INVERSE BOUNDARY VALUE PROBLEM BY MEASURING DIRICHLET DATA AND NEUMANN DATA ON DISJOINT SETS

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Abstract. We discuss the inverse boundary value problem of determining the conductivity in two dimensions from the pair of all input Dirichlet data supported on an open subset Γ_+ and all the corresponding Neumann data measured on an open subset Γ_-. We prove the global uniqueness under some additional geometric condition, in the case where $\overline{\Gamma_+ \cap \Gamma_-} = \emptyset$, and we prove also the uniqueness for a similar inverse problem for the stationary Schrödinger equation.

The key of the proof is the construction of appropriate complex geometrical optics solutions using Carleman estimates with a singular weight.

1. Introduction

In a bounded simply connected domain $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial\Omega$, we consider

$$\begin{aligned} \operatorname{div}(\gamma\nabla u) &= 0 \text{ in } \Omega, \\ u\big|_{\partial\Omega} &= f, \end{aligned} \tag{1.1}$$

where a positive function γ on $\overline{\Omega}$ models the electrical conductivity, $u \in H^1(\Omega)$ is a potential, and $f \in H^{\frac{1}{2}}(\partial\Omega)$ is a given boundary voltage potential and is regarded as input. The inverse problem of determining γ by all Cauchy data is called also the Calderón problem ([4]). It is well known ([9]) that $\gamma \in C^2(\overline{\Omega})$ is uniquely determined by the set of all Cauchy data f and $\frac{\partial u}{\partial \nu}$ on the whole boundary:

$$\left(u|_{\partial\Omega}, \gamma\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} \right).$$

The regularity condition C^2 on γ was relaxed in [2] and [1]. In particular in [1] the uniqueness was shown for arbitrary L^∞ conductivities. Since f is an input and $\frac{\partial u}{\partial \nu}$ is the corresponding output, it is practically desirable that we take inputs and outputs on subboundaries, not on the whole boundary $\partial\Omega$. Let $\partial\Omega = \overline{\Gamma_- \cup \Gamma_+ \cup \Gamma_0}$ where $\Gamma_- \cap \Gamma_+ = \Gamma_0 \cap \Gamma_\pm = \emptyset$. Then it is important to discuss the uniqueness by all pairs of Dirichlet data on subboundary Γ_+ and the corresponding Neumann data on subboundary Γ_-:

$$\mathcal{A}_\gamma = \left\{ \left(u|_{\Gamma_+}, \gamma\frac{\partial u}{\partial \nu}\Big|_{\Gamma_-} \right); \operatorname{div}(\gamma\nabla u) = 0 \text{ in } \Omega, \quad u\big|_{\Gamma_0\cup\Gamma_-} = 0, \quad u|_{\Gamma_+} = f \right\}. \tag{1.2}$$

We consider that the input is located on Γ_+, while the output is measured on Γ_-. In the case where $\Gamma_+ = \Gamma_-$ and is an arbitrary open subset of the boundary, the global uniqueness was shown in [7] within $\gamma \in C^{3+\alpha}(\overline{\Omega})$, with some $\alpha \in (0, 1)$.

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The result in [7] is the best possible within $\Gamma_+ = \Gamma_-$, but it is still desirable to exploit the case $\Gamma_+ \cap \Gamma_- = \emptyset$, because the input subboundary Γ_+ and the output subboundary Γ_- should be separated from the practical viewpoint and one needs the uniqueness supporting such practical configurations for determining a conductivity. To the best knowledge of the authors, there are no publications discussing disjoint subboundaries. The main purpose of this article is to establish the global uniqueness by the Cauchy data defined by (1.2) with $\Gamma_+ \cap \Gamma_- = \emptyset$.

For the statement of the main result, we need the following geometric assumption on the position of the sets $\Gamma_+, \Gamma_-, \Gamma_0$ on $\partial\Omega$.

Assumption A. *Let Γ_+, Γ_-, $\Gamma_0 \subset \partial\Omega$ be non-empty open subsets of the boundary such that $\partial\Omega = \overline{\Gamma_+ \cup \Gamma_- \cup \Gamma_0}$, $\Gamma_+ \cap \Gamma_- = \Gamma_\pm \cap \Gamma_0 = \emptyset$, $\Gamma_\pm = \cup_{j=1}^2 \Gamma_{\pm,j}$, $\Gamma_0 = \cup_{k=1}^4 \Gamma_{0,k}$, where $\Gamma_{\pm,j}$, $j = 1,2$, $\Gamma_{0,k}$, $k = 1,2,3,4$ are not empty open connected subsets of $\partial\Omega$ and mutually disjoint. Then $\partial\Omega$ is separated into*

$$\Gamma_{0,1}, \Gamma_{-,1}, \Gamma_{0,2}, \Gamma_{+,1}, \Gamma_{0,3}, \Gamma_{-,2}, \Gamma_{0,4}, \Gamma_{+,2}$$

with clockwise order.

We note that Γ_+, Γ_- can be arbitrarily small provided that the above separation condition is satisfied.

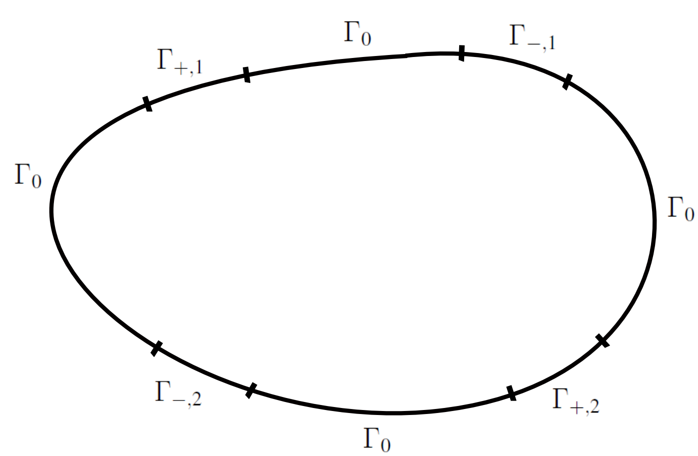

FIGURE 1

Now we are ready to state our main result:

Theorem 1.1. *We suppose Assumption A. Let $\gamma_j > 0$ on $\overline{\Omega}$ and $\gamma_j \in C^{4+\alpha}(\overline{\Omega})$, $j = 1,2$ for some $\alpha > 0$. Assume $\mathcal{A}_{\gamma_1} = \mathcal{A}_{\gamma_2}$ and that $\gamma_1 = \gamma_2$ on $\Gamma_+ \cup \Gamma_-$. Then $\gamma_1 \equiv \gamma_2$ on $\overline{\Omega}$.*

Next for the Schrödinger equation

$$\Delta u + qu = 0 \quad \text{in } \Omega,$$

we discuss the problem of determining a complex-valued potential q by the set of Cauchy data:

$$\mathcal{C}_q = \left\{ \left(u|_{\Gamma_+}, \frac{\partial u}{\partial \nu}\Big|_{\Gamma_-} \right); \, (\Delta + q)u = 0 \,\text{in}\, \Omega, \, u|_{\Gamma_0 \cup \Gamma_-} = 0, \ u \in H^1(\Omega) \right\}. \tag{1.3}$$

Here, when the voltage is applied on Γ_+, the current is measured on subboundary Γ_-.

In the case of full Cauchy data: $\Gamma_+ = \Gamma_- = \partial\Omega$, the uniqueness in determining the potential q in the two dimensional case was initially proved under some restrictions on the potential q in [9], [10], [11]. Recently [3] removed these restrictions for the case of full Cauchy data.

In the case where $\Gamma_+ = \Gamma_-$ and is an arbitrary open subset of the boundary, [7] showed that the potential q can be uniquely determined.

We state our second main result for the Schrödinger equation in a case where $\Gamma_+ \cap \Gamma_- = \emptyset$:

Theorem 1.2. *We suppose Assumption A. Let $q_j \in C^{2+\alpha}(\overline{\Omega})$, $j = 1, 2$ for some $\alpha > 0$ and let q_j be complex-valued. Then if*

$$\mathcal{C}_{q_1} = \mathcal{C}_{q_2},$$

then we have

$$q_1 \equiv q_2 \quad in \ \Omega.$$

In terms of $\gamma_1 = \gamma_2$ on $\Gamma_+ \cup \Gamma_-$, the proof of Theorem 1.1 is reduced to Theorem 1.2 because the change of variables

$$q = -\frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}, \quad u^* = u\sqrt{\gamma}$$

reduces the conductivity equation in u to the Schrödinger equation in u. Therefore we mainly prove Theorem 1.2. A brief outline of the paper is as follows. In section 2 we show some preliminary results and estimates needed in the construction of the appropriate complex geometrical optics solutions. In section 3 we construct these solutions. In section 4 we prove Theorem 2.

2. **Preliminary results**

Throughout the paper we use the following notations.

Notations. $i = \sqrt{-1}$, $x_1, x_2, \xi_1, \xi_2 \in \mathbb{R}^1$, $z = x_1 + ix_2$, $\zeta = \xi_1 + i\xi_2$, $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. We identify $x = (x_1, x_2) \in \mathbb{R}^2$ with $z = x_1 + ix_2 \in \mathbb{C}$. $\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$, $\partial_{\overline{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$. The tangential derivative on the boundary is given by $\partial_{\vec{\tau}} = \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2}$, where $\nu = (\nu_1, \nu_2)$ is the unit outer normal to $\partial\Omega$, $B(\widehat{x}, \delta) = \{x \in \mathbb{R}^2; \, |x - \widehat{x}| < \delta\}$. For $f : \mathbb{R}^2 \to \mathbb{R}^1$, f'' is the Hessian matrix with entries $\frac{\partial^2 f}{\partial x_k \partial x_j}$, $\mathcal{L}(X, Y)$ denotes the Banach space of all bounded linear operators from a Banach space X to another Banach space Y. We set $[P, Q] = PQ - QP$ for operators P and Q.

Let $\Phi(z) = \varphi(x_1, x_2) + i\psi(x_1, x_2) \in C^2(\overline{\Omega})$ be a holomorphic function in Ω with real-valued φ and ψ:

$$\partial_{\overline{z}}\Phi(z) = 0 \quad \text{in } \Omega. \tag{2.1}$$

Denote by $\mathcal{H}$ the set of critical points of the function Φ:

$$\mathcal{H} = \{z \in \overline{\Omega};\ \partial_z\Phi(z) = 0\}.$$

Assume that Φ has no critical points on $\overline{\Gamma_+} \cup \overline{\Gamma_-}$, and that all the critical points are nondegenerate:

$$\mathcal{H} \cap \partial\Omega = \{\emptyset\}, \quad \partial_z^2\Phi(z) \neq 0, \quad \forall z \in \mathcal{H}. \tag{2.2}$$

Then we know that Φ has only a finite number of critical points which

$$\mathcal{H} = \{\widetilde{x}_1, ..., \widetilde{x}_\ell\}.$$

Assume that Φ satisfies

$$\Gamma_0 = \{x \in \partial\Omega;\ (\nu, \nabla\varphi) = 0\}, \quad \Gamma_- = \{x \in \partial\Omega;\ (\nu, \nabla\varphi) < 0\}. \tag{2.3}$$

Consider the boundary value problem

$$\begin{cases} L(x, D)u = \Delta u + qu = f & \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

For this problem we have the following Carleman estimate with boundary terms.

Proposition 2.1. *Suppose that Φ satisfies (2.1) - (2.3) and $u \in H_0^1(\Omega)$, $q \in L^\infty(\Omega)$. Then there exist $\tau_0 = \tau_0(L, \Phi)$ and $C_1 = C_1(L, \Phi)$, independent of u and τ, such that for all $|\tau| > \tau_0$*

$$|\tau| \|ue^{\tau\varphi}\|^2_{L^2(\Omega)} + \|ue^{\tau\varphi}\|^2_{H^1(\Omega)} + \|\frac{\partial u}{\partial \nu}e^{\tau\varphi}\|^2_{L^2(\Gamma_0\cup\Gamma_-)} + \tau^2 \left\| \left| \frac{\partial\Phi}{\partial z}\right| ue^{\tau\varphi}\right\|^2_{L^2(\Omega)}$$
$$\leq C_1 \left(\|(L(x, D)u)e^{\tau\varphi}\|^2_{L^2(\Omega)} + |\tau| \int_{\Gamma_+} \left|\frac{\partial u}{\partial \nu}\right|^2 e^{2\tau\varphi} d\sigma \right). \tag{2.4}$$

Let us introduce the operators:

$$\partial_{\overline{z}}^{-1} g = \frac{1}{2\pi i}\int_\Omega \frac{g(\zeta, \overline{\zeta})}{\zeta - z} d\zeta \wedge d\overline{\zeta} = -\frac{1}{\pi}\int_\Omega \frac{g(\zeta, \overline{\zeta})}{\zeta - z} d\xi_2 d\xi_1,$$
$$\partial_z^{-1} g = -\frac{1}{2\pi i}\overline{\int_\Omega \frac{\overline{g}(\zeta, \overline{\zeta})}{\zeta - z} d\zeta \wedge d\overline{\zeta}} = -\frac{1}{\pi}\int_\Omega \frac{g(\zeta, \overline{\zeta})}{\overline{\zeta} - \overline{z}} d\xi_2 d\xi_1 = \overline{\partial_{\overline{z}}^{-1}\overline{g}}.$$

Then we have (e.g., p.47 and p.56 in [12]):

Proposition 2.2. A) *Let $m \geq 0$ be an integer number and $\alpha \in (0, 1)$. Then $\partial_{\overline{z}}^{-1}, \partial_z^{-1} \in \mathcal{L}(C^{m+\alpha}(\overline{\Omega}), C^{m+\alpha+1}(\overline{\Omega}))$.*
B) *Let $1 \leq p \leq 2$ and $1 < \gamma < \frac{2p}{2-p}$. Then $\partial_{\overline{z}}^{-1}, \partial_z^{-1} \in \mathcal{L}(L^p(\Omega), L^\gamma(\Omega))$.*

We define two other operators:

$$R_{\Phi,\tau} g = e^{\tau(\overline{\Phi}-\Phi)}\partial_{\overline{z}}^{-1}(ge^{\tau(\Phi-\overline{\Phi})}), \quad \widetilde{R}_{\Phi,\tau} g = e^{\tau(\overline{\Phi}-\Phi)}\partial_z^{-1}(ge^{\tau(\Phi-\overline{\Phi})}).$$

In [8] we prove the following:

Proposition 2.3. *Let $g \in C^\alpha(\overline{\Omega})$ for some positive α. The function $R_{\Phi,\tau}g$ is a solution to*

$$\partial_{\overline{z}} R_{\Phi,\tau}g - \tau(\overline{\partial_z \Phi})R_{\Phi,\tau}g = g \quad in\ \Omega. \tag{2.5}$$

The function $\widetilde{R}_{\Phi,\tau}g$ solves

$$\partial_z \widetilde{R}_{\Phi,\tau}g + \tau(\partial_z \Phi)\widetilde{R}_{\Phi,\tau}g = g \quad in\ \Omega. \tag{2.6}$$

Using the stationary phase argument we show

Proposition 2.4. *Let $g \in L^1(\Omega)$ and the function Φ satisfy (2.1), (2.2). Then*

$$lim_{|\tau|\to+\infty} \int_\Omega g e^{\tau(\Phi(z)-\overline{\Phi(z)})}dx = 0.$$

Denote

$$\mathcal{O}_\epsilon = \{x \in \Omega;\ dist(x, \partial\Omega) \le \epsilon\}.$$

We have

Proposition 2.5. *Let $\alpha > 0$, $g \in C^{2+\alpha}(\Omega), g|_{\mathcal{O}_\epsilon} = 0$ and $g|_{\mathcal{H}} = 0$. Then*

$$\left\| R_{\Phi,\tau}g + \frac{g}{\tau\overline{\partial_z\Phi}} \right\|_{L^2(\Omega)} + \left\| \widetilde{R}_{\Phi,\tau}g - \frac{g}{\tau\partial_z\Phi} \right\|_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right) \quad as\ |\tau| \to +\infty. \tag{2.7}$$

Consider the following problem

$$L(x,D)u = fe^{\tau\varphi} \quad \text{in } \Omega, \quad u|_{\Gamma_0\cup\Gamma_-} = ge^{\tau\varphi}. \tag{2.8}$$

We have the following Carleman estimate.

