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DETERMINATION OF STRUCTURES IN THE SPACE-TIME FROM LOCAL MEASUREMENTS: A DETAILED EXPOSITION

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Abstract: *We consider inverse problems for the Einstein equation with a time-depending metric on a 4-dimensional globally hyperbolic Lorentzian manifold* (M, g). *We formulate the concept of active measurements for relativistic models. We do this by coupling the Einstein equation with equations for scalar fields and study the system* $Ein(g) = T$, $T = T(g, \phi) + F_1$, *and* $\square_g \phi = F_2 + S(g, \phi, F_1, F_2)$. *Here* $F = (F_1, F_2)$ *correspond to the perturbations of the physical fields which we control and* S *is a secondary source corresponding to the adaptation of the system to the perturbation so that the conservation law* $div_g(T) = 0$ *will be satisfied.*

The inverse problem we study is the question, do the observation of the solutions (g, ϕ) *in an open subset* $U \subset M$ *of the space-time corresponding to sources* F *supported in* U *determine the properties of the metric in a larger domain* $W \subset M$ *containing* U. *To study this problem we define the concept of light observation sets and show that these sets determine the conformal class of the metric. This corresponds to passive observations from a distant area of the space which is filled by light sources (e.g. we see light from stars varying in time). One can apply the obtained result to solve inverse problems encountered in general relativity and in various practical imaging problems.*

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Contents

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1. Introduction and main results

We consider inverse problems for non-linear 2nd order hyperbolic equations with time-depending metric on a globally hyperbolic Lorentzian manifold (M, g) of dimension $n \geq 2+1$ and in particular, the Einstein equation. Roughly speaking, we study the problem, can an observer on a Lorentzian manifold, satisfying certain natural causality conditions, determine the structure of the surrounding space-time by doing measurements near its world line. Thus the problem we are interested in this paper is the inverse problem with respect to the evolution problem in the general relativity. Let us note that this last problem has recently attracted much interest in the mathematical community with many important results been obtained, see e.g. [11, 13, 15, 17, 18, 19, 51, 53, 54, 58].

We consider two different kind of inverse problems. In the first one we have passive observations: We detect on an open set $U \subset M$ the wave fronts of the waves produced by the point sources located at points q in a relatively compact subset $V \subset M$. We call such observations the light observation sets $\mathcal{O}_U(q) \subset TU$, where TU is the tangent bundle on U.

We ask, can the conformal class of the metric in V be determined from these observations. In the second class of problems we consider active measurements: We consider non-linear hyperbolic partial differential equations on M assuming that we can control sources supported in U which produce waves that we can measure in the same set. We ask, can the properties of the metric (the metric itself or its conformal class) or the coefficients of the equation be determined in a suitable larger set W containing the set U. For instance in the context of relativity theory, these correspond to the following examples: In the first case we consider passive observations from a distant area of the space which is filled by light sources (e.g. we see light from stars varying in time). In the second case we assume that one can cause local perturbations in the stress-energy tensor and measures locally the caused perturbations of the gravitational field.

1.1. **Notations.** Let (M, g) be a C^∞-smooth n-dimensional manifold with C^∞-smooth Lorentzian metric g with a causal structure (For this and other definitions for Lorentzian manifolds, see the next section.) The tangent bundle of M is denoted by TM and the projection to the base is denoted by $\pi : TM \to M$.

Let us introduce next some notations needed below. For $x, y \in M$ we say that x is in the chronological past of y and denote $x \ll y$ if $x \neq y$ and there is a time-like path from x to y. If $x \neq y$ and there is a causal path from x to y, we say that x is in the causal past of y and denote $x < y$. If $x < y$ or $x = y$ we denote $x \leq y$. The chronological future $I^+(p)$ of $p \in M$ consist of all points $x \in M$ such that $p \ll x$, and the causal future $J^+(p)$ of $p \in M$ consist of all points $x \in M$ such that $p \leq x$. Similarly chronological past $I^-(p)$ of $p \in M$ consist of all points $x \in M$ such that $x \ll p$ and the causal past $J^-(p)$ of $p \in M$ consist of all points $x \in M$ such that $x \leq p$. For a set A we denote $J^\pm(A) = \cup_{p \in A} J^\pm(p)$. We also denote $J(p, q) := J^+(p) \cap J^-(q)$ and $I(p, q) := I^+(p) \cap I^-(q)$. If we need to emphasize the metric g which is used to define the causality, we denote by $J_g^\pm(q)$ or $J_{M,g}^\pm(q)$ the past and the future sets of $q \in M$ with respect to a Lorentzian metric g etc.

Let $\gamma_{x,\xi}(t) = \exp_x(t\xi)$ denote the geodesics in (M, g). Also, let $TU = \{(x, \xi) \in TM;\ x \in U\}$. Let $L_x M$ denote the light-like directions of $T_x M$, and $L_x^+ M$ and $L_x^- M$ denote the future and past pointing light-like vectors, correspondingly, and $L_x^{*,+} M$ and $L_x^{*,-} M$ be the future and past pointing light-like co-vectors. Sometimes, to emphasize the metric, we denote $L_x^+ M = L_x^+(M, g)$, etc. We also denote $\mathcal{L}_g^+(x) = \exp_x(L_x^+ M) \cup \{x\}$ the union of the image of the future light-cone in the exponential map of (M, g) and the point x.

By [11], an open Lorenzian manifold (M, g) is globally hyperbolic if and only if it has a causal structure where there are no closed causal paths in M and for all $q^-, q^+ \in M$ such that $q^- < q^+$ the set $J(q^-,q^+)\subset M$ is compact. We assume throughout the paper that (M,g) is globally hyperbolic. Roughly speaking, this means that we have no naked singularities which we could reach by moving along a time-like path starting from a point q^- and ending to a point q^+.

When g is a Lorentzian metric, having eigenvalues $\lambda_j(x)$ and eigenvectors $v_j(x)$ in some local coordinates (U,X), we will use also the corresponding Riemannian metric, denoted $g^+=\text{Riem}(g)$ which has the eigenvalues $|\lambda_j(x)|$ and the eigenvectors $v_j(x)$ in the same local coordinates (U,X). Let $B_{g^+}(x,r)=\{y\in M;\ d_{g^+}(x,y)<r\}$.

All functions $u(x)$ defined on manifold M etc. are real-valued unless otherwise mentioned. Finally, when X is a set, let $P(X)=2^X=\{Z;\ Z\subset X\}$ denote the power set of X.

Let $\mu_g:[-1,1]\to M$ be a freely falling observer, that is, a time-like geodesic on (M,g). Let $-1<s_{-2}<s_{-1}<s_{+1}<s_{+2}<s_{+3}<s_{+4}<1$ and denote $p^-=\mu_g(s_{-1})$ and $p^+=\mu_g(s_{+1})$. Below, we also denote $s_\pm=s_{\pm1}$.

When $z_0=\mu_g(s_{-2})$ and $\eta_0=\partial_s\mu_g(s)|_{s=s_{-2}}$, let $\mathcal{U}_{z_0,\eta_0}(h)$ be the h-neighborhood of (z_0,η_0) in the Sasaki metric of (TM,g^+). We use below a small parameter $\widehat h>0$. For $(z,\eta)\in\mathcal{U}_{z_0,\eta_0}(2\widehat h)$ we define on (M,g) a freely falling observer $\mu_{g,z,\eta}:[-1,1]\to M$, such that $\mu_g(s_{-2})=z$, and $\partial_s\mu_g(s_{-2})=\eta$. We assume that $\widehat h$ is so small that for all $(z,\eta)\in\mathcal{U}_{z_0,\eta_0}(2\widehat h)$ the geodesic $\mu_{g,z,\eta}([-1,s_{+2}])\subset M$ is well defined and time-like and satisfies

$$\begin{aligned}(1)\qquad &\mu_{g,z,\eta}(s_{-2})\in I_g^-(\mu_{g,z_0,\eta_0}(s_{-1})),\quad \mu_{g,z,\eta}(s_{+2})\in I_g^+(\mu_{g,z_0,\eta_0}(s_{+1}),\\ &\mu_{g,z,\eta}(s_{+4})\in I_g^+(\mu_{g,z_0,\eta_0}(s_{+3})).\end{aligned}$$

We denote, see Fig. 1.

$$(2)\qquad U_g=\bigcup_{(z,\eta)\in\mathcal{U}_{z_0,\eta_0}(\widehat h)}\mu_{g,z,\eta}((-1,1)).$$

1.2. **Inverse problem for the light observation sets.** Let us first consider M with a fixed metric g and denote below $U=U_g$. Next we define the light-observation set for point q corresponding to observations from a light source at the point q, see Fig. 2.

Definition 1.1. *The light-observation set corresponding to the point $q\in M$ and the observation set $U\subset M$ is*

$$\mathcal{O}_U(q)\ =\ \{(\gamma_{q,\eta}(r),\dot\gamma_{q,\eta}(r))\in TU;\ r\geq 0,\ \eta\in L_q^+M\}.$$

The set of the light-observation points corresponding to $q\in M$ is

$$\mathcal{P}_U(q):=\{\gamma_{q,\eta}(r)\in U;\ r\geq 0,\ \eta\in L_q^+M\}=\pi(\mathcal{O}_U(q)).$$

In the following, we extend any map $F:A\to A'$ to the map $F:P(A)\to P(A')$ by setting $F(B)=\{F(b);\ b\in B\}$ for $B\subset A$. We call $F:P(A)\to P(A')$ the power set extension of $F:A\to A'$.

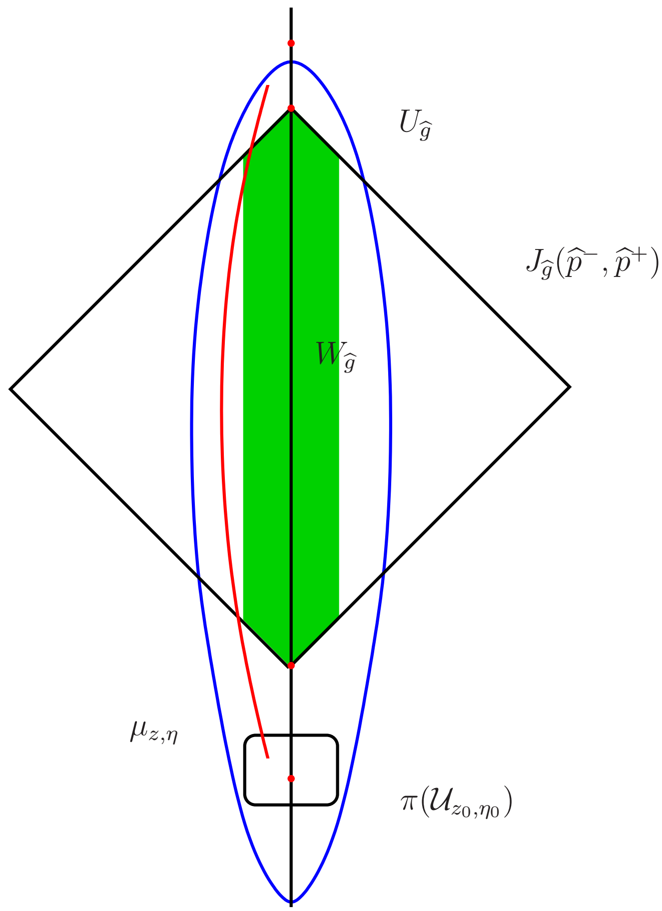

FIGURE 1. Setting throughout the paper: A schematic figure where the space-time is represented as the 2-dimensional set $\mathbb{R}^{1+1}$. In the figure the black vertical line is the freely falling observer $\widehat{\mu}([-1,1])$ and the four red points on it are $z_0 = \widehat{\mu}(s_{-2})$, $\widehat{p}^- = \widehat{\mu}(s_-)$, $\widehat{p}^+ = \widehat{\mu}(s_+)$, and $\widehat{\mu}(s_{+2})$. The rounded black square is $\pi(\mathcal{U}_{z_0,\eta_0})$ that is is a neighborhood of z_0, and the red curve starting from $z \in \pi(\mathcal{U}_{z_0,\eta_0})$ is the time-like geodesic $\mu_{\widehat{g},z,\eta}([s_{-2},1))$. The boundary of the set $U_{\widehat{g}}$ is shown on blue. The green area is the set $W_{\widehat{g}} \subset U_{\widehat{g}}$ where the Fermi-type coordinates are defined, and the black "diamond" is the set $J_{\widehat{g}}(\widehat{p}^-, \widehat{p}^+) = J^+_{\widehat{g}}(\widehat{p}^-) \cap J^-_{\widehat{g}}(\widehat{p}^+)$.

In the following, we will consider the collection $\mathcal{O}_U(V) := \{\mathcal{O}_U(q);\ q \in V\} \subset P(TU)$ of all light observation sets corresponding to the points in an open relatively compact set $V \subset M$. Note that the collection $\mathcal{O}_U(V)$ is considered just as a subset of $P(TU)$ and for a given element $A \in \mathcal{O}_U(V)$ we do not know what is the point q for which $A = \mathcal{O}_U(q)$.

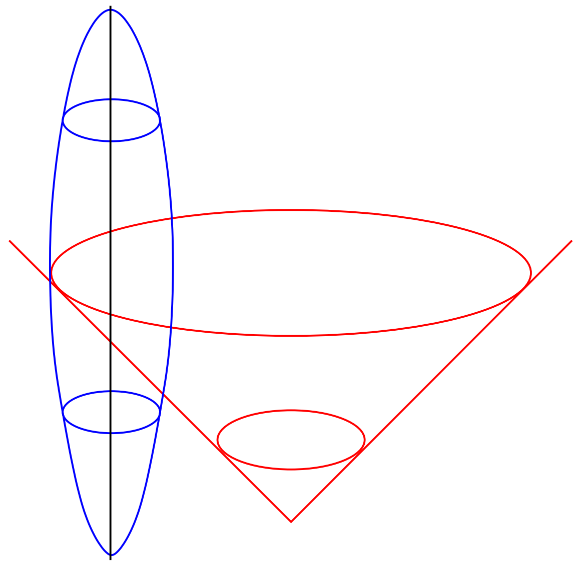

FIGURE 2. A schematic figure where the space-time is represented as the 3-dimensional set $\mathbb{R}^{1+2}$. The future light cone $\mathcal{L}^+_{\widehat g}(q)$ corresponding to the point q is shown as a red cone. The point q is the tip of the cone. The boundary of the set $U_{\widehat g}$ is shown on blue. The set of the light observation points $\mathcal{P}_U(q)$, see Definition 1.1, is the intersection of the set $\mathcal{L}^+_{\widehat g}(q)$ and the set $U_{\widehat g}$.

Later we will introduce a topology on a suitable subset $\mathbb{S}\subset P(TU)$ that contains $\mathcal{O}_U(V)$. Then we can consider the image of the manifold V in the embedding (see Lemma 2.8 below) $\mathcal{O}_U: q\mapsto \mathcal{O}_U(q)$ as an embedded manifold in $\mathbb{S}$, or in $P(TU)$. If we assume that U is given, the set $\mathcal{O}_U(V)$ will be a homeomorphic representation of the unknown manifold V in a known topological space $\mathbb{S}$. For a compact Riemannian manifold N, an analogous representation is the metric space $K(N)\subset \mathrm{Lip}(N)$ is obtained using the Kuratowski-Wojdyslawski embedding, $K: x\mapsto \mathrm{dist}(x,\cdot\,)$, from the metric space N to space of Lipschitz functions $\mathrm{Lip}(N)$ on N. In several inverse problems for Riemannian manifolds with boundary, a homeomorphic image of the compact manifold N has be obtained by using the embedding $R: x\mapsto \mathrm{dist}(x,\cdot\,)$, $R: N\to \mathrm{Lip}(\partial N)$, see [2, 47, 44, 48].

Our first goal is prove that the collection of the light observation sets, $\mathcal{O}_U(V)$, determine the conformal type of the Lorentzian manifold (V,g). As $\pi(\mathcal{O}_U(q))=\mathcal{P}_U(q)$, it is enough to show that $\mathcal{P}_U(V)=\{\mathcal{P}_U(q);\ q\in V\}$ determines the conformal type of (V,g), see Fig. 3.

Theorem 1.2. *Let (M_j,g_j), $j=1,2$ be two open, smooth, globally hyperbolic $(1,n-1)$ Lorentzian manifolds of dimension n, $n\geq 3$ and let $p_j^+,p_j^-\in M_j$ be the endpoints of a time-like geodesic $\mu_{g_j}([s_-,s_+])\subset$*

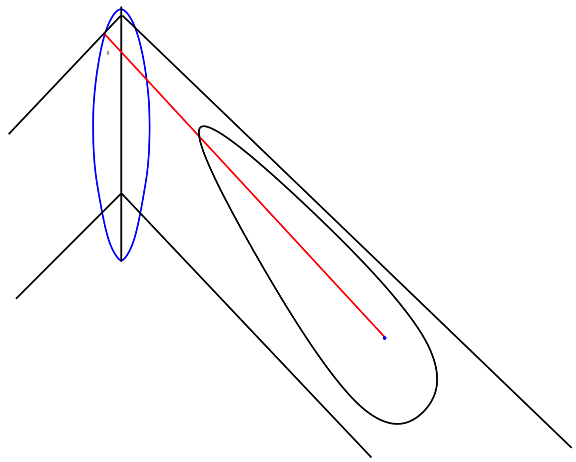

FIGURE 3. A schematic figure where the space-time is represented as the 2-dimensional set $\mathbb{R}^{1+1}$. In Theorem 1.2, we consider a relatively compact set $V \subset I^-(p^+) \setminus J^-(p^-)$. The boundary of the set V is shown in the figure with a black curve. The red curve is a light ray from a point $x \in V$ that intersect the blue set $U_{\widehat{g}}$.

M_j, that is, $p_j^\pm = \mu_{g_j}(s_\pm)$. Let $U_j \subset M_j$ be open relatively compact neighborhood of $\mu_j([s_-, s_+])$ given by (2). Moreover, let V_j be open, relatively compact subsets of $I^-(p_j^+) \setminus J^-(p_j^-) \subset M_j$, $j = 1, 2$.

Let us denote by

$$\mathcal{P}_{U_j}(V_j) = \{\mathcal{P}_{U_j}(q);\ q \in V_j\} \subset P(U_j)$$

the collections of the observation point sets on manifold (M_j, g_j) corresponding to the points in the set V_j. Assume that there is a diffeomorphism $\Phi : U_1 \to U_2$ such that $\Phi(\mu_1(s)) = \mu_2(s)$, $s \in [s_-, s_+]$ and the the power set extension of Φ defines a bijection

$$\Phi : \mathcal{P}_{U_1}(V_1) \to \mathcal{P}_{U_2}(V_2).$$

Then there is a diffeomorphism $\Psi : V_1 \to V_2$ and the metric $\Psi^* g_2$ is conformal to g_1.

Theorem 1.2 can be stated by saying that if an observer moves along a path μ then the diffeomorphism-type of the neighborhood $U \subset M$ of μ and the collection of the observation point sets $\{\mathcal{P}_U(q);\ q \in V\}$ determine uniquely the Lorentzian manifold (V, g) up to a conformal deformation. We note that by the strong hyperbolicity any set $V \subset\subset I(p^-, p^+)$ satisfies also the condition $V \subset I^-(p^+) \setminus J^-(p^-)$.

Note that we do not assume that (M, g) is complete, which is crucial in relativity due to the presence of singularities.

In the case when metric g is known in U_g and g is Ricci flat (i.e. corresponds to vacuum) in a set W that intersects U_g, after constructing the conformal structure we can find the whole metric tensor by constructing the conformal factor along light like geodesics that start at U_g and are subset of W (see Fig. 4). This fact is formulated more precisely in the following corollary.

Corollary 1.3. *Assume that (M_j, g_j) and U_j, V_j, $j = 1, 2$ and the conformal map $\Psi : V_1 \to V_2$ are as in Theorem 1.2. Moreover, assume that $\Phi = \Psi|_{U_1} : (U_1, g_1) \to (U_2, g_2)$ is an isometry and for $j = 1, 2$ there are sets $W_j \subset M_j$ such that the Ricci curvature of g_j is zero in W_j, and $V_j \subset W_j \cup U_j$, $\Psi(U_1 \setminus W_1) = U_2 \setminus W_2$. Moreover, assume that any point $x \in V_1 \cap W_1$ can be connected to some point $y \in U_1 \cap W_1$ with a light-like geodesic $\gamma_{x,\xi}([0, l]) \subset V_1 \cap W_1$. Then the metric $\Psi^* g_2$ is isometric to g_1 in V_1.*

Example 1. Theorem 1.2 concerns passive observations which are idealizations of the measurements used in astronomy and the engineering sciences. For example, consider an event when J astronomical observatories, located at the points $x_1, x_2, \ldots, x_J$ in a neighborhood U of the path of the Earth in the space time M, observe light from a supernova at directions $\xi_j \in T_{x_j} M$, $j = 1, 2, \ldots, J$. By measuring the properties of the observed frequencies (i.e. spectrum), the observatories can draw the conclusion that they are looking at the same supernova and that there is some point q (the unknown location of the supernova) such that $(x_j, \xi_j) \in \mathcal{O}_U(q)$ for all $j = 1, 2, \ldots, J$. Examples of other point source type events which can possibly be observed by astronomic measurements, are novae, quasars, and pulsars as well as eclipses of double stars, pulsating stars or variable stars with extreme sunspots or flares [49]. When one considers the Universe in large scale, the non-flatness of the space time is clearly visible in the observations. For instance, gravitational lenses produced by massive objects [36, 69] cause multiple images of the distant objects and the images have significant time delays. Mathematically, this corresponds to the focal points of geodesics. For instance, in the observations made on the gravitationally lensed quasar Q0957+561, due to the bending of space, light from the quasar arrive to Earth along two different paths and the time delay between these two paths has been measured to be approximately 417 days [56, 79]. In such time scales, many astronomic events can be considered as point sources and thus the light observation sets can be reasonable, although highly idealized, model for the observations. We note that the above described "point sources" in astronomy have very different magnitudes and with the present telescopes and other astronomic instruments only some of those are detectable e.g. from another side of a gravitational lens.

The light observation sets model also observations or theoretical measurements (or thought experiments) which one could do near astronomic systems with black holes. For instance, the light observation sets correspond to observations when a large number of point sources (i.e. matter) emitting time-varying radiation fall in a black hole(s).

The inverse problem for the light observation sets and for the Einstein equation considered in the next section is related also to practical measurements: The detection of small but rapid perturbations of the gravitational field, that is, gravitational waves. The detection of gravitational waves is a very rapidly growing field of physics where new laboratories have be founded in many countries in the recent years, e.g the LIGO interferometer at Washington, US, the GEO-600 detector at Germany, and the VIRGO detector at Italy. We note that the Einstein equation implies that gravitational waves exists if strong enough sources exists. Thus the detection of gravitational waves actually is the question wether strong sources exists. Although gravitational radiation has not been directly detected using present measurement devices, there is indirect evidence for its existence. For example, the 1993 Nobel Prize in Physics was awarded for measurements of the Hulse-Taylor binary system which suggests gravitational waves are more than mathematical anomalies. The detection of gravitational waves can be considered as a far field measurement. The relation of far-field and near-field measurements is a well studied question of inverse problems [9] and in understanding the far-field observations the near field inverse problem needs to be studied. This is just the inverse problem for the coupled Einstein-matter field equations we consider below.

The inverse problems analogous to Theorem 1.2 are encountered also in several mundane applications, for instance one can study if the mobile phone signals can be used to determine the refractive index of the surrounding urban neighborhood.

1.3. **The inverse problem for Einstein equations.**

1.3.1. *How the active measurements could be done.* Let us make a Gedankenexperiment. Assume that we are close to a huge gravitational object and want to measure the distortion of the space time. Let us assume that we can do extremely precise measurements. Let us use several high-power laser sources (like laser pointers). We vary the direction of the laser rays so that some of the rays cross at time $t = T^0$. When the rays cross, we have an increased density of light at the crossing point, just at the crossing time. Because of the non-linearity of the Einstein-Maxwell equations this creates an artificial point source of gravitational waves having a very small amplitude. Using point masses that are located near the sources of the laser rays and observing their movements we can in principle detect the waves. Making lots of such experiments which create artificial sources of gravitational waves at a large set of points, we get data analogous to "boundary distance functions", see e.g. [44], on the Lorentzian manifold. Then we ask, is it possible to use this data to determine the metric up to a conformal deformation in the portion of the space time bordered by the possible event horizons.

For sake of mathematical simplicity, we consider next the Einstein-scalar field equations instead of Maxwell-Einstein equations and replace the laser rays by gravitational and scalar field waves, in fact four waves having only C^k-regularity with some finite k.

1.3.2. *Inverse problems for non-linear wave equations.* Many physical models lead to non-linear differential equations. In small perturbations, these equations can be approximated by linear equations, and most of the previous results on hyperbolic inverse problem concern with these linear models. Moreover, the existing uniqueness results are limited to the time-independent or real-analytic coefficients [2, 7, 8, 23, 44] as these results are based on Tataru's unique continuation principle [83, 84]. Such unique continuation results have been shown to fail for general metric tensors with are not analytic the time variable [1]. Even some linear inverse problem are not uniquely solvable. In fact, the counterexamples for these problems have been used in the so-called transformation optics. This has led to models for fixed frequency invisibility cloaks, see e.g. [31, 30, 32] and references therein. These applications give one more motivation to study inverse problems for non-linear equations.

