_{4,N}, c_{5,N})$  as initial data for (7.1). The following proposition establishes the solvability of (7.1) for a dense set of Cauchy data.

**Proposition 7.1.** *There exists a set  $\mathcal{O} \subset C^6(\overline{\Gamma_0}) \times C^6(\overline{\Gamma_0}) \times \mathbb{C}^{6N}$  such that for each  $(b_1, b_2, \vec{C}) \in \mathcal{O}$ , (7.1) has at least one solution  $(\phi, \psi) \in C^6(\overline{\Omega}) \times C^6(\overline{\Omega})$  and  $\overline{\mathcal{O}} = C^6(\overline{\Gamma_0}) \times C^6(\overline{\Gamma_0}) \times \mathbb{C}^{6N}$ .*

*Proof.* Denote  $B = (b_1, b_2)$  an arbitrary element of the space  $C^7(\overline{\Gamma_0}) \times C^7(\overline{\Gamma_0})$ . Consider the following extremal problem

$$(7.2) \quad J_\epsilon(\phi, \psi) = \|(\phi, \psi) - B\|_{B_4^{\frac{27}{4}}(\Gamma_0)}^4 + \epsilon \sum_{k=0}^3 \left\| \frac{\partial^k(\phi, \psi)}{\partial \nu^k} \right\|_{B_4^{\frac{27}{4}-k}(\partial\Omega)}^4$$

$$+ \frac{1}{\epsilon} \left\| \Delta^3 L(\phi, \psi) \right\|_{L^4(\Omega)}^4 + \sum_{j=1}^N \sum_{k=0}^5 \left| \frac{\partial^k}{\partial z^k} (\phi + i\psi)(\hat{x}_j) - c_{k,j} \right|^2 \rightarrow \inf,$$

$$(7.3) \quad (\phi, \psi) \in W_4^7(\Omega) \times W_4^7(\Omega).$$

Here  $B_k^l$  denotes the Besov space of the corresponding orders.

For each  $\epsilon > 0$  there exists a unique solution to (7.2), (7.3) which we denote as  $(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon)$ . This fact can be proved by standard arguments. We fix  $\epsilon > 0$ . Denote by  $\mathcal{U}_{ad}$  the set of admissible elements of the problem (7.2), (7.3), namely

$$\mathcal{U}_{ad} = \{(\phi, \psi) \in W_4^7(\Omega) \times W_4^7(\Omega) | J_\epsilon(\phi, \psi) < \infty\}.$$

Denote  $\widehat{J}_\epsilon = \inf_{(\phi, \psi) \in W_4^7(\Omega) \times W_4^7(\Omega)} J_\epsilon(\phi, \psi)$ . Clearly the pair  $(0, 0) \in \mathcal{U}_{ad}$ . Therefore there exists a minimizing sequence  $\{(\phi_k, \psi_k)\}_{k=1}^\infty \subset W_4^7(\Omega) \times W_4^7(\Omega)$  such that

$$\widehat{J}_\epsilon = \lim_{k \rightarrow +\infty} J_\epsilon(\phi_k, \psi_k).$$

Observe that the minimizing sequence is bounded in  $W_4^7(\Omega) \times W_4^7(\Omega)$ . Indeed, since the sequence  $\{\Delta^3 L(\phi_k, \psi_k), L(\phi_k, \psi_k)|_{\partial\Omega}, \dots, \frac{\partial^3}{\partial\nu^3} L(\phi_k, \psi_k)|_{\partial\Omega}\}$  is bounded in  $L^4(\Omega) \times L^4(\Omega) \times \prod_{k=0}^3 B_4^{\frac{27}{4}-k}(\partial\Omega) \times B_4^{\frac{27}{4}-k}(\partial\Omega)$  the standard elliptic  $L^p$ -estimate implies that the sequence  $\{L(\phi_k, \psi_k)\}$  is bounded in the space  $W_4^6(\Omega) \times W_4^6(\Omega)$ . Taking into account that the sequence traces of the functions  $(\phi_k, \psi_k)$  is bounded in the Besov space  $B_4^{\frac{27}{4}}(\partial\Omega) \times B_4^{\frac{27}{4}}(\partial\Omega)$  and applying the estimates for elliptic operators one more time we obtain that  $\{(\phi_k, \psi_k)\}$  bounded in  $W_4^7(\Omega) \times W_4^7(\Omega)$ . By the Sobolev imbedding theorem the sequence  $\{(\phi_k, \psi_k)\}$  is bounded in  $C^6(\overline{\Omega}) \times C^6(\overline{\Omega})$ . Then taking if necessary a subsequence, (which we denote again as  $\{(\phi_k, \psi_k)\}$ ) we obtain

$$(\phi_k, \psi_k) \rightarrow (\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon) \quad \text{weakly in } W_4^7(\Omega) \times W_4^7(\Omega),$$

$$\left(\frac{\partial^j \phi_k}{\partial\nu^j}, \frac{\partial^j \psi_k}{\partial\nu^j}\right) \rightarrow \left(\frac{\partial^j \widehat{\phi}_\epsilon}{\partial\nu^j}, \frac{\partial^j \widehat{\psi}_\epsilon}{\partial\nu^j}\right) \quad \text{weakly in } B_4^{\frac{27}{4}-j}(\partial\Omega) \times B_4^{\frac{27}{4}-j}(\partial\Omega) \quad \forall j \in \{0, 1, 2, 3\},$$

$$\frac{\partial^k}{\partial z^k}(\phi + i\psi)(\widehat{x}_j) - c_{k,j} \rightarrow C_{k,j,\epsilon}, \quad k \in \{0, \dots, 5\},$$

$$\Delta^3 L(\phi_k, \psi_k) \rightarrow r_\epsilon \quad \text{weakly in } L^4(\Omega) \times L^4(\Omega), \quad L(\phi_k, \psi_k) \rightarrow \widetilde{r}_\epsilon \quad \text{weakly in } W_4^6(\Omega) \times W_4^6(\Omega).$$

Obviously,  $r_\epsilon = \Delta^3 L(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon)$ ,  $\widetilde{r}_\epsilon = L(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon)$ . Then, since the norms in the spaces  $L^4(\Omega)$  and  $B_4^{\frac{27}{4}-k}(\partial\Omega)$  are lower semicontinuous with respect to weak convergence we obtain that

$$J_\epsilon(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon) \leq \lim_{k \rightarrow +\infty} J_\epsilon(\phi_k, \psi_k) = \widehat{J}_\epsilon.$$

Thus the pair  $(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon)$  is a solution to the extremal problem (7.2), (7.3). Since the set of an admissible elements is convex and the functional  $J_\epsilon$  is strictly convex, this solution is unique.

By Fermat's theorem we have

$$J'_\epsilon(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon)[\widetilde{\delta}] = 0, \quad \forall \widetilde{\delta} \in W_4^7(\Omega) \times W_4^7(\Omega).$$

This equality can be written in the form

$$\begin{aligned}
(7.4) \quad I'_{\Gamma_0, \frac{27}{4}}((\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon) - B)[\widetilde{\delta}] + \epsilon \sum_{k=0}^3 I'_{\partial\Omega, \frac{27}{4}-k} \left( \frac{\partial^k}{\partial \nu^k}(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon) \right) \left[ \frac{\partial^k}{\partial \nu^k} \widetilde{\delta} \right] + (p_\epsilon, \Delta^3 L \widetilde{\delta})_{L^2(\Omega)} \\
+ \sum_{j=1}^N \sum_{k=0}^5 \overline{\left( \frac{\partial^k}{\partial z^k}(\widehat{\phi}_\epsilon + i\widehat{\psi}_\epsilon)(\widehat{x}_j) - c_{k,j} \right)} \frac{\partial^k}{\partial z^k}(\widetilde{\delta}_1 + i\widetilde{\delta}_2)(\widehat{x}_j) \\
+ \left( \frac{\partial^k}{\partial z^k}(\widehat{\phi}_\epsilon + i\widehat{\psi}_\epsilon)(\widehat{x}_j) - c_{k,j} \right) \frac{\partial^k}{\partial z^k}(\widetilde{\delta}_1 + i\widetilde{\delta}_2)(\widehat{x}_j) = 0,
\end{aligned}$$

where  $p_\epsilon = \frac{4}{\epsilon} \left( (\Delta^3 \frac{\partial \widehat{\phi}_\epsilon}{\partial x_1} - \frac{\partial \widehat{\psi}_\epsilon}{\partial x_2})^3, (\Delta^3 (\frac{\partial \widehat{\phi}_\epsilon}{\partial x_2} + \frac{\partial \widehat{\psi}_\epsilon}{\partial x_1}))^3 \right)$  and  $I'_{\Gamma^*, \kappa}(\widehat{w})$  denotes the derivative of the functional  $w \rightarrow \|w\|_{B_4^{\kappa}(\Gamma^*)}^4$  at  $\widehat{w}$ .

Observe that the pair  $J_\epsilon(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon) \leq J_\epsilon(0, 0) = \|B\|_{B_4^{\frac{27}{4}}(\Gamma_0)}^4 + \sum_{j=1}^N \sum_{k=0}^5 |c_{k,j}|^2$ . This implies that the sequence  $\{(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon)\}$  is bounded in  $B_4^{\frac{27}{4}}(\Gamma_0) \times B_4^{\frac{27}{4}}(\Gamma_0)$ , the sequences  $\{\frac{\partial^k}{\partial z^k}(\widehat{\phi}_\epsilon + i\widehat{\psi}_\epsilon)(\widehat{x}_j) - c_{k,j}\}$  are bounded in  $\mathbb{C}$ , the sequence  $\epsilon \sum_{k=0}^3 I'_{\partial\Omega, \frac{27}{4}-k} \left( \frac{\partial^k}{\partial \nu^k}(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon) \right) \left[ \frac{\partial^k}{\partial \nu^k} \widetilde{\delta} \right]$  converges to zero for any  $\widetilde{\delta}$  from  $B_4^{\frac{27}{4}}(\partial\Omega) \times B_4^{\frac{27}{4}}(\partial\Omega)$ . Then (7.4) implies that the sequence  $\{p_\epsilon\}_{\epsilon \in (0,1)}$  is bounded in  $L^{\frac{4}{3}}(\Omega) \times L^{\frac{4}{3}}(\Omega)$ .

Therefore there exist  $\mathcal{B} \in B_4^{\frac{27}{4}}(\Gamma_0) \times B_4^{\frac{27}{4}}(\Gamma_0)$ ,  $C_{0,j}, C_{1,j}, \dots, C_{5,j} \in \mathbb{C}$  and  $p = (p_1, p_2) \in L^{\frac{4}{3}}(\Omega) \times L^{\frac{4}{3}}(\Omega)$  such that

$$(7.5) \quad (\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon) - B \rightharpoonup \mathcal{B} \quad \text{weakly in } B_4^{\frac{27}{4}}(\Gamma_0) \times B_4^{\frac{27}{4}}(\Gamma_0), \quad p_\epsilon \rightharpoonup p \quad \text{weakly in } L^{\frac{4}{3}}(\Omega) \times L^{\frac{4}{3}}(\Omega),$$

$$(7.6) \quad \frac{\partial^k}{\partial z^k}(\widehat{\phi}_\epsilon + i\widehat{\psi}_\epsilon)(\widehat{x}_j) - c_{k,j} \rightharpoonup C_{k,j} \quad k \in \{0, 1, \dots, 5\}, j \in \{1, \dots, N\}.$$

Passing to the limit in (7.4) we obtain

$$(7.7) \quad I'_{\Gamma_0, \frac{27}{4}}(\mathcal{B})[\widetilde{\delta}] + (p, \Delta^3 L \widetilde{\delta})_{L^2(\Omega)} + 2\text{Re} \sum_{j=1}^N \sum_{k=0}^5 C_{k,j} \overline{\frac{\partial^k}{\partial z^k}(\widetilde{\delta}_1 + i\widetilde{\delta}_2)(\widehat{x}_j)} = 0 \quad \forall \widetilde{\delta} \in W_4^7(\Omega) \times W_4^7(\Omega).$$

Next we claim that

$$(7.8) \quad \Delta^3 p = 0 \quad \text{in } \Omega \setminus \cup_{j=1}^N \{\widehat{x}_j\}$$

in the sense of distributions. Suppose that (7.8) is already proved. This implies

$$(p, \Delta^3 L \widetilde{\delta})_{L^2(\Omega)} + 2\text{Re} \sum_{j=1}^N \sum_{k=0}^5 C_{k,j} \overline{\frac{\partial^k}{\partial z^k}(\widetilde{\delta}_1 + i\widetilde{\delta}_2)(\widehat{x}_j)} = 0 \quad \forall \widetilde{\delta}_1, \widetilde{\delta}_2 \in C_0^\infty(\Omega).$$