Proposition 2.6. *(see [8]) Let $q \in L^\infty(\Omega)$. There exists $\tau_0 > 0$ such that for all $\tau > \tau_0$ there exists a solution to the boundary value problem (2.8) such that*

$$\frac{1}{\sqrt{|\tau|}}\|\nabla ue^{-\tau\varphi}\|_{L^2(\Omega)} + \sqrt{|\tau|}\|ue^{-\tau\varphi}\|_{L^2(\Omega)} \le C_2(\|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2},\tau}(\Gamma_0)}). \tag{2.9}$$

Let ϵ be a sufficiently small positive number. If $supp f \subset G_\epsilon = \{x \in \Omega;\ dist(x, \mathcal{H}) > \epsilon\}$ and $g = 0$ then there exists $\tau_0 > 0$ such that for all $\tau > \tau_0$ there exists a solution to the boundary value problem (2.8) such that

$$\|\nabla ue^{-\tau\varphi}\|_{L^2(\Omega)} + |\tau|\|ue^{-\tau\varphi}\|_{L^2(\Omega)} \le C_3(\epsilon)\|f\|_{L^2(\Omega)}. \tag{2.10}$$

We have

Proposition 2.7. *Let $q \in L^\infty(\Omega)$, and let $supp\, g \subset \Gamma_-$ and $g/\sqrt{|\partial_\nu\varphi|} \in L^2(\Gamma_-)$. Then there exists $\tau_0 > 0$ such that for all $\tau > \tau_0$ there exists a solution to (2.8) such that*

$$\sqrt{|\tau|}\|ue^{-\tau\varphi}\|_{L^2(\Omega)} \le C_4\|g/\sqrt{|\partial_\nu\varphi|}\|_{L^2(\Gamma_-)}.$$

For completeness, we give the proof of Proposition 2.7 in Appendix.

3. Complex geometrical optics solutions

In this section, we construct complex geometrical optics solutions for the Schrödinger equation $\Delta + q_j$ with q_j satisfying the conditions of Theorem 1.2. Consider

$$L_1(x,D)u = \Delta u + q_1 u = 0 \quad \text{in } \Omega. \tag{3.1}$$

We will construct solutions to (3.1) of the form

$$u_1(x) = e^{\tau\Phi(z)}(a(z)+a_0(z)/\tau)+e^{\tau\overline{\Phi(z)}}\overline{(a(z)+a_1(z)/\tau)}+e^{\tau\varphi}u_-+e^{\tau\varphi}u_{11}+e^{\tau\varphi}u_{12}, \quad u_1|_{\Gamma_0\cup\Gamma_-}=0. \tag{3.2}$$

Thanks to Assumption A, the set Γ_0 consists of four arcs : $\Gamma_0 = \Gamma_{0,1}\cup\Gamma_{0,2}\cup\Gamma_{0,3}\cup\Gamma_{0,4}$, the set Γ_- consists of two arcs $\Gamma_- = \Gamma_{-,1}\cup\Gamma_{-,2}$ and the set Γ_+ also consists of two arcs $\Gamma_+ = \Gamma_{+,1}\cup\Gamma_{+,2}$. Denote the endpoints of the arc $\Gamma_{0,j}$ as $\hat{x}_{j,\pm}$. Henceforth let a sufficiently large $m\in\mathbb{N}$ be fixed (e.g., $m = 100$).

Proposition 3.1. *Let $\tilde{x}\in\Omega$ be an arbitrary point. There exists a smooth holomorphic function a in Ω such that*

$$a(\tilde{x})\neq 0, \quad Re\, a|_{\Gamma_0}=0, \quad \nabla^k a(\hat{x}_{j,\pm})=0 \quad \forall k\in\{1,\dots,m\}, \quad \forall j\in\{0,\dots,4\}.$$

Proof. Consider the following linear operator

$$\mathcal{R}(v) = (w(\tilde{x}), w(\hat{x}_{j,\pm}),\dots,\partial_z^m w(\hat{x}_{j,\pm})),$$

where

$$\partial_{\bar{z}} w = 0 \quad \text{in } \Omega, \quad \mathrm{Re}\, w = v \quad \text{on } \partial\Omega, \quad supp\, v\subset\Gamma_+.$$

Clearly the image of the operator $\mathcal{R}$ is closed. Let $b(x)$ be a holomorphic function in Ω such that $b(\tilde{x})=1$ and $\mathrm{Re}\, b|_{\Gamma_0\cup\Gamma_-}=0$. By Proposition 5.1 in Appendix, there exists a sequence of holomorphic functions $\{w_k\}_{k=1}^\infty\subset C^{m+\alpha}(\bar{\Omega})$ such that

$$w_k\to 0+i\mathrm{Im}\, b \quad \text{in } C^{m+\alpha}(\Gamma_0\cup\Gamma_-) \quad \text{and} \quad w_k(\tilde{x})\to 0.$$

Using classical results on solvability of the Cauchy-Riemann equations, we construct a sequence of holomorphic functions $\tilde{w}_k$ such that

$$\tilde{w}_k\to 0 \quad \text{in } C^{m+\alpha}(\bar{\Omega}), \quad \mathrm{Re}\,\tilde{w}_k = \mathrm{Re}\, w_k \quad \text{on } \Gamma_0\cup\Gamma_-.$$

Consider the sequence $v_k = b + (\tilde{w}_k - w_k)$. We have $\mathcal{R}(v_k)\to(1,0,\dots,0)$. The proof of the proposition is completed. □

3.1. Construction of the phase function

Without loss of generality, using some conformal mapping if necessary, we may assume that Γ_- and Γ_+ are part of the line $\{x_2=0\}$ and the domain Ω itself is located below the line $x_2=0$.

We construct a holomorphic function Φ with domain $\Omega_\Phi\supset\Omega$ satisfying (2.1), (2.2) and

$$\mathrm{Im}\,\Phi|_{\Gamma_0}=0, \quad \frac{\partial\mathrm{Re}\,\Phi}{\partial\nu}|_{\Gamma_-}<0, \quad \frac{\partial\mathrm{Re}\,\Phi}{\partial\nu}|_{\Gamma_+}>0. \tag{3.3}$$

Here the domain Ω_Φ with sufficiently smooth boundary $\partial\Omega_\Phi$ can be chosen to satisfy:

$$\Omega \subset \Omega_\Phi, \quad \Gamma_0 \subset \partial\Omega_\Phi, \quad (\Gamma_+ \cup \Gamma_-) \cap \partial\Omega_\Phi = \emptyset. \tag{3.4}$$

Therefore, thanks to Assumption A, the set $\partial\Omega_\Phi \setminus \partial\Omega$ consists of four disconnected curves which we denote as $\Gamma_{\Phi,1}, \Gamma_{\Phi,2}, \Gamma_{\Phi,3}, \Gamma_{\Phi,4}$. Counting clockwise, we assume that $\Gamma_{\Phi,1}$ is located between $\Gamma_{0,1}$ and $\Gamma_{0,2}$, $\Gamma_{\Phi,2}$ located between $\Gamma_{0,2}$ and $\Gamma_{0,3}$, $\Gamma_{\Phi,3}$ located between $\Gamma_{0,3}$ and $\Gamma_{0,4}$, $\Gamma_{\Phi,4}$ located between $\Gamma_{0,4}$ and $\Gamma_{0,1}$. Assume in addition that each component $\Gamma_{\Phi,k}$ can be parameterize by the function $\tilde\gamma_k \in C^{12}[\hat x_{k,+}, \hat x_{k+1,-}]$, where $\hat x_{k,+}, \hat x_{k+1,-}$ are the endpoints of the arcs $\Gamma_{0,k}$ and $\hat x_{5,-} = \hat x_{1,-}$.

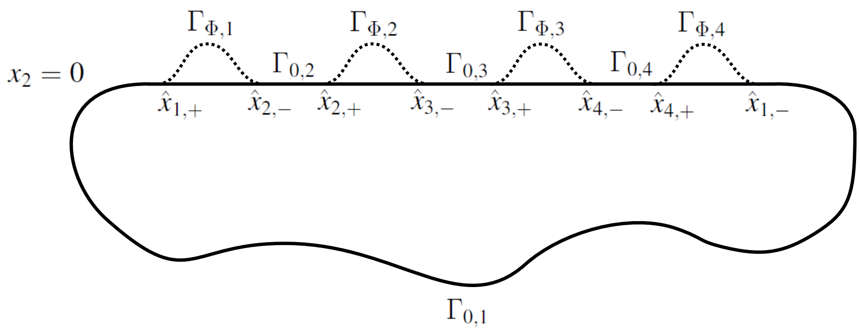

FIGURE 2

Let us start the construction of the function Φ. Consider the functions γ_j with domain $\mathbb{R}^1$ such that γ_j is positive on $(\hat x_{j,+}, \hat x_{j+1,-})$, otherwise γ_j is zero. Moreover we require that

$$\frac{d^k\gamma_j}{dt^k}(\hat x_{j,+}) = \frac{d^k\gamma_j}{dt^k}(\hat x_{j+1,-}) = 0 \quad \forall k \in \{0,\dots,10\}, \qquad \frac{d^{11}\gamma_j}{dt^{11}}(\hat x_{j,+}) \neq 0, \qquad \frac{d^{11}\gamma_j}{dt^{11}}(\hat x_{j+1,-}) \neq 0.$$

There exists some small positive $\hat\epsilon$ such that

$$\gamma_j(x_1) = (x_1 - \hat x_{j,+})^{11} \quad \forall x_1 \in (\hat x_{j,+}, \hat x_{j,+} + \hat\epsilon), \quad \gamma_j(x_1) = (\hat x_{j+1,-} - x_1)^{11} \quad \forall x_1 \in (\hat x_{j+1,-} - \hat\epsilon, \hat x_{j+1,-}). \tag{3.5}$$

We introduce the domain Ω_δ for any small positive δ as follows. From below it is bounded by the boundary of $\partial\Omega$ and from above by segments $\Gamma_{0,k}$ and the graphs of $\delta\gamma_j$.

By ν_δ we denote the outward unit normal derivative to $\partial\Omega_\delta$ and by $\vec\tau_\delta$ we denote the clockwise unit tangential vector on $\partial\Omega_\delta$. We set

$$\Gamma_{\delta,k} = \{(x_1, \delta\gamma_k(x_1)) | x_1 \in [x_{k,+}, x_{k+1,-}]\}.$$

Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ be rational positive numbers:

$$\mathcal{C}_k = \frac{m_k}{n_k} \quad m_k, n_k \in \mathbb{N},\ k = 1,2,3,4, \tag{3.6}$$

where the greatest common divisor of $m_k, n_k \in \mathbb{Z}$ is 1 and $\tilde\psi$ be a harmonic function in Ω such that $\tilde\psi$ is continuous on $\bar\Omega$ and

$$\begin{cases} \tilde\psi = \mathcal{C}_1 \quad \text{on } \Gamma_{0,1}, \quad \tilde\psi = \mathcal{C}_3 \quad \text{on } \Gamma_{0,3}; \\ \tilde\psi = -\mathcal{C}_2 \quad \text{on } \Gamma_{0,2}, \quad \tilde\psi = -\mathcal{C}_4 \quad \text{on } \Gamma_{0,4}; \\ \partial_{\vec\tau}\tilde\psi < 0 \quad \text{on } (\hat x_{1,+}, \hat x_{2,-}) \cup (\hat x_{3,+}, \hat x_{4,-}); \\ \partial_{\vec\tau}\tilde\psi > 0 \quad \text{on } (\hat x_{2,+}, \hat x_{3,-}) \cup (\hat x_{4,+}, \hat x_{1,-}); \\ \tilde\psi \in C^5(\partial\Omega), \quad \tilde\psi \in C^\infty(\partial\Omega \setminus \cup_{k=1}^4 \Gamma_{0,k}). \end{cases} \tag{3.7}$$

Moreover we assume that

$$\begin{array}{ll} \lim_{x_1\to\hat x_{k,+}+0}\partial_{x_1}\tilde\psi(x_1,0)/(\hat x_{k,+}-x_1)^6 < 0 & k=1,3, \\ \lim_{x_1\to\hat x_{k,+}+0}\partial_{x_1}\tilde\psi(x_1,0)/(\hat x_{k,+}-x_1)^6 > 0 & k=2,4, \\ \lim_{x_1\to\hat x_{k,-}+0}\partial_{x_1}\tilde\psi(x_1,0)/(\hat x_{k,-}-x_1)^6 < 0 & k=2,4, \\ \lim_{x_1\to\hat x_{k,-}-0}\partial_{x_1}\tilde\psi(x_1,0)/(\hat x_{k,-}-x_1)^6 > 0 & k=1,3. \end{array}$$

Let function ψ_δ be the harmonic function in Ω_δ such that for any $j \in \{1,2,3,4\}$

$$\psi_\delta = \tilde\psi \quad \text{on } \cup_{k=1}^4 \Gamma_{0,k}, \quad \tilde\psi_\delta(x_1, \delta\gamma_j(x_1)) = \tilde\psi(x_1,0) \quad \text{on } [\hat x_{j,+}, \hat x_{j+1,-}]. \tag{3.8}$$

For all sufficiently small δ, by (3.7), counting clockwise, the function ψ_δ is monotone decreasing on the arcs between $\Gamma_{0,1}$ and $\Gamma_{0,2}$, and $\Gamma_{0,3}$ and $\Gamma_{0,4}$, and ψ_δ is monotone increasing on the arcs between $\Gamma_{0,2}$ and $\Gamma_{0,3}$, and between $\Gamma_{0,4}$ and $\Gamma_{0,1}$. Once the function ψ_δ is constructed, using the Cauchy-Riemann equations, we construct the function φ_δ such that the function $\varphi_\delta + i\psi_\delta$ is holomorphic. The following inequalities are true for all sufficiently small positive δ

$$\frac{\partial\varphi_\delta}{\partial\nu_\delta}|_{\Gamma_{\delta,1}\cup\Gamma_{\delta,3}} < 0, \quad \frac{\partial\varphi_\delta}{\partial\nu_\delta}|_{\Gamma_{\delta,2}\cup\Gamma_{\delta,4}} > 0, \tag{3.9}$$

$$\begin{gathered} \lim_{x_1\to\hat x_{k,+}+0}\frac{\partial\varphi_\delta}{\partial\nu_\delta}(x_1,\delta\gamma_k(x_1))/(\hat x_{k,+}-x_1)^6 > 0 \; k=1,3, \\ \lim_{x_1\to\hat x_{k,+}-0}\frac{\partial\varphi_\delta}{\partial\nu_\delta}(x_1,\delta\gamma_{k-1}(x_1))/(\hat x_{k,-}-x_1)^6 > 0 \; k=2,4. \end{gathered} \tag{3.10}$$

$$\begin{gathered} \lim_{x_1\to\hat x_{k,+}+0}\frac{\partial\varphi_\delta}{\partial\nu_\delta}(x_1,\delta\gamma_{k+1}(x_1))/(\hat x_{k,+}-x_1)^6 < 0 \; k=2,4, \\ \lim_{x_1\to\hat x_{k,+}-0}\frac{\partial\varphi_\delta}{\partial\nu_\delta}(x_1,\delta\gamma_{5-k}(x_1))/(\hat x_{k,-}-x_1)^6 < 0 \; k=1,3. \end{gathered} \tag{3.11}$$

At the endpoints of $\Gamma_{0,2}$ and $\Gamma_{0,4}$, the function ψ_δ reaches its minimum and at the endpoints of $\Gamma_{0,1}$ and $\Gamma_{0,3}$, the function ψ_δ reaches its maximum. By (3.8) we have

$$(\varphi_\delta, \psi_\delta) \to (\tilde\varphi, \tilde\psi) \quad \text{in } C^2(\bar\Omega) \quad \text{as } \delta \to +0. \tag{3.12}$$

Here $\tilde\varphi$ is a harmonic function in Ω such that $\partial_{\bar z}(\tilde\varphi + i\tilde\psi) \equiv 0$.