1.3.3. *Einstein equations.* In the following, the Einstein tensor of a Lorentzian metric $g = g_{jk}(x)$ of type $(-,+,+,+)$ on a 4-dimensional manifold M is

$$\mathrm{Ein}_{jk}(g) = \mathrm{Ric}_{jk}(g) - \frac{1}{2}(\mathrm{tr}\,\mathrm{Ric}(g))g_{jk} = \mathrm{Ric}_{jk}(g) - \frac{1}{2}(g^{pq}\,\mathrm{Ric}_{pq}(g))g_{jk}.$$

Here, $\mathrm{Ric}_{pq}(g)$ is the Ricci curvature of the metric g, $\mathrm{tr}\,\mathrm{Ric}(g) = g^{pq}\,\mathrm{Ric}_{pq}(g)$ is equal to the scalar curvature of g. We define the divergence of a 2-covariant tensor T_{jk} to be in local coordinates $(\mathrm{div}_g T)_k = \nabla_n(g^{nj}T_{jk})$.

Let us consider the Einstein equation in presence of matter,

$$\mathrm{Ein}_{jk}(g) = T_{jk}, \tag{3}$$

$$\mathrm{div}_g T = 0, \tag{4}$$

for a Lorentzian metric g and a stress-energy tensor T related to the distribution of mass and energy. We will consider g which is a small perturbation of some unknown background metric $\widehat{g}$ and the stress-energy tensor T which is a perturbation of an unknown background stress-energy tensor $\widehat{T}$. We recall the essential fact that when (3) has solutions, then due to the Bianchi's identity $\mathrm{div}_g(\mathrm{Ein}(g)) = 0$ and thus the equation (4) follows automatically. Equation (4) is called the conservation law for the stress-energy tensor.

1.3.4. *Global hyperbolicity and the domain of influence* $\mathcal{K}$. Recall that a Lorentzian metric g_1 dominates the metric g_2, if all vectors ξ that are light-like or time-like with respect to the metric g_2 are time-like with respect to the metric g_1, and in this case we denote $g_2 < g_1$. Let $(M, \widehat{g})$ be a C^∞-smooth globally hyperbolic Lorentzian manifold. By [27] it holds that there is a Lorentzian metric $\widetilde{g}$ such that $(M, \widetilde{g})$ is globally hyperbolic and $\widehat{g} < \widetilde{g}$. Note that one can assume that the metric $\widetilde{g}$ is smooth (see Appendix C on details). We use the positive definite Riemannian metric $\widehat{g}^+$ to define norms in the spaces $C^k_b(M)$ of functions with bounded k derivatives, sometimes denoted also by $C^k_b(M; \widehat{g}^+)$ and the Sobolev spaces $H^s(M)$.

By [12], the globally hyperbolic manifold $(M, \widetilde{g})$ has an isometry Φ to the smooth product manifold $(\mathbb{R} \times N, \widetilde{h})$, where N is a 3-dimensional manifold and the metric $\widetilde{h}$ can be written as $\widetilde{h} = -\beta(t, y)dt^2 + \overline{h}(t, y)$ where $\beta : \mathbb{R} \times N \to (0, \infty)$ is a smooth function and $\overline{h}(t, \cdot\,)$ is a Riemannian metric on N depending smoothly on $t \in \mathbb{R}$, and the submanifolds $\{t'\} \times N$ are C^∞-smooth Cauchy surfaces for all $t' \in \mathbb{R}$. We define smooth time function $\mathbf{t} : M \to \mathbb{R}$ by setting $\mathbf{t}(x) = t$ if $\Phi(x) \in \{t\} \times N$. Let us next identify these isometric manifolds, that is, we denote $M = \mathbb{R} \times N$. Then $S_t = \{x \in M;\ \mathbf{t}(x) = t\}$ are space-like Cauchy surfaces also for $\widetilde{g}$ and for all metrics g for which $g < \widetilde{g}$.

For $t \in \mathbb{R}$, let $M(t) = (-\infty, t) \times N$. Let $t_1 > t_0 > 0$ and denote $M_1 = M(t_1)$ and $M_0 = M(t_0)$. When $\widehat{p}^- \in M_0$ and $j \in \{0, 1\}$, it follows from by [5, Cor. A.5.4] that $J^+_{\widetilde{g}}(\widehat{p}^-) \cap M(t_j)$ is compact. We denote below

$$\mathcal{K}_j := J^+_{\widetilde{g}}(\widehat{p}^-) \cap M(t_j). \tag{5}$$

As $\widehat{g} < \widetilde{g}$, we see that there exists $\varepsilon_0 > 0$ such that if $\|g - \widehat{g}\|_{C^0_b(M_1; \widehat{g}^+)} < \varepsilon_0$, then $g|_{\mathcal{K}_1} < \widetilde{g}|_{\mathcal{K}_1}$, and in particular, we have $J^+_g(p) \cap M_1 \subset \mathcal{K}_1$ for all $p \in \mathcal{K}_1$. Next we consider in particular the manifold M_0 and denote $\mathcal{K}_0 = \mathcal{K}$.

1.3.5. *Reduced Einstein tensor.* Let $t_1 > t_0 > t_{-1} = 0$ and g' be a metric on $M(t_1)$ that coincide with $\widehat{g}$ in $M(t_{-1})$ and assume that g' satisfies the Einstein equation $\mathrm{Ein}(g') = T'$ on $M(t_1)$. If g' is a small perturbation of $\widehat{g}$ in a suitable sense (see Appendix A), there is a diffeomorphism $f : M(t_1) \to f(M(t_1)) \subset M$ that is a $(g', \widehat{g})$-wave map $f : M(t_1) \to M$, see (158)-(158) in Appendix A, and satisfies $M(t_0) \subset f(M(t_1))$. The wave map has the property that $\mathrm{Ein}(f_* g') = \mathrm{Ein}_{\widehat{g}}(f_* g')$, where $\mathrm{Ein}_{\widehat{g}}(g)$ is the the $\widehat{g}$-reduced Einstein tensor, given by formula (153) in Appendix A, and has the form

$$(\mathrm{Ein}_{\widehat{g}} g)_{pq} = -\frac{1}{2} g^{jk} \widehat{\nabla}_j \widehat{\nabla}_k g_{pq} + \frac{1}{4}(g^{nm} g^{jk} \widehat{\nabla}_j \widehat{\nabla}_k g_{nm}) g_{pq} + P_{pq}(g, \widehat{\nabla} g),$$

and $\widehat\nabla_j$ is the covariant differentation with respect to the metric $\widehat g$ and P_{pq} is polynomial of g_{nm}, g^{nm}, $\widehat\nabla_j g_{nm}$, $\widehat\nabla_j g^{nm}$, $\widehat g_{nm}$, $\widehat g^{nm}$, and the curvature $\widehat R_{jklm}$ of $\widehat g$. Using this map f, the tensors $g = f_*g'$ and $T = f_*T'$ satisfy the Einstein equations $\text{Ein}(g) = T$ on $M(t_0)$. Moreover, as f is a wave map, the Einstein equation for g can be written as the $\widehat g$-reduced Einstein equations, that is,

$$\text{Ein}_{\widehat g}(g) = T \quad \text{on } M(t_0). \tag{6}$$

In the literature, the above is often stated by saying that the reduced Einstein equation (6) is the Einstein equations written with the wave-gauge corresponding to the metric $\widehat g$. The equation (6) is a quasi-linear hyperbolic system of equations. We emphasize that a solution of the reduced Einstein equation can be a solution of the original Einstein equation only if the stress energy tensor satisfies the conservation law $\nabla^g_j T^{jk} = 0$. Summarizing the above, when g is close to $\widehat g$ and satisfies the Einstein equation, by changing the coordinates appropriately (or physically stated, choosing the correct gauge), the metric tensor g satisfies the $\widehat g$-reduced Einstein equation. Next we formulate a direct problem for the $\widehat g$-reduced Einstein equations.

1.3.6. *Motivation for the direct problem.* We consider metric and physical fields informally on a Lorentzian manifold (M, g). We aim to study an inverse problem with active measurements. As measurements can not be implemented in Vacuum, we have to add matter fields in the model. Since matter fields and the metric tensor have to satisfy a standard conservation law, we add in the model two types of matter fields. In fact, we will start with the standard coupled system of the Einstein equation and scalar filed equations with some sources $\mathcal F_1$ and $\mathcal F_2$, namely

$$\begin{aligned}
&\text{Ein}_{\widehat g}(g) = \mathbf{T}_{jk}(g,\phi) + \mathcal F_1, \quad \text{in } M,\\
&\mathbf{T}_{jk}(g,\phi) = \sum_{\ell=1}^{L} (\partial_j\phi_\ell\, \partial_k\phi_\ell - \frac12 g_{jk} g^{pq}\partial_p\phi_\ell\, \partial_q\phi_\ell - \frac12 m\phi_\ell^2 g_{jk}),\\
&\square_g \phi_\ell - m\phi_\ell = \mathcal F_2, \quad \ell = 1,2,3,\dots,L,
\end{aligned} \tag{7}$$

and make the source term $\mathcal F_1$ correspond to fluids (or fluid fields) consisting of particles which energy and moment vectors are controlled and make $\mathcal F_2$ to adapt the changes of g, ϕ, and $\mathcal F_1$ so that the standard conservation law is satisfied. Note that above we could assume that masses m_ℓ of the fields ϕ_ℓ depend on ℓ, but for simplicity we have assumed that $m_\ell = m$.

By an active measurement we mean a model where we can control some of these fields and the other fields adapt to the changes of all fields so that the conservation law holds. Roughly speaking, we can consider measurement devices as a complicated process that changes one energy form to other forms of energy, like a system of explosives that transform some potential energy to kinetic energy. This process creates a perturbation of the metric and the matter fields that we observe in a subset of the space time. In this paper, our aim is not to consider a physically realistic example but a mathematical model that can be rigorously analyzed.

To motivate such a model, we start with a non-rigorous discussion. Following [13, Ch. III, Sect. 6.4, 7.1, 7.2, 7.3] and [3, p. 36] we start by considering the Lagrangians, associated to gravity, scalar fields $\phi = (\phi_\ell)_{\ell=1}^L$ and non-interacting fluid fields, that is, the number density four-currents $\mathbf{n} = (\mathbf{n}^\kappa)_{\kappa=1}^J$ (where each $\mathbf{n}^\kappa$ is a co-vector, see [3, p. 33]). We also add in to the model a Lagrangian associated with some scalar valued source fields $S = (S_\ell)_{\ell=1}^L$ and $Q = (Q_k)_{k=1}^K$. We consider action corresponding to the coupled Lagrangian

$$\begin{aligned}
\mathcal{A} &= \int_M \Big(L_{grav}(x) + L_{fields}(x) + L_{source}(x)\Big)\, dV_g(x),\\
L_{grav} &= R(g),\\
L_{fields} &= \sum_{\ell=1}^L \Big(g^{jk}\partial_j\phi_\ell\, \partial_k\phi_\ell - \mathcal{V}(\phi_\ell; S_\ell)\Big) + g^{jk}(\sum_{\kappa=1}^J \mathbf{n}_j^\kappa \mathbf{n}_k^\kappa),\\
L_{source} &= \varepsilon \mathcal{H}_\varepsilon(g, S, Q, \mathbf{n}, \phi),
\end{aligned}$$

where $R(g)$ is the scalar curvature, $dV_g = (-\text{det}g)^{1/2}dx$ is the volume form on (M, g),

$$\mathcal{V}(\phi_\ell; S_\ell) = \frac{1}{2}(m-1)\phi_\ell^2 + \frac{1}{2}(\phi_\ell - S_\ell)^2 \tag{8}$$

are energy potentials of the scalar fields ϕ_ℓ that depend on S_ℓ, and $\mathcal{H}_\varepsilon(g, S, Q, \mathbf{n}, \phi)$ is a function modeling the measurement device we use. We assume that $\mathcal{H}_\varepsilon$ is bounded and its derivatives with respect to $S, Q, \mathbf{n}$ are very large (like of order $O((\varepsilon)^{-2})$) and its derivatives with respect of g and ϕ are bounded when $\varepsilon > 0$ is small. We note that the above Lagrangian for the fluid fields is the sum of the single fluid Lagrangians where for all fluids the master function Λ is the identity function, that is, the energy density of each fluid is given by $\rho = -\Lambda(n^2) = -n^2$, $n^2 = g^{jk}\mathbf{n}_j\mathbf{n}_k$. Note that here $\mathbf{n}$ is a time-like vector or zero and thus ρ is non-negative. On fluid Lagrangians, see the discussions in [3, p. 33-37], [13, Ch. III, Sect. 8] and [85] and [24, p. 196].

When we compute the critical points of the Lagrangian L and neglect the $O(\varepsilon)$-terms, the equation $\frac{\delta \mathcal{A}}{\delta g} = 0$ gives the Einstein equation with a stress-energy tensor T_{jk} defined below, see (9), the equation $\frac{\delta \mathcal{A}}{\delta \phi} = 0$ gives the wave equations with sources S_ℓ. We assume that $O(\varepsilon^{-1})$ order equations obtained from the equation $(\frac{\delta \mathcal{A}}{\delta S}, \frac{\delta \mathcal{A}}{\delta Q}, \frac{\delta \mathcal{A}}{\delta \mathbf{n}}) = 0$ fix the values of the scalar functions Q, the tensor

$$\mathcal{P}=\sum_{\kappa=1}^{J}\mathbf{n}^\kappa_j\mathbf{n}^\kappa_k dx^j\otimes dx^k$$

and yields for the sources $S=(S_\ell)_{\ell=1}^L$ equations of the form $S_\ell=\mathcal{S}_\ell(\phi,\nabla^g\phi,Q,\nabla^gQ,\mathcal{P},\nabla^g\mathcal{P},g)$. The function $\mathcal{H}_\varepsilon$ models the way the measurement device works. Due to this we will assume that $\mathcal{H}_\varepsilon$ and thus functions $\mathcal{S}_\ell$ may be quite complicated. The interpretation of the above is that in each measurement event we use a device that fixes the values of the scalar functions Q and the tensor $\mathcal{P}$ and gives the equations $S=\mathcal{S}(\phi,\nabla^g\phi,Q,\nabla^gQ,\mathcal{P},\nabla^g\mathcal{P},g)$ that tell how the sources of the ϕ-fields adapt to these changes so that the physical conservation laws are satisfied.

Now we stop the non-rigorous discussion (where the $O(\varepsilon)$ terms were neglected).

1.3.7. *Formulation of the direct problem.* To start the rigorous analysis, let us define some physical fields and introduce a model as a system of partial differential equations (that hold at a critical point of the above Lagrangian L).

We assume that there are C^∞-background fields $\widehat g$, $\widehat \phi$, $\widehat Q$, and $\widehat P$ on M. We also fix a smooth metric $\widetilde g$ that is a globally hyperbolic metric on M such that $\widehat g<\widetilde g$ and make the the identification $M=\mathbb{R}\times N$ where $\{t\}\times N$ are Cauchy surfaces for $\widetilde g$. Moreover, we fix $t_0>0$ and a point $\widehat{p}^-\in(0,t_0)\times N$ and denote $M_0=M(t_0)=(-\infty,t_0)\times N$.

Let $\mathcal{P}=\mathcal{P}_{jk}(x)dx^jdx^k$ be a symmetric tensor on M_0, corresponding below to a direct perturbation to the stress energy tensor, and $Q=(Q_\ell(x))_{\ell=1}^K$ where $Q_\ell(x)$ are real-valued functions on M_0. Also, we consider a Lorentzian metric g on M_0 and $\phi=(\phi_\ell)_{\ell=1}^L$ where ϕ_ℓ are scalar fields on M_0, $L\le K-1$. The potentials of the fields ϕ_ℓ are $\mathcal{V}(\phi_\ell;S_\ell)$ given in (8). The way how S_ℓ, called below the adaptive source functions, depend on other fields is explained later.

Using the ϕ and $\mathcal{P}$ fields, we define the stress-energy tensor

$$(9)\quad T_{jk}=\sum_{\ell=1}^{L}(\partial_j\phi_\ell\,\partial_k\phi_\ell-\frac12 g_{jk}g^{pq}\partial_p\phi_\ell\,\partial_q\phi_\ell-\mathcal{V}(\phi_\ell;S_\ell)g_{jk})+\mathcal{P}_{jk}.$$

Below, we introduce the notation

$$P_{jk}=\mathcal{P}_{jk}-\sum_{\ell=1}^{L}\frac12 S_\ell^2 g_{jk}.$$

We assume that $P-\widehat P$ and $Q-\widehat Q$ are supported on $\mathcal{K}=J^+_{\widetilde g}(\widehat{p}^-)\cap M_0$. As we will see later, when $P-\widehat P$ and $Q-\widehat Q$ are small enough, in a suitable sense, and the intersection of their support and M_0 is contained in $\mathcal{K}$, then $g < \widetilde{g}$ and $g - \widehat{g}$, considered as a function on M_0, is also supported in $\mathcal{K}$.

Using (8) we can write the stress energy tensor (9) in the form

$$\begin{aligned} T_{jk} &= P_{jk} + Zg_{jk} + \mathbf{T}_{jk}(g,\phi), \quad Z = \sum_{\ell=1}^{L} S_\ell \phi_\ell, \\ \mathbf{T}_{jk}(g,\phi) &= \sum_{\ell=1}^{L} (\partial_j \phi_\ell \, \partial_k \phi_\ell - \frac{1}{2} g_{jk} g^{pq} \partial_p \phi_\ell \, \partial_q \phi_\ell - \frac{1}{2} m \phi_\ell^2 g_{jk}), \end{aligned}$$

where we call Z the stress energy density caused by sources S_ℓ.

Now we are ready to formulate the direct problem for the Einstein-scalar field equations. Let g and ϕ satisfy

$$\begin{aligned} &\mathrm{Ein}_{\widehat{g}}(g) = P_{jk} + Zg_{jk} + \mathbf{T}_{jk}(g,\phi), \quad Z = \sum_{\ell=1}^{L} S_\ell \phi_\ell, \quad \text{in } M_0, \\ &\Box_g \phi_\ell + \mathcal{V}'(\phi_\ell; S_\ell) = 0 \quad \text{in } M_0, \quad \ell = 1,2,3,\dots,L, \\ &S_\ell = \mathcal{S}_\ell(\phi, \nabla^g \phi, Q, \nabla^g Q, P, \nabla^g P, g), \quad \text{in } M_0, \\ &g = \widehat{g}, \quad \phi_\ell = \widehat{\phi}_\ell, \quad \text{in } M_0 \setminus \mathcal{K}. \end{aligned} \tag{10}$$

Above, $\mathcal{V}'(\phi; s) = \partial_\phi \mathcal{V}(\phi; s)$ so that $\mathcal{V}'(\phi_\ell; S_\ell) = m\phi_\ell - S_\ell$. We assume that the background fields $\widehat{g}$, $\widehat{\phi}$, $\widehat{Q}$, and $\widehat{P}$ satisfy these equations.

We consider here $P = (P_{jk})_{j,k=1}^4$ and $Q = (Q_\ell)_{\ell=1}^K$ as fields that we can control. As mathematical idealization we will assume that the fields $P - \widehat{P}$ and $Q - \widehat{Q}$ are compactly supported. To obtain a physically meaningful model, we need to consider how the adaptive source functions $\mathcal{S}_\ell$ should be chosen so that the physical conservation law in relativity,

$$\nabla_k (g^{kp} T_{pq}) = 0 \tag{11}$$

is satisfied. Here $\nabla = \nabla^g$ is the connection corresponding to the metric g. We note that the conservation law is a necessary condition for the equation (10) to have solutions for which $\mathrm{Ein}_{\widehat{g}}(g) = \mathrm{Ein}(g)$, i.e., that the solutions of (10) are solutions of the Einstein field equations.

The functions $\mathcal{S}_\ell(\phi, \nabla^g \phi, Q, \nabla^g Q, P, \nabla^g P, g)$ model the devices that we use to perform active measurements. Thus, even though the Assumption S below may appear quite technical, this assumption can be viewed as the instructions on how to build a device that can be used to measure the structure of the space time far away. Outside the support of the measurement device (i.e. the union of the supports of Q and P) we have just assumed that the standard coupled Einstein-scalar field equations hold, c.f. (12).

Throughout the paper we assume that the following assumption holds.

Assumption S. Throughout the paper we assume that the adaptive source functions $\mathcal{S}_\ell(\phi,\nabla^g\phi,Q,\nabla^gQ,P,\nabla^gP,g)$ have the following properties:

(i) Denoting $c=\nabla^g\phi$, $C=\nabla^gP$, and $H=\nabla^gQ$, we assume that $\mathcal{S}_\ell(\phi,c,Q,H,P,C,g)$ are linear functions of (Q,H,P,C) and in particular satisfy

$$\mathcal{S}_\ell(\phi,c,0,0,0,0,g)=0. \tag{12}$$

We also assume that when $(Q_\ell)_{\ell=1}^K$ and $(P_{jk})_{j,k=1^4}$ are sufficiently close to $(\widehat Q_\ell)_{\ell=1}^K$ and $(\widehat P_{jk})_{j,k=1^4}$, respectively, and ϕ and g are sufficiently close in the C^1-topology to the background fields $\widehat g$ and $\widehat\phi$ then the adaptive source function $\mathcal{S}_\ell(\phi,\nabla^g\phi,Q,\nabla^gQ,P,\nabla^gP,g)$ at $x\in M_0$ is a smooth function of the pointwise values $\phi(x)$, $\nabla^g\phi(x)$, $Q(x)$, $\nabla^gQ(x)$, $P(x)$, $\nabla^gP(x)$ and $g_{jk}(x)$.

(ii) $Q_K=Z$, that is, one of the physical fields we directly control is the density Z of the stress energy tensor caused by the source fields.

(iii) We assume that $\mathcal{S}_\ell$ is independent of $P(x)$ and the dependency of $\mathcal{S}$ on ∇^gP and ∇^gQ is only due to the dependency in the term $g^{pk}\nabla^g_p(P_{jk}+Zg_{jk})=g^{pk}\nabla^g_pP_{jk}+\nabla^g_jQ_K$, associated to the divergence of the perturbation of T, that is, there exist functions $\widetilde{\mathcal{S}}_\ell$ so that

$$\mathcal{S}_\ell(\phi,c,Q,H,P,C,g)=\widetilde{\mathcal{S}}_\ell(\phi,c,Q,R,g),\quad R=(g^{pk}\nabla^g_p(P_{jk}+Q_Kg_{jk}))_{j=1}^4.$$

Let $\widehat R=\widehat g^{pk}\widehat\nabla_p\widehat P_{jk}+\widehat\nabla_j\widehat Q_K$. Moreover, we assume that for all $x\in U_{\widehat g}$ the derivative of $\widetilde{\mathcal{S}}(\widehat\phi,\widehat\nabla\widehat\phi,Q,R,\widehat g)=(\widetilde{\mathcal{S}}_\ell(\widehat\phi,\widehat\nabla\widehat\phi,Q,R,\widehat g))_{\ell=1}^L$ with respect to Q and R, that is, the map

$$D_{Q,R}\widetilde{\mathcal{S}}(\widehat\phi,\widehat\nabla\widehat\phi,Q,R,\widehat g)|_{Q=\widehat Q,R=\widehat R}:\mathbb{R}^{K+4}\to\mathbb{R}^L \tag{13}$$

is surjective.

(iv) We assume that the adaptive source functions $\mathcal{S}_\ell$ are such that if g,ϕ satisfy (10) with any $(Q_\ell)_{\ell=1}^K$ and (P_{jk}) that are sufficiently close to $\widehat Q$ and $\widehat P$ in C^5-topology then the conservation law (11) is valid.

Notice that as sources are small, we need to consider only local existence of solutions, see Appendix C. Above, the assumptions on the smoothness of the sources and solutions are far from optimal. For the local existence results, see [51, 53, 54, 78]. The global existence results are considered e.g. in [16, 17, 58, 59].

Below, expect in Corollary 1.6, we will consider the case when $\widehat Q=0$ and $\widehat P=0$. This implies that for the background fields that adaptive source functions $\mathcal{S}_\ell$ vanish.

Below we will denote $Q=(Q',Q_K)$, $Q'=(Q_\ell)_{\ell=1}^{K-1}$. There are examples when the background fields $(\widehat g,\widehat\phi)$ and the adaptive source functions $\mathcal{S}_\ell(\phi,\nabla^g\phi,Q,\nabla^gQ,P,\nabla^gP,g)$ exists and satisfy the Assumption

S. This is shown in Appendix B in the case the following condition is valid for the background fields:

Condition A: Assume that at any $x \in \widehat{U}$ there is a permutation $\sigma : \{1, 2, \dots, L\} \to \{1, 2, \dots, L\}$, denoted σ_x, such that the 5×5 matrix $[B^\sigma_{jk}(\widehat{\phi}(x), \nabla\widehat{\phi}(x))]_{j,k\leq 5}$ is invertible, where

$$[B^\sigma_{jk}(\phi(x), \nabla\phi(x))]_{k,j\leq 5} = \left[\begin{array}{c} (\, \partial_j \phi_{\sigma(\ell)}(x))_{\ell\leq 5,\ j\leq 4} \\ (\phi_{\sigma(\ell)}(x))_{\ell\leq 5} \end{array} \right].$$

1.3.8. *Inverse problem for the reduced Einstein equations.* Let us next define the measurements precisely.