If  $p = (p_1, p_2)$ , denoting  $P = p_1 - ip_2$ , we have

$$\text{Re}(\Delta^3 P, \partial_{\bar{z}}(\widetilde{\delta}_1 + i\widetilde{\delta}_2))_{L^2(\Omega)} + \text{Re} \sum_{j=1}^N \sum_{k=0}^5 \overline{C_{k,j}} \frac{\partial^k}{\partial z^k}(\widetilde{\delta}_1 + i\widetilde{\delta}_2)(\widehat{x}_j) = 0 \quad \forall \widetilde{\delta}_1, \widetilde{\delta}_2 \in C_0^\infty(\Omega).$$



Since by (7.8)  $\text{supp } \Delta^3 P \subset \cup_{j=1}^N \{\widehat{x}_j\}$  there exist some constants  $m_{\beta,j}$  and  $\widehat{\ell}_j$  such that  $\Delta^3 P = \sum_{j=1}^N \sum_{|\beta|=1}^{\widehat{\ell}_j} m_{\beta,j} D^\beta \delta(x - \widehat{x}_j)$ . The above equality can be written in the form

$$-\sum_{|\beta|=1}^{\widehat{\ell}_j} m_{\beta,j} \frac{\partial}{\partial \bar{z}} D^\beta \delta(x - \widehat{x}_j) = \sum_{k=0}^5 (-1)^k \overline{C_{k,j}} \frac{\partial^k}{\partial z^k} \delta(x - \widehat{x}_j).$$

From this we obtain

$$(7.9) \quad C_{0,j} = C_{1,j} = \dots = C_{5,j} = 0 \quad j \in \{1, \dots, N\}.$$

Therefore

$$(7.10) \quad \Delta^3 p = 0 \quad \text{in } \Omega.$$

This implies

$$(p, \Delta^3 L\widetilde{\delta})_{L^2(\Omega)} = 0 \quad \forall \widetilde{\delta} \in W_4^7(\Omega) \times W_4^7(\Omega), \quad L\widetilde{\delta}|_{\partial\Omega} = \frac{\partial L\widetilde{\delta}}{\partial \nu}|_{\partial\Omega} = \dots = \frac{\partial^5 L\widetilde{\delta}}{\partial \nu^5}|_{\partial\Omega} = 0.$$

This equality and (7.7) yield

$$(7.11) \quad I'_{\Gamma_0, \frac{27}{4}}(\mathcal{B})[\widetilde{\delta}] = 0 \quad \forall \widetilde{\delta} \in W_4^7(\Omega) \times W_4^7(\Omega), \quad L\widetilde{\delta}|_{\partial\Omega} = \frac{\partial L\widetilde{\delta}}{\partial \nu}|_{\partial\Omega} = \dots = \frac{\partial^5 L\widetilde{\delta}}{\partial \nu^5}|_{\partial\Omega} = 0.$$

Then using the trace theorem we conclude that  $\mathcal{B} = 0$ . Using this and (7.5) we obtain

$$(7.12) \quad (\widehat{\phi}_{\epsilon_k}, \widehat{\psi}_{\epsilon_k}) - B \rightarrow 0 \quad \text{weakly in } B_4^{\frac{27}{4}}(\Gamma_0) \times B_4^{\frac{27}{4}}(\Gamma_0).$$

From (7.6) and (7.9) we obtain

$$\frac{\partial^k}{\partial z^k} (\widehat{\phi}_\epsilon + i\widehat{\psi}_\epsilon)(\widehat{x}) \rightarrow c_{k,j} \quad k \in \{0, 1, \dots, 5\}, \quad j \in \{1, \dots, N\}.$$

By the Sobolev embedding theorem  $B_4^{\frac{27}{4}}(\Gamma_0) \subset\subset C^5(\overline{\Gamma_0})$ . Therefore (7.12) implies

$$(7.13) \quad (\widehat{\phi}_{\epsilon_k}, \widehat{\psi}_{\epsilon_k}) - B \rightarrow 0 \quad \text{in } C^5(\overline{\Gamma_0}) \times C^5(\overline{\Gamma_0}).$$

Let the pair  $(\widetilde{\phi}_{\epsilon_k}, \widetilde{\psi}_{\epsilon_k})$  be a solution to the boundary value problem

$$(7.14) \quad L(\widetilde{\phi}_{\epsilon_k}, \widetilde{\psi}_{\epsilon_k}) = L(\widehat{\phi}_{\epsilon_k}, \widehat{\psi}_{\epsilon_k}) \quad \text{in } \Omega, \quad \widetilde{\psi}_{\epsilon_k}|_{\partial\Omega} = \psi_{\epsilon_k}^*.$$

Here  $\psi_{\epsilon_k}^*$  is a smooth function such that  $\psi_{\epsilon_k}^*|_{\Gamma_0} = 0$  and the pair  $(L(\widehat{\phi}_{\epsilon_k}, \widehat{\psi}_{\epsilon_k}), \psi_{\epsilon_k}^*)$  is orthogonal to all solutions of the adjoint problem (see [32]). Moreover since  $L(\widetilde{\phi}_{\epsilon_k}, \widetilde{\psi}_{\epsilon_k}) \rightarrow 0$  in  $W_4^6(\Omega) \times W_4^6(\Omega)$  we may assume  $\psi_{\epsilon_k}^* \rightarrow 0$  in  $C^6(\partial\Omega) \times C^6(\partial\Omega)$ . Among all possible solutions to problem (7.14) (clearly there is no unique solution to this problem) we choose one such that  $\int_{\Omega} \widetilde{\phi}_{\epsilon_k} dx = 0$ . Thus we obtain

$$(7.15) \quad (\widetilde{\phi}_{\epsilon_k}, \widetilde{\psi}_{\epsilon_k}) \rightarrow 0 \quad \text{in } W_4^7(\Omega) \times W_4^7(\Omega).$$

Therefore the sequence  $\{(\widehat{\phi}_{\epsilon_k} - \widetilde{\phi}_{\epsilon_k}, \widehat{\psi}_{\epsilon_k} - \widetilde{\psi}_{\epsilon_k})\}$  represents the desired approximation to the solution of the Cauchy problem (7.1).

Now we prove (7.8). Let  $\widetilde{x}$  be an arbitrary point in  $\Omega \setminus \cup_{j=1}^N \{\widehat{x}_j\}$  and let  $\widetilde{\chi}$  be a smooth function such that it is zero in some neighborhood of  $\Gamma_0 \cup \cup_{j=1}^N \{\widehat{x}_j\}$  and the set  $\mathcal{D} = \{x \in \Omega | \widetilde{\chi}(x) = 1\}$  contains an open connected subset  $\mathcal{F}$  such that  $\widetilde{x} \in \mathcal{F}$  and  $\widetilde{\Gamma} \cap \overline{\mathcal{F}}$  is an open

set in  $\partial\Omega$ . In addition we assume that  $\text{Int}(\text{supp } \chi)$  is a simply connected domain. By (7.7) we have

$$(7.16) \quad 0 = (p, \Delta^3 L(\tilde{\chi}\tilde{\delta}))_{L^2(\Omega)} = (\tilde{\chi}p, \Delta^3 L\tilde{\delta})_{L^2(\Omega)} + (p, [\Delta^3 L, \tilde{\chi}]\tilde{\delta})_{L^2(\Omega)} \quad \forall \tilde{\delta} \in W_4^7(\Omega) \times W_4^7(\Omega).$$

Denote  $L\tilde{\delta} = \hat{\delta}$ . Consider the functional mapping  $\hat{\delta} \in W_4^2(\text{supp } \tilde{\chi})$  to  $(p, [\Delta^3 L, \tilde{\chi}]\hat{\delta})_{L^2(\Omega)}$ , where

$$L\tilde{\delta} = \hat{\delta} \quad \text{in } \Omega, \quad \text{Im } \tilde{\delta}|_{\mathcal{S}} = 0, \quad \int_{\text{supp } \tilde{\chi}} \text{Re } \tilde{\delta} dx = 0,$$

where  $\mathcal{S}$  denotes the boundary of  $\text{supp } \tilde{\chi}$ . For each  $\hat{\delta} \in W_4^2(\text{supp } \tilde{\chi}) \times W_4^2(\text{supp } \tilde{\chi})$ , there exists a unique solution  $\tilde{\delta} \in W_4^3(\text{supp } \tilde{\chi}) \times W_4^3(\text{supp } \tilde{\chi})$ . Hence the functional is well-defined and continuous on  $W_4^2(\text{supp } \tilde{\chi})$ . Therefore there exist  $\mathbf{q}, \mathbf{r}, q_0 \in L^{\frac{4}{3}}(\text{supp } \tilde{\chi})$  such that  $\int_{\text{supp } \tilde{\chi}} (\sum_{j,k=1}^2 r_{jk} \frac{\partial^2}{\partial x_j \partial x_k} \hat{\delta} + (\mathbf{q}, \hat{\delta}) + q_0 \hat{\delta}) dx = (p, [\Delta^3 L, \tilde{\chi}]\hat{\delta})_{L^2(\text{supp } \tilde{\chi})}$ .

Consider the boundary value problem

$$\Delta^3 \tilde{P} = \tilde{f} \quad \text{in } \text{supp } \chi, \quad \tilde{P}|_{\mathcal{S}} = \frac{\partial \tilde{P}}{\partial \nu}|_{\mathcal{S}} = \frac{\partial^2 \tilde{P}}{\partial \nu^2}|_{\mathcal{S}} = 0.$$

Here  $\tilde{f} = 2\text{div}(\nabla \tilde{\mathbf{q}}) - q_0 - \sum_{j,k=1}^2 \frac{\partial^2}{\partial x_j \partial x_k} r_{jk}$ . A solution to this problem exists and is unique, since  $\tilde{f} \in (\dot{W}_4^2(\text{supp } \tilde{\chi}))'$ . Then  $P \in \dot{W}_4^1(\text{supp } \tilde{\chi}) \times \dot{W}_4^1(\text{supp } \tilde{\chi})$ . On the other hand, thanks to (7.16),  $P = \tilde{\chi}p \in \dot{W}_4^1(\text{supp } \tilde{\chi}) \times \dot{W}_4^1(\text{supp } \tilde{\chi})$ .

Next we take another smooth cut off function  $\tilde{\chi}_1$  such that  $\text{supp } \tilde{\chi}_1 \subset \mathcal{D}$  and  $\text{Int}(\text{supp } \tilde{\chi}_1)$  is a simply connected domain. A neighborhood of  $\tilde{x}$  belongs to  $\mathcal{D}_1 = \{x | \tilde{\chi}_1 = 1\}$ , the interior of  $\mathcal{D}_1$  is connected, and  $\overline{\text{Int } \mathcal{D}_1} \cap \tilde{\Gamma}$  contains an open subset  $\mathcal{O}$  in  $\partial\Omega$ . Similarly to (7.16) we have

$$(\tilde{\chi}_1 p, \Delta^3 L\tilde{\delta})_{L^2(\Omega)} - (p, [\Delta^3 L, \tilde{\chi}_1]\tilde{\delta})_{L^2(\Omega)} = 0 \quad \forall \tilde{\delta} \in W_4^7(\Omega) \times W_4^7(\Omega).$$

This equality implies that  $\tilde{\chi}_1 p \in W_{\frac{4}{3}}^2(\Omega) \times W_{\frac{4}{3}}^2(\Omega)$ , using a similar argument to the one above.

Next we take another smooth cut off function  $\tilde{\chi}_2$  such that  $\text{supp } \tilde{\chi}_2 \subset \mathcal{D}_2$  and  $\text{Int}(\text{supp } \tilde{\chi}_2)$  is a simply connected domain. A neighborhood of  $\tilde{x}$  belongs to  $\mathcal{D}_3 = \{x | \tilde{\chi}_2 = 1\}$ , the interior of  $\mathcal{D}_1$  is connected, and  $\overline{\text{Int } \mathcal{D}_3} \cap \tilde{\Gamma}$  contains an open subset  $\mathcal{O}$  in  $\partial\Omega$ . Similarly to (7.16) we have

$$(\tilde{\chi}_2 p, \Delta^3 L\tilde{\delta})_{L^2(\Omega)} - (p, [\Delta^3 L, \tilde{\chi}_2]\tilde{\delta})_{L^2(\Omega)} = 0 \quad \forall \tilde{\delta} \in W_4^7(\Omega) \times W_4^7(\Omega).$$

This equality implies that  $\tilde{\chi}_2 p \in W_{\frac{4}{3}}^3(\Omega) \times W_{\frac{4}{3}}^3(\Omega)$ , using a similar argument to the one above. Let  $\omega$  be a domain such that  $\omega \cap \Omega = \emptyset$ ,  $\partial\omega \cap \partial\Omega \subset \mathcal{O}$  contains an open set in  $\partial\Omega$ .