By (3.9)-(3.11) for all sufficiently small positive δ, the holomorphic function $\varphi_\delta + i\psi_\delta$ satisfies (2.3).

Consider the domain

$$\mathcal{G}_- = \{(x_1,x_2);\, \hat x_{2,+} \le x_1 \le \hat x_{3,-}, -\delta\gamma_2(x_1) \le x_2 \le 0\}$$
$$\cup\{(x_1,x_2);\, \hat x_{4,+} \le x_1 \le \hat x_{1,-}, -\delta\gamma_4(x_1) \le x_2 \le 0\}.$$

We claim that there exists a positive constant C_δ such that

$$\varphi_\delta(x) - \varphi_\delta(x_1,-x_2) \ge C_\delta \ell(x) \quad \forall x \in \mathcal{G}_-,\ \frac{\partial \varphi_\delta}{\partial x_2}(x) \le -C_\delta \ell_1(x) \quad \forall x \in \mathcal{G}_-, \tag{3.13}$$

where $\ell(x) = \min_{y\in\{\hat x_{2,+},\hat x_{4,+}\,\hat x_{3,-},\hat x_{1,-}\}} |x_1-y|^7|x_2|$ and $\ell_1(x) = \min_{y\in\{\hat x_{2,+},\hat x_{4,+}\,\hat x_{3,-},\hat x_{1,-}\}} |x_1-y|^6$. Indeed, suppose that the second inequality in (3.13) fails for all small positive δ. By (3.9) and (3.12), this is possible only for a sequence of the points x_δ such that it converges to the set $\mathcal{D}_- = \{x_{1,+}, x_{2,-}, x_{3,+}, x_{4,-}\}$. Taking a subsequence if necessary, we may assume that x_δ converges to one point of the set $\mathcal{D}_-$. Let it be the point $\hat x_{2,+}$. By the Cauchy-Riemann equations, $\frac{\partial\varphi_\delta}{\partial\nu_\delta} = -\frac{\partial\psi_\delta}{\partial\vec\tau_\delta}$ for any point of $\partial\Omega_\delta$. Therefore by (3.10), there exist positive constants $\hat C$ and ϵ, independent of δ, such that

$$\frac{\partial\varphi_\delta}{\partial\nu_\delta} \le -\hat C(x_1-\hat x_{1,+})^6 \quad \text{on } \{x;\, x\in\Gamma_{\delta,2}, \mathrm{dist}(\hat x_{2,+},x) < \epsilon\}.$$

Taking into account that by (3.5) $\vec\nu_\delta = (8(x_1-\hat x_{2,+})^7,1)/(1+64(x_1-\hat x_{2,+})^{14})^{\frac12}$, we obtain

$$\frac{\partial\varphi_\delta}{\partial x_2}(x) \le -\frac{\hat C}{2}(x_1-\hat x_{2,+})^6 \quad \forall x\in\{(x_1,x_2);\, x_1\in[\hat x_{2,+},\hat x_{2,+}+\epsilon], x_2 = \delta\gamma_2(x_1)\}.$$

Using (3.5), (3.12) and the Taylor's formula for any $x \in \{(x_1,x_2);\, x_1 \in [\hat x_{2,+},\hat x_{2,+}+\epsilon], -\delta\gamma_2(x_1) \le x_2 \le \delta\gamma_2(x_1)\}$ we have

$$\begin{aligned}\frac{\partial\varphi_\delta}{\partial x_2}(x_1,x_2) &= \frac{\partial\varphi_\delta}{\partial x_2}(x_1,\delta\gamma_2(x_1)) + \frac{\partial^2\varphi_\delta}{\partial x_2^2}(x_1,\zeta)(x_2-\delta\gamma_2(x_1))\\ &\le -\frac{\hat C}{2}(x_1-\hat x_{2,+})^6 + 2C_5\gamma_2(x_1)\\ &= -\frac{\hat C}{4}(x_1-\hat x_{2,+})^6 + 2C(x_1-\hat x_{2,+})^{11} \le -\frac{C_6}{4}(x_1-\hat x_{2,+})^6.\end{aligned} \tag{3.14}$$

Therefore we complete the proof of the second inequality in (3.13)

Let $x\in\mathcal{G}_-$. Using (3.14) we have

$$\begin{aligned}&\varphi_\delta(x) - \varphi_\delta(x_1,-x_2) \le \varphi_\delta(x_1,0) - \varphi_\delta(x) =\\ &\int_{x_2}^0 \partial_\xi\varphi_\delta(x_1,\xi)d\xi \le -\frac{C_6}{2}\int_{x_2}^0 (x_1-\hat x_{2,+})^6 d\xi = -\frac{C_6}{2}(x_1-\hat x_{2,+})^6 x_2.\end{aligned} \tag{3.15}$$

The proof of (3.13) is completed.

Consider the domain

$$\mathcal{G}_+ = \{(x_1,x_2);\, \hat x_{2,+} \le x_1 \le \hat x_{3,-}, -\delta\gamma_2(x_1) \le x_2 \le 0\}$$
$$\cup\{(x_1,x_2);\, \hat x_{4,+} \le x_1 \le \hat x_{1,-}, -\delta\gamma_4(x_1) \le x_2 \le 0\}.$$

Similarly one can prove that for all sufficiently small positive δ, there exists a positive constant $\tilde{C}_\delta$ such that for any x in $\mathcal{G}_+$

$$\varphi_\delta(x) - \varphi_\delta(x_1, -x_2) \le -\tilde{C}_\delta \tilde{\ell}(x) \quad \text{and} \quad \frac{\partial \varphi_\delta}{\partial x_2}(x) \ge \tilde{C}_\delta \tilde{\ell}_1(x), \tag{3.16}$$

where $\tilde{\ell}(x) = \min_{y\in\{\hat{x}_{2,+},\hat{x}_{4,+},\hat{x}_{1,-},\hat{x}_{3,-}\}} |x_1 - y|^7 |x_2|$ and $\tilde{\ell}_1(x) = \min_{y\in\{\hat{x}_{2,+},\hat{x}_{4,+},\hat{x}_{1,-},\hat{x}_{3,-}\}} |x_1 - y|^6$. At this point we fix the parameter δ such that (3.13) and (3.16) are valid. The holomorphic function $\varphi_\delta + i\psi_\delta$ satisfies (2.3) and all the interior critical points (if they exist) are nondegenerate. This function may have some critical points in the set $\{\hat{x}_{j,\pm};\ j = 1, 2, 3, 4\}$. Let the tangential derivative of ψ_δ be not equal to zero on some open set $\tilde{\Gamma}$. By Corollary 5.1 in Appendix, there exists a harmonic function $\hat{\varphi} + i\hat{\psi}$ such that $\operatorname{Im}\hat{\psi} = 0$ on $\partial\Omega_\delta$ and $\frac{\partial\hat{\varphi}}{\partial\vec{\tau}}|_{\hat{x}_{j,\pm}}$ is not equal to zero for all j. Then the function $\varphi_\delta + \epsilon\hat{\varphi} + i(\psi_\delta + \hat{\psi})$ does not have critical points on the set $\{\hat{x}_{j,\pm};\ j = 1, 2, 3, 4\}$ for all small positive ϵ. In fact this function can not have more than one interior critical point. Indeed it is known (see e.g., [13]) that if $\hat{x}$ is the interior critical point of the harmonic function ψ, then the set $\{x \in \partial\Omega;\ \psi(x) = \psi(\hat{x})\}$ consists of at least four points. Moreover the set $\{x;\ \psi(x) = \psi(\hat{x})\}$ consists of two continuous curves intersecting at $\hat{x}$. These curves divide Ω into four domains: $\Omega = \cup_{k=1}^4 \Omega_k$. If there exists another interior critical point $\hat{x}_1$, then it belongs to some domain Ω_k. However in this case it is impossible that there exist four different points x_j from $\partial\Omega_k$ such that $\psi(\hat{x}_1) = \psi(x_j)$. The construction of of the weight function Φ is completed. If an interior critical point of Φ exists, then we denote it by $\tilde{x}$.

3.2. Construction of the amplitude

The amplitude function $a(z)$ is not identically zero on $\overline{\Omega}$ and has the following properties:

$$a \in C^6(\overline{\Omega}_\Phi), \quad \partial_{\bar{z}} a \equiv 0,\ \operatorname{Re} a|_{\Gamma_0} = 0,\ |a(x)| \le C_7 |x - \hat{x}_{j,\pm}|^m \quad \forall j \in \{1, 2, 3, 4\}, \tag{3.17}$$

where $m \in \mathbb{N}$ is sufficiently large (e.g., $m = 100$). Such a function can be constructed in the following way: Using a C^4 conformal mapping Π we map the domain Ω_Φ into a bounded domain $\mathcal{O}$ with $\partial\mathcal{O} \in C^\infty$. Applying Proposition 3.1 we construct a holomorphic function $\mathcal{A}$ such that $\mathcal{A}$ is sufficiently smooth on $\overline{\mathcal{O}}$, $\operatorname{Re}\mathcal{A}|_{\Pi(\Gamma_0)} = 0$ and $\partial_z^k \mathcal{A}(\hat{x}_{j,\pm}) = 0$ for $k \in \{0, ..., m\}$. Then we set $a(x) = \mathcal{A} \circ \Pi$.

Let the polynomials $M_1(z)$ and $M_3(\bar{z})$ satisfy

$$\partial_z^j (\partial_{\bar{z}}^{-1}(a q_1) - M_1)(x) = 0, \quad x \in \mathcal{H} \cup \{\hat{x}_{k,\pm}, k = 1, 2, 3, 4\},\ j = 0, 1, 2, \tag{3.18}$$

$$\partial_{\bar{z}}^j (\partial_z^{-1}(\overline{a} q_1) - M_3)(x) = 0, \quad x \in \mathcal{H} \cup \{\hat{x}_{k,\pm}, k = 1, 2, 3, 4\},\ j = 0, 1, 2, \tag{3.19}$$

and

$$\partial_z^k M_1(\hat{x}_{j,\pm}) = \partial_{\bar{z}}^k M_3(\hat{x}_{j,\pm}) = 0 \quad \forall k \in \{3, \dots, m\} \text{ and } \forall j \in \{1, 2, 3, 4\}. \tag{3.20}$$

By (3.18)-(3.20) and (3.17) we have

$$|\partial_{\bar{z}}^{-1}(a q_1) - M_1(z)| \le C_8 |x - \hat{x}_{k,\pm}|^m,\ |\partial_z^{-1}(\overline{a} q_1) - M_3(\bar{z})| \le C_9 |x - \hat{x}_{k,\pm}|^m \quad \forall k \in \{1, 2, 3, 4\}. \tag{3.21}$$

Finally $a_0, a_1 \in C^6(\bar\Omega_\Phi)$ are holomorphic functions such that

$$(a_0+\overline{a_1})|_{\Gamma_0} = \frac{(\partial_{\overline{z}}^{-1}(aq_1)-M_1)}{4\partial_z\Phi} + \frac{(\partial_z^{-1}(\overline{a}q_1)-M_3)}{4\overline{\partial_z\Phi}},$$

and there exists a positive constant C such that

$$|a_k(x)| \le C_9|x-\hat x_{j,\pm}|^3 \ \ \forall j\in\{1,2,3,4\},\ \forall k\in\{0,1\}. \tag{3.22}$$

3.3. Construction of u_-

We introduce the function $u_-(\tau,\cdot)$ by

$$e^{\tau\varphi}u_-(\tau,x) = -\chi_\tau(e^{\tau\overline{\Psi}}\overline{\hat a} + e^{\tau\tilde\Psi}\hat a) + w_\tau(x)e^{\tau\varphi}, \tag{3.23}$$

where $\Psi(z)$ is the holomorphic function defined by

$$\Psi(z) = \varphi(x_1,-x_2) - i\psi(x_1,-x_2) \quad x\in\mathcal{G}_-\cup\mathcal{G}_+. \tag{3.24}$$

In order to construct w_τ, we introduce the following functions

$$\hat a(x_1,x_2) = \mathrm{Re}\, a(x_1,-x_2) - i\mathrm{Im}\, a(x_1,-x_2) \quad x\in\mathcal{G}_-\cup\mathcal{G}_+ \tag{3.25}$$

and

$$\hat a_k(x_1,x_2) = \mathrm{Re}\, a_k(x_1,-x_2) - i\mathrm{Im}\, a_k(x_1,-x_2) \quad x\in\mathcal{G}_-\cup\mathcal{G}_+,\ k\in\{0,1\}. \tag{3.26}$$

The function χ_τ is constructed in the following way. Let $\mu\in C_0^\infty(-2,2)$ and $\mu|_{[-1,1]}=1$. We set

$$\chi_\tau(x) = \begin{cases} (1-\mu((x_1-\hat x_{2,+})\tau^{\frac{1}{80}}) - \mu((x_1-\hat x_{3,-})\tau^{\frac{1}{80}}))\mu(x_2\tau^{\frac17}) \\ \text{for} \quad x\in\mathcal{V}_1 = \{(x_1,x_2)|\hat x_{2,+}\le x_1\le \hat x_{3,-}, -\delta\gamma_2(x_1)\le x_2\le 0\}, \\ (1-\mu((x_1-\hat x_{4,+})\tau^{\frac{1}{80}}) - \mu((x_1-\hat x_{1,-})\tau^{\frac{1}{80}}))\mu(x_2\tau^{\frac17}) \\ \text{for} \quad x\in\mathcal{V}_2 = \{(x_1,x_2)|\hat x_{4,+}\le x_1\le \hat x_{1,-}, -\delta\gamma_4(x_1)\le x_2\le 0\}, \\ 0 \quad \text{for} \quad x\notin \mathcal{V}_1\cup\mathcal{V}_2. \end{cases} \tag{3.27}$$

For all sufficiently large τ

$$\mathrm{supp}\,\chi_\tau\cap\Omega\subset\mathcal{G}_-. \tag{3.28}$$

Let w_τ be a solution to the boundary value problem:

$$\begin{aligned}\Delta(w_\tau e^{\tau\varphi}) + q_1(w_\tau e^{\tau\varphi}) = r_\tau = \chi_\tau q_1(e^{\tau\overline{\Psi}}\overline{(\hat a+\hat a_1/\tau)} + e^{\tau\Psi}(\hat a+\hat a_0/\tau))\\ +[\chi_\tau,\Delta](e^{\tau\overline{\Psi}}\overline{(\hat a+\hat a_1/\tau)} + e^{\tau\Psi}(\hat a+\hat a_0/\tau)) \quad \text{in } \Omega,\end{aligned} \tag{3.29}$$

$$(w_\tau e^{\tau\varphi})|_{\Gamma_0\cup\Gamma_-} = 0. \tag{3.30}$$

Denote $g_\tau = [\chi_\tau,\Delta](e^{\tau\overline{\Psi}}\overline{(\hat a+\hat a_1/\tau)} + e^{\tau\Psi}(\hat a+\hat a_0/\tau))$. We claim that

$$\|g_\tau e^{-\tau\varphi}\|_{L^2(\Omega)} = O(\frac{1}{\tau}) \quad \text{as } \tau\to+\infty. \tag{3.31}$$