Let $\widehat{\mu} : [-1, 1] \to M_0$ be a freely falling observer of $(M_0, \widehat{g})$ and $z_0 = \widehat{\mu}(s_{-2})$ and $\eta_0 = \partial_s\widehat{\mu}(s_{-2})$. We assume that $z_0 \in (-\infty, 0) \times N$ and $\widehat{g}(\eta_0, \eta_0) = -1$.

We will consider Lorentzian metrics g on $M_0 = (-\infty, t_0) \times N$, $t_0 > 0$ that is sufficiently close in $C^2_b(M_0)$ to $\widehat{g}$ and coincides with $\widehat{g}$ in $(-\infty, 0) \times N$. For the metric g we will use the notations of an open set U_g, freely falling observers μ_g and $\mu_{g,z,\eta}$ with $(z, \eta) \in \mathcal{U}_{z_0,\eta_0}$, and numbers $-1 < s_{-2} < s_{-1} = s_- < s_{+1} = s_+ < s_{+2} < 1$, that are defined near the formula (2), with $\widehat{h}$ being small enough and independent of g and so that the set $\mathcal{U}_{z_0,\eta_0}(\widehat{h}) \subset (-\infty, 0) \times N$ is the same for all metric tensors g that we consider. We assume also that $\widehat{p}^- = \widehat{\mu}(s_-) \in (0, t_0) \times N$.

Let $\mu_{g,z,\eta} : [-1, 1] \to M_0$ be geodesics such that $\mu_{g,z,\eta}(s_{-2}) = z$ and $\partial_s\mu_{g,z,\eta}(s_{-2}) = \eta$. Then for $\mu_g := \mu_{g,z_0,\eta_0}$ we have $\mu_g(s) = \mu_{\widehat{g}}(s)$ for all $s \leq s_{-1}$ and $\mu_{\widehat{g}} = \widehat{\mu}$. Moreover, we denote $\widehat{p}^- = \widehat{\mu}(s_-)$ and $\widehat{p}^+ = \widehat{\mu}(s_+)$. We assume that $\widehat{h}$ used to define $\mathcal{U}_{z_0,\eta_0}(\widehat{h})$ is independent of the metric and that it is so small that $\pi(\mathcal{U}_{z_0,\eta_0}(\widehat{h})) \subset (-\infty, t_0) \times N$.

We note that when g is close enough the $\widehat{g}$ in the space $C^2_{loc}(M_0)$, for all $(z, \eta) \in \mathcal{U}_{z_0,\eta_0}(\widehat{h})$ we have $\mu_{g,z,\eta}([s_{-2}, s_{+2}]) \subset U_g$, $\mu_{g,z,\eta}(s_{-2}) \in I^-_g(\mu_g(s_-))$, and $\mu_{g,z,\eta}(s_{+2}) \in I^+_g(\mu_g(s_+))$.

Moreover, for $r > 0$ let

$$W_g(r) = \bigcup_{s_-<s<s_+-r} I_{M,g}(\mu_g(s), \mu_g(s + r)) \tag{14}$$

and let $r_0 \in (0, 1)$ be so small that $W_{\widehat{g}}(2r_0) \subset U_{\widehat{g}}$. We denote next $W_g = W_g(r_0)$.

Let us use causal Fermi-type coordinates: Let $Z_j(s)$, $j = 1, 2, 3, 4$ be a parallel frame of linearly independent time-like vectors on $\mu_g(s)$ such that $Z_1(s) = \dot{\mu}_g(s)$. Let $\Phi_g : (t_j)^4_{j=1} \mapsto \exp_{\mu_g(t_1)}(\sum^4_{j=2} t_j Z_j(t_1))$. We assume that $r_0 > 0$ used above is so small that $\Psi_{\widehat{g}} = \Phi^{-1}_{\widehat{g}}$ defines coordinates in $W_{\widehat{g}}(2r_0)$. Then, when g is sufficiently close to $\widehat{g}$ in the

C^2_b-topology, $\Psi_g=\Phi_g^{-1}$ defines coordinates in $W_g(2r_0)$ that we call the Fermi-type coordinates. We define the norm-like functions

$$\mathcal{N}^{(k)}_{\widehat g}(g)=\|(\Psi_g)_*g-(\Psi_{\widehat g})_*\widehat g)\|_{C^k(\overline{\Psi_{\widehat g}(W_{\widehat g})})},$$
$$\mathcal{N}^{(k)}(F)=\|(\Psi_g)_*F\|_{C^k(\overline{\Psi_{\widehat g}(W_{\widehat g})})},$$

where $k\in\mathbb{N}$, that measures the C^k distance of g from $\widehat g$ and F from zero in a Fermi-type coordinates. As we have assumed that the background metric $\widehat g$ and the field $\widehat\phi$ are C^∞-smooth, we can consider as smooth sources as we wish. Thus, for clarity, we use below smoothness assumptions on sources that are far from the optimal ones.

Let us define (recall that here $\widehat Q=0$ and $\widehat P=0$) the source-to-observation 4-tuples corresponding to measurements in U_g with sources $F=(P,Q)$ supported in U_g. We define

$$\begin{aligned}\mathcal{D}(\widehat g,\widehat\phi,\varepsilon)=\{[(U_g,g|_{U_g},\phi|_{U_g},F|_{U_g})]\ \ ;\ \ &(g,\phi,F)\text{ are smooth solutions}\\ &\text{of (10) with } F=(P,Q),\ F\in C^\infty_0(W_g;\mathcal{B}^K),\\ J^+_g(\mathrm{supp}\,(F))\cap J^-_g(\mathrm{supp}\,(F))\subset W_g,\ &\mathcal{N}^{(16)}(F)<\varepsilon,\ \mathcal{N}^{(16)}_{\widehat g}(g)<\varepsilon\}.\end{aligned}\tag{15}$$

Above, the sources F are considered as sections of the bundle $\mathcal{B}^K$, where $\mathcal{B}^K$ is a vector bundle on M that is the product bundle of the bundle symmetric $(2,0)$-tensors and the trivial vector bundle with the fiber $\mathbb{R}^K$. Above, $[(U_g,g,\phi,F)]$ denotes equivalence class of all Lorenztian manifolds (U',g') and functions $\phi'=(\phi'_\ell)_{\ell=1}^L$ and the tensors F' defined on C^∞-smooth manifold U', such that there is C^∞-smooth diffeomorphism $\Psi:U'\to U_g$ satisfying $\Psi_*g'=g$, $\Psi_*\phi'_\ell=\phi_\ell$ and $\Psi_*F=F$.

Note that as $[(U_{\widehat g},\widehat g,\widehat\phi,0)]$ is the intersection of all collections $\mathcal{D}(\widehat g,\widehat\phi,\varepsilon)$, $\varepsilon>0$, we see that the collection $\mathcal{D}(\widehat g,\widehat\phi,\varepsilon)$ determines the isometry type of $(U_{\widehat g},\widehat g)$.

Our goal is to prove the following result:

Theorem 1.4. *Let $(M_j,\widehat g_j)$, $j=1,2$ be two open, C^∞-smooth, globally hyperbolic Lorentzian manifolds and $\widehat\phi^{(j)}$, $j=1,2$ be background values of the scalar fields on these manifolds, and let $\mathcal{S}_\ell$ be the adaptive source functions satisfying (12) and Assumption S, and assume that background source fields vanish, $\widehat P^{(j)}=0$ and $\widehat Q^{(j)}=0$.*

Let $\mu_{\widehat g_j}:[-1,1]\to M_j$ be freely falling observers on $(M_j,\widehat g_j)$ and $U_{\widehat g_j}$ be the open sets defined by formula (2), and $\widehat p_j^\pm=\mu_{\widehat g_j}(s_\pm)$.

Assume that there is a C^{17}-isometry $\Psi_0:(U_{\widehat g_1},\widehat g_1)\to(U_{\widehat g_2},\widehat g_2)$. We identify these isometric sets and denote

$$\widehat U=U_{\widehat g_1}=U_{\widehat g_2},\quad \widehat g|_{\widehat U}=\widehat g_1|_{U_{\widehat g_1}}=\widehat g_2|_{U_{\widehat g_2}}.$$

Let $\varepsilon > 0$ and assume that source-to-observation 4-tuples $\mathcal{D}(\widehat{g}_j, \widehat{\phi}^{(j)}, \varepsilon)$ for the manifolds $(M_j, \widehat{g}_j)$ and fields $\widehat{\phi}^{(j)}$ satisfy

$$\mathcal{D}(\widehat{g}_1, \widehat{\phi}^{(1)}, \varepsilon) = \mathcal{D}(\widehat{g}_2, \widehat{\phi}^{(2)}, \varepsilon). \tag{16}$$

Then there is a diffeomorphism $\Psi : I_{M_1,\widehat{g}_1}(\widehat{p_1^-}, \widehat{p_1^+}) \to I_{M_2,\widehat{g}_2}(\widehat{p_2^-}, \widehat{p_2^+})$, and the metric $\Psi^*\widehat{g}_2$ is conformal to $\widehat{g}_1$ in $I_{M_1,\widehat{g}_1}(\widehat{p_1^-}, \widehat{p_1^+})$.

Note that above we have assumed that the adaptive source functions $\mathcal{S}_\ell$ are the same on $(M_1, \widehat{g}_1)$ and $(M_2, \widehat{g}_2)$.

Recall that above $I_{M_1,\widehat{g}_1}(\widehat{p_1^-}, \widehat{p_1^+}) = I^+_{M_1,\widehat{g}_1}(\widehat{p_1^-}) \cap I^-_{M_1,\widehat{g}_1}(\widehat{p_1^+})$.

The measurements in Theorem 1.4 provides a subset of all possible sources $(\mathcal{F}_1, \mathcal{F}_2)$ for equations (7). Thus, as we see later, using Theorem 1.4 we can prove the following result where we assume that we have information on measurements with a larger class of sources than was used in Theorem 1.4:

Theorem 1.5. *Assume that $(M, \widehat{g})$ is a globally hyperbolic manifold, and that in the open set $U_{\widehat{g}}$ the Condition A is valid. Assume that we given the set $\mathcal{D}^{alt}(\widehat{g}, \widehat{\phi}, \varepsilon)$ of the the equivalence classes $[(U_g, g|_{U_g}, \phi|_{U_g}, \mathcal{F}|_{U_g})]$ where g and ϕ and $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ satisfy the equations (7), the conservation law $\nabla^g_j(\mathbf{T}^{jk}(g, \phi) + \mathcal{F}_1^{jk}) = 0$, the sources $\mathcal{F}_1$ and $\mathcal{F}_2$ are supported in U_g, and satisfy $\mathcal{N}^{(15)}(\mathcal{F}) < \varepsilon$ and $\mathcal{N}^{(15)}_{\widehat{g}}(g) < \varepsilon$. Then these data determine the conformal type of $\widehat{g}$ in $I_{M,\widehat{g}}(\widehat{p}^-, \widehat{p}^+)$.*

The above result means that if the manifold $(M_0, \widehat{g})$ is unknown, then the source-to-observation pairs corresponding to freely falling sources which are near the freely falling observer $\mu_{\widehat{g}}$ and the measurements of the metric tensor and the scalar fields in a neighborhood $U_{\widehat{g}}$ of $\mu_{\widehat{g}}$, determine the metric tensor up to conformal transformation in the set $I_{M_0,\widehat{g}}(\widehat{p}^-, \widehat{p}^+)$.

We want to point out that by the main theorem, if we have two non-isometric space times, a generic measurement gives different results on these manifolds. In particular, this implies that the perfect space-time cloaking, see [25, 61], with a smooth metric in a globally hyperbolic universe is not possible.

Also, one can ask if one can make an approximative image of the space-time knowing only one measurement. In general, in many inverse problems several measurements can be packed together to one measurement. For instance, for the wave equation with a time-independent simple metric this is done in [35]. Similarly, Theorem 1.4 and its proof make it possible to do approximate reconstructions in a suitable class of manifolds with only one measurement. We will discuss this in detail in forthcoming articles.

Using Theorem 1.4 we will show that the metric tensor can be determined in the domain which can be connected to a measurement set with light-like geodesics through vacuum.

Below, we consider the case when $\widehat{Q}$ and $\widehat{P}$ are non-zero, and we define

$$\begin{aligned}\mathcal{D}^{mod}(\widehat{g},\widehat{\phi},\varepsilon)=\{[(U_g,g|_{U_g},\phi|_{U_g},F|_{U_g})]\ \ ;\ \ &(g,\phi,F)\text{ are smooth solutions,}\\ &\text{of (10) with } F=(P-\widehat{P},Q-\widehat{Q}),\ F\in C_0^\infty(W_g;\mathcal{B}^K),\\ &J_g^+(\text{supp}\,(F))\cap J_g^-(\text{supp}\,(F))\subset W_g,\ \mathcal{N}^{(16)}(F)<\varepsilon,\ \mathcal{N}_{\widehat{g}}^{(16)}(g)<\varepsilon\}.\end{aligned}$$

Next we consider the case when $\widehat{Q}^{(j)}$ and $\widehat{P}^{(j)}$ are not assumed to be zero, see Fig. 4.

Corollary 1.6. *Assume that M_j, $\widehat{g}_j$, $\widehat{\phi}^{(j)}$, $j=1,2$ and Ψ_0 are as in Theorem 1.4 and (16) is not assumed to be valid. Assume that also that for $j=1,2$ there are sets $W_j\subset M_j$ such that $\widehat{\phi}^{(j)}$, $\widehat{Q}^{(j)}$ and $\widehat{P}^{(j)}$ are zero (and thus the metric tensors $\widehat{g}_j$ have vanishing Ricci curvature) in W_j and that $I_{\widehat{g}_j}(\widehat{p}_j^-,\widehat{p}_j^+)\subset W_j\cup U_{\widehat{g}}$. If*

$$\mathcal{D}^{mod}(\widehat{g}^{(1)},\widehat{\phi}^{(1)},\varepsilon)=\mathcal{D}^{mod}(\widehat{g}^{(2)},\widehat{\phi}^{(2)},\varepsilon)$$

then the metric $\Psi^*\widehat{g}_2$ is isometric to $\widehat{g}_1$ in $I_{\widehat{g}_1}(\widehat{p}_1^-,\widehat{p}_1^+)$.

In the setting of Corollary 1.6 the set W is such $I(\widehat{p}^-,\widehat{p}^+)\cap(M\backslash W)\subset U$. This means that if we restrict to the domain $I(\widehat{p}^-,\widehat{p}^+)$ then we have Vacuum Einstein equations in the unknown domain $I(\widehat{p}^-,\widehat{p}^+)\setminus U$ and have matter only in the domain U where we implement our measurement (c.f. a space ship going around in a system of black holes). This could be considered as an "Inverse problem for the vacuum Einstein equations".

Example 3: Consider a black hole modeled by a Schwarzschild metric in $(\mathbb{R}^3\setminus\{0\})\times\mathbb{R}$ with the event horizon $\partial B(R)\times\mathbb{R}$, $B(R)=\{y\in\mathbb{R}^3:\ |y|=R\}$, and add to the stress energy tensor the (small) effect caused by the "space ship" doing the measurements. By Theorem 1.4 the measurements in $U=(\mathbb{R}^3\setminus B(r))\times\mathbb{R}$ with any $r>R$ determines uniquely the metric outside the event horizon, that is, in $W=(\mathbb{R}^3\setminus B(R))\times\mathbb{R}$. Physically a more interesting example could be obtained by considering several black holes or by adding matter to the system.

We note the techniques used to prove Theorem 1.4 are suitable for studying other equations, e.g. Einstein-Maxwell system, where strong electromagnetic waves and the gravitational field have an interaction. However, we do not study these equations in this paper.

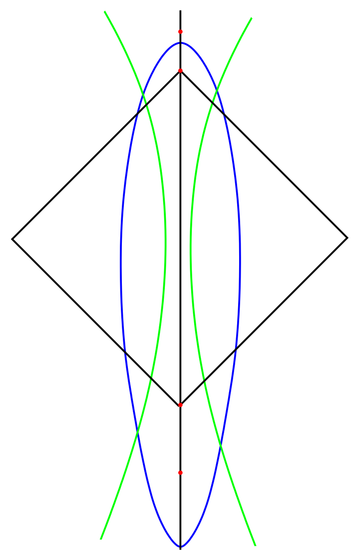

FIGURE 4. A schematic figure where the space-time is represented as the 2-dimensional set $\mathbb{R}^{1+1}$ on the setting in Corollary 1.6. The set $J_{\widehat{g}}(\widehat{p}^-,\widehat{p}^+)$, i.e., the diamond with the black boundary, is contained in the union of the blue set $U_{\widehat{g}}$ and the set W. The set W is in the figure the area outside of the green curves. The sources are controlled in the set $U_{\widehat{g}}\setminus W$ and the set W consists of vacuum.

2. Proofs for inverse problem for light observation sets

2.1. Notations and definitions. A smooth n-dimensional manifold with smooth Lorentzian, type $(1,n-1)$ metric g has a causal structure, if there is a globally defined smooth vector field X_c such that $g(X_c(y),X_c(y))<0$ for all $y\in M$. We say that a piecewise smooth path $\alpha(t)$ is time-like if $g(\dot\alpha(t),\dot\alpha(t))<0$ for almost every t. Also, a piecewise smooth path $\alpha(t)$ is causal (or non-space-like) if $\dot\alpha(t)\neq 0$ and $g(\dot\alpha(t),\dot\alpha(t))\leq 0$ for almost every t.

By [11], a globally hyperbolic manifold, as defined in the introduction, satisfies the strong causality condition:

(17) For every $z\in M$ and every neighborhood $V\subset M$ of z there is a neighborhood $V'\subset M$ of z that if $x,y\in V'$ and $\alpha\subset M$ is a causal path connecting x to y then $\alpha\subset V$.

Recall that g^+ is a Riemannian metric determined by a Lorentzian metric g on M. In the following, for open $V\subset M$, denote $TV=\{(x,\xi)\in TM:\ x\in V\}$ and $S^{g^+}V=\{(x,\xi)\in TM;\ x\in V,\ \|\xi\|_{g^+}=1\}$.

The metric g^+ on M induces a Sasaki metric on TM which we denote by g^+, too.

In the next sections, g will be fixed and we denote $U = U_g$, $J^\pm_g(x) = J^\pm(x)$, etc. Let us next define the earliest observation functions on a geodesic $\gamma_{g,z,\eta}$, $(z,\eta) \in \mathcal{U}_{z_0,\eta_0}$. If $V \subset TU$ and $W = \pi(V) \subset U$ are such that $\mu_{g,z,\eta} \cap W \neq \emptyset$, we define $\mathbf{s}_{z,\eta}(W) = \inf\{s \in [-1,1];\ \mu_{g,z,\eta}(s) \cap W\}$ and

$$
\begin{aligned}
(18) \qquad E_{z,\eta}(V) &= \{(x,\xi) \in \mathrm{cl}\,(V);\ x = \mu_{g,z,\eta}(\mathbf{s}_{z,\eta}(\pi(V)))\},\\
e_{z,\eta}(V) &= \pi(E_{z,\eta}(V)),\\
e_{z,\eta}(W) &= \{x \in \mathrm{cl}\,(W);\ x = \mu_{g,z,\eta}(\mathbf{s}_{z,\eta}(W))\},
\end{aligned}
$$

where $\mathrm{cl}\,(V)$ denotes the closure of V. Moreover, we denote $\mathcal{E}_{z,\eta}(q) = E_{z,\eta}(\mathcal{O}_U(q))$, $\mathcal{P}_{z,\eta}(q) = e_{z,\eta}(\mathcal{O}_U(q))$.

Finally, $^\flat : TM \to T^*M$ is the lowering index operator by the metric tensor $(a^j \frac{\partial}{\partial x^j})^\flat = g_{jk} a^j dx^k$ and $^\sharp : T^*M \to TM$ its inverse.

2.2. **Determination of the conformal class of the metric.**

Lemma 2.1. *Let $z \in M$. Then there is a neighborhood V of z so that*
(i) If the geodesics $\gamma_{y,\eta}([0,s]) \subset V$ and $\gamma_{y,\eta'}([0,s']) \subset V$, $s,s' > 0$ satisfy $\gamma_{y,\eta}(s) = \gamma_{y,\eta'}(s')$, then $\eta = c\eta'$ and $s' = cs$ with some $c > 0$.
(ii) For any $y \in V$, $\eta \in T_yM \setminus 0$ there is $s > 0$ such that $\gamma_{x,\eta}(s) \notin V$.

Proof. The property (i) follows from [71, Prop. 5.7]. Making $s > 0$ so small that $\overline{V} \subset B_{g^+}(z,\rho)$ with a sufficiently small ρ, the claim (ii) from [71, Lem. 14.13]. □

Let $q^-, q^+ \in M$. As M is globally hyperbolic, the set $J(q^-,q^+) = J^-(q^+) \cap J^+(q^-)$ is compact.

Let us consider points $x, y \in M$. If $x < y$, we define the time separation function $\tau(x,y) \in [0,\infty)$ from x to y to be the supremum of the lengths

$$L(\alpha) = \int_0^1 \sqrt{-g(\dot\alpha(s),\dot\alpha(s))}\, ds$$

of the piecewise smooth causal paths $\alpha : [0,1] \to M$ from x to y. If the condition $x < y$ does not hold, we define $\tau(x,y) = 0$. We note that $\tau(x,y)$ satisfies the reverse triangle inequality

$$(19) \qquad \tau(x,y) + \tau(y,z) \le \tau(x,z) \quad \text{for } x \le y \le z.$$

As M is globally hyperbolic, the time separation function $(x,y) \mapsto \tau(x,y)$ is continuous in $M \times M$ by [71, Lem. 14.21].

Let us consider points $x, y \in M$, $x < y$. As the set $J(x,y)$ it compact, by [71, Prop. 14.19] the points x and y can be connected by a causal geodesic whose length is $\tau(x,y)$. In particular, this implies that if y can be connected to x with a causal path which is not a light-like geodesic then $\tau(x,y) > 0$, see [71, Prop. 10.46].

The above facts can be combined as follows: consider a path which is the union of the future going light-like geodesics $\gamma_{x_1,\theta_1}([0,t_1]) \subset M$

and $\gamma_{x_2,\theta_2}([0,t_2]) \subset M$, where $x_2 = \gamma_{x_1,\theta_1}(t_1)$ and $t_1, t_2 > 0$. Let $\zeta = \dot\gamma_{x_1,\theta_1}(t_1)$ and $x_3 = \gamma_{x_2,\theta_2}(t_2)$. Then, if there are no $c > 0$ such that $\zeta = c\theta_2$, the union of these geodesic is not a light-like geodesics and thus $\tau(x_1, x_3) > 0$. In particular, then there exists a time-like geodesic from x_1 to x_3. In the following we call this kind of argument for a union of light-like geodesics a short-cut argument.

When (x, ξ) is a light-like vector, we define $\mathcal{T}(x, \xi)$ to be the length of the maximal interval on which $\gamma_{x,\xi} : [0, \mathcal{T}(x, \xi)) \to M$ is defined.

When (x, ξ_+) is a future pointing light-like vector, and (x, ξ_-) is a past pointing light-like vector, we define the modified cut locus functions, c.f. [6, Def. 9.32], $\rho(x, \xi_\pm) = \rho_g(x, \xi_\pm)$,

$$\begin{aligned}(20)\qquad &\rho(x, \xi_+) = \sup\{s \in [0, \mathcal{T}(x, \xi)) : \ \tau(x, \gamma_{x,\xi_+}(s)) = 0\},\\ &\rho(x, \xi_-) = \sup\{s \in [0, \mathcal{T}(x, \xi)) : \ \tau(\gamma_{x,\xi_-}(s), x) = 0\}.\end{aligned}$$

The point $\gamma_{x,\xi}(s)|_{s=\rho(x,\xi)}$ is called the cut point on the geodesic $\gamma_{x,\xi}$. Below, we say that $s = \rho(x, \xi)$ is a cut value on $\gamma_{x,\xi}([0, a])$.

By [6, Thm. 9.33], the function $\rho(x, \xi)$ is lower semi-continuous on a globally hyperbolic Lorentzian manifold (M, g). We note that by [6, Thm. 9.15], a cut point $\gamma_{x,\xi}(s)|_{s=\rho(x,\xi)}$ is either a conjugate point or a null cut point, that is, an intersection point of two light geodesics starting from the point x. Note that by [6, Cor. 10.73] the infimum of the null cut points is smaller or equal to the first null conjugate point.

Let $p^\pm = \mu_g(s_\pm)$. By [71, Lem. 14.3], the set $I(p^-, p^+) = I^-(p^+) \cap I^+(p^-)$ is open. We need the following first observation time function f^+_μ.

Recall that by formula (1), $p^\pm = \mu_g(s_\pm)$ satisfy $p^\pm \in I^\mp(\mu_{g,z,\eta}(s_{\pm 2}))$ for all $z, \eta \in \mathcal{U}_{z_0,\eta_0}$.

Definition 2.2. *Let* $\mu = \mu_{g,z,\eta}$, $z, \eta \in \mathcal{U}_{z_0,\eta_0}$. *For* $x \in J^-(p^+) \setminus I^-(p^-)$ *we define* $f^+_\mu(x) \in [-1, 1]$ *by setting*

$$\begin{aligned}&h_+(s) = \tau(x, \mu(s)), \quad A^+_\mu(x) = \{s \in (-1, 1) : \ h_+(s) > 0\} \cup \{1\},\\ &f^+_\mu(x) = \inf A^+_\mu(x).\end{aligned}$$

Similarly, for $x \in J^+(p^-) \setminus I^+(p^+)$ *we define* $f^-_\mu(x) \in [-1, 1]$ *by setting*

$$\begin{aligned}&h_-(s) = \tau(\mu(s), x), \quad A^-_\mu(x) = \{s \in (-1, 1) : \ h_-(s) > 0\} \cup \{-1\},\\ &f^-_\mu(x) = \sup A^-_\mu(x).\end{aligned}$$

We need the following simple properties of these functions.