We extend  $p$  on  $\omega$  by zero. Then

$$(\Delta^3(\tilde{\chi}_2 p), L\tilde{\delta})_{L^2(\Omega \cup \omega)} + (p, [\Delta^3 L, \tilde{\chi}_2]\tilde{\delta})_{L^2(\Omega \cup \omega)} = 0.$$

Hence, since  $[\Delta^3 L, \tilde{\chi}_2]|_{\mathcal{D}_1} = 0$  we have

$$L^* \Delta^3(\tilde{\chi}_2 p) = 0 \quad \text{in } \text{Int } \mathcal{D}_2 \cup \omega, \quad p|_{\omega} = 0.$$

By Holmgren's theorem  $\Delta^3(\tilde{\chi}_2 p)|_{\text{Int } \mathcal{D}_1} = 0$ , that is,  $(\Delta^3 p)(\tilde{x}) = 0$ . Thus (7.8) is proved.  $\square$

Consider the Cauchy problem for the Cauchy-Riemann equations

$$(7.17) \quad L(\phi, \psi) = \left( \frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_2}, \frac{\partial \phi}{\partial x_2} + \frac{\partial \psi}{\partial x_1} \right) = 0 \quad \text{in } \Omega, \quad (\phi, \psi)|_{\Gamma_0} = (b(x), 0),$$

$$\frac{\partial^l}{\partial z^l}(\phi + i\psi)(\hat{x}_j) = c_{0,j}, \quad \forall j \in \{1, \dots, N\} \quad \text{and } \forall l \in \{0, \dots, 5\}.$$

Here  $\hat{x}_1, \dots, \hat{x}_N$  be an arbitrary fixed points in  $\Omega$ . We consider the function  $b$  and complex numbers  $\vec{C} = (c_{0,1}, c_{1,1}, c_{2,1}, c_{3,1}, c_{4,1}, c_{5,1} \dots c_{0,N}, c_{1,N}, c_{2,N}, c_{3,N}, c_{4,N}, c_{5,N})$  as initial data for (7.1). The following proposition establishes the solvability of (7.1) for a dense set of Cauchy data.

**Corollary 7.1.** *There exists a set  $\mathcal{O} \subset C^6(\overline{\Gamma_0}) \times \mathbb{C}^{6N}$  such that for each  $(b, \vec{C}) \in \mathcal{O}$ , problem (7.17) has at least one solution  $(\phi, \psi) \in C^6(\overline{\Omega}) \times C^6(\overline{\Omega})$  and  $\overline{\mathcal{O}} = C^6(\overline{\Gamma}) \times \mathbb{C}^{6N}$ .*

The proof of Corollary 7.1 is similar to the proof of Proposition 7.1. The only difference is that instead of extremal problem considered there we have to use the following extremal problem

$$J_\epsilon(\phi, \psi) = \|\phi - b\|_{B_{\frac{27}{4}}(\Gamma_0)}^4 + \epsilon \sum_{k=0}^3 \left\| \frac{\partial^k(\phi, \psi)}{\partial \nu^k} \right\|_{B_{\frac{27}{4}}^{-k}(\partial\Omega)}^4$$

$$+ \frac{1}{\epsilon} \|\Delta^3 L(\phi, \psi)\|_{L^4(\Omega)}^4 + \sum_{j=1}^N \sum_{k=0}^5 \left| \frac{\partial^k}{\partial z^k}(\phi + i\psi)(\hat{x}_j) - c_{k,j} \right|^2 \rightarrow \inf,$$

$$(\phi, \psi) \in W_4^7(\Omega) \times W_4^7(\Omega), \quad \psi|_{\Gamma_0} = 0.$$

We have

**Proposition 7.2.** *Let  $\alpha \in (0, 1)$ ,  $\mathcal{A}, \mathcal{B} \in C^{6+\alpha}(\overline{\Omega})$ ,  $y_1, \dots, y_{\tilde{k}}$  be some arbitrary points from  $\Gamma_0$ ,  $y_{\tilde{k}+1}, \dots, y_{\tilde{k}}$  be some arbitrary points from  $\Omega$  and  $\tilde{x}$  be an arbitrary point from  $\Omega \setminus \{y_1, \dots, y_{\tilde{k}}\}$ . Then there exist a holomorphic function  $a \in C^{5+\alpha}(\overline{\Omega})$  and an antiholomorphic function  $d \in C^{5+\alpha}(\overline{\Omega})$  such that  $(ae^{\mathcal{A}} + de^{\mathcal{B}})|_{\Gamma_0} = 0$ ,*

$$\frac{\partial^{k+j} a}{\partial x_1^k \partial x_2^j}(y_\ell) = 0 \quad k + j \leq 5, \quad \forall \ell \in \{1, \dots, \tilde{k}\},$$

and

$$a(\tilde{x}) \neq 0 \quad \text{and} \quad d(\tilde{x}) \neq 0.$$

*Proof.* Consider the operator

$$R(\gamma) = (a(y_1), \dots, \frac{\partial^5 a}{\partial z^5}(y_1), \dots, a(y_{\tilde{k}}), \dots, \frac{\partial^5 a}{\partial z^5}(y_{\tilde{k}}), d(y_1), \dots, \frac{\partial^5 d}{\partial \bar{z}^5}(y_1), \dots, d(y_{\tilde{k}}), \dots, \frac{\partial^5 d}{\partial \bar{z}^5}(y_{\tilde{k}}), a(\tilde{x}), d(\tilde{x})).$$

Here  $\gamma \in C_0^\infty(\tilde{\Gamma})$  and the functions  $a$  and  $d$  are solutions to the following problem

$$\frac{\partial a}{\partial \bar{z}} = 0, \quad \text{in } \Omega, \quad \frac{\partial d}{\partial z} = 0 \quad \text{in } \Omega, \quad (ae^{\mathcal{A}} + de^{\mathcal{B}})|_{\partial\Omega} = \gamma.$$

Consider the image of the operator  $R$ . Clearly it is closed. Let us show that the point  $(0, \dots, 0, 1, 1)$  belongs to the image of the operator  $R$ . Let a holomorphic function  $a$  satisfy

$$\frac{\partial^\beta a}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}}(y_j) = 0 \quad \forall |\beta| \in \{0, \dots, 5\}, \quad j \in \{1, \dots, \tilde{k}\}, \quad |a(\tilde{x})| > 2.$$

Consider the function  $-e^{A-B}a(z)$  and the pair  $(b_1, b_2) = (\operatorname{Re}\{-e^{A-B}a\}, \operatorname{Im}\{e^{A-B}a\})$ . Using Proposition 7.1 we solve problem (7.1) with  $l = 0$  approximately. Let  $(\phi_\epsilon, \psi_\epsilon)$  be a sequence of functions such that

$$\frac{\partial}{\partial z}(\phi_\epsilon + i\psi_\epsilon) = 0 \quad \text{in } \Omega, \quad (\phi_\epsilon, \psi_\epsilon)|_{\Gamma_0} \rightarrow (b_1, b_2) \quad \text{in } C^{5+\alpha}(\bar{\Gamma}_0), \quad (\phi_\epsilon + i\psi_\epsilon)(\tilde{x}) \rightarrow 1.$$

Denote  $d_\epsilon = \phi_\epsilon - i\psi_\epsilon, \beta_\epsilon = ae^A + d_\epsilon e^B$ . Then the sequence  $\{\beta_\epsilon\}$  converges to zero in the space  $C^{5+\alpha}(\Gamma_0)$ .

By Proposition 2.3 there exists a solution to problem (2.13) with the initial data  $\beta_\epsilon$ , which we denote as  $\{\tilde{a}_\epsilon, \tilde{d}_\epsilon\}$  such that the sequence  $\{\tilde{a}_\epsilon, \tilde{d}_\epsilon\}$  converges to zero in  $(C^5(\bar{\Omega}))^2$ . Denote by  $\gamma_\epsilon = (a + \tilde{a}_\epsilon, d_\epsilon + \tilde{d}_\epsilon)|_{\Gamma_0}$ . Clearly  $R(\gamma_\epsilon)$  converges to  $(0, \dots, 0, 1, 1)$ . The proof of the proposition is completed.  $\square$

## 8. APPENDIX II. ASYMPTOTIC FORMULAS

Here we recall that we identify  $x = (x_1, x_2) \in \mathbf{R}^2$  with  $z = x_1 + ix_2 \in \mathbf{C}$ .

**Proposition 8.1.** *Under the conditions of Theorem 1.3 for any point  $x$  on the boundary of  $\Omega$  we have*

$$(8.1) \quad -\frac{1}{\pi} \int_{\Omega} \frac{e_1 \tilde{g}_1 e^{-\tau(\Phi(\zeta) - \overline{\Phi(\zeta)})}}{\zeta - z} d\xi_1 d\xi_2 = \frac{1}{\tau^2} \frac{e^{-2i\tau\psi(\tilde{x})}}{|\det \psi''(\tilde{x})|^{\frac{1}{2}}} \left( \frac{\frac{\partial_z \tilde{g}_1(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})}}{(\tilde{z} - z)^2} + \frac{\frac{\partial_z \tilde{g}_1(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} \frac{\partial_z^3 \Phi(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} + \frac{\partial_z^2 \tilde{g}_1(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} - \frac{\partial_z^2 \tilde{g}_1(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} \right) + o\left(\frac{1}{\tau^2}\right) \quad \text{as } |\tau| \rightarrow +\infty.$$

$$(8.2) \quad -\frac{1}{\pi} \int_{\Omega} \frac{e_1 \tilde{g}_2 e^{\tau(\Phi(\zeta) - \overline{\Phi(\zeta)})}}{\bar{\zeta} - \bar{z}} d\xi_1 d\xi_2 = \frac{1}{\tau^2} \frac{e^{2i\tau\psi(\tilde{x})}}{|\det \psi''(\tilde{x})|^{\frac{1}{2}}} \left( \frac{\frac{\partial_{\bar{z}} \tilde{g}_2(\tilde{x})}{\partial_{\bar{z}}^2 \Phi(\tilde{x})}}{(\tilde{z} - \bar{z})^2} + \frac{\frac{\partial_{\bar{z}} \tilde{g}_2(\tilde{x})}{\partial_{\bar{z}}^2 \Phi(\tilde{x})} \frac{\partial_{\bar{z}}^3 \Phi(\tilde{x})}{\partial_{\bar{z}}^2 \Phi(\tilde{x})} - \frac{\partial_{\bar{z}}^2 \tilde{g}_2(\tilde{x})}{\partial_{\bar{z}}^2 \Phi(\tilde{x})} + \frac{\partial_{\bar{z}}^2 \tilde{g}_2(\tilde{x})}{\partial_{\bar{z}}^2 \Phi(\tilde{x})} \right) + o\left(\frac{1}{\tau^2}\right) \quad \text{as } |\tau| \rightarrow +\infty.$$

$$(8.3) \quad -\frac{1}{\pi} \int_{\Omega} \frac{e_1 \tilde{g}_3 e^{-\tau(\Phi(\zeta) - \overline{\Phi(\zeta)})}}{\bar{\zeta} - \bar{z}} d\xi_1 d\xi_2 = \frac{1}{\tau^2} \frac{e^{-2i\tau\psi(\tilde{x})}}{|\det \psi''(\tilde{x})|^{\frac{1}{2}}} \left( \frac{\frac{\partial_{\bar{z}} \tilde{g}_3(\tilde{x})}{\partial_{\bar{z}}^2 \Phi(\tilde{x})}}{(\tilde{z} - \bar{z})^2} + \frac{\frac{\partial_{\bar{z}} \tilde{g}_3(\tilde{x})}{\partial_{\bar{z}}^2 \Phi(\tilde{x})} \frac{\partial_{\bar{z}}^3 \Phi(\tilde{x})}{\partial_{\bar{z}}^2 \Phi(\tilde{x})} - \frac{\partial_{\bar{z}}^2 \tilde{g}_3(\tilde{x})}{\partial_{\bar{z}}^2 \Phi(\tilde{x})} + \frac{\partial_{\bar{z}}^2 \tilde{g}_3(\tilde{x})}{\partial_{\bar{z}}^2 \Phi(\tilde{x})} \right) + o\left(\frac{1}{\tau^2}\right) \quad \text{as } |\tau| \rightarrow +\infty.$$

$$(8.4) \quad -\frac{1}{\pi} \int_{\Omega} \frac{e_1 \tilde{g}_4 e^{\tau(\Phi(\zeta) - \overline{\Phi(\zeta)})}}{\zeta - z} d\xi_1 d\xi_2 = \frac{1}{\tau^2} \frac{e^{2i\tau\psi(\tilde{x})}}{|\det \psi''(\tilde{x})|^{\frac{1}{2}}} \left( \frac{\frac{\partial_z \tilde{g}_4(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})}}{(\tilde{z} - z)^2} \right. \\ \left. + \frac{\frac{\partial_z \tilde{g}_4(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} \frac{\partial_z^3 \Phi(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} + \frac{\partial_z^2 \tilde{g}_4(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} - \frac{\partial_z^2 \tilde{g}_4(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})}}{2(\tilde{z} - z)} \right) + o\left(\frac{1}{\tau^2}\right) \quad \text{as } |\tau| \rightarrow +\infty.$$