Indeed the operator $[\chi_\tau,\Delta]$ is a first order operator :$[\chi_\tau,\Delta] = 2(\nabla\chi_\tau,\nabla)+\Delta\chi_\tau$ where

$$\|\nabla\chi_\tau\|_{L^\infty(\Omega)} = O(\tau^{\frac{1}{10}}), \quad \|\Delta\chi_\tau\|_{L^\infty(\Omega)} = O(\tau^{\frac15}) \text{ as } |\tau|\to+\infty. \tag{3.32}$$

By (3.27) there exists τ_0 such that for all $\tau \ge \tau_0$ we have

$$\mathrm{supp}\,\Delta\chi_\tau,\ \mathrm{supp}\nabla\chi_\tau \subset \mathcal{I}_1(\tau)\cup\mathcal{I}_2(\tau),$$

where

$$\mathcal{I}_1(\tau) = \left\{(x_1,x_2);\ \frac{1}{\tau^{\frac{1}{7}}} \le x_2 \le \frac{2}{\tau^{\frac{1}{7}}},\ x_1 \in [\hat{x}_{2,+} + \frac{2}{\tau^{\frac{1}{80}}}, \hat{x}_{3,-} - \frac{2}{\tau^{\frac{1}{80}}}] \cup [\hat{x}_{4,+} + \frac{2}{\tau^{\frac{1}{80}}}, \hat{x}_{1,-} - \frac{2}{\tau^{\frac{1}{80}}}]\right\},$$

$$\mathcal{I}_2(\tau) = \left\{(x_1,x_2);\ 0 \le x_2 \le \frac{2}{\tau^{\frac{1}{7}}},\ x_1 \in [\hat{x}_{2,+} + \frac{1}{\tau^{\frac{1}{80}}}, \hat{x}_{2,+} + \frac{2}{\tau^{\frac{1}{80}}}] \cup [\hat{x}_{3,-} - \frac{2}{\tau^{\frac{1}{80}}}, \hat{x}_{3,-} - \frac{1}{\tau^{\frac{1}{80}}}] \right.$$
$$\left. \cup [\hat{x}_{4,+} + \frac{1}{\tau^{\frac{1}{80}}}, \hat{x}_{4,+} + \frac{2}{\tau^{\frac{1}{80}}}] \cup [\hat{x}_{1,-} - \frac{2}{\tau^{\frac{1}{80}}}, \hat{x}_{1,-} - \frac{1}{\tau^{\frac{1}{80}}}]\right\}.$$

Observe that

$$\mathcal{I}_1(\tau)\cup\mathcal{I}_2(\tau) \subset \mathcal{G}_+. \tag{3.33}$$

Applying (3.17), (3.12), (3.32) and (3.33), we have

$$\begin{aligned}
&\Vert e^{-\tau\varphi}[\chi_\tau,\Delta](e^{\tau\overline{\Psi}}\overline{(\hat{a}+\hat{a}_1/\tau)} + e^{\tau\Psi}(\hat{a}+\hat{a}_0/\tau))\Vert_{L^\infty(\mathcal{I}_1)}\\
&\le \Vert e^{-\tau\varphi}\Delta\chi_\tau(e^{\tau\overline{\Psi}}\overline{(\hat{a}+\hat{a}_1/\tau)} + e^{\tau\Psi}(\hat{a}+\hat{a}_0/\tau))\Vert_{L^\infty(\mathcal{I}_1)}\\
&+2\Vert e^{-\tau\varphi}(e^{\tau\overline{\Psi}}(\nabla\chi_\tau,\nabla)\overline{(\hat{a}+\hat{a}_1/\tau)} + e^{\tau\Psi}(\nabla\chi_\tau,\nabla)(\hat{a}+\hat{a}_0/\tau))\Vert_{L^\infty(\mathcal{I}_1)}\\
&+2\Vert \tau e^{-\tau\varphi}(e^{\tau\overline{\Psi}}(\nabla\chi_\tau,\nabla\overline{\Psi})\overline{(\hat{a}+\hat{a}_1/\tau)} + e^{\tau\Psi}(\nabla\chi_\tau,\nabla\Psi)(\hat{a}+\hat{a}_0/\tau))\Vert_{L^\infty(\mathcal{I}_1)}\\
&\le |\tau|^3 \sup_{x\in\mathcal{I}_1(\tau)} e^{-\tau\varphi+\tau\mathrm{Re}\Psi} \le |\tau|^3 \sup_{x\in\mathcal{I}_1(\tau)} e^{-\tau\tilde{C}_\delta\ell(x)} \le |\tau|^3 e^{-\tau\tilde{C}_\delta\tau^{\frac{-7}{80}}\tau^{\frac{-1}{7}}} = O(\frac{1}{\tau^2}) \quad \text{as } \tau\to+\infty.
\end{aligned}\tag{3.34}$$

Using (3.17), (3.22) and (3.32), we obtain

$$\begin{aligned}
&\Vert e^{-\tau\varphi}[\chi_\tau,\Delta](e^{\tau\overline{\Psi}}\overline{(\hat{a}+\hat{a}_1/\tau)} + e^{\tau\Psi}(\hat{a}+\hat{a}_0/\tau))\Vert_{L^\infty(\mathcal{I}_2)}\\
&\le \Vert e^{-\tau\varphi}\Delta\chi_\tau(e^{\tau\overline{\Psi}}\overline{(\hat{a}+\hat{a}_1/\tau)} + e^{\tau\Psi}(\hat{a}+\hat{a}_0/\tau))\Vert_{L^\infty(\mathcal{I}_2)}\\
&+2\Vert e^{-\tau\varphi}(e^{\tau\overline{\Psi}}(\nabla\chi_\tau,\nabla)\overline{(\hat{a}+\hat{a}_1/\tau)} + e^{\tau\Psi}(\nabla\chi_\tau,\nabla)(\hat{a}+\hat{a}_0/\tau))\Vert_{L^\infty(\mathcal{I}_2)}\\
&+2\Vert \tau e^{-\tau\varphi}(e^{\tau\overline{\Psi}}(\nabla\chi_\tau,\nabla\overline{\Psi})\overline{(\hat{a}+\hat{a}_1/\tau)} + e^{\tau\Psi}(\nabla\chi_\tau,\nabla\Psi)(\hat{a}+\hat{a}_0/\tau))\Vert_{L^\infty(\mathcal{I}_2)}\\
&\le \Vert \Delta\chi_\tau(\overline{(\hat{a}+\hat{a}_1/\tau)} + (\hat{a}+\hat{a}_0/\tau))\Vert_{L^\infty(\mathcal{I}_2)}\\
&+2\Vert (\nabla\chi_\tau,\nabla)\overline{(\hat{a}+\hat{a}_1/\tau)} + (\nabla\chi_\tau,\nabla)(\hat{a}+\hat{a}_0/\tau)\Vert_{L^\infty(\mathcal{I}_2)}\\
&+2\Vert \tau((\nabla\chi_\tau,\nabla\overline{\Psi})\overline{(\hat{a}+\hat{a}_1/\tau)} + (\nabla\chi_\tau,\nabla\Psi)(\hat{a}+\hat{a}_0/\tau))\Vert_{L^\infty(\mathcal{I}_2)} = O(\frac{1}{\tau^2}) \quad \text{as } \tau\to+\infty.
\end{aligned}\tag{3.35}$$

The inequalities (3.34) and (3.35) imply (3.31) immediately. By (3.13) and (3.24)

$$\Vert e^{-\tau\varphi}\chi_\tau q_1(e^{\tau\overline{\Psi}}\overline{(\hat{a}+\hat{a}_1/\tau)} + e^{\tau\Psi}(\hat{a}+\hat{a}_0/\tau))\Vert_{L^2(\Omega)} = o(1) \quad \text{as } \tau\to+\infty. \tag{3.36}$$

Using (3.31), (3.36) and $\mathrm{supp}\,\chi_\tau\cap\mathcal{H} = \emptyset$, we can apply Proposition 2.6 to obtain a solution to the boundary value problem (3.29), (3.30) such that

$$\Vert w_\tau\Vert_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as} \quad \tau\to+\infty. \tag{3.37}$$

3.4. Completion of the construction of the complex geometrical optics solution u_1

The function u_{11} is given by

$$u_{11} = -\frac{1}{4}e^{i\tau\psi}\widetilde{R}_{\Phi,\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_1) - M_1)) - \frac{1}{4}e^{-i\tau\psi}R_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a}q_1) - M_3)) - \frac{e^{i\tau\psi}}{\tau}\frac{e_2(\partial_{\overline{z}}^{-1}(aq_1) - M_1)}{4\partial_z\Phi} - \frac{e^{-i\tau\psi}}{\tau}\frac{e_2(\partial_z^{-1}(\overline{a}q_1) - M_3)}{4\overline{\partial_z\Phi}}, \tag{3.38}$$

where $e_1, e_2 \in C^\infty(\Omega)$ are constructed so that

$$e_1 + e_2 \equiv 1 \text{ on } \overline{\Omega}, e_2 \text{ vanishes in some neighborhood of } \mathcal{H} \text{ and } e_1 \text{ vanishes in a neighborhood of } \partial\Omega. \tag{3.39}$$

Let u_{12} be a solution to the inhomogeneous problem

$$\Delta(u_{12}e^{\tau\varphi}) + q_1u_{12}e^{\tau\varphi} = -q_1u_{11}e^{\tau\varphi} + h_1e^{\tau\varphi} \quad \text{in } \Omega, \tag{3.40}$$

$$u_{12} = d_{1,\tau} + d_{2,\tau} + d_{3,\tau} \quad \text{on } \Gamma_0 \cup \Gamma_-, \tag{3.41}$$

where

$$h_1 = e^{\tau i\psi}\Delta\left(\frac{e_2(\partial_{\overline{z}}^{-1}(aq_1) - M_1)}{4\tau\partial_z\Phi}\right) + e^{-\tau i\psi}\Delta\left(\frac{e_2(\partial_z^{-1}(\overline{a}q_1) - M_3)}{4\tau\overline{\partial_z\Phi}}\right) - a_0q_1e^{i\tau\psi}/\tau - \overline{a_1}q_1e^{-i\tau\psi}/\tau, \tag{3.42}$$

and $d_{1,\tau} = (\frac{e^{i\tau\psi}}{4}\tilde{R}_{\Phi,\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_1) - M_1)) + \frac{e^{-i\tau\psi}}{4}R_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a}q_1) - M_3)))$,
$d_{2,\tau} = \chi_{\Gamma_-}(1-\chi_\tau)\mathrm{Re}\,\{e^{i\tau\psi}a\}$, $d_{3,\tau} = \frac{e^{i\tau\psi}}{\tau}\frac{e_2(\partial_{\overline{z}}^{-1}(aq_1)-M_1)}{4\partial_z\Phi} + \frac{e^{-i\tau\psi}}{\tau}\frac{e_2(\partial_z^{-1}(\overline{a}q_1)-M_3)}{4\overline{\partial_z\Phi}} - \frac{a_0e^{\tau i\psi}+\overline{a_1}e^{-\tau i\psi}}{\tau}$.

By (3.17) and (3.22), there exists a constant C_{10}, independent of τ, such that

$$\left\|d_{3,\tau}/\sqrt{|\frac{\partial\varphi}{\partial\nu}|}\right\|_{L^2(\Gamma_-)} \le \frac{C_{10}}{|\tau|}. \tag{3.43}$$

Consequently applying Proposition 2.7, we obtain a solution for the initial value problem $L_1(x,D)(e^{\tau\varphi}u_{12,I}) = 0, u_{12,I}|_{\Gamma_0} = 0, u_{12,I}|_{\Gamma_-} = d_{3,\tau}$ which satisfies the estimate

$$\|u_{12,I}\|_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as} \quad \tau \to +\infty. \tag{3.44}$$

Since

$$\|q_1u_{11} + h_1\|_{L^2(\Omega)} \le C_{11}(\delta)/|\tau|^{1-\delta} \quad \forall\delta \in (0,1)$$

and by the stationary phase argument $\|d_{1,\tau}\|_{L^2(\Gamma_0\cup\Gamma_-)} = O(\frac{1}{\tau^2})$, there exists a solution to the initial value problem $L_1(x,D)(e^{\tau\varphi}u_{12,II}) = 0, u_{12,II}|_{\Gamma_0\cup\Gamma_-} = d_{1,\tau}$ which satisfies the estimate

$$\|u_{12,II}\|_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as} \quad \tau \to +\infty. \tag{3.45}$$

Finally $\|d_{1,\tau}\|_{L^2(\Gamma_0\cup\Gamma_-)} = O(\frac{1}{\tau^2})$ by (3.17). Therefore applying Proposition 2.6, we obtain a solution to the initial value problem $L_1(x,D)(e^{\tau\varphi}u_{12,III}) = 0, u_{12,III}|_{\Gamma_0\cup\Gamma_-} = d_{2,\tau}$ which satisfies the estimate

$$\|u_{12,III}\|_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as} \quad \tau \to +\infty. \tag{3.46}$$

Setting $u_{12} = u_{12,III} + u_{12,II} + u_{12,I}$, we obtain a solution to (3.40), (3.41) satisfying

$$\|u_{12}\|_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as } \tau \to +\infty. \tag{3.47}$$

Now consider a sequence of τ_j such that

$$\tau_j = 2\pi j n_1 n_2 n_3 n_4, \tag{3.48}$$

where $n_1, n_2, n_3, n_4 \in \mathbb{N}$ are defined in (3.6). For each τ_j from this sequence, the solution u_1 satisfies the zero Dirichlet boundary condition on $\Gamma_0 \cup \Gamma_-$.

3.5. Construction of the complex geometrical optics solution v

Consider now the Schrödinger equation

$$L_2(x, D)v = \Delta v + q_2 v = 0 \quad \text{in } \Omega. \tag{3.49}$$

We will construct solutions to (3.49) of the form

$$v(x) = e^{-\tau\Phi}(a + b_0/\tau) + e^{-\tau\overline{\Phi}}\overline{(a + b_1/\tau)} + e^{-\tau\varphi}v_+ + e^{-\tau\varphi}v_{11} + e^{-\tau\varphi}v_{12}, \quad v|_{\Gamma_0} = 0. \tag{3.50}$$

The construction of v repeats the corresponding steps of the construction of u_1. The only difference is that instead of q_1 and τ, we use q_2 and $-\tau$ respectively. We provide details of the construction of v for the sake of completeness. Let polynomials $M_2(z), M_4(\bar z)$ satisfy

$$\partial_z^j(\partial_{\bar z}^{-1}(aq_1) - M_2)(x) = 0, \quad x \in \mathcal{H} \cup \{\hat x_{k,\pm}, k = 1, 2, 3, 4\},\ j = 0, 1, 2, \tag{3.51}$$

$$\partial_{\bar z}^j(\partial_z^{-1}(\overline{a}q_1) - M_4)(x) = 0, \quad x \in \mathcal{H} \cup \{\hat x_{k,\pm}, k = 1, 2, 3, 4\},\ j = 0, 1, 2, \tag{3.52}$$

and

$$\partial_z^k M_2(\hat x_{j,\pm}) = \partial_{\bar z}^k M_4(\hat x_{j,\pm}) = 0 \quad \forall k \in \{3, \dots, m\} \text{ and } \forall j \in \{1, 2, 3, 4\}, \tag{3.53}$$

where $m \in \mathbb{N}$ can be chosen for example as 100. Finally b_0, b_1 are holomorphic functions such that

$$(b_0 + \overline{b}_1)|_{\Gamma_0} = -\frac{(\partial_{\bar z}^{-1}(aq_2) - M_2)}{4\partial_z\Phi} - \frac{(\partial_z^{-1}(\overline{a}q_2) - M_4)}{4\overline{\partial_z\Phi}}$$

and there exists a positive constant C_{12} such that

$$|b_k(x)| \le C_{12}|x - \hat x_{j,\pm}|^3\ \forall j \in \{1, 2, 3, 4\},\ \forall k \in \{0, 1\}. \tag{3.54}$$