Lemma 2.3. *Let* $\mu = \mu_{z,\eta}$, $z, \eta \in \mathcal{U}_{z_0,\eta_0}$, *and* $x \in J^-(p^+) \setminus I^-(p^-)$. *Then*

(i) The function $s \mapsto \tau(\mu(s), x)$ *is non-decreasing on the interval* $s \in [-1, 1]$ *and strictly increasing on* $s \in [f^+_\mu(x), 1]$.

(ii) It holds that $s_{-2} < f^+_\mu(x) < s_{+2}$.

(iii) Let $y = \mu(f_\mu^+(x))$. Then $\tau(x, y) = 0$. Also, if $x \notin \mu$, there is a light-like geodesic $\gamma([0, s])$ in M from x to y with no conjugate points on $\gamma([0, s))$.

(iv) The map $f_\mu^+ : J^-(p^+) \setminus I^-(p^-) \to (-1, 1)$ is continuous.

(v) For $q \in J^-(p^+)\setminus I^-(p^-)$ the map $F : \mathcal{U}_{z_0,\eta_0} \to \mathbb{R}$; $F(z, \eta) = f^+_{\mu(z,\eta)}(q)$ is continuous.

Note that above we consider also a single point path as a light-like geodesic.

Proof. (i) As μ is a time like-path, we have $\tau(\mu(s), \mu(s')) > 0$ for $s < s'$. When $x \le \mu(s)$ and $s < s'$, the reverse triangle inequality (19) yields that $\tau(x, \mu(s)) < \tau(x, \mu(s'))$. As $\tau(x, \mu(t)) = 0$ for $t \le f_\mu^+(x)$, (i) follows.

(ii) Recall that $p^\pm \in I^\mp(\mu(s_{\pm 2}))$ by (1). As $x \in I^-(p^+)$, we see that $\tau(x, \mu(s_{+2})) > 0$. Due to the global hyperbolicity, τ is continuous in $M \times M$, see [71, Lem. 14.21]. Therefore, $h_+(s) = \tau(x, \mu(s)) > 0$ for some $s < s_{+2}$ and we see that $s_+ = f_\mu^+(x) < s_{+2}$.

Assume next that $f_\mu^+(x) \le s_{-2}$. Since then h_+ is strictly increasing, we have $h_+(s) = \tau(x, \mu(s)) > 0$ for all $s \in (s_{+2}, 1]$. Let $s_j > s_{-2}$ be such that $s_j \to s_{-2}$ as $j \to \infty$ and $h_+(s_j) > 0$, so that $x < \mu(s_j)$. As $J^+(x) \cap J^-(p^+)$ is closed, then $x \le q_- = \mu(s_{-2}) < p^-$. This is not possible since $x \in J^-(p^+) \setminus I^-(p^-)$. Thus (ii) is proven.

(iii) Let $s_x^+ = f_\mu^+(x)$ and $s_j^+ < s_x^+$ be such that $s_j^+ \to s_x^+$ as $j \to \infty$. Then, $\tau(x, \mu(s_j^+)) = 0$ and by continuity of τ, $\tau(x, y) = 0$. On the other hand, let $s_j^+ \in A_+(x)$ be such that $s_j^+ \to s_x^+$ as $j \to \infty$. Then $x \le \mu(s_j^+)$, and by closedness of $J^+(x)$ (see [71, Lem. 14.22]), $x \le y = \mu(s_x^+)$. As $x \notin \mu$, $x < y$ and by [71, Lem. 10.51] there is a light-like geodesic from x to y with no conjugate points before y. This proves (iii).

(iv) Assume that $x_j \to x$ in $J^-(p^+) \setminus I^-(p^-)$ as $j \to \infty$. Let $s_j = f_\mu^+(x_j)$ and $s = f_\mu^+(x)$. As τ is continuous, for any $\varepsilon > 0$ we have $\lim_{j\to\infty} \tau(x_j, \mu(s+\varepsilon)) = \tau(x, \mu(s+\varepsilon)) > 0$ and thus for j large enough $x_j < \mu(s + \varepsilon)$. Thus $\lim_{j\to\infty} s_j \le s$. Assume next that $\liminf_{j\to\infty} s_j = \widetilde{s} < s$ and denote $\varepsilon = \tau(\mu(\widetilde{s}), \mu(s)) > 0$. Then $\liminf_{j\to\infty} \tau(x_j, \mu(s)) \ge \varepsilon$, and as τ is continuous in $M \times M$, we obtain $\tau(x, \mu(s)) \ge \varepsilon$, which is not possible as $s = f_\mu^+(x)$. Hence $s_j \to s$ as $j \to \infty$. This proves (iv).

(v) Observe that as $J^+(q)$ is a closed set, $F(z, \eta)$ is equal to the smallest value $s \in [-1, 1]$ such that $\mu_{z,\eta}(s) \in J^+(q)$. Let $(z_j, \eta_j) \to (z, \eta)$ in (TM, g^+) as $j \to \infty$ and $s_j = F(z_j, \eta_j)$ and $\underline{s} = \liminf_{j\to\infty} s_j$. As the map $(z, \eta, s) \mapsto \mu_{z,\eta}(s)$ is continuous, we see that for a suitable subsequence $\mu_{z,\eta}(\underline{s}) = \lim_{k\to\infty} \mu_{z_{j_k},\eta_{j_k}}(s_{j_k}) \in J^+(q)$ and hence $F(z, \eta) \le \underline{s} = \liminf_{j\to\infty} F(z_j, \eta_j)$. This shows that F is lower-semicontinuous.

On the other hand, let $\overline{s} = F(z, \eta)$. As $\mu_{z,\eta}$ is a time-like geodesic, we see that for any $\varepsilon \in (0, 1 - \overline{s})$ we have $\tau(q, \mu_{z,\eta}(\overline{s} + \varepsilon)) > 0$. As τ

and the map $(z,\eta,s)\mapsto \mu_{z,\eta}(s)$ are continuous, we see that there is j_0 such that if $j>j_0$ then $\tau(q,\mu_{z_j,\eta_j}(\overline{s}+\varepsilon))>0$. Hence, $F(z_j,\eta_j)\le \overline{s}+\varepsilon$. Thus $\limsup_{j\to\infty} F(z_j,\eta_j)\le \overline{s}+\varepsilon$, and as $\varepsilon>0$ can be chosen to be arbitrarily small, we have $\limsup_{j\to\infty} F(z_j,\eta_j)\le \overline{s}=F(z,\eta)$. Thus F is also upper-semicontinuous that proves (v). □

Similarly, under the assumptions of Lemma 2.3, we see that if $x'\in I^+(p^-)\setminus J^+(p^+)$ then the function $s\mapsto \tau(\mu(s),x')$ is non-increasing on the interval $s\in[-1,1]$ and strictly decreasing on $[-1,f^-_\mu(x)]$. In addition, for $s_-=f^-_\mu(x)$ we have $\tau(\mu(s_-),x')=0$ and if $x\not\in\mu$, there is a light-like geodesic from $\mu(s^-)$ to x'. Moreover, $f^-_\mu: J^+(p^-)\setminus I^+(p^+)\to\mathbb{R}$ is continuous.

In the following, let us consider the light observation set $\mathcal{O}_U: q\mapsto \mathcal{O}_U(q)$ as a map $\mathcal{O}_U: I^-(p^+)\setminus J^-(p^-)\to P(L^+U)$.

Let $q\in I^-(p^+)\setminus J^-(p^-)$ and $\mu=\mu_{z,\eta}$, $(z,\eta)\in\mathcal{U}_{z_0,\eta_0}$. By Lemma 2.3 (iii), $Z=\mathcal{O}_U(q)\subset P(TU)$ is non-empty.

First, let us consider the case when $q\in M\setminus\mu$. Then there is a sequence $s_j\in A_-(q)$, $s_j\to f^+_\mu(q)$ as $j\to\infty$, and $\zeta_j\in S^{g^+}_{z_j}M$, $z_j=\mu(s_j)$, such that $(z_j,\zeta_j)\in O_U(q)$. Then z_j converge in the metric of (M,g^+) to $z_0=\mu(f^+_\mu(q))$ as $j\to\infty$. As the set $J^+(q)\cap J^-(p^+)$ is closed, we see that $q\le z_0$. Moreover, we see as above that $\tau(q,z_0)=\lim_{j\to\infty}\tau(q,\mu(s_j))=0$. Hence there is a light-like geodesic $\gamma_{q,\theta}([0,l])$ from q to z_0. Second, in the case when $q\in\mu$ we see that $L^+_qM\subset O_U(q)$ and $z_0=\mu(f^+_\mu(q))=q$. These cases show that for any $q\in M$ there are $z_0=\mu(f^+_\mu(q))$ and $\zeta_0\in L^+_{z_0}M$ such that $(z_0,\zeta_0)\in O_U(q)$. Also, $z_0\in\mathcal{P}_U(q)$.

Definition 2.4. *Let $\mu=\mu_{z,\eta}$, $(z,\eta)\in\mathcal{U}_{z_0,\eta_0}$ and $q\in I^-(p^+)\setminus J^-(p^-)$. We define the set of the earliest observations, c.f. (18),*

$$\mathcal{E}_U(q) \;=\; \bigcup_{(z,\eta)\in\mathcal{U}_{z_0,\eta_0}} \mathcal{E}_{z,\eta}(q),$$

and the map $\mathbf{e}_U: J(p^-,p^+)\to\mathbb{P}(U)$ given by $\mathbf{e}_U(q)=\bigcup_{(z,\eta)\in\mathcal{U}_{z_0,\eta_0}}\mathbf{e}_{z,\eta}(q)$, where $\mathbf{e}_{z,\eta}(q)=e_{z,\eta}(\mathcal{P}_U(q))$.

Above we have seen that $z_0=\mu(f^+_\mu(q))\in\mathcal{P}_U(q)\subset\pi(O_U(q))$ satisfies $\tau(q,z_0)=0$ and on the other hand, $\tau(q,\gamma_{q,\xi}(s))>0$ for $s<\rho(q,\xi)$. Using these, we see that

$$\mathcal{E}(q)=\{\dot\gamma_{q,\xi}(s)\in U_g;\ 0\le s\le\rho(q,\xi),\ \xi\in L^+_qM\}. \tag{21}$$

Below, denote $T(I^-(x))=\{(x,\xi)\in TM_0;\ x\in I^-(x_0)\}$.

Lemma 2.5. *Assume that we are given (U,g), $U=U_g$, $x_0\in U$, and a set $F\subset U$ satisfying $\mathbf{e}_U(q_0)\cap I^-(x_0)\subset F\subset\mathcal{P}_U(q_0)$ for some $q_0\in I^-(p^+)\setminus J^-(p^-)$. These data determine the set $\mathcal{E}_U(q_0)\cap T(I^-(x_0))$. If we are given the above data for all $x_0\in U$, we can determine $\mathcal{E}_U(q_0)$.*

Proof. Given F, we can find $\mathbf{e}_U(q_0)\cap I^-(x_0)$ by taking union of points that can be represented as $e_{z,\eta}(F)\in I^-(x_0)$ with some $(z,\eta)\in\mathcal{U}_{z_0,\eta_0}$. Thus we may assume that we are given $\mathbf{e}_U(q_0)\cap I^-(x_0)$.

Below, let $y=\mathbf{e}_U(q_0)\cap I^-(x_0)$ be such that $y\in I^-(x_0)$.

Consider a vector $\theta\in L^-_yM$ such that $(y,\theta)\in\mathcal{E}_U(q_0)$. If $r>0$ is so small that $\widetilde y=\gamma_{y,\theta}(-r)\in U_g$, a short cut argument shows that $\gamma_{y,\theta}([-r,0])$ is the only causal geodesic in U connecting $\widetilde y$ and y. Moreover, there is $(\widetilde z,\widetilde \eta)\in\mathcal{U}_{z_0,\eta_0}$ such that for $\widetilde\mu=\mu_{\widetilde z,\widetilde\eta}$ we have $\widetilde y\in\widetilde\mu$. Moreover, as $\widetilde y\in U$, using the reverse triangle inequality we see that then $\widetilde y=\pi(\mathcal{E}_{\widetilde z,\widetilde\eta}(q_0))\in\mathbf{e}_U(q_0)$. Clearly, $\widetilde y\in I^-(x_0)$.

On the other hand, if there exists $\theta_1\in L^-_yM$ such that for some $r>0$ we have $\gamma_{y,\theta_1}([-r,0])\subset U$ and $\widetilde y_1=\gamma_{y,\theta_1}(-r)$ is such that $\widetilde y_1\in\widetilde\mu_1\cap(\mathbf{e}_U(q_0)\cap I^-(x_0))$ for some $(\widetilde z_1,\widetilde\eta_1)\in\mathcal{U}_{z_0,\eta_0}$ and $\widetilde\mu_1=\mu_{\widetilde z_1,\widetilde\eta_1}$, a short cut argument shows that then we must have $(y,\theta_1)\in\mathcal{O}(q_0)$. This shows that knowing $\mathbf{e}_U(q_0)\cap I^-(x_0)$ we can determine $\mathcal{E}_U(q)\cap T(I^-(x_0))$.

Finally, we observe that if we can find $\mathcal{E}_U(q)\cap T(I^-(x_0))$ for all $x_0\in U$, by taking union of these sets we find $\mathcal{E}_U(q)$. □

Lemma 2.6. *Let $q_1,q_2\in I^-(p^+)\setminus J^-(p^-)$ be such that $e_{z,\eta}(\mathcal{O}_U(q_1))=e_{z,\eta}(\mathcal{O}_U(q_2))$ where $(z,\eta)\in\mathcal{U}_{z_0,\eta_0}$.*

If $e_{z,\eta}(\mathcal{O}_U(q_1))$ has a neighborhood $V\subset M$ such that

$$\mathcal{E}_U(q_1)\cap TV=\mathcal{E}_U(q_2)\cap TV,$$

then $q_1=q_2$.

In particular, if $q_1,q_2\in I^-(p^+)\setminus J^-(p^-)$ are such that $\mathcal{O}_U(q_1)=\mathcal{O}_U(q_2)$, then $q_1=q_2$.

The above lemma can be also stated as follows: a germ of the set $\mathcal{O}_U(q)$ near $e_{z,\eta}(\mathcal{O}_U(q))$ determine $q\in I^-(p^+)\setminus J^-(p^-)$ uniquely.

Proof. Let $\mu=\mu_{g,z,\eta}$ and $Z:=\mathcal{E}_U(q_1)\cap TV=\mathcal{E}_U(q_2)\cap TV$.

Let us assume that $q_1\neq q_2$. Let $(y,-\zeta)\in E_{z,\eta}(Z)$. Then $f^+_\mu(q_1)=f^+_\mu(q_2)$ and $y=\mu(f^+_\mu(q_1))$. Let $s_1,s_2\geq 0$ be such that $q_1=\gamma_{y,\zeta}(s_1)$ and $q_2=\gamma_{y,\zeta}(s_2)$. Without loss of generality, we can assume that $s_1<s_2$. Let $\theta_j=-\dot\gamma_{y,\zeta}(s_j)\in L^+_{q_j}M$, $j=1,2$, and let consider a vector $\theta_2'\in L^+_{q_2}M$ satisfying $\theta_2'\neq\theta_2$ and $\|\theta_2'\|_{g^+}=\|\theta_2\|_{g^+}$. We assume that θ_2' is so close to θ_2 in the Sasaki metric of (TM,g^+) and that $s_2'<s_2$ is so close to s_2 that $y'=\gamma_{q_2,\theta_2'}(s_2')\in V$. Moreover, as the function $(x,\xi)\mapsto\rho(x,\xi)$ is lower semicontinuous, and $\rho(q_2,\theta_2)\geq s_2$ we can assume that θ_2' and s_2' are so close to θ_2' and s_2', correspondingly, that that $\rho(q_2,\theta_2')>s_2'$.

Let us define $\zeta'=-\gamma_{q_2,\theta_2'}(s_2')\in L^-_{y'}M$. Then $(y',-\zeta')\in Z=\mathcal{O}_U(q_2)\cap TV$, and $q_2=\gamma_{y',\zeta'}(s_2')$. As then $(y',-\zeta')\in Z=\mathcal{O}_U(q_1)\cap TV$, too, we see that there is $s_1'\geq 0$ such that

$$q_1=\gamma_{y',\zeta'}(s_1').$$

Then $s_1' < s_2'$, as otherwise the union of $\gamma_{q_2,\theta_2}([0, s_2 - s_1])$ and $\gamma_{y',\zeta'}([s_2', s_1'])$, oriented in the opposite direction, would be a closed causal path.

Let us consider a path which is the union of the light-like geodesics $\gamma_{q_2,\theta_2'}([0, s_2' - s_1'])$ and $\gamma_{q_1,\theta_1}([0, s_1])$. As $\theta_2' \neq \theta_2$, this path is not a light-like geodesics and using a short-cut argument we see that $\tau(q_2, y) > 0$. Hence $f^+_\mu(q_2) < f^+_\mu(q_1)$ which is a contradiction with the fact that $\mathcal{E}_U(q_1) \cap TV = \mathcal{E}_U(q_2) \cap TV$, proving that $q_1 = q_2$.

Finally, consider the case when $q_1 \in \mu$ and $\mathcal{E}_U(q_1) \cap TV = \mathcal{E}_U(q_2) \cap TV$. Let $(q_1, \xi) \in \mathcal{E}_U(q_1)$. Then, if $q_2 \neq q_1$, we have $(q_1, \xi) \in \mathcal{E}_U(q_2)$ and thus there is $s > 0$ so that $q_2 = \gamma_{q_1,\xi}(-s)$. When $\widetilde{s} > 0$ is sufficiently small, we see that $(\widetilde{x}, \widetilde{\xi}) = (\gamma_{q_1,\xi}(-\widetilde{s}), \dot{\gamma}_{q_1,\xi}(-\widetilde{s})) \in \mathcal{E}_U(q_2) \cap TV$ but $\gamma_{q_1,\xi}(-\widetilde{s}) < q_1$ and thus $(\widetilde{x}, \widetilde{\xi}) \notin \mathcal{E}_U(q_1) \cap TV$ that is is a contradiction. Hence $q_1 = q_2$.

The last claim follows from the fact that if $\mathcal{O}_U(q_1) = \mathcal{O}_U(q_2)$ then $\mathcal{E}_U(q_1) \cap TV_1 = \mathcal{E}_U(q_2) \cap TV_1$ for some neighborhood $V_1 \subset U$ of $e_{z,\eta}(\mathcal{O}_U(q_1))$.
□

Lemma 2.7. *Let $K \subset M$ be a compact set. Then there is $R_1 > 0$ such that if $\gamma_{y,\theta}([0, l]) \subset K$ is a light-like geodesic with $\|\theta\|_{g^+} = 1$, then $l \leq R_1$. In the case when $K = J(p^-, p^+)$, with $q^-, q^+ \in M$ we have $\gamma_{y,\theta}(t) \notin J(q^-, q^+)$ for $t > R_1$.*

The proof of this lemma is standard, but we include it for the convenience of the reader.

Proof. Assume that there are no such R_1. Then there are geodesics $\gamma_{y_j,\theta_j}([0, l_j]) \subset K$, $j \in \mathbb{Z}_+$ such that $\|\theta_j\|_{g^+} = 1$ and $l_j \to \infty$ as $j \to \infty$. Let us choose a subsequence (y_j, θ_j) which converges to some point (y, θ) in (TM, g^+). As θ_j are light-like, also θ is light-like.

Then, we observe that for all $R_0 > 0$ the functions $t \mapsto \gamma_{y_j,\theta_j}(s)$, converge in $C^1([0, R_0]; M)$ to $s \mapsto \gamma_{y,\theta}(t)$, as $j \to \infty$. As $\gamma_{y_j,\theta_j}([0, l_j]) \subset K$ for all j, we see that $\gamma_{y,\theta}([0, R_0]) \subset K$ for all $R_0 > 0$. Let $z_n = \gamma_{y,\theta}(n)$, $n \in \mathbb{Z}_+$. As K is compact, we see that there is a subsequence z_{n_k} which converges to a point z as $n_k \to \infty$. Let now $V \subset M$ be a small convex neighborhood of z such that each geodesic starting from V exits the set V (cf. Lemma 2.1). Let $V' \subset M$ be a neighborhood of z so that the strong causality condition (17) is satisfied for V and V'. Then we see that there is k_0 such that if $k \geq k_0$ then $z_{n_k} \in V'$, implying that $\gamma_{y,\theta}([n_{k_0}, \infty)) \subset V$. This is a contradiction and thus the claimed $R_1 > 0$ exists.

Finally, in the case when $K = J(q^-, q^+)$, $q^-, q^+ \in M$ we see that if $q(s) = \gamma_{y,\theta}(s) \in K$ for some $s > R_1$, then for all $\widetilde{s} \in [0, s]$ we have $q(\widetilde{s}) \leq q(s) \leq q^+$ and $q_- \leq q(0) \leq q(\widetilde{s})$. Thus $q(\widetilde{s}) \in K$ for all $\widetilde{s} \in [0, s]$. As $t > R_1$, this is not possible by the above reasoning, and thus the last assertion follows. □

Assume next that $y_j = \gamma_{q,\eta_j}(s_j)$ and $\zeta_j = \dot\gamma_{q,\eta_j}(s_j)$, where $q \in M$, $\eta_j \in L^+_q M$, $\|\eta_j\|_{g^+} = 1$, and $t_j > 0$, are such that $(y_j, \zeta_j) \to (y, \zeta)$ in TM as $j \to \infty$. There exists $p \in M$ such that $p \in I^+(y)$. Then for sufficiently large j we have $y_j \in J(q, p)$ and we see that by Lemma 2.7, s_j are uniformly bounded. Thus there exist subsequences s_{j_k} and η_{j_k} satisfying $s_{j_k} \to t$ and $\eta_{j_k} \to \eta$ as $k \to \infty$. Then $(y, \zeta) = (\gamma_{q,\eta}(s), \dot\gamma_{q,\eta}(s))$. This shows that the light observation set $\mathcal{O}_U(q)$ is a closed subset TU.

Let $\mathbb{S}$ be the collection of relatively closed sets K in U_g that intersect all geodesics $\gamma_{z,\eta}$, $(z, \eta) \in \mathcal{U}_{z_0,\eta_0}$ precisely once. We endow $\mathbb{S}$ with the topology τ_e that is the weakest topology for which all maps $e_{z,\eta}$, $K \mapsto e_{z,\eta}(K)$, parametrized by $(z, \eta) \in \mathcal{U}_{z_0,\eta_0}$, are continuous. Note that $\mathcal{P}_U(q) \in \mathbb{S}$ for all $q \in I^-(p^+)\setminus J^-(p^-)$ by formula (1). We use below the continuous map $F : \mathbb{S} \to \prod_{(z,\eta)\in\mathcal{U}_{z_0,\eta_0}} \mathbb{R} = \mathbb{R}^{\mathcal{U}_{z_0,\eta_0}}$ that is defined for $Z \in \mathbb{S}$ by setting $F(Z) = (s_{z,\eta})_{(z,\eta)\in\mathcal{U}_{z_0,\eta_0}}$ where $\mu_{z,\eta}(s_{z,\eta}) \in \pi(Z)$.

Lemma 2.8. *Let $V \subset I^-(p^+) \setminus J^-(p^-)$ be relatively compact open set Then the map*

$$\mathbf{e}_U : \overline{V} \to \mathbb{S}$$

defines a homeomorphism $\mathbf{e}_U : \overline{V} \to \mathbf{e}_U(\overline{V})$.

Proof. Let $\mu = \mu_{g,z,\eta}$. First we note that as the map $x \mapsto f^+_\mu(x)$ is continuous in $I^-(p^+) \setminus J^-(p^-)$ for all $\mu = \mu(z, \eta)$, the map $\mathbf{e}_U$ is continuous.

Let us next show that the relative topology on $\mathbf{e}_U(J(p^-, p^+))$ determined by $(\mathbb{S}, \tau_e)$ is a Hausdorff topology.

If $(\mathbf{e}_U(J(p^-, p^+)), \tau_e)$ is not Hausdorff space then there are $q_1, q_2 \in J(p^-, p^+)$, $q_1 \neq q_2$ such that $f^+_{\mu(z,\eta)}(q_1) = f^+_{\mu(z,\eta)}(q_2)$ for all $(z, \eta) \in \mathcal{U}_{z_0,\eta_0}$ and there is $(x, \zeta_1) \in \mathcal{E}_U(q_1) \setminus \mathcal{E}_U(q_2)$. By Lemma 2.6 , we have either that the set $\mathcal{E}_U(q_1) \setminus \mathcal{E}_U(q_2)$ or $\mathcal{E}_U(q_1) \setminus \mathcal{E}_U(q_2)$ is non-empty. Next, consider the case where $\mathcal{E}_U(q_1) \setminus \mathcal{E}_U(q_2) \neq \emptyset$.