*Proof.* Let  $\delta > 0$  be a sufficiently small number and  $\tilde{e} \in C_0^\infty(B(\tilde{x}, \delta))$ ,  $\tilde{e}|_{B(\tilde{x}, \delta/2)} \equiv 1$ . Let  $\tilde{g} \in C^2(\overline{\Omega})$  be some function such that  $\tilde{g}(\tilde{x}) = \frac{\partial \tilde{g}}{\partial \tilde{z}}(\tilde{x}) = 0$ . We compute the asymptotic formulae of the following integral as  $|\tau|$  goes to infinity.

$$(8.5) \quad -\frac{1}{\pi} \int_{\Omega} \frac{e_1 \tilde{g} e^{-\tau(\Phi(\zeta) - \overline{\Phi(\zeta)})}}{\zeta - z} d\xi_1 d\xi_2 = -\frac{1}{\pi} \int_{B(\tilde{x}, \delta)} \frac{\tilde{e} \tilde{g} e^{-\tau(\Phi(\zeta) - \overline{\Phi(\zeta)})}}{\zeta - z} d\xi_1 d\xi_2 + o\left(\frac{1}{\tau^2}\right) = \\ -\frac{1}{\pi} \int_{B(\tilde{x}, \delta)} \tilde{e} \left\{ \frac{\partial_z \tilde{g}(\tilde{x})(\zeta - \tilde{z}) + \frac{1}{2} \partial_z^2 \tilde{g}(\tilde{x})(\bar{\zeta} - \tilde{z})^2}{\zeta - z} \right. \\ \left. + \frac{\partial_z \partial_z \tilde{g}(\tilde{x})(\zeta - \tilde{z})(\bar{\zeta} - \tilde{z}) + \frac{1}{2} \partial_z^2 \tilde{g}(\tilde{x})(\zeta - \tilde{z})^2}{\zeta - z} \right\} e^{-\tau(\Phi(\zeta) - \overline{\Phi(\zeta)})} d\xi_1 d\xi_2 + o\left(\frac{1}{\tau^2}\right) = \\ -\frac{1}{\pi} \int_{B(\tilde{x}, \delta)} \tilde{e} \left\{ \frac{\frac{\partial_z \tilde{g}(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} (\partial_\zeta \Phi - \frac{1}{2} \partial_z^3 \Phi(\tilde{x})(\zeta - \tilde{z})^2) + \frac{1}{2} \frac{\partial_z^2 \tilde{g}(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} \partial_{\bar{\zeta}} \overline{\Phi}(\bar{\zeta} - \tilde{z})}{\zeta - z} \right. \\ \left. + \frac{\frac{\partial_z \partial_z \tilde{g}(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} (\zeta - \tilde{z}) \partial_{\bar{\zeta}} \overline{\Phi} + \frac{1}{2} \frac{\partial_z^2 \tilde{g}(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} \partial_\zeta \Phi(\zeta - \tilde{z})}{\zeta - z} \right\} e^{-\tau(\Phi(\zeta) - \overline{\Phi(\zeta)})} d\xi_1 d\xi_2 + o\left(\frac{1}{\tau^2}\right) = \\ -\frac{1}{\pi} \int_{B(\tilde{x}, \delta)} \tilde{e} \left\{ \frac{\frac{\partial_z \tilde{g}(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} (\partial_\zeta \Phi - \frac{1}{2} \frac{\partial_z^3 \Phi(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} \partial_\zeta \Phi(\zeta - \tilde{z})) + \frac{1}{2} \frac{\partial_z^2 \tilde{g}(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} \partial_{\bar{\zeta}} \overline{\Phi}(\bar{\zeta} - \tilde{z})}{\zeta - z} \right. \\ \left. + \frac{\frac{\partial_z \partial_z \tilde{g}(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} (\zeta - \tilde{z}) \partial_{\bar{\zeta}} \overline{\Phi} + \frac{1}{2} \frac{\partial_z^2 \tilde{g}(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} \partial_\zeta \Phi(\zeta - \tilde{z})}{\zeta - z} \right\} e^{-\tau(\Phi(\zeta) - \overline{\Phi(\zeta)})} d\xi_1 d\xi_2 + o\left(\frac{1}{\tau^2}\right) = \\ -\frac{1}{\pi \tau} \int_{B(\tilde{x}, \delta)} \tilde{e} \left\{ \frac{-\frac{\partial_z \tilde{g}(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} \frac{\partial_z^3 \Phi(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} - \frac{\partial_z^2 \tilde{g}(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})}}{2(\zeta - z)} - \frac{\frac{\partial_z \tilde{g}(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} (1 - \frac{1}{2} \frac{\partial_z^3 \Phi(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} (\zeta - \tilde{z}))}{(\zeta - z)^2} \right. \\ \left. + \frac{\frac{\partial_z^2 \tilde{g}(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})}}{2(\zeta - z)} - \frac{\frac{\partial_z^2 \tilde{g}(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} (\zeta - \tilde{z})}{2(\zeta - z)^2} \right\} e^{-\tau(\Phi(\zeta) - \overline{\Phi(\zeta)})} d\xi_1 d\xi_2 + o\left(\frac{1}{\tau^2}\right).$$

(8.5)

Here we used

$$\int_{B(\tilde{x}, \delta)} \tilde{e} \frac{(\zeta - \tilde{z})}{(\zeta - z)^2} e^{-\tau(\Phi(\zeta) - \overline{\Phi(\zeta)})} d\xi_1 d\xi_2 = o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty,$$

which is obtained by stationary phase. Another asymptotic calculation is

$$\begin{aligned}
& -\frac{1}{\pi} \int_{\Omega} \frac{e_1 \tilde{g} e^{\tau(\Phi(\zeta) - \overline{\Phi(\zeta)})}}{\zeta - z} d\xi_1 d\xi_2 = \frac{1}{\pi\tau} \int_{B(\tilde{x}, \delta)} \tilde{e} \frac{\frac{\partial_z \tilde{g}(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})}}{(\zeta - z)^2} e^{-\tau(\Phi(\zeta) - \overline{\Phi(\zeta)})} dx \\
& + \frac{1}{2\pi\tau} \int_{B(\tilde{x}, \delta)} \tilde{e} \frac{\frac{\partial_z \tilde{g}(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} \frac{\partial_z^3 \Phi(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} + \frac{\partial_z^2 \tilde{g}(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} - \frac{\partial_z^2 \tilde{g}(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})}}{(\zeta - z)} e^{-\tau(\Phi(\zeta) - \overline{\Phi(\zeta)})} dx + o\left(\frac{1}{\tau^2}\right) = \\
& \frac{1}{\tau^2} \frac{e^{-2i\tau\psi(\tilde{x})}}{|\det \psi''(\tilde{x})|^{\frac{1}{2}}} \left( \frac{\frac{\partial_z \tilde{g}(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})}}{(\tilde{z} - z)^2} + \frac{\frac{\partial_z \tilde{g}(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} \frac{\partial_z^3 \Phi(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} + \frac{\partial_z^2 \tilde{g}(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})} - \frac{\partial_z^2 \tilde{g}(\tilde{x})}{\partial_z^2 \Phi(\tilde{x})}}{2(\tilde{z} - z)} \right) \\
& + o\left(\frac{1}{\tau^2}\right) \quad \text{as } |\tau| \rightarrow +\infty.
\end{aligned} \tag{8.6}$$

Taking  $\tilde{g} = g_1$  and  $\tilde{g} = \bar{g}_3$  we obtain (8.1) and (8.3) from the above formula. Taking  $\tilde{g} = g_4$  or  $\tilde{g} = \bar{g}_2$  and replacing  $\tau$  by  $-\tau$  we obtain (8.4) and (8.2).  $\square$

**Proposition 8.2.** *For any  $x$  from the boundary of  $\Omega$ , the following asymptotic formulae holds true as  $|\tau|$  goes to  $+\infty$  :*

$$\mathfrak{G}_1(x, \tau) = -\frac{e^{-2i\tau\psi(\tilde{x})}}{2\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \left( \frac{\frac{\partial(g_1 e^{-\mathcal{A}_1})}{\partial z}(\tilde{x}) + \frac{\partial \Phi}{\partial z} m_1(\tilde{x})}{\tilde{z} - z} + \frac{\sigma_1(\tilde{x}) \frac{\partial \Phi}{\partial z}}{(\tilde{z} - z)^2} \right) + o\left(\frac{1}{\tau}\right), \tag{8.7}$$

$$\mathfrak{G}_2(x, \tau) = -\frac{e^{2i\tau\psi(\tilde{x})}}{2\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \left( \frac{\frac{\partial(g_2 e^{-\mathcal{B}_1})}{\partial \bar{z}}(\tilde{x}) + \frac{\partial \bar{\Phi}}{\partial \bar{z}} \tilde{m}_1(\tilde{x})}{\tilde{z} - \bar{z}} + \frac{\tilde{\sigma}_1(\tilde{x}) \frac{\partial \bar{\Phi}}{\partial \bar{z}}}{(\tilde{z} - \bar{z})^2} \right) + o\left(\frac{1}{\tau}\right), \tag{8.8}$$

$$\mathfrak{G}_3(x, \tau) = -\frac{e^{-2i\tau\psi(\tilde{x})}}{2\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \left( \frac{\frac{\partial(g_3 e^{-\mathcal{A}_2})}{\partial \bar{z}}(\tilde{x}) - \frac{\partial \bar{\Phi}}{\partial \bar{z}} t_1(\tilde{x})}{\tilde{z} - \bar{z}} - \frac{r_1(\tilde{x}) \frac{\partial \bar{\Phi}}{\partial \bar{z}}}{(\tilde{z} - \bar{z})^2} \right) + o\left(\frac{1}{\tau}\right), \tag{8.9}$$

$$\mathfrak{G}_4(x, \tau) = -\frac{e^{2i\tau\psi(\tilde{x})}}{2\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \left( \frac{\frac{\partial(g_4 e^{-\mathcal{B}_2})}{\partial z}(\tilde{x}) - \frac{\partial \Phi}{\partial z} \tilde{t}_1(\tilde{x})}{\tilde{z} - z} - \frac{\tilde{r}_1(\tilde{x}) \frac{\partial \Phi}{\partial z}}{(\tilde{z} - z)^2} \right) + o\left(\frac{1}{\tau}\right). \tag{8.10}$$

Here  $\tilde{z} = \tilde{x}_1 + i\tilde{x}_2$  and  $m_1, \tilde{m}_1, \sigma_1, \tilde{\sigma}_1, t_1, \tilde{t}_1, r_1, \tilde{r}_1$  are introduced in (4.12), (4.13), (4.40) and (4.41). Moreover for a sufficiently small positive  $\epsilon$  the following asymptotic formula holds true

$$\left\| \frac{\partial \mathfrak{G}_1(\cdot, \tau)}{\partial \bar{z}} \right\|_{C(\overline{\mathcal{O}_\epsilon})} + \left\| \frac{\partial \mathfrak{G}_2(\cdot, \tau)}{\partial z} \right\|_{C(\overline{\mathcal{O}_\epsilon})} + \left\| \frac{\partial \mathfrak{G}_3(\cdot, \tau)}{\partial z} \right\|_{C(\overline{\mathcal{O}_\epsilon})} + \left\| \frac{\partial \mathfrak{G}_4(\cdot, \tau)}{\partial \bar{z}} \right\|_{C(\overline{\mathcal{O}_\epsilon})} = o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty. \tag{8.11}$$