Let

$$\hat b_j(x_1, x_2) = \mathrm{Re}\, b_j(x_1, -x_2) - i\mathrm{Im}\, b_j(x_1, -x_2) \quad \forall x \in \mathcal{G}_+,\ j \in \{0, 2\}. \tag{3.55}$$

We set

$$e^{-\tau\varphi}v_+(\tau, x) = -\tilde\chi_\tau(e^{-\tau\overline{\Psi}}\overline{(a + \hat b_0/\tau)} + e^{-\tau\Psi}(a + \hat b_1/\tau)) + \tilde w_\tau(x)e^{-\tau\varphi}. \tag{3.56}$$

The function $\tilde\chi_\tau$ is constructed in the following way. We set

$$\tilde\chi_\tau(x)=\begin{cases}(1-\mu((x_1-\hat x_{1,+})\tau^{\frac{1}{80}})-\mu((x_1-\hat x_{2,-})\tau^{\frac{1}{80}}))\mu(x_2\tau^{\frac17})\\ \text{for}\quad x\in\mathcal V_3=\{(x_1,x_2);\,\hat x_{1,+}\le x_1\le\hat x_{2,-},-\delta\gamma_1(x_1)\le x_2\le 0\},\\ (1-\mu((x_1-\hat x_{3,+})\tau^{\frac{1}{80}})-\mu((x_1-\hat x_{4,-})\tau^{\frac{1}{80}}))\mu(x_2\tau^{\frac17})\\ \text{for}\quad x\in\mathcal V_4=\{(x_1,x_2);\,\hat x_{3,+}\le x_1\le\hat x_{4,-},-\delta\gamma_3(x_1)\le x_2\le 0\},\\ 0\quad\text{for}\quad x\notin\mathcal V_3\cup\mathcal V_4.\end{cases}\tag{3.57}$$

Let $\tilde w_\tau$ be a solution to the following boundary value problem:

$$\begin{aligned}\Delta(\tilde w_\tau e^{-\tau\varphi})+q_2(\tilde w_\tau e^{-\tau\varphi})&=\tilde\chi_\tau q_2(e^{-\tau\overline{\Psi}}\overline{(a+\hat b_0/\tau)}+e^{-\tau\Psi}(a+\hat b_1/\tau))\\ &+[\tilde\chi_\tau,\Delta](e^{-\tau\overline{\Psi}}\overline{(a+\hat b_0/\tau)}+e^{-\tau\Psi}(a+\hat b_1/\tau))\quad\text{in }\Omega,\end{aligned}\tag{3.58}$$

$$\begin{aligned}(\tilde w_\tau e^{-\tau\varphi})|_{\Gamma_0\cup\Gamma_-}&=-e^{-\tau\Phi}(a+\hat b_0/\tau)+e^{-\tau\overline{\Phi}}\,\overline{(a+\hat b_1/\tau)}\\ &+\tilde\chi_\tau(e^{-\tau\overline{\Psi}}\overline{(a+\hat b_0/\tau)}+e^{-\tau\Psi}(a+\hat b_1/\tau)).\end{aligned}\tag{3.59}$$

Denote $\tilde g_\tau=[\tilde\chi_\tau,\Delta](e^{-\tau\overline{\Psi}}\overline{(a+\hat b_0/\tau)}+e^{-\tau\Psi}(a+\hat b_1/\tau))$. We claim that

$$\|\tilde g_\tau e^{\tau\varphi}\|_{L^2(\Omega)}=o(\frac{1}{\tau})\quad\text{as }|\tau|\to+\infty.\tag{3.60}$$

Indeed the operator $[\tilde\chi_\tau,\Delta]$ is the first order operator :$[\tilde\chi_\tau,\Delta]=2(\nabla\tilde\chi_\tau,\nabla)+\Delta\tilde\chi_\tau$ where

$$\|\nabla\tilde\chi_\tau\|_{L^\infty(\Omega)}=O(\tau^{\frac{1}{10}}),\quad\|\Delta\tilde\chi_\tau\|_{L^\infty(\Omega)}=O(\tau^{\frac15})\text{ as }|\tau|\to+\infty.\tag{3.61}$$

By (3.57) we have

$$\operatorname{supp}\nabla\tilde\chi_\tau,\ \operatorname{supp}\Delta\tilde\chi_\tau\subset\tilde{\mathcal I}_1(\tau)\cup\tilde{\mathcal I}_2(\tau),$$

where

$$\tilde{\mathcal I}_1(\tau)=\left\{(x_1,x_2);\,\frac{1}{\tau^{\frac17}}\le x_2\le\frac{2}{\tau^{\frac17}},\ x_1\in[\hat x_{1,+}+\frac{2}{\tau^{\frac{1}{80}}},\hat x_{2,-}-\frac{2}{\tau^{\frac{1}{80}}}]\cup[\hat x_{3,+}+\frac{2}{\tau^{\frac{1}{80}}},\hat x_{4,-}-\frac{2}{\tau^{\frac{1}{80}}}]\right\},$$

$$\begin{aligned}\tilde{\mathcal I}_2(\tau)=\Big\{(x_1,x_2);\,0\le x_2\le\frac{2}{\tau^{\frac17}},\ x_1\in[\hat x_{1,+}+\frac{1}{\tau^{\frac{1}{80}}},\hat x_{1,+}+\frac{2}{\tau^{\frac{1}{80}}}]\cup[\hat x_{2,-}-\frac{2}{\tau^{\frac{1}{80}}},\hat x_{2,-}-\frac{1}{\tau^{\frac{1}{80}}}]\\ \cup[\hat x_{3,+}+\frac{1}{\tau^{\frac{1}{80}}},\hat x_{3,+}+\frac{2}{\tau^{\frac{1}{80}}}]\cup[\hat x_{4,-}-\frac{2}{\tau^{\frac{1}{80}}},\hat x_{4,-}-\frac{1}{\tau^{\frac{1}{80}}}]\Big\}.\end{aligned}$$

Observe that

$$\tilde{\mathcal I}_1(\tau)\cup\tilde{\mathcal I}_2(\tau)\subset\Gamma_-.\tag{3.62}$$

Applying (3.17), (3.32) and (3.62), we have

$$\begin{aligned}
(3.63)\qquad & \Vert e^{\tau\varphi}[\tilde\chi_\tau,\Delta](e^{-\tau\overline{\Psi}}\overline{(\hat a+\hat b_0/\tau)}+e^{-\tau\Psi}(\hat a+\hat b_1/\tau))\Vert_{L^\infty(\tilde{\mathcal I}_1)}\\
&\le \Vert e^{\tau\varphi}\Delta\tilde\chi_\tau(e^{-\tau\overline{\Psi}}\overline{(\hat a+\hat b_0/\tau)}+e^{-\tau\Psi}(\hat a+\hat b_1/\tau))\Vert_{L^\infty(\tilde{\mathcal I}_1)}\\
&+2\Vert e^{\tau\varphi}(e^{-\tau\overline{\Psi}}(\nabla\tilde\chi_\tau,\nabla)\overline{(\hat a+\hat b_0/\tau)}+e^{-\tau\Psi}(\nabla\tilde\chi_\tau,\nabla)(\hat a+\hat b_1/\tau))\Vert_{L^\infty(\tilde{\mathcal I}_1)}\\
&+2\Vert\tau e^{\tau\varphi}(e^{-\tau\overline{\Psi}}(\nabla\tilde\chi_\tau,\nabla\overline{\Psi})\overline{(\hat a+\hat b_0/\tau)}+e^{-\tau\Psi}(\nabla\tilde\chi_\tau,\nabla\Psi)(\hat a+\hat b_1/\tau))\Vert_{L^\infty(\tilde{\mathcal I}_1)}\\
&\le |\tau|^3\sup_{x\in\tilde{\mathcal I}_1(\tau)}e^{\tau\varphi-\tau\mathrm{Re}\Psi}\le|\tau|^3\sup_{x\in\tilde{\mathcal I}_1(\tau)}e^{-\tau\tilde C_\delta\tilde\ell(x)}\le|\tau|^3e^{-\tau\tau^{\frac{-7}{80}}\tilde C_\delta\tau^{\frac{-1}{7}}}=O(\frac{1}{\tau^2})\quad\text{as } |\tau|\to+\infty.
\end{aligned}$$

Using (3.17), (3.22) and (3.32), we have

$$\begin{aligned}
(3.64)\qquad & \Vert e^{\tau\varphi}[\tilde\chi_\tau,\Delta](e^{\tau\overline{\Psi}}\overline{(\hat a+\hat b_0/\tau)}+e^{\tau\Psi}(\hat a+\hat b_1/\tau)\Vert_{L^\infty(\tilde{\mathcal I}_2)}\\
&\le \Vert e^{\tau\varphi}\Delta\tilde\chi_\tau(e^{-\tau\overline{\Psi}}\overline{(\hat a+\hat b_0/\tau)}+e^{-\tau\Psi}(\hat a+\hat b_1/\tau))\Vert_{L^\infty(\tilde{\mathcal I}_2)}\\
&+2\Vert e^{\tau\varphi}(e^{-\tau\overline{\Psi}}(\nabla\tilde\chi_\tau,\nabla)\overline{(\hat a+\hat b_0/\tau)}+e^{-\tau\Psi}(\nabla\tilde\chi_\tau,\nabla)(\hat a+\hat b_1/\tau))\Vert_{L^\infty(\tilde{\mathcal I}_2)}\\
&+2\Vert\tau e^{\tau\varphi}(e^{-\tau\overline{\Psi}}(\nabla\tilde\chi_\tau,\nabla\overline{\Psi})\overline{(\hat a+\hat b_0/\tau)}+e^{-\tau\Psi}(\nabla\tilde\chi_\tau,\nabla\Psi)(\hat a+\hat b_1/\tau))\Vert_{L^\infty(\tilde{\mathcal I}_2)}\\
&\le \Vert\Delta\tilde\chi_\tau(\overline{(\hat a+\hat b_0/\tau)}+(\hat a+\hat b_1/\tau))\Vert_{L^\infty(\tilde{\mathcal I}_2)}\\
&+2\Vert(\nabla\tilde\chi_\tau,\nabla)\overline{(\hat a+\hat b_0/\tau)}+(\nabla\tilde\chi_\tau,\nabla)(\hat a+\hat b_1/\tau)\Vert_{L^\infty(\tilde{\mathcal I}_2)}\\
&+2\Vert\tau((\nabla\tilde\chi_\tau,\nabla\overline{\Psi})\overline{(\hat a+\hat b_0/\tau)}+(\nabla\tilde\chi_\tau,\nabla\Psi)(\hat a+\hat b_1/\tau))\Vert_{L^\infty(\tilde{\mathcal I}_2)}=O(\frac{1}{\tau^2})\quad\text{as } |\tau|\to+\infty.
\end{aligned}$$

The inequalities (3.63) and (3.64) imply (3.60) immediately. Using (3.31), (3.36) and $\mathrm{supp}\,\chi_\tau\cap\mathcal H=\emptyset$, we can apply Proposition 2.6 to obtain a solution to the boundary value problem (3.58) and (3.59) such that

$$(3.65)\qquad \Vert\tilde w_\tau\Vert_{L^2(\Omega)}=o(\frac{1}{\tau})\quad\text{as}\quad|\tau|\to+\infty.$$

The function v_{11} is given by

$$\begin{aligned}
(3.66)\qquad v_{11}=&-\frac14 e^{-i\tau\psi}\widetilde R_{\Phi,-\tau}(e_1(\partial_{\overline z}^{-1}(q_2a)-M_2))-\frac14 e^{i\tau\psi}R_{\Phi,\tau}(e_1(\partial_z^{-1}(q_2\overline a)-M_4))\\
&+\frac{e^{-i\tau\psi}}{\tau}\frac{e_2(\partial_{\overline z}^{-1}(aq_2)-M_2)}{4\partial_z\Phi}+\frac{e^{i\tau\psi}}{\tau}\frac{e_2(\partial_z^{-1}(\overline a q_2)-M_4)}{4\overline{\partial_z\Phi}}.
\end{aligned}$$

Denote

$$\begin{aligned}
h_2=e^{-\tau i\psi}\Delta\left(\frac{e_2(\partial_{\overline z}^{-1}(aq_2)-M_2)}{4\tau\partial_z\Phi}\right)+e^{\tau i\psi}\Delta\left(\frac{e_2(\partial_z^{-1}(\overline a q_2)-M_4)}{4\tau\overline{\partial_z\Phi}}\right)\\
-\frac{b_0}{\tau}q_2e^{-i\tau\psi}-\frac{\overline{b_1}}{\tau}q_2e^{i\tau\psi}.
\end{aligned}$$

The function v_{12} is a solution to

$$(3.67)\qquad \Delta(v_{12}e^{-\tau\varphi})+q_2v_{12}e^{-\tau\varphi}=-q_2v_{11}e^{-\tau\varphi}-h_2e^{-\tau\varphi}\quad\text{in }\Omega,$$

$$(3.68)\qquad v_{12}\vert_{\Gamma_0\cup\Gamma_+}=\tilde d_{1,\tau}+\tilde d_{2,\tau}+\tilde d_{3,\tau},$$

where $\tilde{d}_{1,\tau} = \frac{e^{i\tau\psi}}{4}\tilde{R}_{\Phi,-\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_2) - M_2)) + \frac{e^{-i\tau\psi}}{4}R_{\Phi,\tau}(e_1(\partial_z^{-1}(\overline{a}q_2) - M_4))$,
$\tilde{d}_{2,\tau} = \chi_{\Gamma_+}(1-\chi_\tau)\mathrm{Re}\,\{e^{-\tau i\psi}a\}$, $\tilde{d}_{3,\tau} = \frac{e^{i\tau\psi}}{\tau}\frac{e_2(\partial_{\overline{z}}^{-1}(aq_2)-M_2)}{4\partial_z\Phi} + \frac{e^{-i\tau\psi}}{\tau}\frac{e_2(\partial_z^{-1}(\overline{a}q_2)-M_4)}{4\partial_{\overline{z}}\Phi} - \frac{b_0e^{-\tau i\psi}+\overline{b_1}e^{\tau i\psi}}{\tau}$.

By (3.17) and (3.22), there exists a constant C_{13}, independent of τ, such that

$$\left\|\tilde{d}_{3,\tau}/\sqrt{\frac{\partial\varphi}{\partial\nu}}\right\|_{L^2(\Gamma_+)} \le \frac{C_{13}}{|\tau|}. \tag{3.69}$$

Applying Proposition 2.7, we obtain a solution to the boundary value problem $L_2(x,D)(e^{-\tau\varphi}v_{12,I}) = 0, v_{12,I}|_{\Gamma_0} = 0, v_{12,I}|_{\Gamma_+} = \tilde{d}_{3,\tau}$ which satisfies the estimate

$$\|v_{12,I}\|_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as} \quad \tau\to+\infty. \tag{3.70}$$

Since

$$\|q_2v_{11} + h_2\|_{L^2(\Omega)} \le C_{14}(\delta)/|\tau|^{1-\delta} \quad \forall\delta\in(0,1)$$

and by the stationary phase argument $\|\tilde{d}_{1,\tau}\|_{L^2(\Gamma_0\cup\Gamma_+)} = O(\frac{1}{\tau^2})$, there exists a solution to the initial value problem $L_2(x,D)(e^{-\tau\varphi}v_{12,II}) = 0, v_{12,II}|_{\Gamma_0\cup\Gamma_+} = \tilde{d}_{1,\tau}$ which satisfies the estimate

$$\|v_{12,II}\|_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as} \quad \tau\to+\infty. \tag{3.71}$$

Finally $\|\tilde{d}_{1,\tau}\|_{L^2(\Gamma_0\cup\Gamma_+)} = O(\frac{1}{\tau^2})$ by (3.17). Therefore, applying Proposition 2.6, we obtain a solution to the initial value problem $L_2(x,D)(e^{-\tau\varphi}v_{12,III}) = 0, v_{12,III}|_{\Gamma_0\cup\Gamma_+} = \tilde{d}_{2,\tau}$ which satisfies the estimate

$$\|v_{12,III}\|_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as} \quad \tau\to+\infty. \tag{3.72}$$

Setting $v_{12} = v_{12,III} + v_{12,II} + v_{12,I}$ we obtain a solution to (3.40), (3.41) satisfying

$$\|v_{12}\|_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as } \tau\to+\infty. \tag{3.73}$$

For each τ_j defined by (4.19), the solution v satisfies the zero Dirichlet boundary condition on $\Gamma_0\cup\Gamma_-$.