Let $t > 0$ be so small that $\gamma_{x_1,\zeta_1}([-t, 0]) \subset U_g$ and $\gamma_{x_1,\zeta_1}([-t, 0])$ is the only light-like geodesic from x_1 to $x'_1 = \gamma_{x_1,\zeta_1}(-t)$, see Lemma 2.1 (i). Let $\zeta'_1 = \dot\gamma_{x_1,\zeta_1}(-t)$. Then $(x'_1, \zeta'_1) \in \mathcal{E}_U(q_1)$ and $\tau(x'_1, x_1) = 0$. Let $\mu_k(s) = \mu(z_k, \eta_k; s)$, $k = 1, 2$ be such that $x_1 = \mu_1(s_1)$ and $x'_1 = \mu_2(s_2)$. Then $f^+_{\mu_k}(q_2) = f^+_{\mu_k}(q_1) = s_k$ for $k = 1, 2$ implies that $\tau(q_2, x_1) = \tau(q_2, x'_1) = 0$. As $\tau(x'_1, x_1) = 0$, this implies by a short cut argument that the union of a light-like geodesic $\widetilde\gamma$ from q_2 to x'_1 and $\gamma_{x_1,\zeta_1}([-t, 0])$ from x'_1 to x_1 is a light-like geodesic and there is s_2 such that $\gamma_{x_1,\zeta_1}(-s_2) = q_2$. As $\tau(q_2, x_1) = 0$ we see that $\gamma_{x_1,\zeta_1}([-s_2, 0])$ is a longest possible curve between its end points and thus we have to have $(x_1, \zeta_1) \in \mathcal{E}_U(q_2)$. This contradiction shows that τ_e induces on $\mathcal{E}_U(J(p^-, p^+))$ a Hausdorff topology.

The claim follows then from the general fact that a continuous bijective map from a compact Hausdorff space onto a Hausdorff space is a homeomorphism. $\square$

In the next lemma we consider coordinates associated with light observations, see Fig. 5.

Lemma 2.9. *Let $q_0 \in I^-(p^+) \setminus J^-(p^-)$. Then there is a neighborhood $W \subset M$ of q_0 and time-like paths $\mu_{z_j,\eta_j}((-1,1)) \subset U_g$, $(z_j,\eta_j) \in \mathcal{U}_{z_0,\eta_0}$, $j = 1, 2, \dots, n$, such that if $Y^j(q) \in \mathbb{R}$ are determined by the equations*

$$\mu_j(Y^j(q)) = e_{z_j,\eta_j}(\mathcal{O}_U(q)), \quad \textit{or equivalently,} \quad Y^j(q) = f^+_{\mu_j}(q),$$

then $Y(q) = (Y^j(q))_{j=1}^n$ define coordinates $Y : W \to \mathbb{R}^n$ which are compatible with the differentiable structure of M. Moreover, if $(\widetilde{z}, \widetilde{\eta}) \in \mathcal{U}_{z_0,\eta_0}$ is given, the points $(z_j, \eta_j) \in \mathcal{U}_{z_0,\eta_0}$, $j = 1, 2, \dots, n$ can be chosen in an arbitrary open neighborhood of $(\widetilde{z}, \widetilde{\eta})$.

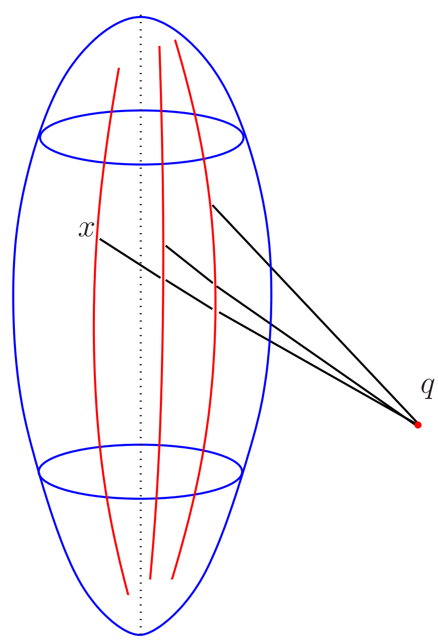

FIGURE 5. A schematic figure where the space-time is represented as the 3-dimensional set $\mathbb{R}^{2+1}$. The red curves are paths $\mu_{z_j,\eta_j} = \mu(z_j, \eta_j)$. In a neighborhood of a point $q_0 \in I^-(\widehat{p}^+) \setminus I^-(\widehat{p}^-)$ we can choose $(z_j, \eta_j) \in \mathcal{U}_{z_0,\eta_0}$, $j = 1, 2, 3, 4$ such that $q \mapsto (f^+_{\mu(z_j,\eta_j)}(q))_{j=1}^4$ defines coordinates near q_0. In the figure, light lines are light-like geodesics and $x = \mu_{z_j,\eta_j}(f^+_{\mu(z_j,\eta_j)}(q))$.

Proof. Let $q_0 \in I^-(p^+) \setminus J^-(p^-)$, $(y_0, \xi_0) \in E_\mu(\mathcal{O}_U(q_0))$, and $\gamma_{y_0,-\xi_0}([0, t_0])$ be a light-like geodesic from y_0 to q_0. Let $\vartheta_0 = -\dot{\gamma}_{y_0,-\xi_0}(t_0)$. As $\gamma_{q_0,\vartheta_0}([0, t_0])$ is a longest causal curve between its end points (note that it needs to be unique), for any $t_1 \in (0, t_0)$ (see [71, Thm. 10.51] or [39, Prop 4.5.12]) there are no conjugate points on the geodesic $\gamma_{q_0,\vartheta_0}([0, t_1])$. Let $\mathcal{V}_r = \mathcal{V}_r(q_0, t_1\vartheta_0) \subset TM$ be the r-neighborhood of $(q_0, t_1\vartheta_0)$ in the Sasaki metric of (TM, g^+). We see by using [71, Prop. 10.10] that, for all $0 < t_1 < t_0$, there is $r_1 = r_1(t_1) > 0$ such that the exponential map $\Phi : (x,\zeta) \mapsto (x, \gamma_{x,\zeta}(1))$ is a diffeomorphism $\Phi : \mathcal{V}_{r_1} \to \Phi(\mathcal{V}_{r_1}) \subset M \times M$.

In the following, let $t_1 < t_0$ be so close to t_0 and $r_2 \in (0, r_1(t_1))$ be so small that for all $(x,\xi) \in \mathcal{V}_{r_2}$ we have $\gamma_{x,\xi}(1) \in U_g$.

Next we show that there is $r_3 \in (0, r_2)$ such that, if $(x,\xi) \in L^+M \cap \mathcal{V}_{r_3}$, then the geodesic $\gamma_{x,\xi}([0,1])$ is the unique causal curve between its endpoints x and $y = \gamma_{x,\xi}(1) \in U$. To show this, assume that there are no such r_3. Then there is a sequence $r_j \to 0$, points $(x_j, \xi_j) \in \mathcal{V}_{r_j} \cap L^+M$, and $\zeta_j \in T_{x_j}M$, $\zeta_j \neq \xi_j$ so that $\gamma_{x_j,\zeta_j}([0,1])$ is some other causal geodesic between the points $\gamma_{x_j,\xi_j}(0)$ and $\gamma_{x_j,\xi_j}(1)$ having at least the same length as $\gamma_{x_j,\xi_j}([0,1])$. Note that $\gamma_{x_j,\xi_j}(0) \to q_0$ and $\gamma_{x_j,\xi_j}(1) \to \gamma_{q_0,\vartheta_0}(t_1)$ as $j \to \infty$. Then, the sequence (x_j, ζ_j) has a subsequence which converges to some point (q_0, ζ) in the Sasaki metric, see Lemma 2.7. If $\zeta = t_1\vartheta_0$, we see that there the map Φ is not a local diffeomorphism near the point $(q_0, t_1\vartheta_0)$ which is not possible. On the other hand, if $\zeta \neq t_1\vartheta_0$, we see using a short cut argument for the union of the geodesics $\gamma_{q_0,\zeta}([0,1])$ and $\gamma_{q_0,\vartheta_0}([t_1, t_0])$, that the geodesic $\gamma_{q_0,\vartheta_0}([0, t_0])$ is not a longest causal curve between its end points q_0 and y_0. This is in contradiction with the earlier assumptions, and thus we see that the claimed $r_3 > 0$ exists.

For $(x, y) \in \Phi(\mathcal{V}_{r_4})$, $r_4 = r_3/2$ we denote $(x, \zeta_{x,y}) = \Phi^{-1}(x, y)$. If $\zeta = \zeta_{x,y} \in L^+M$, then the light-like geodesic $\gamma_{x,\zeta}([0,1])$ is the unique causal curve between x and y. Note that for $\varepsilon > 0$ small enough, e.g., $\varepsilon = r_4$, also the geodesic $\gamma_{x,\zeta}([0, 1+\varepsilon])$ with endpoints in $\Phi(\mathcal{V}_{r_3})$ is the unique causal curve between its end points and then by [71, Thm. 10.51] or [39, Prop 4.5.12]) there are no conjugate points on the geodesic $\gamma_{x,\zeta}([0,1])$.

Let us next fix $t_1 < t_0$ and r_3 as above, and consider $t_1\vartheta_1 \in L^+_{q_0}M \cap \mathcal{V}_{r_3}$, $\|\vartheta_1\|_{g^+} = \|\vartheta_0\|_{g^+}$, $\vartheta_1 \neq \vartheta_0$, and $y_1 = \gamma_{q_0,\vartheta_1}(t_1) \in U_g$. Let $(x, \eta) \in \mathcal{U}_{z_0,\eta_0}$ be such the $y_1 \in \mu_{z,\eta}$. Then $\dot\mu_{z,\eta}$ and $\theta_1 = \dot\gamma_{q_0,\vartheta_1}(t_1)$ are both light-like vectors in $T_{y_1}M$ and hence $g(\dot\mu_{z,\eta}(0), \theta_1) \neq 0$.

Let us recall that the energy of a piecewise smooth curve $\alpha : [0, l] \to M$ is defined by

$$E(\alpha) = \frac{1}{2}\int_0^l g(\dot\alpha(t), \dot\alpha(t))\, dt.$$

Then the energy of the geodesic $\gamma_{x,\zeta}([0,1])$, $\zeta = \zeta_{x,y}$ connecting the points x and y is equal to $g(\zeta_{x,y}, \zeta_{x,y})$, and thus is a C^∞ smooth function in $(x, y) \in \Phi(\mathcal{V}_r)$.

Next, let $X \in T_{q_0}M$, and $z(s) = \gamma_{q_0,X}(s)$, $s \in [-s_0, s_0]$, $s_0 > 0$. Let $T : \mathbb{R} \to \mathbb{R}$ be a smooth function so that $T(0) = f^+_{\mu_{z,\eta}}(q_0)$. Let $\gamma_s(t) = \gamma_{z(s),\zeta(s)}(t)$, where $(z(s), \zeta(s)) = \Phi^{-1}(z(s), \mu_{z,\eta}(T(s)))$. Denote also $\theta(s) = \dot\gamma_{z(s),\zeta(s)}(1) \in T_{\mu_{z,\eta}(T(s))}M$.

Using [71, Prop. 10.39] for the first variation of the energy of the geodesics $\gamma_s(t)$, we see that

$$\frac{d}{ds}E(\gamma_s) = g(\theta(s), \dot\mu_{z,\eta}(T(s)))T'(s) - g(\dot\gamma_{q_0,X}(s), \zeta(s)).$$

Thus, $\mathcal{E}(\gamma_s) = 0$ for all $s \in [-s_0, s_0]$ if and only if

$$(22)\quad T'(s) = \frac{g(\dot\gamma_{q_0,X}(s), \zeta(s))}{g(\theta(s), \dot\mu_{z,\eta}(T(s)))} = \frac{g(\dot\gamma_{q_0,X}(s), \zeta(s))}{g(\dot\gamma_{z(s),\zeta(s)}(1), \dot\mu_{z,\eta}(T(s)))}\bigg|_{(z(s),\zeta(s))=\Phi^{-1}(z(s),\mu_{z,\eta}(T(s)))}$$

for all $s \in [-s_0, s_0]$. Now the equation (22) can be considered as an ordinary differential equation for $T(s)$, which has a unique solution with initial data $T(0) = f^+_{\mu_{z,\eta}}(q_0)$ on the interval $s \in [-s_0, s_0]$ when $s_0 > 0$ is sufficiently small. Let us denote the solution by $T_1(s)$. Using in the above $T(s) = T_1(s)$, we see that $E(\gamma_s) = 0$, implying $\zeta(s) \in L^+_{z(s)}M$. Thus, when $|s|$ is small enough $(z(s), \mu_{z,\eta}(T(s))) \in \Phi(\mathcal{V}_{r_3})$ and the light-like geodesic $\gamma_{z(s),\zeta(s)}([0, 1])$ is a longest geodesic between $z(s)$ and $\mu_{z,\eta}(T(s))$. Hence, for $s_0 > 0$ small enough $T = T_1(s)$, $s \in [-s_0, s_0]$ is the solution of the equation

$$\mu_{z,\eta}(T) = e_{z,\eta}(\mathcal{O}_U(z(s))),$$

that is, $T_1(s) = f^+_{\mu_{z,\eta}}(z(s))$. In particular, we see that for $s = 0$,

$$0 = \frac{d}{ds}E(\gamma_s)|_{s=0} = g(\theta_1, \dot\mu_{z,\eta}(0))T_1'(0) - g(X, \vartheta_1), \quad \text{i.e. } T_1'(0) = \frac{g(X, \vartheta_1)}{g(\theta_1, \dot\mu_{z,\eta}(0))}.$$

Let us now denote $Y^1(z(s)) := T_1(s)$. Above $T_1'(0) = g(\operatorname{grad}_g Y^1|_{q_0}, X)$ where $X = \dot z(0)$. Thus, by varying the vector $X \in T_{q_0}M$ in the above construction, we see that q_0 has a neighborhood W in M, $Y^1 : W \to \mathbb{R}$ is well defined function and

$$\operatorname{grad}_g Y^1|_{q_0} := g^{jk}\frac{\partial Y^1(x)}{\partial x^j}\frac{\partial}{\partial x^k} = c_1\vartheta_1,$$
$$c_1 = \frac{1}{g(\theta_1, \dot\mu_{z,\eta}(0))}, \quad \vartheta_1 = \vartheta_1^j\frac{\partial}{\partial x^j}.$$

Now, choose such $\vartheta_j \in L^+_{q_0}M$, $\|\vartheta_j\|_{g^+} = \|\vartheta_0\|_{g^+}$ and $t_j < t_0$ so close to t_0, $j = 2, 3, \dots, n$, such that $(q_0, t_j\vartheta_j) \in \mathcal{V}_{r_3}$ and $\vartheta_1, \vartheta_2, \dots, \vartheta_n$ are linearly independent. Then $y_j = \gamma_{q_0,\vartheta_j}(t_j) \in U_g$ and there are no conjugate points on the geodesic $\gamma_{q_0,\vartheta_j}([0, t_j + \varepsilon])$ with some $\varepsilon > 0$. Let $(z_j, \eta_j) \in \mathcal{U}_{z_0,\eta_0}$, $j = 2, 3, \dots, n$ and $\mu_j = \mu_{g,z_j,\eta_j}$ be a smooth timelike curve going through the point $y_j = \mu_j(0)$. Again, $g(\dot\mu_j(f^+_{\mu_j}(q_0)), \theta_j) \neq 0$, $\theta_j = \dot\gamma_{q_0,\vartheta_j}(t_j)$. Finally, let $Y^j(z)$ be such that

$$\mu_j(Y^j(z)) = e_{z_j,\eta_j}(\mathcal{O}_U(z)), \quad j = 2, 3, \dots n.$$

Since the vectors $\text{grad}_g Y_j|_{q_0} = c_j\vartheta_j$, $j = 1, 2, \dots, n$, $c_j \neq 0$ are linearly independent, the function $Y : W \to \mathbb{R}^n$, $Y(z) = (Y^1(z), Y^2(z), \dots, Y^n(z))$ gives coordinates near q_0.

Finally, we note that as (y_0, ξ_0) was in above an arbitrary point of $E_\mu(\mathcal{O}_U(q_0))$, we see easily that in the above construction $(z_j, \eta_j) \in \mathcal{U}_{z_0,\eta_0}$, $j = 1, 2, \dots, n$, can be chosen in an arbitrary open neighborhood of $(\widetilde{z}, \widetilde{\eta})$. □

Let $W \subset V$ be a neighborhood of q_0 and $Y : W \to \mathbb{R}^n$ be local coordinates considered in Lemma 2.9. Let us consider the metric $\mathbf{g} = (\mathcal{E}_U)_* g$ which makes $\mathcal{E}_U : (W, g) \to (\mathcal{E}_U(W), \mathbf{g})$ an isometry. Next we show that we can determine the conformal class of $\mathbf{g}$.

Let $(z, \zeta) \in TM$ and define

$$\Gamma(z, \zeta) = \{\mathcal{E}_U(q') \in \mathcal{E}_U(W);\ (z, \zeta) \in \mathcal{E}_U(q')\}.$$

Then $\Gamma(z, \zeta) = \mathcal{E}_U(\gamma_{z,\zeta}((-\infty, 0]) \cap W)$, that is, $\Gamma(z, \zeta)$ is the image of a geodesic $\gamma_{z,\zeta}((-\infty, 0])$ on $\mathcal{E}_U(W)$. Thus when $\mathcal{E}_U(W)$ is given, we can find in the Y-coordinates all light-like geodesics in $\mathcal{E}_U(W)$ that go through a specified point $\mathcal{E}_U(q)$. Consider the tangent space $T_Q(\mathcal{E}_U(W))$ of the manifold $\mathcal{E}_U(W)$ at $Q = \mathcal{E}_U(q)$. Then $\mathcal{E}_U(W)$ determines an open subset of the light cone (with respect to the metric $\mathbf{g}$) in $T_Q(\mathcal{E}_U(W))$. As the light cone is a quadratic surface in $T_Q(\mathcal{E}_U(W))$, we can determine the whole light cone in $T_Q(\mathcal{E}_U(W))$ in local coordinates. Thus we can determine all light-like vectors in the tangent space at the point $\mathcal{E}_U(q)$, where $q \in W$ is arbitrary. By [6, Thm. 2.3], these collections of light-like vectors determine uniquely the conformal class of the tensor $\mathbf{g}$ in $\mathcal{E}_U(W)$. This proves Theorem 1.2. □

2.3. **Geometric preparations for analytic results.** Next we prove some auxiliary geometrical results needed later in the analysis Einstein equations.

As the modified cut locus function $\rho(x, \xi)$ on (M, g) is lower semi-continuous, there is $\rho_0 > 0$, depending on (M, g) and p^+, p^-, such that $\rho(x, \xi) \geq \rho_0$ for all $x \in J_g(p^-, p^+)$ and $\xi \in L_x^+(M, g)$ with $\|\xi\|_{g^+} = 1$. Below, let $R_1 > 0$ be given in Lemma 2.7. On the setting of the lemma, see Fig. 6.

Lemma 2.10. *Denote $\mu_{g,z_0,\eta_0} = \mu$. There are $\vartheta_0, \kappa_0, \kappa_1, \kappa_2, \kappa_3 \in (0, \rho_0)$ such that for all $y \in \mu([s_-, s_+])$, $y = \mu(r_1)$ with $r_1 \in [s_-, s_+]$, $\zeta \in L_y^+ M$, $\|\zeta\|_{g^+} = 1$ and $(x, \xi) \in L^+M$ satisfying $d_{g^+}((y, \zeta), (x, \xi)) \leq \vartheta_0$ we have $x \in J(\mu(-1), \mu(+1))$ and the following holds:*

(i) If $t > R_1$ and $t < \mathcal{T}(x, \xi)$, then $\gamma_{x,\xi}(t) \notin J^-(\mu(s_{+2}))$. Moreover, if $0 < t \leq R_1$ and $\gamma_{y,\zeta}(t) \in J^-(\mu(s_{+2}))$, then $\gamma_{x,\xi}([0, t + \kappa_0]) \subset I^-(\mu_g(s_{+3}))$. Finally, if $0 < t \leq 10\kappa_1$, then $\gamma_{x,\xi}(t) \in U_g$.

(ii) Let $t_0 \in [\kappa_1, 6\kappa_1]$ and $t_2(x, \xi, t_0) := \rho(\gamma_{x,\xi}(t_0), \dot\gamma_{x,\xi}(t_0)) \in (0, \infty]$. If $\rho(y, \zeta) \geq R_1 + \kappa_0$, then $t_2(x, \xi, t_0) + t_0 > R_1$. If $\rho(y, \zeta) \leq R_1 + \kappa_0$, then $t_2(x, \xi, t_0) + t_0 \geq \rho(y, \zeta) + 3\kappa_2$.

(iii) Let $t_1 = \rho(y, \zeta)$. *Assume that* $t_2 \in [0, \mathcal{T}(y, \zeta))$ *is such that* $t_2 - t_1 \geq \kappa_2$ *and* $p_2 = \gamma_{y,\zeta}(t_2) \in J^-(\mu(s_{+2}))$. *Then* $r_2 = f_\mu^-(p_2)$ *satisfies* $r_2 - r_1 > 3\kappa_3$.

Moreover, let $p_3 = \gamma_{x,\xi}(t_2)$. *Then either* $p_3 \notin J^-(\mu(s_{+1}))$ *or* $r_3 = f_\mu^-(p_3)$ *satisfies* $r_3 - r_1 > 2\kappa_3$.

(iv) Above, $\kappa_1 > 0$ *can be chosen so that*

$$\gamma_{\mu(s'),\xi}([0, 4\kappa_1]) \cap J^+(\mu(s'')) = \emptyset \quad \text{when } s_- \leq s' < s'' \leq s_+, \quad \xi \in L^+_{\mu(s')}M_0. \tag{23}$$

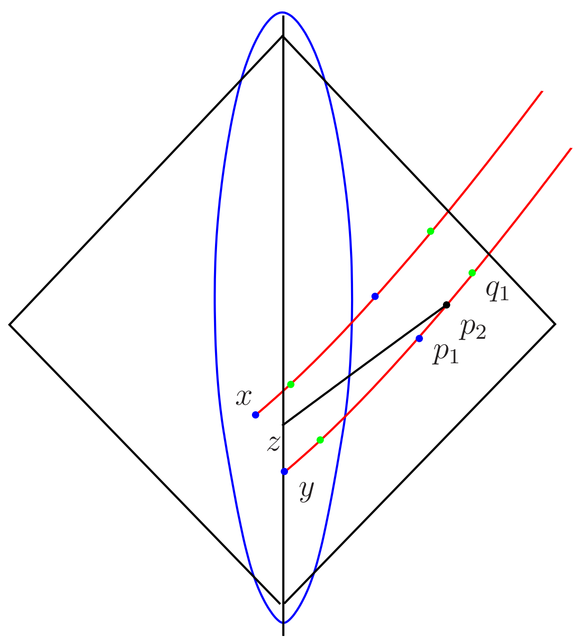

FIGURE 6. A schematic figure where the space-time is represented as the 2-dimensional set $\mathbb{R}^{1+1}$. The figure shows the situation in Lemma 2.10. The point $y = \widehat{\mu}(r_1)$ is on the time-like geodesic $\widehat{\mu}$ shown as a black line. The black diamond is the set $J_{\widehat{g}}(\widehat{p}^-, \widehat{p}^+)$. The point y is the starting point of the light-like geodesic $\gamma_{y,\xi}$ and (x, ζ) is a light-like direction close to (y, ξ). The geodesics $\gamma_{y,\xi}$ and $\gamma_{x,\zeta}$ are shown as red curves and the blue points on them are the first cut-point on $\gamma_{y,\xi}([0, \infty))$, that is, $p_1 = \gamma_{y,\xi}(t_1)$, $t_1 = \rho(y, \xi)$ and the first cut point on $\gamma_{x,\zeta}([0, \infty))$. The green points are $\gamma_{y,\xi}(t_0)$ and $\gamma_{x,\zeta}(t_0)$ and the first cut point $q_1 = \gamma_{y,\xi}(t_c)$ on $\gamma_{y,\xi}([t_0, \infty))$ and the first cut point on $\gamma_{x,\zeta}([t_0, \infty))$. In Lemma 2.10, $t_0 + t_c \geq t_1 + 2\kappa_2$. The black point $p_2 = \gamma_{y,\xi}(t_2)$ is such that $t_2 \geq t_1 + \kappa_2$. Note that if $z = \widehat{\mu}(r_2)$ is such that $r_2 = f^-_{\widehat{\mu}}(p_2)$, then $r_2 - r_1 > 3\kappa_3$.

Proof. Below we denote $\mu = \mu_{g,z_0,\eta_0}$. We start with the claim (iv).

(iv) We observe that if $\kappa_1 > 0$ is small enough (v) follows from the definition of the Fermi type coordinates. Below we assume that κ_1 is so small that (iv) is valid.

Next we prove (i)-(iii).

(i) ϑ_0, κ_0, and κ_1 can be chosen using the compactness of the sets $J(p^-, \mu(s_{+j}))$, $j \le 3$ and $S = \{(y, \zeta) \in L^+M;\ y \in \mu_g([s_-, s_+]),\ \|\zeta\|_{g^+} = 1\}$.

(ii) Let us fix $\kappa_0 > 0$. As $\rho(x, \xi)$ is lower semi-continuous, we see that if $\vartheta_0 > 0$ is chosen first enough, the first claim in (ii) holds.