*Proof.* Observe that the functions  $\mathfrak{G}_1(x, \tau), \dots, \mathfrak{G}_4(x, \tau)$  are given by

$$\mathfrak{G}_1(x, \tau) = -\frac{1}{2\pi} \int_{\Omega} \frac{\frac{\partial \mathcal{A}_1(\zeta, \bar{\zeta})}{\partial \zeta} + \tau \frac{\partial \Phi}{\partial \zeta}(\zeta) - \left( \frac{\partial \mathcal{A}_1(z, \bar{z})}{\partial z} + \tau \frac{\partial \Phi}{\partial z}(z) \right)}{\zeta - z} (e_1 g_1 e^{-\mathcal{A}_1})(\xi_1, \xi_2) e^{\tau(\overline{\Phi(\zeta)} - \Phi(\zeta))} d\xi_1 d\xi_2,$$

$$\mathfrak{G}_2(x, \tau) = -\frac{1}{2\pi} \int_{\Omega} \frac{\frac{\partial \mathcal{B}_1(\zeta, \bar{\zeta})}{\partial \zeta} + \tau \frac{\partial \overline{\Phi}(\zeta)}{\partial \zeta} - \left( \frac{\partial \mathcal{B}_1(z, \bar{z})}{\partial \bar{z}} + \tau \frac{\partial \overline{\Phi}(\bar{z})}{\partial \bar{z}} \right)}{\bar{\zeta} - \bar{z}} (e_1 g_2 e^{-\mathcal{B}_1})(\xi_1, \xi_2) e^{\tau(\Phi(\zeta) - \overline{\Phi}(\bar{\zeta}))} d\xi_1 d\xi_2,$$

$$\mathfrak{G}_3(x, \tau) = \frac{1}{2\pi} \int_{\Omega} \frac{\tau \frac{\partial \overline{\Phi}(\bar{\zeta})}{\partial \bar{\zeta}} - \frac{\partial \mathcal{A}_2(\zeta, \bar{\zeta})}{\partial \zeta} - \left( \tau \frac{\partial \overline{\Phi}(\bar{z})}{\partial \bar{z}} - \frac{\partial \mathcal{A}_2(z, \bar{z})}{\partial \bar{z}} \right)}{\bar{\zeta} - \bar{z}} (e_1 g_3 e^{-\mathcal{A}_2})(\xi_1, \xi_2) e^{\tau(\overline{\Phi}(\bar{\zeta}) - \Phi(\zeta))} d\xi_1 d\xi_2,$$

$$\mathfrak{G}_4(x, \tau) = \frac{1}{2\pi} \int_{\Omega} \frac{\tau \frac{\partial \Phi(\zeta)}{\partial \zeta} - \frac{\partial \mathcal{B}_2(\zeta, \bar{\zeta})}{\partial \zeta} - \left( \tau \frac{\partial \Phi(z)}{\partial z} - \frac{\partial \mathcal{B}_2(z, \bar{z})}{\partial z} \right)}{\zeta - z} (e_1 g_4 e^{-\mathcal{B}_2})(\xi_1, \xi_2) e^{\tau(\Phi(\zeta) - \overline{\Phi}(\bar{\zeta}))} d\xi_1 d\xi_2.$$

Let  $z = x_1 + ix_2$  where  $x = (x_1, x_2) \in \partial\Omega$ . By Proposition 3.5, the following asymptotics are valid:

$$\begin{aligned} \frac{\tau}{2\pi} \int_{\Omega} \frac{\partial_{\zeta} \overline{\Phi}(\zeta)}{\zeta - z} (e_1 g_1 e^{-\mathcal{A}_1}) e^{\tau(\overline{\Phi}(\bar{\zeta}) - \Phi(\zeta))} d\xi_1 d\xi_2 &= -\frac{1}{2\pi} \int_{\Omega} \frac{e_1 g_1 e^{-\mathcal{A}_1}}{\zeta - z} \frac{\partial}{\partial \zeta} e^{\tau(\overline{\Phi}(\bar{\zeta}) - \Phi(\zeta))} d\xi_1 d\xi_2 = \\ \frac{1}{2\pi} \int_{\Omega} \frac{\partial}{\partial \zeta} \left( \frac{e_1 g_1 e^{-\mathcal{A}_1}}{\zeta - z} \right) e^{\tau(\overline{\Phi}(\bar{\zeta}) - \Phi(\zeta))} d\xi_1 d\xi_2 &= \frac{1}{2\tau} \frac{e^{-2i\tau\psi(\tilde{x})}}{|\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial_z (g_1 e^{-\mathcal{A}_1})(\tilde{x})}{\tilde{z} - z} + o\left(\frac{1}{\tau}\right), \end{aligned}$$

$$\begin{aligned} \frac{\tau}{2\pi} \int_{\Omega} \frac{\partial_{\bar{\zeta}} \overline{\Phi}(\bar{\zeta})}{\bar{\zeta} - \bar{z}} (e_1 g_2 e^{-\mathcal{B}_1}) e^{\tau(\Phi(\zeta) - \overline{\Phi}(\bar{\zeta}))} d\xi_1 d\xi_2 &= -\frac{1}{2\pi} \int_{\Omega} \frac{e_1 g_2 e^{-\mathcal{B}_1}}{\bar{\zeta} - \bar{z}} \frac{\partial}{\partial \bar{\zeta}} e^{\tau(\Phi(\zeta) - \overline{\Phi}(\bar{\zeta}))} d\xi_1 d\xi_2 = \\ \frac{1}{2\pi} \int_{\Omega} \frac{\partial}{\partial \bar{\zeta}} \left( \frac{e_1 g_2 e^{-\mathcal{B}_1}}{\bar{\zeta} - \bar{z}} \right) e^{\tau(\Phi(\zeta) - \overline{\Phi}(\bar{\zeta}))} d\xi_1 d\xi_2 &= \frac{1}{2\tau} \frac{e^{2i\tau\psi(\tilde{x})}}{|\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial_{\bar{z}} (g_2 e^{-\mathcal{B}_1})(\tilde{x})}{\tilde{z} - \bar{z}} + o\left(\frac{1}{\tau}\right), \end{aligned}$$

$$\begin{aligned} \frac{\tau}{2\pi} \int_{\Omega} \frac{\partial_{\bar{\zeta}} \overline{\Phi}(\bar{\zeta})}{\bar{\zeta} - \bar{z}} (e_1 g_3 e^{-\mathcal{A}_2}) e^{\tau(\overline{\Phi}(\bar{\zeta}) - \Phi(\zeta))} d\xi_1 d\xi_2 &= \frac{1}{2\pi} \int_{\Omega} \frac{e_1 g_3 e^{-\mathcal{A}_2}}{\bar{\zeta} - \bar{z}} \frac{\partial}{\partial \bar{\zeta}} e^{\tau(\overline{\Phi}(\bar{\zeta}) - \Phi(\zeta))} d\xi_1 d\xi_2 = \\ -\frac{1}{2\pi} \int_{\Omega} \frac{\partial}{\partial \bar{\zeta}} \left( \frac{e_1 g_3 e^{-\mathcal{A}_2}}{\bar{\zeta} - \bar{z}} \right) e^{\tau(\overline{\Phi}(\bar{\zeta}) - \Phi(\zeta))} d\xi_1 d\xi_2 &= -\frac{1}{2\tau} \frac{e^{-2i\tau\psi(\tilde{x})}}{|\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial_{\bar{z}} (g_3 e^{-\mathcal{A}_2})(\tilde{x})}{\tilde{z} - \bar{z}} + o\left(\frac{1}{\tau}\right) \end{aligned}$$

and

$$\begin{aligned} \frac{\tau}{2\pi} \int_{\Omega} \frac{\partial_{\zeta} \Phi(\zeta)}{\zeta - z} (e_1 g_4 e^{-\mathcal{B}_2}) e^{\tau(\Phi(\zeta) - \overline{\Phi}(\bar{\zeta}))} d\xi_1 d\xi_2 &= \frac{1}{2\pi} \int_{\Omega} \frac{e_1 g_4 e^{-\mathcal{B}_2}}{\zeta - z} \frac{\partial}{\partial \zeta} e^{\tau(\Phi(\zeta) - \overline{\Phi}(\bar{\zeta}))} d\xi_1 d\xi_2 = \\ -\frac{1}{2\pi} \int_{\Omega} \frac{\partial}{\partial \zeta} \left( \frac{e_1 g_4 e^{-\mathcal{B}_2}}{\zeta - z} \right) e^{\tau(\Phi(\zeta) - \overline{\Phi}(\bar{\zeta}))} d\xi_1 d\xi_2 &= -\frac{1}{2\tau} \frac{e^{2i\tau\psi(\tilde{x})}}{|\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial_z (g_4 e^{-\mathcal{B}_2})(\tilde{x})}{\tilde{z} - z} + o\left(\frac{1}{\tau}\right). \end{aligned}$$

Noting that  $\tilde{g}_1 = e^{-\mathcal{A}_1} g_1$ ,  $\tilde{g}_2 = e^{-\mathcal{B}_1} g_2$ ,  $\tilde{g}_3 = e^{-\mathcal{A}_2} g_3$  and  $\tilde{g}_4 = e^{-\mathcal{B}_2} g_4$ , and taking into account Proposition 8.1, we obtain (8.7)-(8.10) for the functions  $\mathfrak{G}_1(x, \tau), \dots, \mathfrak{G}_4(x, \tau)$ .

For proving the estimate (8.11), it suffices to show that

$$\left\| \frac{\partial \mathfrak{G}_1(\cdot, \tau)}{\partial \bar{z}} \right\|_{C(\overline{\mathcal{O}_\epsilon})} \leq o\left(\frac{1}{\tau}\right).$$

In fact,

$$\partial_{\bar{z}} \mathfrak{G}_1(x, \tau) = -\frac{1}{4\pi} \frac{\partial A_1}{\partial z} \int_{\Omega} \frac{(e_1 g_1 e^{-\mathcal{A}_1})(\xi_1, \xi_2) e^{\tau(\overline{\Phi}(\bar{\zeta}) - \Phi(\zeta))}}{\zeta - z} d\xi_1 d\xi_2.$$

Then Proposition 3.5 and (4.6) imply  $\left\| \frac{\partial \mathfrak{G}_1(\cdot, \tau)}{\partial \bar{z}} \right\|_{C(\overline{\mathcal{O}_\epsilon})} = o\left(\frac{1}{\tau}\right)$ .  $\square$

**Proof of Proposition 5.2.** Using (5.13) and (5.16) we have

$$\begin{aligned}
\mathfrak{L}_0 &\equiv (2(A_1 - A_2) \frac{\partial U_1}{\partial z}, b_\tau e^{\mathcal{B}_2 - \tau\Phi} + c_\tau e^{\mathcal{A}_2 - \tau\bar{\Phi}})_{L^2(\Omega)} \\
&+ (2(B_1 - B_2) \frac{\partial U_1}{\partial \bar{z}}, b_\tau e^{\mathcal{B}_2 - \tau\Phi} + c_\tau e^{\mathcal{A}_2 - \tau\bar{\Phi}})_{L^2(\Omega)} \\
&= 2((A_1 - A_2) e^{\tau\bar{\Phi}} (-\mathcal{R}_{-\tau, A_1} \{ \frac{\partial(e_1 g_1)}{\partial z} \}) + e^{\mathcal{A}_1 - \tau(\bar{\Phi} - \Phi)} \mathfrak{G}_1(\cdot, \tau), c_\tau e^{\mathcal{A}_2 - \tau\bar{\Phi}})_{L^2(\Omega)} \\
&- 2((A_1 - A_2) e^{\tau\bar{\Phi}} \frac{\partial}{\partial z} \mathcal{R}_{-\tau, A_1} \{e_1 g_1\}, b_\tau e^{\mathcal{B}_2 - \tau\Phi})_{L^2(\Omega)} \\
(8.12) \quad &+ ((B_1 - B_2)(-e_1 g_1 + A_1 \mathcal{R}_{-\tau, A_1} \{e_1 g_1\}) e^{\tau\bar{\Phi}}, b_\tau e^{\mathcal{B}_2 - \tau\Phi} + c_\tau e^{\mathcal{A}_2 - \tau\bar{\Phi}})_{L^2(\Omega)} + o(\frac{1}{\tau}).
\end{aligned}$$