4. **Proof of Theorem 1.2**

Proposition 4.1. *Let the function Ψ defined in (3.24) and the holomorphic function Φ constructed in Section 3 have an internal critical point $\tilde{x}$. Then for any potentials $q_1, q_2 \in C^{2+\alpha}(\overline{\Omega})$, $\alpha>0$ satisfying $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$ and for any holomorphic function a satisfying (3.17) and $M_1(z), M_2(z), M_3(\overline{z}), M_4(\overline{z})$ as in Section 3, we have*

$$2\frac{\pi(q|a|^2)(\widetilde{x})Re\,e^{2i\tau_j Im\Phi(\widetilde{x})}}{|(det\,Im\Phi'')(\widetilde{x})|^{\frac{1}{2}}} + \int_\Omega q(a(a_0+b_0)+\overline{a}(\overline{a}_1+\overline{b}_1))dx \tag{4.1}$$

$$+ \quad \frac{1}{4}\int_\Omega \left(qa\frac{\partial_{\overline{z}}^{-1}(aq_2)-M_2}{\partial_z\Phi} + q\overline{a}\frac{\partial_z^{-1}(q_2\overline{a})-M_4}{\overline{\partial_z\Phi}}\right)dx$$
$$- \quad \frac{1}{4}\int_\Omega \left(qa\frac{\partial_{\overline{z}}^{-1}(aq_1)-M_1}{\partial_z\Phi} + q\overline{a}\frac{\partial_z^{-1}(\overline{a}q_1)-M_3}{\overline{\partial_z\Phi}}\right)dx$$
$$+ \quad \int_{\Gamma_-} q|a|^2 Re\left\{\frac{1}{\partial_{x_2}(\overline{\Psi}-\Phi)}\right\}d\sigma - \int_{\Gamma_+} q|a|^2 Re\left\{\frac{1}{\partial_{x_2}(\overline{\Psi}-\Phi)}\right\}d\sigma = o(1) \quad \text{as } \tau_j\to+\infty$$

where $q = q_1 - q_2$ and the sequence τ_j is given by (4.19).

Proof. Let u_1 be a solution to (3.1) and satisfy (3.2), and u_2 be a solution to

$$\Delta u_2 + q_2u_2 = 0 \quad \text{in } \Omega, \quad u_2|_{\partial\Omega} = u_1|_{\partial\Omega}.$$

Since $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$, we have

$$\nabla u_2 = \nabla u_1 \quad \text{on } \Gamma_-.$$

Denoting $u = u_1 - u_2$, we obtain

$$\Delta u + q_2 u = -qu_1 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = \frac{\partial u}{\partial\nu}|_{\Gamma_-} = 0. \tag{4.2}$$

Let v satisfy (3.49) and (3.50). We multiply (4.2) by v, integrate over Ω and we use $v|_{\Gamma_0} = 0$ and $\frac{\partial u}{\partial\nu} = 0$ on $\widetilde{\Gamma}$ to obtain $\int_\Omega qu_1vdx = 0$. By (3.2), (3.50) and (3.47), (3.73), we have

$$\begin{aligned}
0 = \int_\Omega qu_1vdx = \int_\Omega q(a^2+\overline{a}^2+|a|^2e^{\tau_j(\Phi-\overline{\Phi})}+|a|^2e^{\tau_j(\overline{\Phi}-\Phi)}\\
+\frac{1}{\tau_j}(a(a_0+b_0)+\overline{a}(\overline{a}_1+\overline{b}_1)) + u_{11}e^{\tau_j\varphi}(ae^{-\tau_j\Phi}+\overline{a}e^{-\tau_j\overline{\Phi}})\\
+(ae^{\tau_j\Phi}+\overline{a}e^{\tau_j\overline{\Phi}})v_{11}e^{-\tau_j\varphi})dx\\
+\int_\Omega q(e^{-\tau_j\Phi}a+e^{-\tau_j\overline{\Phi}}\overline{a})u_-e^{\tau_j\varphi}dx\\
+\int_\Omega q(e^{\tau_j\Phi}a+e^{\tau_j\overline{\Phi}}\overline{a})v_+e^{-\tau_j\varphi}dx + o\left(\frac{1}{\tau_j}\right), \quad \tau_j>0.
\end{aligned} \tag{4.3}$$

The first and second terms in the asymptotic expansion of (4.3) are independent of τ_j, so that

$$\int_\Omega q(a^2+\overline{a}^2)dx = 0. \tag{4.4}$$

Let the functions e_1, e_2 be defined in (3.39). We have

$$\int_\Omega q(|a|^2e^{\tau_j(\Phi-\overline{\Phi})}+|a|^2e^{\tau_j(\overline{\Phi}-\Phi)})dx = \int_\Omega e_1q(|a|^2e^{\tau_j(\Phi-\overline{\Phi})}+|a|^2e^{\tau_j(\overline{\Phi}-\Phi)})dx$$
$$+ \quad \int_\Omega e_2q(|a|^2e^{\tau_j(\Phi-\overline{\Phi})}+|a|^2e^{\tau_j(\overline{\Phi}-\Phi)})dx.$$

By the Cauchy-Riemann equations, we see that $\text{sgn}(\text{Im}\,\Phi''(\widetilde{x}_k)) = 0$, where sgn A denotes the signature of the invertible matrix A, that is, the number of positive eigenvalues of A

minus the number of negative eigenvalues (e.g., [5], p.210). Moreover we note that

$$\det \operatorname{Im} \Phi''(z) = -(\partial_{x_1}\partial_{x_2}\varphi)^2 - (\partial^2_{x_1}\varphi)^2 \neq 0.$$

To see this, suppose that $\det \operatorname{Im} \Phi''(z) = 0$. Then $\partial_{x_1}\partial_{x_2}\varphi(\operatorname{Re} z, \operatorname{Im} z) = \partial^2_{x_1}\varphi(\operatorname{Re} z, \operatorname{Im} z) = 0$ and the Cauchy-Riemann equations imply that all the second order partial derivatives of the functions φ, ψ at the point z are zero. This contradicts the assumption that the critical points of the function Φ are nondegenerate.

Observe that if Φ has a critical point on Ω, then it can not have any critical points on Γ_0. Then by (2.2) $\tilde{x}$ is the only critical point of this function on $\bar{\Omega}$. Using the stationary phase (see p.215 in [5]), we obtain

$$\int_\Omega e_1 q(|a|^2 e^{\tau_j(\Phi-\overline{\Phi})} + |a|^2 e^{\tau_j(\overline{\Phi}-\Phi)})dx = 2\frac{\pi q|a|^2(\widetilde{x})\operatorname{Re} e^{2\tau_j i \operatorname{Im}\Phi(\widetilde{x})}}{\tau_j|(\det \operatorname{Im}\Phi'')(\widetilde{x})|^{\frac{1}{2}}} + o\left(\frac{1}{\tau_j}\right). \tag{4.5}$$

Integrating by parts we have

$$\begin{aligned}
&\int_\Omega e_2 q(|a|^2 e^{\tau_j(\Phi-\overline{\Phi})} + |a|^2 e^{\tau_j(\overline{\Phi}-\Phi)})dx \\
= &\int_\Omega e_2 q|a|^2 \left(\frac{(\nabla\psi, \nabla e^{\tau_j(\Phi-\overline{\Phi})})}{2i\tau_j|\nabla\psi|^2} - \frac{(\nabla\psi, \nabla e^{\tau_j(\overline{\Phi}-\Phi)})}{2i\tau_j|\nabla\psi|^2}\right)dx \\
= &-\int_\Omega \operatorname{div}\left(\frac{e_2 q|a|^2\nabla\psi}{2i\tau_j|\nabla\psi|^2}\right)(e^{\tau_j(\Phi-\overline{\Phi})} - e^{\tau_j(\overline{\Phi}-\Phi)})dx \\
+ &\int_{\partial\Omega} \frac{q|a|^2}{2i\tau_j|\nabla\psi|^2}\frac{\partial\psi}{\partial\nu}(e^{\tau_j(\Phi-\overline{\Phi})} - e^{\tau_j(\overline{\Phi}-\Phi)})d\sigma \\
= &-\int_{\operatorname{supp} e_2} \operatorname{div}\left(\frac{e_2 q|a|^2\nabla\psi}{2i\tau_j|\nabla\psi|^2}\right)(e^{\tau_j(\Phi-\overline{\Phi})} - e^{\tau_j(\overline{\Phi}-\Phi)})dx \\
+ &\int_{\Gamma_-\cup\Gamma_+} \frac{q|a|^2}{2i\tau_j|\nabla\psi|^2}\frac{\partial\psi}{\partial\nu}(e^{2\tau_j i\psi} - e^{-2\tau_j i\psi})d\sigma + o(\frac{1}{\tau_j}) \quad \text{as } \tau_j \to +\infty.
\end{aligned}$$

In the last equality, we used that $e^{\tau_j(\Phi-\overline{\Phi})} - e^{\tau_j(\overline{\Phi}-\Phi)} = 0$ on Γ_0 which follows by (2.3) and $\operatorname{Im}\Phi = 0$ on Γ_0, and applied (3.17) in order to show that $\operatorname{div}\left(\frac{e_2 q|a|^2\nabla\psi}{2i\tau_j|\nabla\psi|^2}\right)$ and $\frac{q|a|^2}{2i\tau_j|\nabla\psi|^2}$ are bounded functions. Applying Proposition 2.4 we obtain

$$\int_\Omega e_2 q(|a|^2 e^{\tau_j(\Phi-\overline{\Phi})} + |a|^2 e^{\tau_j(\overline{\Phi}-\Phi)})dx = o(\frac{1}{\tau_j}) \quad \text{as } \tau_j \to +\infty.$$

Since the function ψ is strictly monotone on $\Gamma_-\cup\Gamma_+$, we have

$$\int_{\Gamma_-\cup\Gamma_+} \frac{q|a|^2}{2i\tau_j|\nabla\psi|^2}\frac{\partial\psi}{\partial\nu}(e^{2\tau_j i\psi} - e^{-2\tau_j i\psi})d\sigma = o(\frac{1}{\tau_j}) \quad \text{as } \tau_j \to +\infty.$$

Therefore

$$\int_\Omega q(|a|^2 e^{\tau_j(\Phi-\overline{\Phi})} + |a|^2 e^{\tau_j(\overline{\Phi}-\Phi)})dx = o\left(\frac{1}{\tau_j}\right). \tag{4.6}$$

Next we claim that

$$\int_\Omega q(e^{-\tau_j\Phi}a+e^{-\tau_j\overline{\Phi}}\overline{a})u_-e^{\tau_j\varphi}dx=\int_{\Gamma_-}\frac{q|a|^2}{\tau_j}\mathrm{Re}\left\{\frac{1}{\partial_{x_2}(\overline{\Psi}-\Phi)}\right\}d\sigma+o(\frac{1}{\tau_j}) \tag{4.7}$$

and

$$\int_\Omega q(e^{\tau_j\Phi}a+e^{\tau_j\overline{\Phi}}\overline{a})v_+e^{-\tau_j\varphi}dx=\int_{\Gamma_+}\frac{q|a|^2}{\tau_j}\mathrm{Re}\left\{\frac{1}{\partial_{x_2}(\overline{\Psi}-\Phi)}\right\}d\sigma+o(\frac{1}{\tau_j}) \tag{4.8}$$

as $\tau_j\to+\infty$.

Indeed, by (3.56) and (3.24)

$$\mathcal{K}=\int_\Omega q(e^{-\tau_j\Phi}a+e^{-\tau_j\overline{\Phi}}\overline{a})u_-e^{\tau_j\varphi}dx=\int_\Omega q(e^{-\tau_j\Phi}a+e^{-\tau_j\overline{\Phi}}\overline{a})\chi_{\tau_j}(e^{\tau_j\overline{\Psi}}\overline{(a+\hat{a}_0/\tau_j)}$$
$$+e^{\tau_j\Psi}(a+\hat{a}_1/\tau_j))dx=\int_\Omega q\chi_{\tau_j}(a\overline{(a+\hat{a}_0/\tau_j)}e^{\tau_j(\overline{\Psi}-\Phi)}+\bar{a}\overline{(a+\hat{a}_0/\tau_j)}e^{\tau_j(\overline{\Psi}-\bar{\Phi})}$$
$$+a(a+\hat{a}_1/\tau_j)e^{\tau_j(\Psi-\Phi)}+\bar{a}(a+\hat{a}_1/\tau_j)e^{\tau_j(\Psi-\bar{\Phi})})dx=$$
$$\int_{\partial\Omega}q\chi_{\tau_j}(a\overline{(a+\hat{a}_0/\tau_j)}\frac{\nu_2}{\tau_j\partial_{x_2}(\overline{\Psi}-\Phi)}+\bar{a}\overline{(a+\hat{a}_0/\tau_j)}e^{2\tau_j i\psi}\frac{1}{2}\frac{(\nu_1+i\nu_2)}{\tau_j\partial_{\bar{z}}(\overline{\Psi}-\Phi)}$$
$$+a(a+\hat{a}_1/\tau_j)e^{-2\tau_j i\psi}\frac{1}{2}\frac{(\nu_1-i\nu_2)}{\tau_j\partial_z(\Psi-\Phi)}+\bar{a}(a+\hat{a}_1/\tau_j)\frac{\nu_2}{\tau_j\partial_{x_2}(\Psi-\bar{\Phi})})d\sigma-$$
$$-\frac{1}{\tau_j}\int_\Omega(B_1(x,D)^*(q\chi_{\tau_j}a\overline{(a+\hat{a}_0/\tau_j)})e^{\tau_j(\overline{\Psi}-\Phi)}+B_2(x,D)^*(q\chi_{\tau_j}\bar{a}\overline{(a+\hat{a}_0/\tau_j)})e^{\tau_j(\overline{\Psi}-\bar{\Phi})}$$
$$+B_3(x,D)^*(q\chi_{\tau_j}a(a+\hat{a}_1/\tau_j))e^{\tau_j(\Psi-\Phi)}+B_4(x,D)^*(q\chi_{\tau_j}\bar{a}(a+\hat{a}_1/\tau_j))e^{\tau_j(\Psi-\bar{\Phi})})dx,$$

where

$$B_1(x,D)=\frac{\partial_{x_2}}{\partial_{x_2}(\overline{\Psi}-\Phi)},\ B_2(x,D)=\frac{\partial_{\bar{z}}}{\partial_{\bar{z}}(\overline{\Psi}-\overline{\Phi})},$$

$$B_3(x,D)=\frac{\partial_z}{\partial_z(\Psi-\Phi)},\ B_4(x,D)=\frac{\partial_{x_2}}{\partial_{x_2}(\Psi-\bar{\Phi})}.$$

Obviously

$$\int_{\partial\Omega}q\chi_{\tau_j}(a\overline{(a+\hat{a}_0/\tau_j)}\frac{\nu_2}{\tau_j\partial_{x_2}(\overline{\Psi}-\Phi)}+\bar{a}\overline{(a+\hat{a}_0/\tau_j)}e^{2\tau_j i\psi}\frac{1}{2}\frac{(\nu_1+i\nu_2)}{\tau_j\partial_{\bar{z}}(\overline{\Psi}-\Phi)}$$

$$+\overline{a(a+\hat{a}_1/\tau_j)}e^{-2\tau_j i\psi}\frac{1}{2}\frac{(\nu_1-i\nu_2)}{\tau_j\partial_z(\Psi-\Phi)}+\bar{a}(a+\hat{a}_1/\tau_j)\frac{\nu_2}{\tau_j\partial_{x_2}(\Psi-\bar{\Phi})})d\sigma=$$

$$\int_{\Gamma_-}q\left(\frac{a\bar{a}}{\tau_j\partial_{x_2}(\overline{\Psi}-\Phi)}+\frac{\bar{a}a}{\tau_j\partial_{x_2}(\Psi-\bar{\Phi})}\right)d\sigma+o(\frac{1}{\tau_j}).$$