To show the second claim in (ii), assume that the opposite holds. Then there are $(y^n, \zeta^n) \in L^+(\mu([s_-, s_+]), \|\zeta^n\|_{g^+} = 1$, and $t_0^n \ge \kappa_1$, $n \in \mathbb{Z}_+$ such that $\lim_{n\to\infty}(t_2(y^n, \zeta^n, t_0^n) + t_0^n) = \lim_{n\to\infty} \rho(y^n, \zeta^n)$. As $\rho(x, \zeta)$ is lower semi-continuous, we see that for some subsequences of (y^n, ζ^n) and t_0^n have limits $(\overline{y}, \overline{\zeta})$ and $\overline{t}_0$ such that $t_2(\overline{y}, \overline{\zeta}, \overline{t}_0) + \overline{t}_0 = \rho(\overline{y}, \overline{\zeta})$. This is not possible and hence (ii) is proven.

(iii) Denote $p_{+2} = \mu(s_{+2})$. Let $T_+(x, \zeta) = \sup\{t \ge 0\ ;\ \gamma_{x,\zeta}(t) \in J^-(p_{+2})\}$ and

$$K = \{(r, \xi);\ r \in [s_-, s_+],\ \xi \in L^+_{\mu(r)}M_0,\ \|\xi\|_{g^+} = 1\},$$
$$K_0 = \{(r, \xi) \in K;\ \rho(\mu(r), \xi) + \kappa_2 \le T_+(\mu(r), \xi)\}, \quad K_1 = K \setminus K_0.$$

Then the map $L : G_0 = \{(r, \xi, t) \in K \times \mathbb{R}_+;\ t \le T_+(\mu(r), \xi)\} \to \mathbb{R}$,

$$L(r, \xi, t) = f^-_\mu(\gamma_{\mu(r),\xi}(t)) - r$$

is continuous. Also, we define a map $H : K \to \mathbb{R}$,

$$H(r, \xi) = \begin{cases} L(r, \xi, \rho(\mu(r), \xi) + \kappa_2), & \text{for } (r, \xi) \in K_0, \\ 3, & \text{for } (r, \xi) \in K_1. \end{cases}$$

Note that if $(r, \xi) \in K_0$ then

$$L(r, \xi, \rho(\mu(r), \xi) + \kappa_2) = f^-_\mu(P(r, \xi)) - r \le 2,$$

where $P(r, \xi) = \gamma_{\mu(r),\xi}(\rho(\mu(r), \xi) + \kappa_2) \in J^-(p_{+2})$. Above, $\rho(x, \xi)$ is lower semi-continuous and $T_+(x, \xi)$ is upper semi-continuous on the set $L^+(J(p^-, p_{+2}))$ and thus sets K_0 and G_0 are closed. Moreover, as $\rho(x, \xi)$ is lower semi-continuous and the function $t \mapsto L(r, \xi, t)$ is increasing, we see that $H : K \to \mathbb{R}$ is a lower semi-continuous function.

For $(r, \xi) \in K_0$, we see that $\tau(q, \gamma_{q,\xi}(\rho(q, \xi) + \kappa_2)) > 0$, where $q = \mu(r)$. Hence H is strictly positive. If $K_0 = \emptyset$, it is obvious that the claim is valid as the condition $p_2 \in J^-(p_{+2})$ never holds. Assume next that $K_0 \ne \emptyset$. Then H obtains it minimum $\varepsilon_1 := H(r_0, \xi_0) \in (0, 3)$ at some point $(r_0, \xi_0) \in K_0$. As $t \mapsto L(r, \xi, t)$ is increasing, the first claim follows by choosing $\kappa_3 < \varepsilon_1/3$.

As $f^-_\mu : J(\mu(-1), \mu(1)) \to \mathbb{R}$ is continuous and $J(\mu(-1), \mu(1))$ is compact, this function is uniformly continuous. Thus there exists $\delta_0 > 0$ such that if $d_{g^+}(x_1, x_2) < \delta$, then $|f^-_\mu(x_1) - f^-_\mu(x_2)| < \kappa_3$.

Let

$$\begin{aligned} B_0 \;=\; & \{(y',\zeta',t_1',x',\xi',t_2');\ (r',\zeta',t_1')\in G_0,\ y'=\mu(r'),\\ & (x',\xi')\in L^+M,\ d_{g^+}((y',\zeta'),(x',\xi'))\le \vartheta_0,\ |t_1'-t_2'|\le \kappa_0\}, \end{aligned}$$

where G_0 is the closed set defined above. The set B_0 is compact and by claim (i) for $(y',\zeta',t_1',x',\xi',t_2')\in B_0$ we have $\gamma_{x',\xi'}(t_2')\in J(\mu(-1),\mu(+1))$. Thus by using Lipschitz continuity of exponential map in $T(J(\mu(-1),\mu(+1)))$, we see that there is $L_0>0$ such that for $(y',\zeta',t_1',x',\xi',t_2')\in B_0$ we have

$$|d_{g^+}(\gamma_{y',\zeta'}(t_1'),\gamma_{x',\xi'}(t_2'))|\le L_0(d_{g^+}((y',\zeta'),(x',\xi'))+|t_1'-t_2'|), \tag{24}$$

and moreover, there are $L_1,L_2>0$ such that

$$\begin{aligned} &|\mathbf{t}(\gamma_{y',\zeta'}(t_1'))-\mathbf{t}(\gamma_{x',\xi'}(t_2'))|\le L_1(d_{g^+}((y',\zeta'),(x',\xi'))+|t_1'-t_2'|),\\ &\frac{\partial}{\partial s}\mathbf{t}(\gamma_{x',\xi'}(s))|_{s=t_1'}\ge L_2 \end{aligned} \tag{25}$$

where $\mathbf{t}:M\to\mathbb{R}$ is the smooth time function on M used to introduce the identification $M=\mathbb{R}\times N$.

Let $\delta_1=\mathrm{dist}_{g^+}(J(\mu(s_{-1}),\mu(s_{+1}),M\setminus J(\mu(s_{-2}),\mu(s_{+2}))$. Let us now assume that $\vartheta_0<\min(\delta_0,\delta_1)/L_0$.

Then we see that if $\gamma_{y,\zeta}(t_2)\not\in J^-(\mu(s_{-2}))$, then $\gamma_{x,\xi}(t_2)\not\in J^-(\mu(s_{-1}))$. On the other hand, if $p_2=\gamma_{y,\zeta}(t_2)\in J^-(\mu(s_{-2}))$, then $f^-_\mu(p_2)>r_1+3\kappa_3$ and we see that $p_3=\gamma_{x,\xi}(t_2)$ satisfies $f^-_\mu(p_3)>r_1+3\kappa_3-\kappa_3=r_1+2\kappa_3$. □

Note that for proving the unique solvability of the inverse problem we need to consider two manifolds, $(M^{(1)},\widehat g^{(1)})$ and $(M^{(2)},\widehat g^{(2)})$ with same data. For these manifolds, we can choose R_1,ϑ_0,κ_j so that they are same for the both manifolds.

Proof. (of Corollary 1.3) By our assumptions, $\Psi:(V_1,g^{(1)})\to(V_1,g^{(2)})$ is conformal, the Ricci curvature of $g^{(j)})$ is zero in W_j, and $V_j\subset W_j\cup U_j$, $\Psi(U_1\setminus W_1)=U_2\setminus W_2$. Moreover, any point $x\in V_1\cap W_1$ can be connected to some point $y\in U_1\cap W_1$ with a light-like geodesic $\gamma_{x,\xi}([0,l])\subset V_1\cap W_1$. As $\Psi:V_1\to V_2$ and $\Psi:U_1\to U_2$ are bijections and $\Psi(U_1\setminus W_1)=U_2\setminus W_2$, we have $\Psi(V_1\cap W_1)=V_2\cap W_2$. This implies that any point $\Psi(x)\in V_2\cap W_2$ can be connected to the point $\Psi(y)\in U_2\cap W_2$ with a light-like geodesic $\Psi(\gamma_{x,\xi}([0,l]))\subset V_2\cap W_2$.

By the above there is $f:V_1\to\mathbb{R}$ such that $g^{(1)}=e^{2f}\Psi^*g^{(2)}$ on V_1. To simplify notations we denote $\widehat g=g^{(1)}$ and $g=\Psi^*g^{(2)}$. Next consider how function $f:V_1\to\mathbb{R}$ can be constructed when $\widehat g$ and g are given in U_1 and $\widehat g_{jk}=e^{2f}g_{jk}$ corresponds to the vacuum, i.e., its Ricci curvature vanishes in a set W_1. By [77, formula (2.73)], the Ricci tensors $\mathrm{Ric}_{jk}(g)$ of g and $\mathrm{Ric}_{jk}(\widehat g)$ of $\widehat g$ satisfy in W_1

$$\begin{aligned} 0=\mathrm{Ric}_{jk}(\widehat g)=\mathrm{Ric}_{jk}(g)_{jk}-2\nabla_j\nabla_k f+2(\nabla_j f)(\nabla_k f)\\ -(g^{pq}\nabla_p\nabla_q f+2g^{pq}(\nabla_p f)(\nabla_q f))g_{jk} \end{aligned}$$

where $\nabla = \nabla^g$. For scalar curvature this yield

$$0 = e^{2f}\widehat{g}^{jk}\text{Ric}_{jk}(\widehat{g}) = g^{jk}\text{Ric}_{jk}(g) - 3g^{jk}\nabla_j\nabla_k f.$$

Combining these, we obtain

$$2\nabla_j\nabla_k f - 2(\nabla_j f)(\nabla_k f) + 2g^{pq}(\nabla_p f)(\nabla_q f)g_{jk} = \text{Ric}_{jk}(g) - \frac{1}{3}g^{pq}\text{Ric}_{pq}(g)g_{jk}.$$

Let us next use local coordinate neighborhood that cover the geodesic $\gamma(t) = (\gamma^j(t))_{j=1}^4$. The above equation gives a system of second order ordinary differential equations which can be solved along the light-like geodesics $\gamma(t)$ in W_1 starting from $U_{\widehat{g}}$. Indeed, we can take a contraction of the above equations with $\dot\gamma^j(t)$ and obtain, on each piece of the geodesic that belongs in one coordinate neighborhood, an ordinary differential equation for $[(\nabla_k f)(\gamma(t))]_{k=1}^n$. Since by our assumption any point $x \in V_1 \cap W_1$ can be connected to $y \in U_1 \cap W_1$ with a light-like geodesic that is a subset of $V_1 \cap W_1$, we can use the above system of the second order ordinary differential equations and the fact that we know f in U_1 to determine $f(x)$. This proves the claim. □

3. Analysis of Einstein equations in wave coordinates

3.1. **Notations.** Let X be a manifold of dimension n and $\Lambda \subset T^*X \setminus \{0\}$ be a Lagrangian submanifold. Let $\phi(x,\theta)$, $\theta \in \mathbb{R}^N$ be a non-degenerate phase function that locally parametrizes Λ. We say that a distribution $u \in \mathcal{D}'(X)$ is a Lagrangian distribution associated with Λ and denote $u \in \mathcal{I}^m(X;\Lambda)$, if it can locally be represented as

$$u(x) = \int_{\mathbb{R}^N} e^{i\phi(x,\theta)}a(x,\theta)\,d\theta,$$

where $a(x,\theta) \in S^{m+n/4-N/2}(X;\mathbb{R}^N \setminus 0)$, see [28, 42, 65].

In particular, when $S \subset X$ be a submanifold, its conormal bundle $N^*S = \{(x,\xi) \in T^*X \setminus \{0\};\ x \in S,\ \xi \perp T_xS\}$ is a Lagrangian submanifold.

Let us next consider the case when $X = \mathbb{R}^n$ and let $(x^1, x^2, \dots, x^n) = (x', x'', x''')$ be the Euclidean coordinates with $x' = (x_1, \dots, x_{d_1})$, $x'' = (x_{d_1+1}, \dots, x_{d_1+d_2})$, $x''' = (x_{d_1+d_2+1}, \dots, x_n)$. If $S_1 = \{x' = 0\} \subset \mathbb{R}^n$, $\Lambda_1 = N^*S_1$ and $u \in \mathcal{I}^m(X;\Lambda_1)$, then

$$u(x) = \int_{\mathbb{R}^{d_1}} e^{ix'\cdot\theta'}a(x,\theta')\,d\theta', \quad a(x,\theta') \in S^\mu(X;\mathbb{R}^{d_1} \setminus 0)$$

where $\mu = m - d_1/2 + n/4$.

Next we recall the definition of $\mathcal{I}^{p,l}(X;\Lambda_1,\Lambda_2)$, the space of the distributions u in $\mathcal{D}'(X)$ associated to two cleanly intersecting Lagrangian manifolds $\Lambda_1, \Lambda_2 \subset T^*X \setminus \{0\}$, see [28, 65]. Let us start on the case when $X = \mathbb{R}^n$.

Let $S_1, S_2 \subset \mathbb{R}^n$ be the linear subspaces of codimensions d_1 and d_1+d_2, respectively, and $S_2 \subset S_1$, given by $S_1 = \{x' = 0\}$, $S_2 = \{x' = x'' = 0\}$. Let us denote $\Lambda_1 = N^*S_1$, $\Lambda_2 = N^*S_2$. Then $u \in \mathcal{I}^{p,l}(\mathbb{R}^n; N^*S_1, N^*S_2)$ if and only if

$$u(x) = \int_{\mathbb{R}^{d_1+d_2}} e^{i(x'\cdot\theta' + x''\cdot\theta'')} a(x,\theta',\theta'')\, d\theta' d\theta'',$$

where $a(x,\theta',\theta'')$ belongs to the product type symbol class $S^{\mu',\mu''}(\mathbb{R}^n; (\mathbb{R}^{d_1}\setminus 0)\times\mathbb{R}^{d_2})$ that is the space of function $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ that satisfy

$$(26) \quad |\partial_x^\gamma \partial_{\theta'}^\alpha \partial_{\theta''}^\beta a(x,\theta',\theta'')| \le C_{\alpha\beta\gamma K}(1+|\theta'|+|\theta''|)^{\mu-|\alpha|}(1+|\theta''|)^{\mu'-|\beta|}$$

for all $x \in K$, multi-indexes α, β, γ, and compact sets $K \subset \mathbb{R}^n$. Above, $\mu = p + l - d_1/2 + n/4$ and $\mu' = -l - d_2/2$.

When X is a manifold of dimension n and $\Lambda_1, \Lambda_2 \subset T^*X \setminus \{0\}$ are two cleanly intersecting Lagrangian manifolds, we define the class $\mathcal{I}^{p,l}(M;\Lambda_1,\Lambda_2) \subset \mathcal{D}'(X)$ to consist of locally finite sums of functions $u = Au_0$, where $u_0 \in \mathcal{I}^{p,l}(\mathbb{R}^n; N^*S_1, N^*S_2)$ and $S_1, S_2 \subset \mathbb{R}^n$ are the linear subspace of codimensions d_1 and d_1+d_2, respectively, such that $S_2 \subset S_1$, and A is a Fourier integral operator of order zero with a canonical relation Σ for which $\Sigma \circ (N^*S_1)' \subset \Lambda_1'$ and $\Sigma \circ (N^*S_2)' \subset \Lambda_2'$. Here, $\Lambda' = \{(x,-\xi) \in T^*X;\ (x,\xi) \in \Lambda\}$.

In most cases, below $X = M$. We denote then $\mathcal{I}^p(M;\Lambda_1) = \mathcal{I}^p(\Lambda_1)$ and $\mathcal{I}^{p,l}(M;\Lambda_1,\Lambda_2) = \mathcal{I}^{p,l}(\Lambda_1,\Lambda_2)$, etc. Also, $\mathcal{I}(\Lambda_1) = \cup_{p\in\mathbb{R}}\mathcal{I}^p(\Lambda_1)$ etc.

By [28, 65], microlocally away from Λ_2 and Λ_1,

$$(27) \quad \mathcal{I}^{p,l}(\Lambda_1,\Lambda_2) \subset \mathcal{I}^{p+l}(\Lambda_1\setminus\Lambda_2) \quad \text{and} \quad \mathcal{I}^{p,l}(\Lambda_1,\Lambda_2) \subset \mathcal{I}^{p}(\Lambda_2\setminus\Lambda_1),$$

correspondingly. Thus the principal symbol of $u \in \mathcal{I}^{p,l}(\Lambda_0,\Lambda_1)$ is well defined on $\Lambda_0 \setminus \Lambda_1$ and $\Lambda_1 \setminus \Lambda_0$.

Below, when $\Lambda_j = N^*S_j$, $j = 1, 2$ are conormal bundles of smooth cleanly intersecting submanifolds $S_j \subset M$ of codimension d_j, where $\dim(M) = n$, we use the traditional notations,

$$(28) \quad \mathcal{I}^{\mu}(S_1) = \mathcal{I}^{\mu+1/2-n/4}(N^*S_1), \quad \mathcal{I}^{\mu_1,\mu_2}(S_1,S_2) = \mathcal{I}^{p,l}(N^*S_1,N^*S_2),$$

where $p = \mu_1 + \mu_2 + d_1/2 - n/4$ and $l = -\mu_2 - d_2/2$, and call such distributions the conormal distributions associated to S_1 or product type conormal distributions associated to S_1 and S_2. We note that $\mathcal{I}^\mu(X;S_1) \subset L^p_{loc}(X)$ for $\mu < -d_1(p-1)/p$, $1 \le p < \infty$, see [28].

By [65], see also [52], a classical pseudodifferential operator P of real principal type and order m on M has a parametrix $Q \in \mathcal{I}^{p,l}(\Delta'_{T^*M},\Lambda_P)$, $p = \frac{1}{2} - m$, $l = -\frac{1}{2}$, where $\Delta_{T^*M} = N^*(\{(x,x);\ x \in M\})$ and $\Lambda_P \subset T^*M \times T^*M$ is the Lagrangian manifold associated to the canonical relation of the operator P, that is,

$$\Lambda_P = \{(x,\xi,y,-\eta);\ (x,\xi) \in \mathrm{Char}\,(P),\ (y,\eta) \in \Theta_{x,\xi}\},$$

where $\Theta_{x,\xi} \subset T^*M$ is the bicharacteristic of P containing (x,ξ).

For the wave operator $\square_g$ on the globally hyperbolic manifold (M, g) $\text{Char}\,(\square_g)$ is the set of light-like vectors with respect to g, and $(y, \eta) \in \Theta_{x,\xi}$ if and only if there is $t \in \mathbb{R}$ such that $(y, a) = (\gamma^g_{x,b}(t), \dot\gamma^g_{x,b}(t))$ where $\gamma^g_{x,b}$ is a light-like geodesic with resect to the metric g with the initial data $(x, b) \in TM$, $a = \eta^\flat$, $b = \xi^\flat$. For $P = \square_g + B^0 + B^j \partial_j$, where B^0 and B_j are tensors, we denote $\Lambda_P = \Lambda_g$. When (M, g) is globally hyperbolic manifold, the operator P has a causal inverse operator, see e.g. [5, Thm. 3.2.11]. We denote it by P^{-1} and by [65], we have $P^{-1} \in \mathcal{I}^{-3/2,-1/2}(\Delta'_{T^*M}, \Lambda_g)$. We will repeatedly use the fact (see [28, Prop. 2.1]) that if $F \in \mathcal{I}^p(\Lambda_0)$ and Λ_0 intersects $\text{Char}(P)$ transversally so that all bicharacterestics of P intersect Λ_0 only finitely many times, then $(\square_g + B^0 + B^j \partial_j)^{-1} F \in \mathcal{I}^{p-3/2,-1/2}(\Lambda_0, \Lambda_1)$ where $\Lambda'_1 = \Lambda_g \circ \Lambda'_0$ is called the flowout from Λ_0 on $\text{Char}(P)$, that is,

$$\Lambda_1 = \{(x, -\xi);\ (x, \xi, y, -\eta) \in \Lambda_g,\ (y, \eta) \in \Lambda_0\}.$$

3.1.1. *Notations used to consider Einstein equations.* For Einstein equation, we will consider a smooth background metric $\widehat{g}$ on M and the smooth metric $\widetilde{g}$ for which $\widehat{g} < \widetilde{g}$ and $(M, \widetilde{g})$ is globally hyperbolic. We also use the notations defined in Section 1.3.4. In particular, we identify $M = \mathbb{R} \times N$ and consider the metric tensor g on $M_0 = (-\infty, t_0) \times N$, $t_0 > 0$ that coincide with $\widehat{g}$ in $(-\infty, 0) \times N$. Recall also that we consider a freely falling observer $\widehat{\mu} = \mu_{\widehat{g}} : [-1, 1] \to M_0$ for which $\widehat{\mu}(s_-) = \widehat{p}^- \in (0, t_0) \times N$. We denote the cut locus function on $(M_0, \widehat{g})$ by $\rho(x, \xi) = \rho_{\widehat{g}}(x, \xi)$, denote $L^+_x M_0 = L^+_x(M_0, \widehat{g})$ and $L^+ M_0 = L^+(M_0, \widehat{g})$, and denote by $\widehat{U} = U_{\widehat{g}}$ the neighborhood of the geodesic $\widehat{\mu} = \mu_{\widehat{g}}$, and denote by $\widehat{\gamma}_{x,\xi}(t)$ the geodesics of $(M_0, \widehat{g})$. To simplify notations, we study below the case when $\widehat{P} = 0$ and $\widehat{Q} = 0$, that corresponds to the setting of Theorem 1.4.

3.2. **Direct problem.** Let we consider the solutions (g, ϕ) of the equations (10) with source $F = (P, Q)$. To consider their existence, let us write them in the form

$$(g, \phi) = (\widehat{g}, \widehat{\phi}) + u, \quad \text{where } u = ((G_{jk})^4_{j,k=1}, (\Phi_\ell)^L_{\ell=1}). \tag{29}$$

Next we formulate an appropriate equation for u and review the existence and uniqueness results for it.

Let us assume that F is small enough in the norm $C^5_b(M_0)$ and that it is supported in a compact set $\mathcal{K} = J_{\widetilde{g}}(\widehat{p}^-) \cap [0, t_0] \times N \subset M_1$. Then using the fact that $\partial_j g^{nm} = -g^{na}(\partial_j g_{ab}) g^{bm}$ we can write the equations (10) for u appearing in (29) in the form

$$\begin{aligned} &P_{g(u)}(u) = R_{\widehat{g}}(x, u(x), \partial u(x)) F, \quad x \in M_0, \\ &\text{supp}\,(u) \subset \mathcal{K}, \end{aligned} \tag{30}$$

where

$$P_{g(u)}(u) := g^{jk}(x;u)\partial_j\partial_k u(x) + H(x,u(x),\partial u(x)), \tag{31}$$
$$(g^{jk}(x;u))_{j,k=1}^4 = ((\widehat{g}_{jk}(x) + G_{jk}(x))_{j,k=1}^4)^{-1},$$

and $(x,v,w) \mapsto H(x,v,w)$ is a smooth function which is a second order polynomial in w with coefficients being smooth functions of v and derivatives of $\widehat{g}$ and $R_{\widehat{g}}(x,u(x),\partial u(x))F$, when represented in local coordinates, is a first order linear differential operator in F, i.e., at the point x it depends linearly on $F(x)$ and $\partial F(x)$, which coefficients are smooth functions depending on $u(x)$, $\partial u(x)$, and derivatives of $\widehat{g}$.

Below we endow N with the Riemannian metric $h := \iota^*\widetilde{g}$ inherited from the embedding $\iota : N \to \{0\} \times N \subset M$ and use it to define the Sobolev spaces $H^s(N)$.

Let $s_0 \geq 5$ be an odd integer. Below we will consider the solutions $u = (g,\phi)$ as sections of the bundle $\mathcal{B}^L$ on M_0 and the sources $F = (Q,P)$ as sections of the bundle $\mathcal{B}^K$ on M_0. We will consider these functions as elements of section valued Sobolev spaces $H^s(M_0;\mathcal{B}^L)$ and $H^s(M_0;\mathcal{B}^K)$ etc. Below, we omit the bundle $\mathcal{B}^K$ in these notations and denote just $H^s(M_0;\mathcal{B}^L) = H^s(M_0)$ etc. We use the same convention for the spaces

$$E^s = \bigcap_{j=0}^{s} C^j([0,t_0];H^{s-j}(N)), \quad s \in \mathbb{N}.$$

Note that $E^s \subset C^p([0,t_0] \times N))$ when $0 \leq p < s-2$. Using standard techniques for quasi-linear equations developed e.g. in [46] or [38], or [80, Section 9] (for details, see Appendix C), we see that when F is supported in the compact set $\mathcal{K}$ and $\|F\|_{E^{s_0}} < c_0$, then there exists a unique function u satisfying equation (30) on M_0 with the source F and

$$\|u\|_{E^{s_0-1}} \leq C_1\|F\|_{E^{s_0}}. \tag{32}$$

For convenience of the reader we give the proofs of these facts in Appendixes A and C.