By (4.10) and Proposition 3.4 we have

$$\begin{aligned}
2((A_1 - A_2) e^{\tau\bar{\Phi}} \frac{\partial}{\partial z} \mathcal{R}_{-\tau, A_1} \{e_1 g_1\}, b_\tau e^{\mathcal{B}_2 - \tau\Phi})_{L^2(\Omega)} &= 2((A_1 - A_2) \frac{\partial}{\partial z} \mathcal{R}_{-\tau, A_1} \{e_1 g_1\}, b_\tau e^{\mathcal{B}_2})_{L^2(\Omega)} = \\
&((A_1 - A_2)(\nu_1 - i\nu_2) \mathcal{R}_{-\tau, A_1} \{e_1 g_1\}, b_\tau e^{\mathcal{B}_2})_{L^2(\partial\Omega)} - 2(\mathcal{R}_{-\tau, A_1} \{e_1 g_1\}, \frac{\partial}{\partial \bar{z}} \{b_\tau e^{\mathcal{B}_2} \overline{(A_1 - A_2)}\})_{L^2(\Omega)} \\
(8.13) \quad &= -(\frac{e_1 g_1}{\tau \frac{\partial}{\partial z} \Phi}, \frac{\partial}{\partial \bar{z}} \{b_\tau e^{\mathcal{B}_2} \overline{(A_1 - A_2)}\})_{L^2(\Omega)} + o(\frac{1}{\tau}) \quad \text{as } |\tau| \rightarrow +\infty.
\end{aligned}$$

Using the stationary phase argument we obtain

$$\begin{aligned}
&-2((A_1 - A_2) e^{\tau\bar{\Phi}} \mathcal{R}_{-\tau, A_1} \{ \frac{\partial(e_1 g_1)}{\partial z} \}, c_\tau e^{\mathcal{A}_2 - \tau\bar{\Phi}})_{L^2(\Omega)} \\
&= -((A_1 - A_2) e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2} \partial_{\bar{z}}^{-1} \{ \frac{\partial(e_1 g_1)}{\partial z} e^{-\mathcal{A}_1 - \tau(\Phi - \bar{\Phi})} \}, c_\tau)_{L^2(\Omega)} = \\
&\frac{1}{\pi} \int_{\Omega} (A_1 - A_2) e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2} \bar{c}_\tau \left( \int_{\Omega} \frac{\frac{\partial(e_1 g_1)}{\partial \zeta} e^{-\mathcal{A}_1 - \tau(\Phi - \bar{\Phi})}}{\zeta - z} d\xi_1 d\xi_2 \right) dx_1 dx_2 = \\
&\frac{1}{\pi} \int_{\Omega} \frac{\partial(e_1 g_1)}{\partial \zeta} e^{-\mathcal{A}_1 - \tau(\Phi - \bar{\Phi})} \left( \int_{\Omega} \frac{(A_1 - A_2) \bar{c}_\tau e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2}}{\zeta - z} dx_1 dx_2 \right) d\xi_1 d\xi_2 = \\
(8.14) \quad &\frac{e^{-2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_1}{\partial z}(\tilde{x}) e^{-\mathcal{A}_1(\tilde{x})} \left( \int_{\Omega} \frac{(A_1 - A_2) \bar{c}_\tau e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2}}{\tilde{z} - z} dx_1 dx_2 \right) + o(\frac{1}{\tau}).
\end{aligned}$$

Integrating by parts and using (8.11), (3.2),  $2\frac{\partial \mathcal{A}_1}{\partial \bar{z}} = -A_1$  and  $2\frac{\partial \mathcal{A}_2}{\partial z} = \bar{A}_2$ , we have

$$\begin{aligned}
(8.15) \quad &2((A_1 - A_2) e^{\tau\bar{\Phi}} e^{\mathcal{A}_1} e^{-\tau(\bar{\Phi} - \Phi)} \mathfrak{G}_1, c_\tau e^{\mathcal{A}_2 - \tau\bar{\Phi}})_{L^2(\Omega)} = 2((A_1 - A_2) e^{\mathcal{A}_1} \mathfrak{G}_1, c_\tau e^{\mathcal{A}_2})_{L^2(\Omega)} \\
&= \int_{\Omega} 2(A_1 - A_2) e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2} \bar{c}_\tau(\bar{z}) \mathfrak{G}_1(x, \tau) dx = -4 \int_{\Omega} \frac{\partial}{\partial \bar{z}} (e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2} \bar{c}_\tau(\bar{z})) \mathfrak{G}_1(x, \tau) dx \\
&= 4 \int_{\Omega} e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2} \bar{c}_\tau(\bar{z}) \frac{\partial}{\partial \bar{z}} \mathfrak{G}_1(x, \tau) dx - 2 \int_{\partial\Omega} (\nu_1 + i\nu_2) e^{\mathcal{A}_1 + \bar{\mathcal{A}}_2} \bar{c}_\tau(\bar{z}) \mathfrak{G}_1(x, \tau) d\sigma + o(\frac{1}{\tau}) \quad \text{as } |\tau| \rightarrow +\infty.
\end{aligned}$$



Since  $e^{A_1} \frac{\partial}{\partial \bar{z}} \mathfrak{G}_1(x, \tau) = -\frac{e^{A_1}}{4\pi} \frac{\partial A_1}{\partial z} \int_{\Omega} \frac{(e_1 g_1 e^{-A_1})(\xi_1, \xi_2) e^{\tau(\overline{\Phi(\zeta)} - \Phi(\zeta))}}{\zeta - z} d\xi_1 d\xi_2 = \frac{1}{2} \frac{\partial A_1}{\partial z} e^{\tau(\overline{\Phi} - \Phi)} \mathcal{R}_{-\tau, A_1} \{e_1 g_1\}$ , applying the Proposition 3.4 and Proposition 3.3 we obtain

$$(8.16) \quad \int_{\Omega} e^{A_1 + \overline{A_2}} c_{\tau}(\bar{z}) \frac{\partial}{\partial \bar{z}} \mathfrak{G}_1(x, \tau) dx = o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty.$$

By (8.7), (8.14)-(8.16) and Propositions 3.4 and 3.5, we conclude

$$(8.17) \quad \begin{aligned} \mathfrak{L}_0 &= ((B_1 - B_2)(-e_1 g_1 + \frac{A_1 e_1 g_1}{2\tau \overline{\partial_z \Phi}}), b_{\tau} e^{\mathcal{B}_2})_{L^2(\Omega)} \\ &+ (\frac{e_1 g_1}{\tau \overline{\partial_z \Phi}}, \frac{\partial}{\partial \bar{z}} \{b e^{\mathcal{B}_2} \overline{(A_1 - A_2)}\})_{L^2(\Omega)} \\ &+ \frac{e^{-2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_1}{\partial z}(\tilde{x}) e^{-A_1(\tilde{x})} \left( \int_{\Omega} \frac{(A_1 - A_2) \bar{c} e^{A_1 + \overline{A_2}}}{\tilde{z} - z} dx_1 dx_2 \right) \\ &- 2 \int_{\partial\Omega} (\nu_1 + i\nu_2) e^{A_1 + \overline{A_2}} c_{\tau}(\bar{z}) \mathfrak{G}_1(x, \tau) d\sigma + o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty. \end{aligned}$$

Using (5.14) and (5.18) we obtain after simple computations:

$$(8.18) \quad \begin{aligned} \mathfrak{L}_1 &\equiv (2(A_1 - A_2) \frac{\partial U_2}{\partial z}, b_{\tau} e^{\mathcal{B}_2 - \tau\Phi} + c_{\tau} e^{A_2 - \tau\overline{\Phi}})_{L^2(\Omega)} \\ &+ (2(B_1 - B_2) \frac{\partial U_2}{\partial \bar{z}}, b_{\tau} e^{\mathcal{B}_2 - \tau\Phi} + c_{\tau} e^{A_2 - \tau\overline{\Phi}})_{L^2(\Omega)} \\ &= ((A_1 - A_2)(-e_1 g_2 + B_1 \tilde{\mathcal{R}}_{\tau, B_1} \{e_1 g_2\}) e^{\tau\Phi}, b_{\tau} e^{\mathcal{B}_2 - \tau\Phi} + c_{\tau} e^{A_2 - \tau\overline{\Phi}})_{L^2(\Omega)} \\ &+ 2((B_1 - B_2)(-\tilde{\mathcal{R}}_{\tau, B_1} \{\frac{\partial(e_1 g_2)}{\partial \bar{z}}\}) e^{\tau\Phi} + e^{\mathcal{B}_1 + \tau(\overline{\Phi} - \Phi)} \mathfrak{G}_2(\cdot, \tau), b_{\tau} e^{\mathcal{B}_2 - \tau\Phi})_{L^2(\Omega)} \\ &- 2((B_1 - B_2) \frac{\partial}{\partial \bar{z}} \tilde{\mathcal{R}}_{\tau, B_1} \{e_1 g_2\} e^{\tau\Phi}, c_{\tau} e^{A_2 - \tau\overline{\Phi}})_{L^2(\Omega)}. \end{aligned}$$

Integrating by parts and using (4.11) we have

$$(8.19) \quad \begin{aligned} &- 2((B_1 - B_2) \frac{\partial}{\partial \bar{z}} \tilde{\mathcal{R}}_{\tau, B_1} \{e_1 g_2\} e^{\tau\Phi}, c_{\tau} e^{A_2 - \tau\overline{\Phi}})_{L^2(\Omega)} = -2((B_1 - B_2) \frac{\partial}{\partial \bar{z}} \tilde{\mathcal{R}}_{\tau, B_1} \{e_1 g_2\}, c_{\tau} e^{A_2})_{L^2(\Omega)} = \\ &- 2 \int_{\Omega} \tilde{\mathcal{R}}_{\tau, B_1} \{e_1 g_2\} \frac{\partial}{\partial \bar{z}} ((B_1 - B_2) \bar{c}_{\tau} e^{\overline{A_2}}) dx - \int_{\partial\Omega} (B_1 - B_2) (\nu_1 + i\nu_2) \tilde{\mathcal{R}}_{\tau, B_1} \{e_1 g_2\} \bar{c}_{\tau} e^{\overline{A_2}} d\sigma = \\ &\int_{\Omega} \frac{e_1 g_2}{\tau \overline{\partial_z \Phi}} \frac{\partial}{\partial \bar{z}} ((B_1 - B_2) \bar{c} e^{\overline{A_2}}) dx + o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty. \end{aligned}$$

The stationary phase argument implies the formula

$$(8.20) \quad \begin{aligned} -2((B_1 - B_2) \tilde{\mathcal{R}}_{\tau, B_1} \{\frac{\partial(e_1 g_2)}{\partial \bar{z}}\}) e^{\tau\Phi}, b_{\tau} e^{\mathcal{B}_2 - \tau\Phi})_{L^2(\Omega)} &= - \int_{\Omega} (B_1 - B_2) \partial_z^{-1} \left\{ \frac{\partial(e_1 g_2)}{\partial \bar{z}} e^{-\mathcal{B}_1 + \tau(\Phi - \overline{\Phi})} \right\} \bar{b}_{\tau} e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}} dx = \\ &\frac{1}{\pi} \int_{\Omega} \frac{\partial(e_1 g_2)}{\partial \bar{\zeta}} e^{-\mathcal{B}_1 + \tau(\Phi - \overline{\Phi})} \left( \int_{\Omega} \frac{(B_1 - B_2) \bar{b}_{\tau} e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}}}{\zeta - \bar{z}} dx \right) d\xi_1 d\xi_2 = \\ &\frac{e^{2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_2(\tilde{x})}{\partial \bar{z}} e^{-\mathcal{B}_1(\tilde{x})} \int_{\Omega} \frac{(B_1 - B_2) \bar{b}_{\tau} e^{\mathcal{B}_1 - \overline{\mathcal{B}_2}}}{\tilde{z} - \bar{z}} dx + o\left(\frac{1}{\tau}\right). \end{aligned}$$