Using this equality and integrating one more time, we have

$$\mathcal{K}=\int_{\Gamma_-} q\left(\frac{a\bar a}{\tau_j\partial_{x_2}(\overline{\Psi}-\Phi)}+\frac{\bar a a}{\tau_j\partial_{x_2}(\Psi-\bar\Phi)}\right)d\sigma \tag{4.9}$$

$$+\frac{1}{\tau_j}\int_\Omega (B_1(x,D)^*(q\chi_{\tau_j}(a\overline{(a+\hat a_0/\tau_j)})e^{\tau_j(\overline{\Psi}-\Phi)}+B_2(x,D)^*(q\chi_{\tau_j}\bar a\overline{(a+\hat a_0/\tau_j)})e^{\tau_j(\overline{\Psi}-\bar\Phi)}$$

$$+B_3(x,D)^*(q\chi_{\tau_j}a(a+\hat a_1/\tau_j))e^{\tau_j(\Psi-\Phi)}+B_4(x,D)^*(q\chi_{\tau_j}\bar a(a+\hat a_1/\tau_j))e^{\tau_j(\Psi-\bar\Phi)})dx=$$

$$\mathcal{K}_1+o(\frac{1}{\tau_j})+\int_{\Gamma_-} q\left(\frac{a\bar a}{\tau_j\partial_{x_2}(\overline{\Psi}-\Phi)}+\frac{\bar a a}{\tau_j\partial_{x_2}(\Psi-\bar\Phi)}\right)d\sigma$$

$$+\int_{\partial\Omega}(B_1(x,D)^*(q\chi_{\tau_j}a\overline{(a+\hat a_0/\tau_j)})\frac{\nu_2}{\tau_j{}^2\partial_{x_2}(\overline{\Psi}-\Phi)}+B_2(x,D)^*(q\chi_{\tau_j}\bar a\overline{(a+\hat a_0/\tau_j)})e^{2\tau_j i\psi}\frac{1}{2}\frac{(\nu_1+i\nu_2)}{\tau_j{}^2\partial_{\bar z}(\overline{\Psi}-\overline{\Phi})}$$

$$+B_3(x,D)^*(q\chi_{\tau_j}a(a+\hat a_1/\tau_j))e^{-2\tau_j i\psi}\frac{1}{2}\frac{(\nu_1-i\nu_2)}{\tau_j{}^2\partial_z(\Psi-\Phi)}+B_4(x,D)^*(q\chi_{\tau_j}\bar a(a+\hat a_1/\tau_j))\frac{\nu_2}{\tau_j{}^2\partial_{x_2}(\Psi-\bar\Phi)})d\sigma,$$

where

$$\mathcal{K}_1$$

$$=\frac{1}{\tau_j{}^2}\int_{\operatorname{supp}\chi_{\tau_j}\cap\mathcal{G}_-}((B_1(x,D)^*)^2(q\chi_{\tau_j}a\overline{(a+\hat a_0/\tau_j)})e^{\tau_j(\bar\Psi-\Phi)}+(B_2(x,D)^*)^2(q\chi_{\tau_j}\bar a\overline{(a+\hat a_0/\tau_j)})e^{\tau_j(\bar\Psi-\bar\Phi)}$$

$$+(B_3(x,D)^*)^2(q\chi_{\tau_j}a(a+\hat a_1/\tau_j))e^{\tau_j(\Psi-\Phi)}+(B_4(x,D)^*)^2(q\chi_{\tau_j}\bar a(a+\hat a_1/\tau_j))e^{\tau_j(\Psi-\bar\Phi)})dx.$$

Since

$$\operatorname{Re}(\Psi-\Phi)\le 0\quad\forall x\in\mathcal{G}_- \tag{4.10}$$

by (3.13), we have

$$|\mathcal{K}_1|\le\frac{1}{\tau_j{}^2}\int_{\operatorname{supp}\chi_{\tau_j}\cap\mathcal{G}_-}(|(B_1(x,D)^*)^2(q\chi_{\tau_j}a\overline{(a+\hat a_0/\tau_j)})|+|(B_2(x,D)^*)^2(q\chi_{\tau_j}\bar a\overline{(a+\hat a_0/\tau_j)})|$$

$$+|(B_3(x,D)^*)^2(q\chi_{\tau_j}a(a+\hat a_1/\tau_j))|+|(B_4(x,D)^*)^2(q\chi_{\tau_j}\bar a(a+\hat a_1/\tau_j))|)dx$$

$$\le\frac{C}{\tau_j{}^2}\int_{\operatorname{supp}\chi_{\tau_j}\cap\mathcal{G}_-}\frac{1}{|\ell_1(x)|^2}dx\le\frac{C\tau_j^{\frac{12}{80}}}{\tau_j{}^2}=o(\frac{1}{\tau_j})\quad\text{as }\tau_j\to+\infty. \tag{4.11}$$

Again, by (3.17) the last boundary integral in (4.9) can be estimated by $o(\frac{1}{\tau_j})$. Then from (4.9) and (4.11) we obtain (4.7).

We prove (4.8) in the similar way.

$$\tilde{\mathcal{K}} = \int_\Omega q(e^{\tau_j\Phi}a + e^{\tau_j\overline{\Phi}}\overline{a})v_+e^{-\tau_j\varphi}dx = \int_\Omega q(e^{\tau_j\Phi}a + e^{\tau_j\overline{\Phi}}\overline{a})\tilde{\chi}_{\tau_j}(e^{-\tau_j\overline{\Psi}}\overline{(a+\hat{b}_0/\tau_j)} \tag{4.12}$$
$$+e^{-\tau_j\Psi}(a+\hat{b}_1/\tau_j))dx = \int_\Omega q\tilde{\chi}_{\tau_j}(a\overline{(a+\hat{b}_0/\tau_j)}e^{-\tau_j(\overline{\Psi}-\Phi)} + \bar{a}\overline{(a+\hat{b}_0/\tau_j)}e^{-\tau_j(\overline{\Psi}-\bar{\Phi})}$$
$$+a(a+\hat{b}_1/\tau_j)e^{-\tau_j(\Psi-\Phi)} + \bar{a}(a+\hat{b}_1/\tau_j)e^{-\tau_j(\Psi-\bar{\Phi})})dx + o(\frac{1}{\tau_j}) =$$
$$-\int_{\partial\Omega} q\tilde{\chi}_{\tau_j}(a\overline{(a+\hat{b}_0/\tau_j)}\frac{\nu_2}{\tau_j\partial_{x_2}(\overline{\Psi}-\Phi)} + \bar{a}\overline{(a+\hat{b}_0/\tau_j)}e^{-2\tau_j i\psi}\frac{1}{2}\frac{(\nu_1+i\nu_2)}{\tau_j\partial_{\bar{z}}(\overline{\Psi}-\bar{\Phi})}$$
$$+a(a+\hat{b}_1/\tau_j)e^{2\tau_j i\psi}\frac{1}{2}\frac{(\nu_1-i\nu_2)}{\tau_j\partial_z(\Psi-\Phi)} + \bar{a}(a+\hat{b}_1/\tau_j)\frac{\nu_2}{\tau_j\partial_{x_2}(\Psi-\bar{\Phi})})d\sigma -$$
$$+\frac{1}{\tau_j}\int_\Omega(B_1(x,D)^*(q\tilde{\chi}_{\tau_j}a\overline{(a+\hat{b}_0/\tau_j)})e^{-\tau_j(\overline{\Psi}-\Phi)} + B_2(x,D)^*(q\tilde{\chi}_{\tau_j}\bar{a}\overline{(a+\hat{b}_0/\tau_j)})e^{-\tau_j(\overline{\Psi}-\bar{\Phi})}$$
$$+B_3(x,D)^*(q\tilde{\chi}_{\tau_j}a(a+\hat{b}_1/\tau_j))e^{-\tau_j(\Psi-\Phi)} + B_4(x,D)^*(q\tilde{\chi}_{\tau_j}\bar{a}(a+\hat{b}_1/\tau_j))e^{-\tau_j(\Psi-\bar{\Phi})})dx$$
$$+o(\frac{1}{\tau_j}) \quad \text{as } \tau_j\to+\infty.$$

Obviously

$$\int_{\partial\Omega} q\tilde{\chi}_{\tau_j}(a\overline{(a+\hat{b}_0/\tau_j)}\frac{\nu_2}{\tau_j\partial_{x_2}(\overline{\Psi}-\Phi)} + \bar{a}\overline{(a+\hat{b}_0/\tau_j)}e^{-2\tau_j i\psi}\frac{1}{2}\frac{(\nu_1+i\nu_2)}{\tau_j\partial_{\bar{z}}(\overline{\Psi}-\bar{\Phi})}$$
$$+a(a+\hat{b}_1/\tau_j)e^{2\tau_j i\psi}\frac{1}{2}\frac{(\nu_1-i\nu_2)}{\tau_j\partial_z(\Psi-\Phi)} + \bar{a}(a+\hat{b}_1/\tau_j)\frac{\nu_2}{\tau_j\partial_{x_2}(\Psi-\bar{\Phi})})d\sigma =$$
$$\int_{\Gamma_+} q\left(\frac{a\overline{a}}{\tau_j\partial_{x_2}(\overline{\Psi}-\Phi)} + \frac{\bar{a}a}{\tau_j\partial_{x_2}(\Psi-\bar{\Phi})}\right)d\sigma + o(\frac{1}{\tau_j}).$$

Using this equality and integrating by parts in (4.12) once more, we have

$$\tilde{\mathcal{K}} = -\int_{\Gamma_+} q\left(\frac{a\overline{a}}{\tau_j\partial_{x_2}(\overline{\Psi}-\Phi)} + \frac{\bar{a}a}{\tau_j\partial_{x_2}(\Psi-\bar{\Phi})}\right)d\sigma \tag{4.13}$$
$$-\frac{1}{\tau_j}\int_\Omega(B_1(x,D)^*(q\tilde{\chi}_{\tau_j}(a\overline{(a+\hat{b}_0/\tau_j)})e^{-\tau_j(\overline{\Psi}-\Phi)} + B_2(x,D)^*(q\tilde{\chi}_{\tau_j}\bar{a}\overline{(a+\hat{b}_0/\tau_j)})e^{-\tau_j(\overline{\Psi}-\bar{\Phi})}$$
$$+B_3(x,D)^*(q\tilde{\chi}_{\tau_j}a(a+\hat{b}_1/\tau_j))e^{-\tau_j(\Psi-\Phi)} + B_4(x,D)^*(q\tilde{\chi}_{\tau_j}\bar{a}(a+\hat{b}_1/\tau_j))e^{-\tau_j(\Psi-\bar{\Phi})})dx$$
$$=\tilde{\mathcal{K}}_1 + o(\frac{1}{\tau_j}) - \int_{\Gamma_+} q\left(\frac{a\overline{a}}{\tau_j\partial_{x_2}(\overline{\Psi}-\Phi)} + \frac{\bar{a}a}{\tau_j\partial_{x_2}(\Psi-\bar{\Phi})}\right)d\sigma$$
$$-\int_{\partial\Omega}\Bigg(B_1(x,D)^*(q\tilde{\chi}_{\tau_j}a\overline{(a+b_0/\tau_j)})\frac{\nu_2}{\tau_j{}^2\partial_{x_2}(\overline{\Psi}-\Phi)} + B_2(x,D)^*(q\tilde{\chi}_{\tau_j}\bar{a}\overline{(a+\hat{b}_0/\tau_j)})e^{-2\tau_j i\psi}\frac{1}{2}\frac{(\nu_1+i\nu_2)}{\tau_j{}^2\partial_{\bar{z}}(\Psi-\Phi)}$$
$$+B_3(x,D)^*(q\tilde{\chi}_{\tau_j}a(a+\hat{b}_1/\tau_j))e^{2\tau_j i\psi}\frac{1}{2}\frac{(\nu_1-i\nu_2)}{\tau_j{}^2\partial_z(\Psi-\Phi)} + B_4(x,D)^*(q\tilde{\chi}_{\tau_j}\bar{a}(a+\hat{b}_1/\tau_j))\frac{\nu_2}{\tau_j{}^2\partial_{x_2}(\Psi-\bar{\Phi})}d\sigma,$$

where

$$\tilde{\mathcal{K}}_1 = \frac{1}{{\tau_j}^2}\int_{\mathrm{supp}\,\tilde\chi_{\tau_j}\cap\mathcal{G}_+}((B_1(x,D)^*)^2(q\tilde\chi_{\tau_j}a\overline{(a+\hat b_0/\tau_j)})e^{-\tau_j(\overline{\Psi}-\Phi)}+(B_2(x,D)^*)^2(q\tilde\chi_{\tau_j}\bar a\overline{(a+\hat b_0/\tau_j)})e^{-\tau_j(\overline{\Psi}-\bar\Phi)}$$
$$+(B_3(x,D)^*)^2(q\tilde\chi_{\tau_j}a(a+\hat b_1/\tau_j))e^{-\tau_j(\Psi-\Phi)}+(B_4(x,D)^*)^2(q\tilde\chi_{\tau_j}\bar a(a+\hat b_1/\tau_j))e^{-\tau_j(\Psi-\bar\Phi)})dx.$$

Observe that

$$\mathrm{Re}\,(\Psi-\Phi)\ge 0,\quad \forall x\in\mathcal{G}_+. \tag{4.14}$$

By (4.14) and (3.16), we have

$$|\tilde{\mathcal{K}}_1|\le\frac{1}{{\tau_j}^2}\int_{\mathrm{supp}\,\tilde\chi_{\tau_j}\cap\mathcal{G}_+}(|(B_1(x,D)^*)^2(q\tilde\chi_{\tau_j}a\overline{(a+\hat b_0/\tau_j)})|+|(B_2(x,D)^*)^2(q\tilde\chi_{\tau_j}\bar a\overline{(a+\hat b_0/\tau_j)})|$$
$$+|(B_3(x,D)^*)^2(q\tilde\chi_{\tau_j}a(a+\hat b_1/\tau_j))|+|(B_4(x,D)^*)^2(q\tilde\chi_{\tau_j}\bar a(a+\hat b_1/\tau_j))|)dx\le$$
$$\frac{C}{{\tau_j}^2}\int_{\mathrm{supp}\,\tilde\chi_{\tau_j}\cap\mathcal{G}_+}\frac{1}{|\tilde\ell_1(x)|^2}dx\le\frac{C{\tau_j}^{\frac{12}{80}}}{{\tau_j}^2}=o(\frac{1}{\tau_j}). \tag{4.15}$$

Applying (4.15) and using the fact that the last boundary integral in (4.13) is $o(\frac{1}{\tau_j})$ we obtain the formula (4.8).