3.3. **Asymptotic expansion for non-linear wave equation.** Let us consider a small parameter $\varepsilon > 0$ and the sources $P = \varepsilon\mathbf{p}$ and $Q' = (Q_\ell)_{\ell=1}^{K-1} = \varepsilon\mathbf{q}'$, $Q_K = \varepsilon\mathbf{z}$ (cf. Assumption S, (ii)), and denote $\mathbf{q} = (\mathbf{q}',\mathbf{z})$. This corresponds to the source $F = \varepsilon f = (\varepsilon\mathbf{p},\varepsilon\mathbf{q}',\varepsilon\mathbf{z})$ in (30). Below, we always assume that $\mathbf{p}$ and $\mathbf{q}$ are supported in $\mathcal{K}$ and $\mathbf{p},\mathbf{q} \in E^s$, where $s \geq 13$ is an odd integer. We consider the solution $u = u_\varepsilon$ of (30) and write it in the form

$$u_\varepsilon(x) = \sum_{j=1}^{4} \varepsilon^j w^j(x) + w^{res}(x,\varepsilon). \tag{33}$$

To obtain equations for w^j, we use the representation for the $\widehat{g}$-reduced Ricci tensor given in the Appendix A, see (154) to write an analogous representation for the $\widehat{g}$-reduced Einstein tensor, see (153) or (156), and substitute the expansion (33) in to the equation (30). This, Assumption S and equation (12) imply that w^j, $j = 1,2,3,4$ are given in term of sources of $\mathcal{H}^1 = \mathcal{H}^1(\widehat{g}, f)$, $\mathcal{H}^2 = \mathcal{H}^2(\widehat{g}, f, w^1)$, $\mathcal{H}^3 = \mathcal{H}^3(\widehat{g}, f, w^1, w^2)$, and $\mathcal{H}^4 = \mathcal{H}^4(\widehat{g}, f, w^1, w^2, w^3)$ that satisfy

$$\begin{aligned}
w^j &= (g^j, \phi^j) = \mathbf{Q}_{\widehat{g}}\mathcal{H}^j, \quad j = 1,2,3,4, \text{ where}\\
\mathcal{H}^1 &= R_{\widehat{g}}(x,0,0)f,\\
\mathcal{H}^2 &= (\mathcal{G}_2, 0) + \mathcal{A}_2^{(2)}(w^1, \partial w^1; f, \partial f),\\
\mathcal{G}_2 &= 2\widehat{g}^{jp} w^1_{pq} \widehat{g}^{qk} \partial_j \partial_k w^1,\\
\mathcal{H}^3 &= (\mathcal{G}_3, 0) + \mathcal{A}_2^{(3)}(w^1, \partial w^1, w^2, \partial w^2; f, \partial f) +\\
&\quad + \mathcal{A}_3^{(3)}(w^1, \partial w^1; f, \partial f),\\
\mathcal{G}_3 &= -6\widehat{g}^{jl} w^1_{li} \widehat{g}^{ip} w^1_{pq} \widehat{g}^{qk} \partial_j \partial_k w^1 +\\
&\quad + 3\widehat{g}^{jp} (w^2)_{pq} \widehat{g}^{qk} \partial_j \partial_k w^1 + 3\widehat{g}^{jp} w^1_{pq} \widehat{g}^{qk} \partial_k \partial_j w^2
\end{aligned} \tag{34}$$

where $f \mapsto R_{\widehat{g}}(x,0,0)f$ is a first order linear differential operator appearing in (30), and

$$\begin{aligned}
\mathcal{H}^4 &= (\mathcal{G}_4, 0) + \mathcal{A}_2^{(4)}(w^1, \partial w^1, w^2, \partial w^2, w^3, \partial w^3; f, \partial f)\\
&\quad + \mathcal{A}_3^{(4)}(w^1, \partial w^1, w^2, \partial w^2; f, \partial f) + \mathcal{A}_4^{(4)}(w^1, \partial w^1; f, \partial f),\\
\mathcal{G}_4 &= 24\widehat{g}^{js} w^1_{sr} \widehat{g}^{rl} w^1_{li} \widehat{g}^{ip} w^1_{pq} \widehat{g}^{qk} \partial_k \partial_j w^1 +\\
&\quad - 18\widehat{g}^{jl} w^1_{li} \widehat{g}^{ip} w^2_{pq} \widehat{g}^{qk} \partial_k \partial_j w^1 - 12\widehat{g}^{jl} w^1_{li} \widehat{g}^{ip} w^1_{pq} \widehat{g}^{qk} \partial_k \partial_j w^2 +\\
&\quad + 3\widehat{g}^{jp} w^3_{pq} \widehat{g}^{qk} \partial_k \partial_j w^1 + 3\widehat{g}^{jp} w^1_{pq} \widehat{g}^{qk} \partial_k \partial_j w^3 +\\
&\quad + 6\widehat{g}^{jp} w^2_{pq} \widehat{g}^{qk} \partial_k \partial_j w^2.
\end{aligned} \tag{35}$$

Here, we have used the notation $w^j = (((w^j)_{pq})_{p,q=1}^4, ((w^j)_\ell)_{\ell=1}^L)$ with $((w^j)_{pq})_{p,q=1}^4$ being the g-component of w^j and $((w^j)_\ell)_{\ell=1}^L$ being the ϕ-component of w^j. Moreover, $\mathbf{Q}_{\widehat{g}} = (\square_{\widehat{g}} + V(x,D))^{-1}$ is the causal inverse of the operator $\square_{\widehat{g}} + V(x,D)$ where $V(x,D)$ is a first order differential operator with coefficients depending on $\widehat{g}$ and its derivatives, $R(x,D)$ is a first order linear differential operator with coefficients depending on $\widehat{g}$ and its derivatives, and $\mathcal{A}_m^{(\alpha)}$, $\alpha = 2,3,4$ denotes a multilinear operators of order m having at a point x the representation

$$\begin{aligned}
&(\mathcal{A}_m^{(\alpha)}(v^1, \partial v^1, v^2, \partial v^2, v^3, \partial v^3; f, \partial f))(x) \\
&= \sum \Big(a^{(\alpha)}_{abcijkP_1P_2P_3pqn\ell_1\ell_2}(x)\, (v^1_a(x))^i (v^2_b(x))^j (v^3_c(x))^k \cdot \\
&\qquad \cdot P_1(\partial v^1(x)) P_2(\partial v^2(x)) P_3(\partial v^3(x)) (f_p(x))^{\ell_1} (\partial_n f_q(x))^{\ell_2} \Big)
\end{aligned} \tag{36}$$

where $(v^1_a(x))^i$ denotes the i-th power of a-th component of $v^1(x)$, etc, and the sum is taken over the indexes a, b, c, p, q, n, integers i, j, k, ℓ_1, ℓ_2, and the homogeneous monomials $P_d(y) = y^{\beta_d}$, $\beta_d = (b_1, b_2, \dots, b_{4(10+L)}) \in \mathbb{N}^{4(10+L)}$, $d = 1, 2, 3$ having orders $|\beta_d|$, correspondingly, that satisfy

$$\ell_1 + \ell_2 \le 1, \tag{37}$$

$$i + 2j + 3k + |\beta_1| + 2|\beta_2| + 3|\beta_3| + \ell_1 + \ell_2 = \alpha, \tag{38}$$

$$i + j + k + |\beta_1| + |\beta_2| + |\beta_3| + \ell_1 + \ell_2 = m, \quad \text{and} \tag{39}$$

$$\text{if } \ell_1 = 0 \text{ and } \ell_2 = 0 \text{ then } |\beta_1| + |\beta_2| + |\beta_3| \le 2. \tag{40}$$

Here, condition (37) means that $\mathcal{A}^{(\alpha)}_m$ is an affine function of f and its first derivative, condition (38) implies that the term $\mathcal{A}^{(\alpha)}_m$ produces a term of order $O(\varepsilon^\alpha)$ when $v^j = w^j$, condition (39) that $\mathcal{A}^{(\alpha)}_m$ is multilinear of order m, and condition (40) means that for $x \notin \operatorname{supp}(f)$ the non-vanishing terms in $\mathcal{A}^{(\alpha)}_m(v^1, \partial v^1, v^2, \partial v^2, v^3, \partial v^3; f, \partial f)$ contain only terms where the total power of derivatives of v^1, v^2, and v^3 is at most two. We note that the inequalities (37)-(40) follow from Assumption S and equation (12).

By [13, App. III, Thm. 3.7], or alternatively, the proof of [38, Lemma 2.6] adapted for manifolds, for details, see (176) and (177) in the Appendix C, the estimate $\|\mathbf{Q}_{\widehat{g}} H\|_{E^{s_1}} \le C_{s_1} \|H\|_{E^{s_1}}$ holds for all H that are supported in $\mathcal{K} = \mathcal{K}_0$, having the form (5) and satisfy $H \in E^{s_1}$, $s_1 \in \mathbb{Z}_+$. Note that we are interested only on the local solvability of the Einstein equations and that singularities can appear on sufficiently large time intervals, see [18].

Recall that $s \ge 13$ is an odd integer and $f \in E^{s+1}$. By defining w^j via the above equations with $f \in E^{s+1}$ we obtain $w^j \in E^{s+2-2j}$, for $j = 1, 2, 3, 4$. Thus, by using Taylor's expansion of the coefficients in the equation (30) we see that the approximate 4th order expansion $u^{app}_\varepsilon = \varepsilon w^1 + \varepsilon^2 w^2 + \varepsilon^3 w^3 + \varepsilon^4 w^4$ satisfies an equation of the form

$$\begin{aligned} &P_{g(u^{app}_\varepsilon)}(u^{app}_\varepsilon) = R(x, u^{app}_\varepsilon(x), \partial u^{app}_\varepsilon(x))(\varepsilon f) + H^{res}(\,\cdot\,, \varepsilon), \quad x \in M_0, \\ &\operatorname{supp}(u^{app}_\varepsilon) \subset \mathcal{K}, \end{aligned} \tag{41}$$

such that

$$\|H^{res}(\,\cdot\,, \varepsilon)\|_{E^{s-8}} \le c_1 \varepsilon^5.$$

Using the Lipschitz continuity of the solution of the equation (41), see Appendix C, we see that there are $c_1, c_2 > 0$ such that for all $0 < \varepsilon < c_1$ the function $w^{res}(x, \varepsilon) = u_\varepsilon(x) - u^{app}_\varepsilon(x)$ satisfies

$$\|u_\varepsilon - u^{app}_\varepsilon\|_{E^{s-8}} \le c_2 \varepsilon^5.$$

Thus $w^j = \partial^j_\varepsilon u_\varepsilon|_{\varepsilon=0} \in E^{s-8}$, $j = 1, 2, 3, 4$.

3.4. **Linearized conservation law and divergence condition.**

3.4.1. *Linearized Einstein equation.* We will below consider sources $Q' = (Q_\ell)_{\ell=1}^{K-1} = \varepsilon \mathbf{q}'$, $Q_K = \varepsilon \mathbf{z}$ and $P = \varepsilon \mathbf{p}$. We denote $\mathbf{q} = (\mathbf{q}', \mathbf{z})$.

To analyze the linearized waves, we denote $w^1 = u^{(1)}$. We see that $u^{(1)}$ satisfies the linear equation

$$\Box_{\widehat{g}} u^{(1)} + V(x, \partial_x) u^{(1)} = \mathbf{h}, \tag{42}$$

where $v \mapsto V(x, \partial_x)v$ is a linear first order partial differential operator with coefficients depending on the derivatives of $\widehat{g}$ and, $\mathbf{h} = H(x; \mathbf{p}, \mathbf{q})$, where

$$\begin{aligned} H(x; \mathbf{p}, \mathbf{q}) &= \begin{pmatrix} \mathbf{p} \\ 0 \end{pmatrix} + \begin{pmatrix} M_{(1)}(x)\, \mathbf{q}'(x) \\ M_{(2)}(x)\, \mathbf{q}'(x) \end{pmatrix} + \\ &+ \begin{pmatrix} L_{(1)}(x)\, \mathbf{z}(x) \\ L_{(2)}(x)\, \mathbf{z}(x) \end{pmatrix} + \begin{pmatrix} N^j_{(1)}(x)\, \widehat{g}^{lk} \widehat{\nabla}_l (\mathbf{p}_{jk} + \mathbf{z}\widehat{g}_{jk}) \\ N^j_{(2)}(x)\, \widehat{g}^{lk} \widehat{\nabla}_l (\mathbf{p}_{jk} + \mathbf{z}\widehat{g}_{jk}) \end{pmatrix}, \end{aligned} \tag{43}$$

where $M_{(k)} = M_{(k)}(\widehat{\phi}(x), \widehat{\nabla}\widehat{\phi}(x), \widehat{g}(x))$, $L_{(k)} = L_{(k)}(\widehat{\phi}(x), \widehat{\nabla}\widehat{\phi}(x), \widehat{g}(x))$, and $N^j_{(k)} = N^j_{(k)}(\widehat{\phi}(x), \widehat{\nabla}\widehat{\phi}(x), \widehat{g}(x))$ are, in local coordinates, matrices whose elements are smooth functions of $\widehat{\phi}(x), \widehat{\nabla}\widehat{\phi}(x)$, and $\widehat{g}(x)$. Later we use the fact that by the form of the equation (10), $M_{(1)} = 0$, $N^j_{(1)} = 0$, and $L_{(1)} = \widehat{g}$. We see using Assumption S (iii) that the union of the image spaces of the matrices $M_{(2)}(x)$ and $L_{(2)}(x)$, and $N^j_{(2)}(x)$, $j = 1, 2, 3, 4$, span the space $\mathbb{R}^L$ for all $x \in \widehat{U}$.

3.4.2. *The linearized conservation law for sources.* Assume that $Y \subset M_0$ is a 2-dimensional space-like submanifold and consider local coordinates defined in $V \subset M_0$. Moreover, assume that in these local coordinates $Y \cap V \subset \{x \in \mathbb{R}^4;\ x^j b_j = 0,\ x^j b'_j = 0\}$, where b'_j are some constants and let $\mathbf{p} \in \mathcal{I}^{n-3/2}(Y)$, $n \le n_0 = -15$, be defined by

$$\mathbf{p}_{jk}(x^1, x^2, x^3, x^4) = \mathrm{Re} \int_{\mathbb{R}^2} e^{i(\theta_1 b_m + \theta_2 b'_m) x^m} c_{jk}(x, \theta_1, \theta_2)\, d\theta_1 d\theta_2. \tag{44}$$

Here, we assume that $c_{jk}(x, \theta)$, $\theta = (\theta_1, \theta_2)$ are classical symbols and we denote their principal symbols by $\sigma_p(\mathbf{p}_{jk})(x, \theta)$. When $x \in Y$ and $\xi = (\theta_1 b_m + \theta_2 b'_m) dx^m$ so that $(x, \xi) \in N^*Y$, we denote the value of the principal symbol $\sigma_p(\mathbf{p})$ at (x, θ_1, θ_2) by $\widetilde{c}^{(a)}_{jk}(x, \xi)$, that is, $\widetilde{c}^{(a)}_{jk}(x, \xi) = \sigma_p(\mathbf{p}_{jk})(x, \theta_1, \theta_2)$, and say that it is the principal symbol of $\mathbf{p}_{jk}$ at (x, ξ), associated to the local X-coordinates and the phase function $\phi(x, \theta_1, \theta_2) = (\theta_1 b_m + \theta_2 b'_m) x^m$. The above defined principal symbols can be defined invariantly, see [34].

We assume that also $\mathbf{q}', \mathbf{z} \in \mathcal{I}^{n-3/2}(Y)$ have

representations (44) with classical symbols. Let us denote the principal symbols of $\mathbf{p}, \mathbf{q}', \mathbf{z} \in \mathcal{I}^{n-3/2}(Y)$ by $\widetilde{c}^{(a)}(x, \xi)$, $\widetilde{d}_1(x, \xi)$, $\widetilde{d}_2^{(a)}(x, \xi)$, correspondingly and let $\widetilde{c}^{(b)}$ and $\widetilde{d}_2^{(b)}(x, \xi)$ denote the sub-principal symbols of $\mathbf{p}$ and $\mathbf{z}$, correspondingly, at $(x, \xi) \in N^*Y$.

We will below consider what happens when $(\mathbf{p}_{jk}+\mathbf{z}\widehat{g}_{jk}) \in \mathcal{I}^{n-3/2}(Y)$ satisfies

$$\widehat{g}^{lk}\nabla_l^{\widehat{g}}(\mathbf{p}_{jk}^{(a)}+\mathbf{z}\widehat{g}_{jk}) \in \mathcal{I}^{n-3/2}(Y), \quad j=1,2,3,4. \tag{45}$$

When (45) is valid, we say that the leading order of singularity of the wave satisfies the *linearized conservation law.* This corresponds to the assumption that the principal symbol of the sum of divergence of the first two terms appearing in the stress energy tensor on the right hand side of (10) vanishes.

When (45) is valid, we have

$$\widehat{g}^{lk}\xi_l(\widetilde{c}_{kj}^{(a)}(x,\xi)+\widehat{g}_{kj}(x)\widetilde{d}_2^{(a)}(x,\xi))=0, \quad \text{for } j\le 4 \text{ and } \xi\in N_x^*Y. \tag{46}$$

We say that this is the *linearized conservation law for principal symbols.*

3.4.3. *A divergence condition for linearized solutions.* Above, we have considered the $\widehat{g}$-reduced Einstein equations. As discussed in the Appendix A, if we pull back a solution of the Einstein equation via a $(\widehat{g},g)$-wave map, the metric tensor will satisfy the $\widehat{g}$-reduced Einstein equations. Roughly speaking, this means that we are in fact considering the metric tensor g in special coordinates associated to the metric $\widehat{g}$. Next we recall some well known consequences of this.

Assume now that (g,ϕ) satisfy equations (10). By Assumption S (iv), the conservation law (11) is valid. As discussed in Appendix A.5, the conservation law (11) and the $\widehat{g}$-reduced Einstein equations (10) imply that the harmonicity functions $\Gamma^j=g^{nm}\Gamma^j_{nm}$ satisfy

$$g^{nm}\Gamma^j_{nm}=g^{nm}\widehat{\Gamma}^j_{nm}. \tag{47}$$

Next we discuss the implications of this for the metric component of the solution of the linearized Einstein equation.

Let us next do calculations in local coordinates of M_0 and denote $\partial_k=\frac{\partial}{\partial x^k}$. Direct calculations show that $h^{jk}=g^{jk}\sqrt{-\det(g)}$ satisfies $\partial_k h^{kq}=-\Gamma^q_{kn}h^{nk}$. Then (47) implies that

$$\partial_k h^{kq}=-\widehat{\Gamma}^q_{kn}h^{nk}. \tag{48}$$

We call (48) the *harmonicity condition* for metric g.

Assume now that g_ε and ϕ_ε satisfy (10) with sources $P=\varepsilon\mathbf{p}$ and $Q=\varepsilon\mathbf{q}$ where $\varepsilon>0$ is a small parameter. We define $h^{jk}_\varepsilon=g^{jk}_\varepsilon\sqrt{-\det(g_\varepsilon)}$ and denote $\dot g_{jk}=\partial_\varepsilon(g_\varepsilon)_{jk}|_{\varepsilon=0}$, $\dot g^{jk}=\partial_\varepsilon(g_\varepsilon)^{jk}|_{\varepsilon=0}$, and $\dot h^{jk}=\partial_\varepsilon h^{jk}_\varepsilon|_{\varepsilon=0}$.

The equation (48) yields then[1]

$$\partial_k\dot h^{kq}=-\widehat{\Gamma}^q_{kn}\dot h^{nk}. \tag{49}$$

[1]The treatment on this de Donder-type gauge condition is known in folklore of the field. For a similar gauge condition to (49) in harmonic coordinates, see [60, pages 6 and 250], or [57, formulas 107.5, 108.7, 108.8], or [39, p. 229-230].

A direct computation shows that

$$\dot{h}^{ab} \;=\; (-\det(\widehat{g}))^{1/2}\kappa^{ab},$$

where $\kappa^{ab} = \dot{g}^{ab} - \frac{1}{2}\widehat{g}^{ab}\widehat{g}_{qp}\dot{g}^{pq}$. Thus (49) gives

$$\partial_a((-\det(\widehat{g}))^{1/2}\kappa^{ab}) = -\widehat{\Gamma}^b_{ac}(-\det(\widehat{g}))^{1/2}\kappa^{ac} \tag{50}$$

that implies $\partial_a\kappa^{ab} + \kappa^{nb}\widehat{\Gamma}^a_{an} + \kappa^{an}\widehat{\Gamma}^b_{an} = 0$, or equivalently,

$$\widehat{\nabla}_a\kappa^{ab} = 0. \tag{51}$$

We call (51) the *linearized divergence condition* for g. Writing this for $\dot{g}$, we obtain

$$-\widehat{g}^{an}\partial_a\dot{g}_{nj} + \frac{1}{2}\widehat{g}^{pq}\partial_j\dot{g}_{pq} = m_j^{pq}\dot{g}_{pq} \tag{52}$$

where m_j depend on $\widehat{g}_{pq}$ and its derivatives. On similar conditions for the polarization tensor, see [74, form. (9.58) and example 9.5.a, p. 416].

3.4.4. *Properties of the principal symbols of the waves.* Let $K \subset M_0$ be a light-like submanifold of dimension 3. We use below local coordinates $X : V \to \mathbb{R}^4$ defined in a neighborhood V of a point $x_0 \in M_0$. We use the Euclidian coordinates $x = (x^k)_{k=1}^4$ on $X(V)$. We assume these coordinates are such that in V the submanifold K is given by $K \cap V \subset \{x \in \mathbb{R}^4;\ b_kx^k = 0\}$, where $b_k \in \mathbb{R}$ are constants. Assume that the solution $u^{(1)} = (\dot{g}, \dot{\phi})$ of the linear wave equation (42) with the right hand side vanishing in V is such that $\dot{g}_{jk} \in \mathcal{I}^\mu(K)$ with a suitable $\mu \in \mathbb{R}$. Let us write $\dot{g}_{jk}$ as an oscillatory integral using a phase function $\varphi(x,\theta) = b_kx^k\theta$, and a symbol $a_{jk}(x,\theta) \in S^\mu(\mathbb{R}^4, \mathbb{R})$,

$$\dot{g}_{pq}(x^1, x^2, x^3, x^4) = \mathrm{Re}\int_{\mathbb{R}} e^{i(\theta b_m x^m)}a_{pq}(x,\theta)\,d\theta. \tag{53}$$

We assume that $a_{jk}(x,\theta)$ is a classical symbol and denote its (positively homogeneous) principal symbol by $\sigma_p(\dot{g}_{pq})(x,\theta)$. When $x \in K$ and $\xi = \theta b_k dx^k$ so that $(x,\xi) \in N^*K$, we denote the value of $\sigma_p(\dot{g}_{pq})$ at (x,θ) by $\widetilde{a}_{jk}(x,\xi)$, that is, $\widetilde{a}_{jk}(x,\xi) = \sigma_p(\dot{g}_{pq})(x,\theta)$ and say that it is the principal symbol of $\dot{g}_{pq}$ at (x,ξ), associated to the local X-coordinates and the phase function $\phi(x,\theta) = \theta b_m x^m$.

Then, if $\dot{g}_{jk}$ satisfies the divergence condition (52), its principal symbol $\widetilde{a}_{jk}(x,\theta)$ satisfies

$$-\widehat{g}^{mn}(x)\xi_m v_{nj} + \frac{1}{2}\xi_j(\widehat{g}^{pq}(x)v_{pq}) = 0, \quad v_{pq} = \widetilde{a}_{pq}(x,\xi), \tag{54}$$

where $j = 1, 2, 3, 4$ and $\xi = \theta b_k dx^k \in N^*_xK$. If (54) holds, we say that the *divergence condition for the symbol* is satisfied for $\widetilde{a}(x,\xi)$ at $(x,\xi) \in N^*K$.

3.5. **Pieces of spherical waves satisfying linear wave equation.**

3.5.1. *Solutions which are singular on hypersurfaces.* Next we consider a piece of spherical wave whose singular support is concentrated near a geodesic.

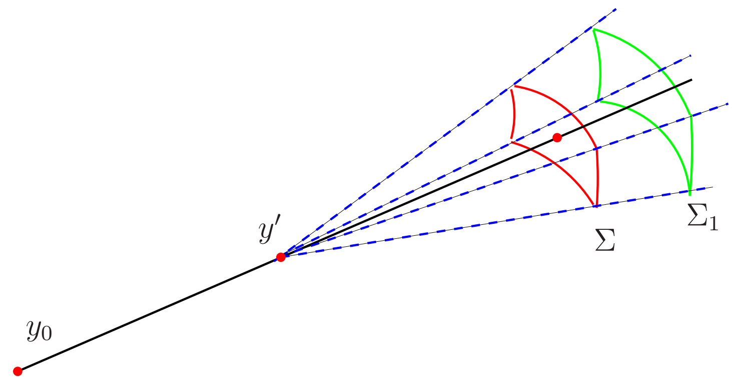

FIGURE 7. A schematic figure where in the 3-dimensional Euclidean space $\mathbb{R}^3$ we describe the route of the piece of the spherical wave that propagates near the geodesic $\gamma_{x_0,\zeta_0}((0,\infty))$. The geodesic is the black line in the figure. The spherical wave is the solution u_1 that is singular on the surface $K(x_0,\zeta_0;t_0,s_0)\subset\mathbb{R}^{1+3}$ that is a subset of a light cone centered at $x'=\gamma_{x_0,\zeta_0}(t_0)$. When $P:\mathbb{R}^{1+3}\to\mathbb{R}^3$ is the projection to the space component, $P(t,y)=y$, the figure shows $P(K(x_1,\xi_1;t_0,s_0))$ and the points $y_0=P(x_0)$ and $y'=P(x')$. The piece of spherical wave is sent from the surface $\Sigma=P(Y(x_0,\zeta_0;t_0,s_0))$, (the surface with red boundary) at time $T=\mathbf{t}(\gamma_{x_0,\zeta_0}(2t_0))$. It starts to propagate and at a later time $T_1>T$ its singular support is the surface Σ_1 shown in the figure in green.