By (8.11) we have the asymptotic formula

$$\begin{aligned}
(8.21) \quad & 2((B_1 - B_2)(e^{\mathcal{B}_1} e^{\tau \bar{\Phi}} \mathfrak{G}_2, b_\tau e^{\mathcal{B}_2 - \tau \Phi})_{L^2(\Omega)} \\
& = 2((B_1 - B_2)e^{\mathcal{B}_1} \mathfrak{G}_2, b_\tau e^{\mathcal{B}_2})_{L^2(\Omega)} = 2 \int_{\Omega} (B_1 - B_2) e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} \overline{b_\tau(z)} \mathfrak{G}_2(x, \tau) dx \\
& = -4 \int_{\Omega} \frac{\partial}{\partial z} e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} \overline{b_\tau(z)} \mathfrak{G}_2(x, \tau) dx = -2 \int_{\partial \Omega} (\nu_1 - i\nu_2) e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} \overline{b_\tau(z)} \mathfrak{G}_2(x, \tau) d\sigma \\
& \quad + 4 \int_{\Omega} e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} \overline{b_\tau(z)} \frac{\partial}{\partial z} \mathfrak{G}_2(x, \tau) dx.
\end{aligned}$$

Observe that  $\frac{\partial}{\partial z} \mathfrak{G}_2(x, \tau) = -\frac{1}{4\pi} \frac{\partial B_1}{\partial \bar{z}} \int_{\Omega} \frac{(e_1 g_2 e^{-\mathcal{B}_1})(\xi_1, \xi_2)}{\bar{\zeta} - \bar{z}} e^{\tau(\Phi(\zeta) - \bar{\Phi}(\bar{\zeta}))} d\xi_1 d\xi_2 = \frac{e^{-\mathcal{B}_1}}{2} e^{\tau(\Phi - \bar{\Phi})} \tilde{\mathcal{R}}_{\tau, B_1} \{e_1 g_2\}$ . Then by Proposition 3.4

$$\begin{aligned}
(8.22) \quad & 4 \int_{\Omega} e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} \overline{b_\tau(z)} \frac{\partial}{\partial z} \mathfrak{G}_2(x, \tau) dx = 2 \int_{\Omega} e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} \overline{b_\tau(z)} e^{-\mathcal{B}_1} e^{\tau(\Phi - \bar{\Phi})} \tilde{\mathcal{R}}_{\tau, B_1} \{e_1 g_2\} dx \\
& = \int_{\Omega} e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} \overline{b_\tau(z)} e^{-\mathcal{B}_1} e^{\tau(\Phi - \bar{\Phi})} \frac{e_1 g_2}{\tau \partial_z \Phi} dx = o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty.
\end{aligned}$$

By (8.21) -(8.22) we have

$$\begin{aligned}
(8.23) \quad \mathfrak{L}_1 & = ((A_1 - A_2)(-e_1 g_2 + \frac{B_1 e_1 g_2}{2\tau \partial_z \Phi}), c_\tau e^{A_2})_{L^2(\Omega)} \\
& + \int_{\Omega} \frac{e_1 g_2}{\tau \partial_z \Phi} \frac{\partial}{\partial \bar{z}} ((B_1 - B_2) \bar{c}_\tau e^{\bar{A}_2}) dx \\
& \quad \frac{e^{2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_2(\tilde{x})}{\partial \bar{z}} e^{-\mathcal{B}_1(\tilde{x})} \int_{\Omega} \frac{(B_1 - B_2) \bar{b} e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2}}{\bar{\zeta} - \bar{z}} dx \\
& - 2 \int_{\partial \Omega} (\nu_1 - i\nu_2) e^{\mathcal{B}_1 + \bar{\mathcal{B}}_2} \overline{b_\tau(z)} \mathfrak{G}_2(x, \tau) d\sigma + o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty.
\end{aligned}$$

Recall that  $V_1 = -e^{-\tau\Phi} \tilde{\mathcal{R}}_{-\tau, -\bar{A}_2} \{e_1 g_3\}$  and  $V_2 = -e^{-\tau\bar{\Phi}} \mathcal{R}_{\tau, -\bar{B}_2} \{e_1 g_4\}$ .

By Proposition 3.2 we conclude

$$(8.24) \quad 2 \frac{\partial V_1}{\partial z} = (-e_1 g_3 + \bar{A}_2 \tilde{\mathcal{R}}_{-\tau, \bar{A}_2} \{e_1 g_3\}) e^{-\tau\Phi}$$

and

$$(8.25) \quad 2 \frac{\partial V_2}{\partial \bar{z}} = (-e_1 g_4 + \bar{B}_2 \mathcal{R}_{\tau, -\bar{B}_2} \{e_1 g_4\}) e^{-\tau\bar{\Phi}}.$$

Similarly to (5.15) and (5.17) we calculate  $\frac{\partial V_1}{\partial \bar{z}}$  and  $\frac{\partial V_2}{\partial z}$ :

$$(8.26) \quad \frac{\partial V_1}{\partial \bar{z}} = -e^{-\tau\Phi} \tilde{\mathcal{R}}_{-\tau, -\bar{A}_2} \left\{ \frac{\partial(e_1 g_3)}{\partial \bar{z}} \right\} + e^{-\tau\bar{\Phi} + A_2} \mathfrak{G}_3(\cdot, \tau)$$

and

$$(8.27) \quad \frac{\partial V_2}{\partial z} = -e^{-\tau\bar{\Phi}} \mathcal{R}_{\tau, -\bar{B}_2} \left\{ \frac{\partial(e_1 g_4)}{\partial z} \right\} + e^{-\tau\Phi + B_2} \mathfrak{G}_4(\cdot, \tau).$$

Using (3.2) and integrating by parts we obtain

$$\begin{aligned}
\mathfrak{L}_2 &\equiv \left( 2(A_1 - A_2) \frac{\partial}{\partial z} (a_\tau e^{A_1 + \tau\Phi} + d_\tau e^{B_1 + \tau\bar{\Phi}}), V_1 + V_2 \right)_{L^2(\Omega)} \\
&= -\left( (A_1 - A_2) d_\tau B_1 e^{B_1} e^{\tau\bar{\Phi}}, V_1 + V_2 \right)_{L^2(\Omega)} \\
&+ \left( (A_1 - A_2) (\nu_1 - i\nu_2) a_\tau e^{A_1 + \tau\Phi}, V_1 + V_2 \right)_{L^2(\partial\Omega)} \\
&- \left( 2 \frac{\partial}{\partial z} (A_1 - A_2) a_\tau e^{A_1 + \tau\Phi}, V_1 + V_2 \right)_{L^2(\Omega)} \\
&- \left( (A_1 - A_2) a_\tau e^{A_1 + \tau\Phi}, 2 \left( \frac{\partial V_1}{\partial \bar{z}} + \frac{\partial V_2}{\partial \bar{z}} \right) \right)_{L^2(\Omega)}.
\end{aligned}$$

We observe that by (8.26), (8.11), Proposition 3.4 and Proposition 3.3 and the equality  $\frac{\partial \mathfrak{G}_3}{\partial z} = -\frac{1}{2} e^{-A_2} \frac{\partial \bar{A}_2}{\partial \bar{z}} e^{\tau(\bar{\Phi} - \Phi)} \tilde{\mathcal{R}}_{-\tau, -\bar{A}_2}(e_1 g_3)$  :

$$\begin{aligned}
&\left( (A_1 - A_2) a_\tau e^{A_1 + \tau\Phi}, 2 \frac{\partial V_1}{\partial \bar{z}} \right)_{L^2(\Omega)} = -4 \int_{\Omega} a_\tau(z) \frac{\partial}{\partial \bar{z}} e^{A_1 + \bar{A}_2} \overline{\mathfrak{G}_3(x, \tau)} dx \\
&+ \left( (A_1 - A_2) a_\tau e^{A_1 + \tau\Phi}, -e^{-\tau\Phi} \tilde{\mathcal{R}}_{-\tau, -\bar{A}_2} \left\{ \frac{\partial(e_1 g_3)}{\partial \bar{z}} \right\} \right)_{L^2(\Omega)} + o\left(\frac{1}{\tau}\right) = \\
&\quad -2 \int_{\partial\Omega} a_\tau(z) \overline{\mathfrak{G}_3(x, \tau)} (\nu_1 + i\nu_2) e^{A_1 + \bar{A}_2} d\sigma \\
&+ \frac{1}{\pi} \int_{\Omega} \frac{\partial(e_1 g_3)}{\partial \bar{z}} e^{\tau(\Phi - \bar{\Phi}) - \bar{A}_2} \left( \int_{\Omega} \frac{(A_1 - A_2) a_\tau e^{A_1 + \bar{A}_2}}{\zeta - z} dx \right) d\xi_1 d\xi_2 + o\left(\frac{1}{\tau}\right) = \\
&\quad \frac{e^{2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_3(\tilde{x})}{\partial \bar{z}} e^{-\bar{A}_2(\tilde{x})} \int_{\Omega} \frac{(A_1 - A_2) a e^{A_1 + \bar{A}_2}}{\tilde{z} - z} dx \\
(8.28) \quad &-2 \int_{\partial\Omega} a_\tau(z) \overline{\mathfrak{G}_3(x, \tau)} (\nu_1 + i\nu_2) e^{A_1 + \bar{A}_2} d\sigma + o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty.
\end{aligned}$$

Hence, using (8.25) we have

$$\begin{aligned}
\mathfrak{L}_2 &= \left( (A_1 - A_2) d_\tau B_1 e^{B_1}, \tilde{\mathcal{R}}_{-\tau, -\bar{A}_2} \{e_1 g_3\} \right)_{L^2(\Omega)} \\
&+ \left( (A_1 - A_2) (\nu_1 - i\nu_2) a_\tau e^{A_1 + \tau\Phi}, V_1 + V_2 \right)_{L^2(\partial\Omega)} \\
&+ \left( 2 \frac{\partial}{\partial z} (A_1 - A_2) a_\tau e^{A_1}, \mathcal{R}_{\tau, -\bar{B}_2} \{e_1 g_4\} \right)_{L^2(\Omega)} \\
&- \frac{e^{2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_3(\tilde{x})}{\partial \bar{z}} e^{-\bar{A}_2(\tilde{x})} \int_{\Omega} \frac{(A_1 - A_2) a e^{A_1 + \bar{A}_2}}{\tilde{z} - z} dx \\
&+ \left( (A_1 - A_2) a_\tau e^{A_1}, -e_1 g_4 - \bar{B}_2 \mathcal{R}_{\tau, -\bar{B}_2} \{e_1 g_4\} \right)_{L^2(\Omega)} + o\left(\frac{1}{\tau}\right) \\
&+ 2 \int_{\partial\Omega} a_\tau(z) \overline{\mathfrak{G}_3(x, \tau)} (\nu_1 + i\nu_2) e^{A_1 + \bar{A}_2} d\sigma \quad \text{as } |\tau| \rightarrow +\infty.
\end{aligned}$$

By (4.39) and (4.38) we obtain

$$\left( (A_1 - A_2) (\nu_1 - i\nu_2) a_\tau e^{A_1 + \tau\Phi}, V_1 + V_2 \right)_{L^2(\partial\Omega)} = o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty.$$

Therefore, by Proposition 3.4

$$\begin{aligned}
\mathfrak{L}_2 = & - \left( (A_1 - A_2)d_\tau B_1 e^{\mathcal{B}_1}, \frac{e_1 g_3}{2\tau \partial_z \Phi} \right)_{L^2(\Omega)} \\
& - \left( \frac{\partial}{\partial z} (A_1 - A_2) a_\tau e^{\mathcal{A}_1}, \frac{e_1 g_4}{\tau \partial_z \Phi} \right)_{L^2(\Omega)} \\
& + \left( (A_1 - A_2) a_\tau e^{\mathcal{A}_1}, -e_1 g_4 + \overline{B_2} \frac{e_1 g_4}{2\tau \partial_z \Phi} \right)_{L^2(\Omega)} \\
& - \frac{e^{2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \overline{\frac{\partial g_3(\tilde{x})}{\partial \bar{z}}} e^{-\overline{\mathcal{A}_2}(\tilde{x})} \int_\Omega \frac{(A_1 - A_2) a e^{\mathcal{A}_1 + \overline{\mathcal{A}_2}}}{\tilde{z} - z} dx \\
(8.29) \quad & + 2 \int_{\partial\Omega} a_\tau(z) \overline{\mathfrak{G}_3(x, \tau)} (\nu_1 + i\nu_2) e^{\mathcal{A}_1 + \overline{\mathcal{A}_2}} d\sigma + o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty.
\end{aligned}$$