We calculate the two remaining terms in (4.3). By (3.38) and Proposition 2.5 we have:

$$\int_\Omega qu_{11}e^{\tau_j\varphi}(ae^{-\tau_j\Phi}+\bar ae^{-\tau_j\overline{\Phi}})dx=o(\frac{1}{\tau_j}) \tag{4.16}$$
$$-\int_\Omega\left(\frac{e^{\tau_j\Phi}}{\tau_j}\frac{(\partial_{\bar z}^{-1}(aq_1)-M_1)}{4\partial_z\Phi}+\frac{e^{\tau_j\overline{\Phi}}}{\tau_j}\frac{(\partial_z^{-1}(\bar aq_1)-M_3)}{4\overline{\partial_z\Phi}}\right)q(ae^{-\tau_j\Phi}+\bar ae^{-\tau_j\overline{\Phi}})dx=$$
$$-\int_\Omega q\left(\frac{e^{\tau_j(\Phi-\overline{\Phi})}}{\tau_j}\frac{\bar a(\partial_{\bar z}^{-1}(aq_1)-M_1)}{4\partial_z\Phi}+\frac{e^{\tau_j(\overline{\Phi}-\Phi)}}{\tau_j}\frac{a(\partial_z^{-1}(\bar aq_1)-M_3)}{4\overline{\partial_z\Phi}}\right)dx$$
$$-\int_\Omega q\left(\frac{a}{\tau_j}\frac{(\partial_{\bar z}^{-1}(aq_1)-M_1)}{4\partial_z\Phi}+\frac{\bar a}{\tau_j}\frac{(\partial_z^{-1}(\bar aq_1)-M_3)}{4\overline{\partial_z\Phi}}\right)dx+o(\frac{1}{\tau_j})=$$
$$-\int_\Omega q\left(\frac{a}{\tau_j}\frac{(\partial_{\bar z}^{-1}(aq_1)-M_1)}{4\partial_z\Phi}+\frac{\bar a}{\tau_j}\frac{(\partial_z^{-1}(\bar aq_1)-M_3)}{4\overline{\partial_z\Phi}}\right)dx+o(\frac{1}{\tau_j})\text{ as }\tau_j\to+\infty.$$

Similarly by (3.66) and Proposition 2.5

$$\int_\Omega qv_{11}e^{-\tau_j\varphi}(ae^{\tau_j\Phi}+\bar ae^{\tau_j\overline{\Phi}})dx= \tag{4.17}$$
$$+\int_\Omega q\left(\frac{e^{-\tau_j\Phi}}{\tau_j}\frac{(\partial_{\bar z}^{-1}(aq_2)-M_2)}{4\partial_z\Phi}+\frac{e^{-\tau_j\overline{\Phi}}}{\tau_j}\frac{(\partial_z^{-1}(\bar aq_2)-M_4)}{4\overline{\partial_z\Phi}}\right)(ae^{\tau_j\Phi}+\bar ae^{\tau_j\overline{\Phi}})dx+o(\frac{1}{\tau_j})=$$
$$\int_\Omega q\left(\frac{e^{-\tau_j(\Phi-\overline{\Phi})}}{\tau_j}\frac{\bar a(\partial_{\bar z}^{-1}(aq_2)-M_2)}{4\partial_z\Phi}+\frac{e^{\tau_j(\Phi-\overline{\Phi})}}{\tau_j}\frac{a(\partial_z^{-1}(\bar aq_2)-M_4)}{4\overline{\partial_z\Phi}}\right)dx$$
$$+\int_\Omega q\left(\frac{a}{\tau_j}\frac{\partial_{\bar z}^{-1}(aq_2)-M_2}{4\partial_z\Phi}+\frac{\bar a}{\tau_j}\frac{\partial_z^{-1}(\bar aq_2)-M_4}{4\overline{\partial_z\Phi}}\right)dx+o(\frac{1}{\tau_j})=$$
$$\int_\Omega q\left(\frac{a}{\tau_j}\frac{\partial_{\bar z}^{-1}(aq_2)-M_2}{4\partial_z\Phi}+\frac{\bar a}{\tau_j}\frac{\partial_z^{-1}(\bar aq_2)-M_4}{4\overline{\partial_z\Phi}}\right)dx+o(\frac{1}{\tau_j})\quad\text{as }\tau_j\to+\infty.$$

Therefore, applying (4.4), (4.6), (4.7), (4.8), (4.17) and (4.16) in (4.3), we conclude that

$$
\begin{aligned}
&2\frac{\pi(q|a|^2)(\widetilde{x})\mathrm{Re}\, e^{2i\tau_j \mathrm{Im}\Phi(\widetilde{x})}}{|(\det \mathrm{Im}\Phi'')(\widetilde{x})|^{\frac{1}{2}}} + \int_\Omega q(a(a_0+b_0)+\overline{a}(\overline{a}_1+\overline{b}_1))dx\\
+\ &\frac{1}{4}\int_\Omega \left(qa\frac{\partial_{\overline{z}}^{-1}(aq_2)-M_2}{\partial_z\Phi} + q\overline{a}\frac{\partial_z^{-1}(q_2\overline{a})-M_4}{\overline{\partial_z\Phi}}\right)dx\\
&-\frac{1}{4}\int_\Omega \left(qa\frac{\partial_{\overline{z}}^{-1}(q_1a)-M_1}{\partial_z\Phi} + q\overline{a}\frac{\partial_z^{-1}(q_1\overline{a})-M_3}{\overline{\partial_z\Phi}}\right)dx\\
+\ &\int_{\Gamma_-} q\left(\frac{a\overline{a}}{\tau_j\partial_{x_2}(\overline{\Psi}-\Phi)} + \frac{\bar{a}a}{\tau_j\partial_{x_2}(\Psi-\bar{\Phi})}\right)d\sigma\\
-\ &\int_{\Gamma_+} q\left(\frac{a\overline{a}}{\tau_j\partial_{x_2}(\overline{\Psi}-\Phi)} + \frac{\bar{a}a}{\tau_j\partial_{x_2}(\Psi-\bar{\Phi})}\right)d\sigma = o(1)
\end{aligned}
$$

as $\tau_j \to +\infty$. The proof of the proposition is finished. □

Completion of the proof of Theorem 1.2. First we observe that any smooth holomorphic function $\Phi = \varphi + i\psi$ satisfying (2.3) can be approximated by the sequence of harmonic functions constructed in Section 3. Moreover the function satisfying (2.3) has at most one interior critical point. Therefore by Proposition 4.1 the function q is zero at this critical point. Consider the set of harmonic functions ψ such that

$$\psi \text{ is equal to some constant on each connected component of the set } \Gamma_0;$$

$$\frac{\partial\psi}{\partial\vec{\tau}}|_{\Gamma_+} < 0;$$

$$\frac{\partial\psi}{\partial\vec{\tau}}|_{\Gamma_-} > 0.$$

We show that the set of critical points of a harmonic function ψ with the above properties is dense in Ω. In order to see that, it suffices to consider the following case. Let $\partial\Omega = \overline{\cup_{k=1}^4 \Gamma_k}$, where Γ_k is an arc and $\Gamma_j \cap \Gamma_k = \emptyset$ for any $k \ne j$ and Ω is the unit ball centered at zero. Consider the set of harmonic functions ψ with the boundary data $\psi|_{\Gamma_k} = C_k$. We claim that for a generic choice of Γ_k we can find constants C_k such that $\nabla\psi(0) = 0$. Indeed since $\psi(x) = \frac{1}{2\pi}\int_0^{2\pi}\psi(e^{it})\frac{1-|z|^2}{|e^{it}-z|^2}dt$, we have $\partial_z\psi(0) = \frac{1}{2\pi}\int_0^{2\pi}\psi(e^{it})e^{it}dt$.

To see this, let $C_1 = 0, C_4 = 1$ and the endpoints of the arcs Γ_k on the complex plane are given by $e^0, e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}$ with $0 < \theta_1 < \theta_2 < \theta_3 < 2\pi$. Then

$$i\partial_z\psi(0) = C_2(e^{-i\theta_1} - e^{-i\theta_2}) + C_3(e^{-i\theta_2} - e^{-i\theta_3}) + (e^{-i\theta_3} - 1).$$

The equation $\partial_z\psi(0) = 0$ is equivalent to

$$C_2 = -\frac{C_3(e^{-i\theta_2} - e^{-i\theta_3})(e^{i\theta_1} - e^{i\theta_2}) + (e^{-i\theta_3} - 1)(e^{i\theta_1} - e^{i\theta_2})}{|e^{-i\theta_1} - e^{-i\theta_2}|}.$$

The existence of real valued solutions C_2, C_3 to this equation is equivalent to

$$\mathrm{Im}(e^{i(\theta_1-\theta_2)} + e^{i(\theta_2-\theta_3)} - e^{i(\theta_1-\theta_3)}) \ne 0.$$

This clearly is valid for a generic position of θ_j.

In the set $\Gamma_+ \cup \Gamma_-$ we make the choice of four points $\hat{x}_1, \dots, \hat{x}_4$ such that $\hat{x}_1 \in \Gamma_{1,+}, \hat{x}_2 \in \Gamma_{1,-}, \hat{x}_3 \in \Gamma_{2,+}, \hat{x}_2 \in \Gamma_{2,-}$. Denote by $\hat{\Gamma}_1, \dots \hat{\Gamma}_4$ the arcs connecting these points. Consider the conformal mapping Π which transforms the domain Ω into the unit ball and the point $\tilde{x}$ into the center of the coordinate system. Above we show that with a generic choice of the points $\hat{x}_j$, there exists a harmonic function ψ_0 which is equal to some constant on each arc $\Pi(\hat{\Gamma}_k)$. Consider the boundary data $\psi_0(\Pi)$. By $\hat{\psi}$, we denote the corresponding harmonic function. The function $\hat{\psi}$ is equal to constant C_j on each arc $\hat{\Gamma}_j$ and it has only one nondegenerate critical point $\tilde{x}$. Without loss of generality, we may assume that $C_0 = 0$ and $C_4 = -1$ by multiplying, if necessary, the function $\psi_0 \circ \Pi$ by a nonzero constant. Observe that $C_2 < 0$ and $C_3 > C_2$. (Otherwise if at least one of these inequalities fails, then the function $\psi_0 \circ \Pi$ can not have the internal critical point.) In a small neighborhood $\mathcal{F} \subset \cup_{j=1}^2 \Gamma_{j,\pm}$ of the points of discontinuity of the function $\psi_0 \circ \Pi$ we approximate it by a sequence $\{\mu_k\}$ of strictly monotone decreasing or strictly monotone increasing functions. Outside of $\mathcal{F}$ the function μ_k are equal to the corresponding constants.

Moreover

$$\mu_k \to \psi_0 \circ \Pi \quad \text{in } L^2(\partial\Omega).$$

We claim that for all sufficiently large k the harmonic functions ψ_k such that $\psi_k|_{\partial\Omega} = \mu_k$ have a unique interior critical point which we denote as $\tilde{x}_j$. Moreover $\tilde{x}_j \to \tilde{x}$. Our proof is by contradiction. Suppose that for large j, the functions ψ_j do not have interior critical point or the sequence converges to some point $y \neq \tilde{x}$. Indeed for any $\Omega_0 \subset\subset \Omega$

$$\psi_k \to \psi_0 \circ \Pi \quad \text{in } C^2(\Omega_0). \tag{4.18}$$

On the other hand, it is known that the number N of zeros of a holomorphic function $f(z)$ on a domain G is given by

$$N = \frac{1}{2\pi i} \int_{\partial G} \frac{\partial_z f}{f(z)} dz. \tag{4.19}$$

Solving the system of Cauchy-Riemann equations we construct a holomorphic function $\Phi_j = \varphi_j + i\psi_j$. By (4.18) for all sufficiently small positive δ and all large k, we have $\frac{1}{2\pi i} \int_{S(\tilde{x},\delta)} \frac{\partial_z^2(\varphi_k + i\psi_k)}{\partial_z(\varphi_k + i\psi_k)} dz = 1$. This means that the function $\varphi_k + i\psi_k$ has a critical point in the ball $B(\tilde{x}, \delta)$. However this function can not have more than one critical point. Therefore $y = \tilde{x}$. The proof of the theorem is completed. $\square$

5. **Appendix.**

Consider the Cauchy problem for the Cauchy-Riemann equations

$$L(\phi, \psi) = \left(\frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_2}, \frac{\partial \phi}{\partial x_2} + \frac{\partial \psi}{\partial x_1} \right) = 0 \quad \text{in } \Omega, \quad (\phi, \psi)\,|_{\Gamma_0} = (b_1(x), b_2(x)), \tag{5.1}$$
$$(\phi + i\psi)(\hat{x}_j) = c_{0,j},\ j \in \{1, \dots N\}.$$

Here $\hat{x}_1, \dots \hat{x}_N$ be arbitrary fixed points in Ω. We consider the pair b_1, b_2 and the complex numbers $\vec{C} = (c_{0,1}, ..., c_{0,N})$ as initial data for (5.1). The following proposition establishes the solvability of (5.1) for a dense set of Cauchy data.

Proposition 5.1. *There exists a set* $\mathcal{O} \subset C^{100}(\overline{\Gamma_0})^2 \times \mathbb{C}^N$ *such that for each* $(b_1, b_2, \vec{C}) \in \mathcal{O}$, *(5.1) has at least one solution* $(\phi, \psi) \in (C^{100}(\overline{\Omega}))^2$ *and* $\overline{\mathcal{O}} = C^{100}(\overline{\Gamma_0})^2 \times \mathbb{C}^N$.

Consider the Cauchy problem for the Cauchy-Riemann equations

$$L(\phi, \psi) = \left(\frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_2}, \frac{\partial \phi}{\partial x_2} + \frac{\partial \psi}{\partial x_1}\right) = 0 \quad \text{in } \Omega, \quad (\phi, \psi)\,|_{\Gamma_0} = (b(x), 0), \tag{5.2}$$

$$\frac{\partial^l}{\partial z^l}(\phi + i\psi)(\hat{x}_j) = c_{l,j}, \quad \forall j \in \{1, \dots N\} \quad \text{and } \forall l \in \{0, \dots, 5\}.$$

Here $\hat{x}_1, \dots \hat{x}_N$ be arbitrary fixed points in Ω. We consider the function b and the complex numbers $\vec{C} = (c_{0,1}, c_{0,2}, ..., c_{0,N}, ..., c_{5,1}, c_{5,2}, ..., c_{5,N})$ as initial data for (5.2). The following proposition establishes the solvability of (5.2) for a dense set of Cauchy data.

Corollary 5.1. *There exists a set* $\mathcal{O} \subset C^6(\overline{\Gamma_0}) \times \mathbb{C}^{6N}$ *such that for each* $(b, \vec{C}) \in \mathcal{O}$, *the problem (5.2) has at least one solution* $(\phi, \psi) \in C^6(\overline{\Omega}) \times C^6(\overline{\Omega})$ *and* $\overline{\mathcal{O}} = C^6(\overline{\Gamma}) \times \mathbb{C}^{6N}$.

Now we give the proof of Proposition 2.7.

Proof. Let us introduce the space

$$H = \left\{ v \in H_0^1(\Omega);\ \Delta v + q_0 v \in L^2(\Omega), \frac{\partial v}{\partial \nu}|_{\Gamma_+} = 0 \right\}$$

with the scalar product

$$(v_1, v_2)_H = \int_\Omega e^{2\tau\varphi}(\Delta v_1 + q_0 v_1)\overline{(\Delta v_2 + q_0 v_2)}dx.$$

By Proposition 2.1, H is a Hilbert space. Consider the linear functional on $H : v \to \int_\Omega v\overline{f}dx + \int_{\Gamma_-} g\overline{\frac{\partial v}{\partial \nu}}d\sigma$. By (2.4) this is a continuous linear functional with the norm estimated by a constant $C_{12}(\|fe^{\tau\varphi}\|_{L^2(\Omega)}/\tau^{\frac{1}{2}} + \|ge^{\tau\varphi}/\sqrt{|\partial_\nu \varphi|}\|_{L^2(\Gamma_-)})$. Therefore by the Riesz representation theorem there exists an element $\widehat{v} \in H$ so that

$$\int_\Omega v\overline{f}dx + \int_{\Gamma_-} g\overline{\frac{\partial v}{\partial \nu}}d\sigma = \int_\Omega e^{2\tau\varphi}(\Delta\widehat{v} + q_0\widehat{v})\overline{(\Delta v + q_0 v)}dx.$$

Then, as a solution to (2.8), we take the function $u = e^{2\tau\varphi}(\Delta\widehat{v} + q_0\widehat{v})$. □

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