Integrating by parts we compute

$$\begin{aligned}
\mathfrak{L}_3 \equiv & \left( 2(B_1 - B_2) \frac{\partial}{\partial \bar{z}} (a_\tau e^{\mathcal{A}_1 + \tau\Phi} + d_\tau e^{\mathcal{B}_1 + \tau\overline{\Phi}}), V_1 + V_2 \right)_{L^2(\Omega)} = \\
& - \left( 2 \frac{\partial}{\partial \bar{z}} (B_1 - B_2) d_\tau e^{\mathcal{B}_1 + \tau\overline{\Phi}}, V_1 + V_2 \right)_{L^2(\Omega)} \\
& - \left( (B_1 - B_2) A_1 a_\tau e^{\mathcal{A}_1 + \tau\Phi}, V_1 + V_2 \right)_{L^2(\Omega)} \\
& + \left( (\nu_1 + i\nu_2) (B_1 - B_2) d_\tau e^{\mathcal{B}_1 + \tau\overline{\Phi}}, V_1 + V_2 \right)_{L^2(\partial\Omega)} \\
& - \left( (B_1 - B_2) d_\tau e^{\mathcal{B}_1 + \tau\overline{\Phi}}, 2 \left( \frac{\partial V_1}{\partial z} + \frac{\partial V_2}{\partial z} \right) \right)_{L^2(\Omega)}.
\end{aligned}$$

We observe that by (8.11), (8.27), Proposition 3.4 and Proposition 3.3 and the equality  $\frac{\partial \mathfrak{G}_4}{\partial \bar{z}} = -\frac{1}{2} e^{-\mathcal{B}_2} \frac{\partial \overline{\mathcal{B}_2}}{\partial z} e^{\tau(\Phi - \overline{\Phi})} \mathcal{R}_{\tau, -\overline{\mathcal{B}_2}}(e_1 g_4)$  :

$$\begin{aligned}
& 2 \left( (B_1 - B_2) d_\tau e^{\mathcal{B}_1 + \tau\overline{\Phi}}, \frac{\partial V_2}{\partial z} \right)_{L^2(\Omega)} = -4 \int_\Omega \frac{\partial}{\partial z} e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}} d_\tau(\bar{z}) \overline{\mathfrak{G}_4(x, \tau)} dx \\
& + 2 \left( (B_1 - B_2) d_\tau e^{\mathcal{B}_1 + \tau\overline{\Phi}}, -e^{-\tau\overline{\Phi}} \mathcal{R}_{\tau, -\overline{\mathcal{B}_2}} \left\{ \frac{\partial(e_1 g_4)}{\partial z} \right\} \right)_{L^2(\Omega)} + o\left(\frac{1}{\tau}\right) = \\
& \quad -2 \int_{\partial\Omega} (\nu_1 - i\nu_2) e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}} d_\tau(\bar{z}) \overline{\mathfrak{G}_4(x, \tau)} d\sigma \\
& + \int_\Omega \left( \int_\Omega \frac{1}{\pi} \frac{(B_1 - B_2) d_\tau e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}}}{\bar{\zeta} - \bar{z}} dx \right) \overline{\left\{ \frac{\partial(e_1 g_4)}{\partial z} \right\}} e^{-\overline{\mathcal{B}_2} + \tau(\overline{\Phi} - \Phi)} d\xi_1 d\xi_2 + o\left(\frac{1}{\tau}\right) = \\
& \quad \frac{e^{-2i\tau\psi(\tilde{x})}}{\tau |\det \psi''(\tilde{x})|^{\frac{1}{2}}} \int_\Omega \frac{(B_1 - B_2) d_\tau e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}}}{\tilde{z} - \bar{z}} dx \overline{\left\{ \frac{\partial g_4(\tilde{x})}{\partial \zeta} \right\}} e^{-\overline{\mathcal{B}_2}(\tilde{x})} \\
(8.30) \quad & -2 \int_{\partial\Omega} (\nu_1 - i\nu_2) e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}} d_\tau(\bar{z}) \overline{\mathfrak{G}_4(x, \tau)} d\sigma + o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \rightarrow +\infty.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathfrak{L}_3 &= (2\frac{\partial}{\partial\bar{z}}(B_1 - B_2)d_\tau e^{\mathcal{B}_1}, \widetilde{\mathcal{R}}_{-\tau, -\bar{A}_2}\{e_1 g_3\})_{L^2(\Omega)} \\
&- ( (B_1 - B_2)A_1 a_\tau e^{\mathcal{A}_1}, \mathcal{R}_{\tau, -\bar{B}_2}\{e_1 g_4\})_{L^2(\Omega)} \\
&+ ( (B_1 - B_2)d_\tau e^{\mathcal{B}_1}, -e_1 g_3 + \bar{A}_2 \widetilde{\mathcal{R}}_{-\tau, -\bar{A}_2}\{e_1 g_3\})_{L^2(\Omega)} \\
&+ ( (\nu_1 + i\nu_2)(B_1 - B_2)d_\tau e^{\mathcal{B}_1 + \tau\Phi}, V_1 + V_2)_{L^2(\partial\Omega)} \\
&- \frac{e^{-2i\tau\psi(\bar{x})}}{\tau|\det\psi''(\bar{x})|^{\frac{1}{2}}}\left\{\overline{\frac{\partial g_4(\bar{x})}{\partial z}}\right\}e^{-\bar{B}_2(\bar{x})}\int_{\Omega}\frac{(B_1 - B_2)de^{\mathcal{B}_1 + \bar{B}_2}}{\bar{z} - z}dx \\
(8.31) \quad &+ 2\int_{\partial\Omega}(\nu_1 - i\nu_2)e^{\mathcal{B}_1 + \bar{B}_2}d_\tau(\bar{z})\overline{\mathfrak{G}_4(x, \tau)}d\sigma + o\left(\frac{1}{\tau}\right) \text{ as } |\tau| \rightarrow +\infty.
\end{aligned}$$

By (4.39), (4.38) and the stationary phase argument

$$((\nu_1 + i\nu_2)(B_1 - B_2)d_\tau e^{\mathcal{B}_1 + \tau\Phi}, V_1 + V_2)_{L^2(\partial\Omega)} = o\left(\frac{1}{\tau}\right) \text{ as } |\tau| \rightarrow +\infty.$$

Therefore, applying Proposition 3.4, we finally conclude that

$$\begin{aligned}
\mathfrak{L}_3 &= - 2\left(\frac{\partial}{\partial\bar{z}}(B_1 - B_2)d_\tau e^{\mathcal{B}_1}, \frac{e_1 g_3}{2\tau\partial_z\Phi}\right)_{L^2(\Omega)} \\
&- ( (B_1 - B_2)A_1 a_\tau e^{\mathcal{A}_1}, \frac{e_1 g_4}{2\tau\partial_z\Phi})_{L^2(\Omega)} \\
&+ ( (B_1 - B_2)d_\tau e^{\mathcal{B}_1}, -e_1 g_3 - \frac{\bar{A}_2 e_1 g_3}{2\tau\partial_z\Phi})_{L^2(\Omega)} \\
&- \frac{e^{-2i\tau\psi(\bar{x})}}{\tau|\det\psi''(\bar{x})|^{\frac{1}{2}}}\left\{\overline{\frac{\partial g_4(\bar{x})}{\partial z}}\right\}e^{-\bar{B}_2(\bar{x})}\int_{\Omega}\frac{(B_1 - B_2)de^{\mathcal{B}_1 + \bar{B}_2}}{\bar{z} - z}dx \\
(8.32) \quad &+ 2\int_{\partial\Omega}(\nu_1 - i\nu_2)e^{\mathcal{B}_1 + \bar{B}_2}d_\tau(\bar{z})\overline{\mathfrak{G}_4(x, \tau)}d\sigma + o\left(\frac{1}{\tau}\right) \text{ as } |\tau| \rightarrow +\infty.
\end{aligned}$$

The sum  $\sum_{k=0}^3 \mathfrak{L}_k$  is equal to the left hand side of (5.19). Observe that

$$\begin{aligned}
& - \frac{e^{-2i\tau\psi(\tilde{x})}}{\tau|\det\psi''(\tilde{x})|^{\frac{1}{2}}} \overline{\left\{ \frac{\partial g_4(\tilde{x})}{\partial z} \right\}} e^{-\overline{\mathcal{B}_2}(\tilde{x})} \int_{\Omega} \frac{(B_1 - B_2)de^{\mathcal{B}_1 + \overline{\mathcal{B}_2}}}{\tilde{z} - \bar{z}} dx \\
& - \frac{e^{2i\tau\psi(\tilde{x})}}{\tau|\det\psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\overline{\partial g_3(\tilde{x})}}{\partial \bar{z}} e^{-\overline{\mathcal{A}_2}(\tilde{x})} \int_{\Omega} \frac{(A_1 - A_2)ae^{\mathcal{A}_1 + \overline{\mathcal{A}_2}}}{\tilde{z} - z} dx \\
& + \frac{e^{2i\tau\psi(\tilde{x})}}{\tau|\det\psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_2(\tilde{x})}{\partial \bar{z}} e^{-\mathcal{B}_1(\tilde{x})} \int_{\Omega} \frac{(B_1 - B_2)\bar{b}e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}}}{\tilde{z} - \bar{z}} dx \\
& + \frac{e^{-2i\tau\psi(\tilde{x})}}{\tau|\det\psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_1(\tilde{x})}{\partial z} e^{-\mathcal{A}_1(\tilde{x})} \int_{\Omega} \frac{(A_1 - A_2)\bar{c}e^{\mathcal{A}_1 + \overline{\mathcal{A}_2}}}{\tilde{z} - z} dx = \\
& \frac{e^{-2i\tau\psi(\tilde{x})}}{\tau|\det\psi''(\tilde{x})|^{\frac{1}{2}}} \overline{\left\{ \frac{\partial g_4(\tilde{x})}{\partial z} \right\}} e^{-\overline{\mathcal{B}_2}(\tilde{x})} \int_{\partial\Omega} \frac{(\nu_1 - i\nu_2)de^{\mathcal{B}_1 + \overline{\mathcal{B}_2}}}{\tilde{z} - \bar{z}} d\sigma \\
& - \frac{e^{-2i\tau\psi(\tilde{x})}}{\tau|\det\psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_1(\tilde{x})}{\partial z} e^{-\mathcal{A}_1(\tilde{x})} \int_{\partial\Omega} \frac{(\nu_1 + i\nu_2)\bar{c}e^{\mathcal{A}_1 + \overline{\mathcal{A}_2}}}{\tilde{z} - z} d\sigma \\
& + \frac{e^{2i\tau\psi(\tilde{x})}}{\tau|\det\psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\overline{\partial g_3(\tilde{x})}}{\partial \bar{z}} e^{-\overline{\mathcal{A}_2}(\tilde{x})} \int_{\partial\Omega} \frac{(\nu_1 + i\nu_2)ae^{\mathcal{A}_1 + \overline{\mathcal{A}_2}}}{\tilde{z} - z} d\sigma \\
& - \frac{e^{2i\tau\psi(\tilde{x})}}{\tau|\det\psi''(\tilde{x})|^{\frac{1}{2}}} \frac{\partial g_2(\tilde{x})}{\partial \bar{z}} e^{-\mathcal{B}_1(\tilde{x})} \int_{\partial\Omega} \frac{(\nu_1 - i\nu_2)\bar{b}e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}}}{\tilde{z} - \bar{z}} d\sigma \\
(8.33) \quad & - \frac{2\pi(\mathcal{Q}_+ \bar{a} \bar{b} e^{\mathcal{A}_1 + \overline{\mathcal{B}_2} + 2i\tau\psi} + \mathcal{Q}_- \bar{c} \bar{d} e^{\mathcal{B}_1 + \overline{\mathcal{A}_2} - 2i\tau\psi})(\tilde{x})}{\tau|\det\psi''(\tilde{x})|^{\frac{1}{2}}}
\end{aligned}$$

By (8.17), (8.23), (8.29) and (8.32), (8.33) there exist numbers  $\kappa, \kappa_0$  such that the asymptotic formula (5.19) holds true.  $\square$

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DEPARTMENT OF MATHEMATICS, COLORADO STATE UNIVERSITY, 101 WEBER BUILDING, FORT COLLINS  
CO, 80523 USA, E-MAIL: OLEG@MATH.COLOSTATE.EDU

DEPARTMENT MATHEMATICS, UC IRVINE, IRVINE CA 92697, DEPARTMENT OF MATHEMATICS, UNI-  
VERSITY OF WASHINGTON, SEATTLE, WA 98195 USA, E-MAIL: GUNTHER@MATH.WASHINGTON.EDU

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF TOKYO, KOMABA, MEGURO, TOKYO  
153, JAPAN, E-MAIL: MYAMA@MS.U-TOKYO.AC.JP