We define the 3-submanifold $K(x_0,\zeta_0;t_0,s_0)\subset M_0$ associated to $(x_0,\zeta_0)\in L^+(M_0,\widehat{g})$, $x_0\in U_{\widehat{g}}$ and parameters $t_0,s_0\in\mathbb{R}_+$ as

$$K(x_0,\zeta_0;t_0,s_0)=\{\gamma_{x',\eta}(t)\in M_0;\ \eta\in\mathcal{W},\ t\in(0,\infty)\}. \tag{55}$$

where $(x',\zeta')=(\gamma_{x_0,\zeta_0}(t_0),\dot{\gamma}_{x_0,\zeta_0}(t_0))$ and $\mathcal{W}\subset L^+_{x'}(M_0,\widehat{g})$ is a neighborhood of ζ' consisting of vectors $\eta\in L^+_{x'}(M_0,\widehat{g})$ that satisfy

$$\|\eta-\zeta'\|_{\widehat{g}^+}<s_0,$$

where $\widehat{g}^+$ is the Riemannian metric corresponding to the Lorentzian metric $\widehat{g}$. Note that $K(x_0,\zeta_0;t_0,s_0)\subset\mathcal{L}^+_{\widehat{g}}(x')\cup\mathcal{L}^-_{\widehat{g}}(x')\cup\{x'\}$ is a subset of the light cone starting from $x'=\gamma_{x_0,\xi_0}(t_0)$ and that it is singular near the point x'. Moreover, $\bigcap_{s_0>0}K(x_0,\zeta_0;t_0,s_0)=\gamma_{x_0,\zeta_0}((-t_0,\infty))\cap M_0$. Let $S=\{x\in M_0;\mathbf{t}(x)=\mathbf{t}(\gamma_{x_0,\zeta_0}(2t_0))\}$ be a Cauchy surface which intersects $\gamma_{x_0,\zeta_0}(\mathbb{R})$ transversally at the point $\gamma_{x_0,\xi_0}(2t_0)$. When $t_0>0$

is small enough, $Y(x_0,\zeta_0;t_0,s_0)=S\cap K(x_0,\zeta_0;t_0,s_0)$ is a smooth 2-surface that is a subset of $U_{\widehat g}$.

Let $\Lambda(x_0,\zeta_0;t_0,s_0)$ be the lagranginan manifold that is the flowout from $N^*Y(x_0,\zeta_0;t_0,s_0)\cap N^*K(x_0,\zeta_0;t_0,s_0)$ on $\mathrm{Char}(\square_{\widehat g})$ in the future direction.

Note that if K_s is the smooth 3-dimensional manifold that is subset of $K(x_0,\zeta_0;t_0,s_0)$, then $N^*K_s\subset \Lambda(x_0,\zeta_0;t_0,s_0)$.

Lemma 3.1. *Let $n\le n_0=-15$ be an integer, $t_0,s_0>0$, $Y=Y(x_0,\zeta_0;t_0,s_0)$, $K=K(x_0,\zeta_0;t_0,s_0)$, and $\Lambda_1=\Lambda(x_0,\zeta_0;t_0,s_0)$. Assume that $\mathbf{h}=(\mathbf{h}_1,\mathbf{h}_2)\in \mathcal{I}^{n-3/2}(Y)$, is a $\mathcal{B}^L$-valued conormal distribution that is supported in a neighborhood $V\subset M_0$ of $\gamma_{x_0,\zeta_0}\cap Y=\{\gamma_{x_0,\zeta_0}(2t_0)\}$ and has a $\mathbb{R}^{10+L}$-valued classical symbol. Denote the principal symbol of $\mathbf{h}$ by $\widetilde h(x,\xi)=(\widetilde h_k(x,\xi))_{k=1}^{10+L}$. Assume that the symbol of $\mathbf{h}$ vanishes near the light-like directions in $N^*Y\setminus N^*K$.*

Let $u^{(1)}=(\dot g,\dot\phi)$ be a solution of the linear wave equation (42) with the source $\mathbf{h}$. Then $u^{(1)}$, considered as a vector valued lagrangian distribution on the set $M_0\setminus Y$, satisfies $u^{(1)}\in \mathcal{I}^{n-1/2}(M_0\setminus Y;\Lambda_1)$, and its principal symbol $\widetilde a(y,\eta)=(\widetilde a_1(y,\eta),\widetilde a_2(y,\eta))$ at $(y,\eta)\in\Lambda_1$ is given by

$$\widetilde a(y,\eta)=\sum_{k=1}^{10+L} R^k_j(y,\eta,x,\xi)\widetilde h_k(x,\xi), \tag{56}$$

where the pairs (x,ξ) and (y,η) are on the same bicharacteristics of $\square_{\widehat g}$, and $x<y$, that is, $((x,\xi),(y,\eta))\in\Lambda'_{\widehat g}$, and in addition, $(x,\xi)\in N^*Y$. Moreover, the matrix $(R^k_j(y,\eta,x,\xi))_{j,k=1}^{10+L}$ is invertible.

We call the solution $u^{(1)}$ a piece of spherical wave that is associated to the submanifold $K(x_0,\zeta_0;t_0,s_0)$. Below, we will consider interaction of spherical waves $u^{(1)}\in\mathcal{I}^{n-1/2}(M_0\setminus Y;\Lambda_1)$. The interaction terms involve several derivatives, and thus these solutions need to be smooth enough. This is why we chose above $n_0=-15$.

Proof.[2] By [65], the parametrix of the scalar wave equation satisfies $(\square_{\widehat g}+v(x,D))^{-1}\in\mathcal{I}^{-3/2,-1/2}(\Delta'_{T^*M_0},\Lambda_{\widehat g})$, where $v(x,D)$ is a 1st order differential operator, $\Delta_{T^*M_0}$ is the conormal bundle of the diagonal of $M_0\times M_0$ and $\Lambda_{\widehat g}$ is the flow-out of the canonical relation of $\square_{\widehat g}$.

Let $x=(x^1,x')\in\mathbb{R}^4$ denote local coordinates of $\mathbb{R}^4$ and $\partial_1=\frac{\partial}{\partial x^1}$. By [42, Prop. 26.1.3] there are elliptic Fourier integral operators Φ_1 and Φ_0, of order 1 and 0, having the same canonical relation so that $\square_{\widehat g}=\Phi_1\partial_1\Phi_0^{-1}$. Thus an operator matrix $A=(A_{jk})_{j,k=1}^{10+L}$ with $A_{jk}=\square_{\widehat g}\delta_{jk}+B^p_{jk}\nabla_p+C_{jk}$ can be written as $A=\Phi_1\widetilde A\Phi_0^{-1}$, $\widetilde A=(\widetilde A_{jk})_{j,k=1}^{10+L}$ with $\widetilde A_{jk}=\partial_1\delta_{jk}+\widetilde R_{jk}$, $\widetilde R_{jk}=\Phi_0(B^p_{jk}\nabla_p+C_{jk})\Phi_1^{-1}$.

[2]We note that Nils Dencker's results for the polarization sets, see [21] are closely related to this result.

Furthermore, Φ_1 and Φ_0 have the same canonical relation and we conclude that $\widetilde{R}$ is zeroth order pseudodifferential operator. Thus the parametrix for A is

$$A^{-1} = \Phi_0(\partial_1 I + \widetilde{R})^{-1}\Phi_1^{-1},$$

where $(\partial_1 I + \widetilde{R})^{-1} \sim \sum_{j=0}^{\infty}(\partial_1 I)^{-1}(\widetilde{R}(\partial_1 I)^{-1})^j$. Here $\sim$ denotes an asymptotic sum. This implies for also the matrix valued wave operator, $\square_{\widehat{g}} I + V(x,D)$, when $V(x,D)$ is the 1st order differential operator, that $(\square_{\widehat{g}} I + V(x,D))^{-1} \in \mathcal{I}^{-3/2,-1/2}(\Delta'_{T^*M_0}, \Lambda_{\widehat{g}})$. By [28, Prop. 2.1], this yields that $u^{(1)} \in \mathcal{I}^{n-1/2}(\Lambda_1)$ and the formula (56) where $R = (R_j^k(y,\eta,x,\xi))_{j,k=1}^{10+L}$ is related to the symbol of $(\square_{\widehat{g}} I + V(x,D))^{-1}$ on $\Lambda_{\widehat{g}}$. Making a similar consideration for the adjoint of the $(\square_{\widehat{g}} I + V(x,D))^{-1}$, i.e., considering the propagation of singularities using reversed causality, we see that the matrix R is invertible.

$\square$

Let $\mathfrak{S}_{y,\eta}^n$ be the space of the elements in $\mathcal{B}_y^L$ satisfying the divergence condition for the symbols (54) at (y,η).

Lemma 3.2. *Let $n \le n_0$, $t_0, s_0 > 0$, $Y = Y(x_0,\zeta_0;t_0,s_0)$, $K = K(x_0,\zeta_0;t_0,s_0)$, and $\Lambda_1 = \Lambda(x_0,\zeta_0;t_0,s_0)$. Let us consider $\mathbf{p},\mathbf{q}',\mathbf{z} \in \mathcal{I}^{n-3/2}(Y)$ that have classical symbols with principal symbols $\widetilde{c}^{(a)}(x,\xi)$, $\widetilde{d}_1(x,\xi)$, $\widetilde{d}_2^{(a)}(x,\xi)$, correspondingly, at $(x,\xi) \in N^*Y$. Moreover, assume that the principal symbols of $\mathbf{p}$ and $\mathbf{z}$ satisfy the linearized conservation law for the principal symbols, that is, (46), at a light-like co-vector $(x,\xi) \in N^*Y$.*

Let $(y,\eta) \in T^*M_0$, $y \notin Y$ be a light-like co-vector such that $(x,\xi) \in \Theta_{y,\eta} \cap N^*Y$. Then the principal symbol $\widetilde{a}(y,\eta) = (\widetilde{a}_1(y,\eta), \widetilde{a}_2(y,\eta))$ of the wave $u^{(1)} \in \mathcal{I}^{n-1/2}(M_0 \setminus Y;\Lambda_1)$, defined in Lemma 3.1, varies in $\mathfrak{S}_{y,\eta}$. Moreover, by varying $\mathbf{p},\mathbf{q}',\mathbf{z}$ so that the linearized conservation law (46) for principal symbols is satisfied, the principal symbol $\widetilde{a}(y,\eta)$ at (y,η) achieves all values in the $(L+6)$ dimensional space $\mathfrak{S}_{y,\eta}$.

Below, we denote $\mathbf{f} \in {\mathcal{I}_S}^{n-3/2}(Y(x_0,\zeta_0;t_0,s_0))$ when

$$\mathbf{f} = (\mathbf{p},\mathbf{q}), \quad \mathbf{q} = (\mathbf{q}',\mathbf{z}) \tag{57}$$

and $\mathbf{p},\mathbf{q}',\mathbf{z} \in \mathcal{I}^{n-3/2}(Y(x_0,\zeta_0;t_0,s_0))$ and the principal symbols of $\mathbf{p}$ and $\mathbf{z}$ satisfy the linearized conservation law for principal symbols, that is, equation (46).

Proof. Let us use local coordinates $X : V \to \mathbb{R}^4$ where $V \subset M_0$ is a neighborhood of x. In these coordinates, let $\widetilde{c}^{(b)}(x,\xi)$ and $\widetilde{d}_2^{(b)}(x,\xi)$ denote the sub-principal symbols of $\mathbf{p}$ and $\mathbf{z}$, respectively, at $(x,\xi) \in N^*Y$. Moreover, let $\widetilde{c}_j^{(c)}(x,\xi) = \frac{\partial}{\partial x^j}\widetilde{c}_j^{(a)}(x,\xi)$ and $\widetilde{d}_j^{(c)}(x,\xi) = \frac{\partial}{\partial x^j}\widetilde{d}_2^{(a)}(x,\xi)$, $j = 1,2,3,4$ be the x-derivatives of the principal symbols and let us denote $\widetilde{c}^{(c)}(x,\xi) = (\widetilde{c}_j^{(c)}(x,\xi))_{j=1}^4$ and $\widetilde{d}^{(c)}(x,\xi) = (\widetilde{d}_j^{(c)}(x,\xi))_{j=1}^4$.

Let $\mathbf{h}=(\mathbf{h}_1,\mathbf{h}_2)=H(x;\mathbf{p},\mathbf{q})$ be defined by (43), where we recall that $M_{(1)}=0$, $N^j_{(1)}=0$, and $L_{(1)}=\widehat g$. Then $\mathbf{h}\in \mathcal{I}^{n-3/2}(Y)$ has the principal symbol $\widetilde h(x,\xi)=(\widetilde h_1(x,\xi),\widetilde h_2(x,\xi))$ at (x,ξ), given by

$$\begin{pmatrix}\widetilde h_1(x,\xi)\\ \widetilde h_2(x,\xi)\end{pmatrix}=\begin{pmatrix}(\widetilde c^{(a)}+\widehat g\widetilde d_2^{(a)})(x,\xi)\\ K_{(2)}(x,\xi)(\widetilde c^{(a)}+\widehat g\widetilde d_2^{(a)})(x,\xi)\end{pmatrix}+\begin{pmatrix}0\\ M_{(2)}(x)\widetilde d_1(x,\xi)\end{pmatrix}$$

$$+\begin{pmatrix}0\\ J_{(2)}(x,\xi)(\widetilde c^{(c)}+\widehat g\widetilde d^{(c)})(x,\xi)\end{pmatrix}+\begin{pmatrix}0\\ L_{(2)}(x)\widetilde d_2^{(a)}(x,\xi)\end{pmatrix}+ \tag{58}$$

$$+\begin{pmatrix}0\\ N^j_{(2)}(x)\,\widehat g^{lk}\xi_l(\widetilde c_1^{(b)}+\widehat g\widetilde d_2^{(b)})_{jk}(x,\xi)\end{pmatrix},$$

where $K_{(2)}(x,\xi)(\widetilde c^{(a)}+\widehat g\widetilde d_2^{(a)})(x,\xi)$ and $J_{(2)}(x,\xi)(\widetilde c^{(c)}+\widehat g\widetilde d_2^{(c)})(x,\xi)$ is related to the sub-principal symbol of the term $\widehat g^{lk}\nabla^{\widehat g}_l(\mathbf{p}_{jk}+\mathbf{z}\widehat g_{jk})$. Observe that the map $(c^{(b)}_{jk})\mapsto(\widehat g^{lk}\xi_l c^{(b)}_{jk})_{j=1}^4$, defined as $\text{Symm}(\mathbb{R}^{4\times 4})\to\mathbb{R}^4$, is surjective. Denote $\widetilde m^{(a)}=(\widetilde c^{(a)}+\widehat g\widetilde d_2^{(a)})(x,\xi)$, $\widetilde m^{(b)}=(\widetilde c^{(b)}+\widehat g\widetilde d_2^{(b)})(x,\xi)$, and $\widetilde m^{(c)}=(\widetilde c^{(c)}+\widehat g\widetilde d^{(c)})(x,\xi)$. As noted above, by Assumption S (iii), the union of the image spaces of the matrices $M_{(2)}(x)$ and $L_{(2)}(x)$, and $N^j_{(2)}(x)$, $j=1,2,3,4$, span the space $\mathbb{R}^L$ for all $x\in\widehat U$. Hence the map

$$\mathbf{A}:(\widetilde m^{(a)},\widetilde m^{(b)},\widetilde m^{(c)},\widetilde d_1,\widetilde d_2^{(a)})|_{(x,\xi)}\mapsto(\widetilde h_1(x,\xi),\widetilde h_2(x,\xi)),$$

given by (58), considered as a map $\mathbf{A}:Y=(\text{Symm}(\mathbb{R}^{4\times4}))^{1+1+4}\times\mathbb{R}^K\times\mathbb{R}\to\text{Symm}(\mathbb{R}^{4\times4})\times\mathbb{R}^L$, is surjective. Let $\mathcal{X}$ be the set of elements $(\widetilde m^{(a)}(x,\xi),\widetilde m^{(b)}(x,\xi),\widetilde m^{(c)}(x,\xi),\widetilde d_1(x,\xi),\ \widetilde d_2^{(a)}(x,\xi))\in Y$ where $\widetilde m^{(a)}(x,\xi)=(\widetilde c^{(a)}+\widehat g\widetilde d_2^{(a)})(x,\xi)$ is such that the pair $(\widetilde c^{(a)}(x,\xi),\widetilde d_2^{(a)}(x,\xi))$ satisfies the linearized conservation law for principal symbols, see (46). Then $\mathcal{X}$ has codimension 4 in Y, we see that the image $\mathbf{A}(\mathcal{X})$ has codimension less or equal to 4, that is, it has at least dimension $(L+6)$.

Let $(\mathbf{h}_1,\mathbf{h}_2)$ be a source with the principal symbol $\widetilde h(x,\xi)$ that corresponds to functions $(\mathbf{p},\mathbf{q})$. When $(P,Q)=(\varepsilon\mathbf{p},\varepsilon\mathbf{q})$ and $\varepsilon>0$ is small enough, it follows from Assumption S (iv), that the $\widehat g$-reduced Einstein equations (10) have a solution $(g_\varepsilon,\phi_\varepsilon)$ that satisfies the conservation law (11) and thus g_ε satisfies the harmonicity condition (47). Hence $\dot g=\partial_\varepsilon g_\varepsilon|_{\varepsilon=0}$ satisfies the linearized divergence condition (52). Observe that the metric component of the solution $u^{(1)}$ of the linearized Einstein equation corresponding to the functions $(\mathbf{p},\mathbf{q})$ is equal to $\dot g$. Thus the principal symbol $\widetilde a_1(y,\eta)$ of $u^{(1)}$ at (y,η) satisfies the divergence condition for the symbols (54), that is, four linear conditions, and also satisfies $\widetilde a_1(y,\eta)\in\mathfrak{S}_{y,\eta}$.

By Lemma 3.1 the $R:\widetilde h(x,\xi)\mapsto\widetilde a(y,\eta)$ maps a subspace of $\mathbb{R}^{10+L}$ whose has codimension 4 onto some subspace of $\mathbb{R}^{10+L}$ whose codimension is 4.

The above imply that the map $R\circ \mathbf{A}$, that maps the principal and sub-principal symbols $(\widetilde m^{(a)}(x,\xi),\widetilde m^{(b)}(x,\xi),\widetilde m^{(c)}(x,\xi),\widetilde d_1(x,\xi),\widetilde d_2^{(a)}(x,\xi))$ to the principal symbol $\widetilde a(y,\eta)$ of the solution $u^{(1)}$ at (y,η), is such that $R\circ \mathbf{A}$ maps $\mathcal{X}$ onto a subspace of $\mathbb{R}^{10+L}$ which codimension is at most 4 and is a subspace of $\mathfrak{S}_{y,\eta}$. As $\mathfrak{S}_{y,\eta}$ has codimension 4, we see that $R\circ \mathbf{A}(\mathcal{X})$ coincides with $\mathfrak{S}_{y,\eta}$ and has thus dimension $(L+6)$. □

3.6. **Interaction of non-linear waves.** Next we consider interaction of four C^k-smooth waves on a $(1+3)$ dimensional manifold having conormal singularities, where $k\in\mathbb{Z}_+$ is sufficiently large. Interaction of such waves produces a "corner point" in the space time. On the related microlocal tools to consider scattering by corners, see [86, 87]. Earlier related interaction of three waves has been studied by Melrose and Ritter [66, 67] and Rauch and Reed, [76] for a non-linear hyperbolic equations in $\mathbb{R}^{1+2}$ where the non-linearity appears in the lower order terms.

3.6.1. *Interaction of non-linear waves on a general manifold.* Next, we introduce a vector of four ε variables denoted by $\vec\varepsilon=(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)\in\mathbb{R}^4_+$. Let $s_0,t_0>0$ and consider the solution $u_{\vec\varepsilon}=(g_{\vec\varepsilon}-\widehat g,\phi_{\vec\varepsilon}-\widehat\phi)$ where $g_{\vec\varepsilon}$ and $\phi_{\vec\varepsilon}$ solve the equations (10) with $(P,Q)=\mathbf{f}_{\vec\varepsilon}$ being

$$
\mathbf{f}_{\vec\varepsilon}:=\sum_{j=1}^4\varepsilon_j\mathbf{f}_j,\quad \mathbf{f}_j\in{\mathcal{I}_S}^{n-3/2}(Y(x_j,\zeta_j;t_0,s_0)),\tag{59}
$$

see (57), where (x_j,ζ_j) are light-like vectors with $x_j\in\widehat U$. Moreover, we assume that for some $0<r_1<r_0$ and $s_-+r_1<s_1<s_+$ the sources satisfy

$$
\begin{aligned}
&\operatorname{supp}(\mathbf{f}_j)\cap J^+_{\widehat g}(\operatorname{supp}(\mathbf{f}_k))=\emptyset,\quad \text{for all } j\neq k,\\
&\operatorname{supp}(\mathbf{f}_j)\subset I_{\widehat g}(\mu_{\widehat g}(s_1-r_1),\mu_{\widehat g}(s_1)),\quad \text{for all } j=1,2,3,4,
\end{aligned}\tag{60}
$$

where r_0 is the parameter introduced after (14) to define $W_{\widehat g}=W_{\widehat g}(r_0)$. The first condition implies that the supports of the sources are causally independent.

The sources $\mathbf{f}_j$ give raise for $\mathcal{B}^L$-section valued solutions of the linearized wave equations, denoted

$$
u_j:=u_j^{(1)}=QR\,\mathbf{f}_j\in\mathcal{I}(\Lambda(x_0^{(j)},\zeta_0^{(j)};t_0,s_0)),
$$

where R is a first order differential operator depending on $\widehat g$ and $\mathbf{Q}=\mathbf{Q}_{\widehat g}=(\square_{\widehat g}+V(x,D))^{-1}$ is the causal inverse of the wave equation where $V(x,D)$ is a first order differential operator.

In the following we use the notations

$$\begin{aligned}\partial_{\vec\varepsilon}^1 u_{\vec\varepsilon}|_{\vec\varepsilon=0} &:= \partial_{\varepsilon_1} u_{\vec\varepsilon}|_{\vec\varepsilon=0},\\ \partial_{\vec\varepsilon}^2 u_{\vec\varepsilon}|_{\vec\varepsilon=0} &:= \partial_{\varepsilon_1}\partial_{\varepsilon_2} u_{\vec\varepsilon}|_{\vec\varepsilon=0},\\ \partial_{\vec\varepsilon}^3 u_{\vec\varepsilon}|_{\vec\varepsilon=0} &:= \partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3} u_{\vec\varepsilon}|_{\vec\varepsilon=0},\\ \partial_{\vec\varepsilon}^4 u_{\vec\varepsilon}|_{\vec\varepsilon=0} &:= \partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3}\partial_{\varepsilon_4} u_{\vec\varepsilon}|_{\vec\varepsilon=0}.\end{aligned}$$

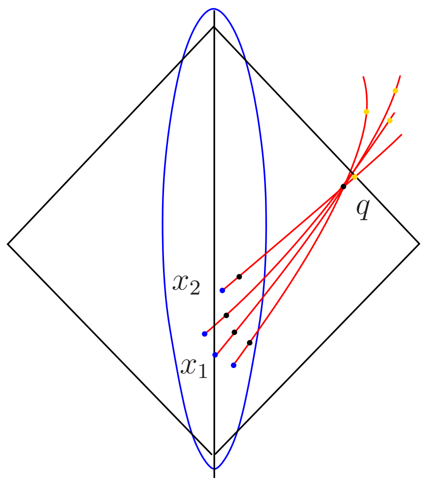

FIGURE 8. A schematic figure where the space-time is represented as the 2-dimensional set $\mathbb{R}^{1+1}$. The four light-like geodesics $\gamma_{x_j,\xi_j}([0,\infty))$, $j = 1,2,3,4$ starting at the blue points x_j intersect at q before the first cut points of $\gamma_{x_j,\xi_j}([t_0,\infty))$, denoted by the golden points. We send pieces of spherical waves from the black points $\gamma_{x_j,\xi_j}(t_0)$ that propagate near these geodesics and produce an artificial point source at the point q.

Next we denote the waves produced by the ℓ-th order interaction by

$$\mathcal{M}^{(\ell)} := \partial_{\vec\varepsilon}^{\ell} u_{\vec\varepsilon}|_{\vec\varepsilon=0}, \quad \ell \in \{1,2,3,4\}$$

in $\widehat{U}$, see Fig. 8. Below, $\mathcal{B}_j^\beta$, $j,\beta \in \mathbb{Z}_+$ are differential operators having in local coordinates the form

$$\begin{aligned}\mathcal{B}_j^\beta &: (v_p)_{p=1}^{10+L} \mapsto (b_p^{r,(j,\beta)}(x)\partial_x^{\vec k(\beta,j)} v_r(x))_{p=1}^{10+L}, \text{ and}\\ S_j^\beta &= \mathbf{Q} \text{ or } S_j^\beta = I,\end{aligned} \tag{61}$$

where the coefficients $b_p^{r,(j,\beta)}(x)$ depend on the derivatives of $\widehat{g}_{jk}$.

Computing ε_j derivatives of the equations (36) with the sources $\mathbf{f}_{\vec{\varepsilon}}$, and taking into account the condition (60), we obtain

$$\begin{aligned}
(62)\quad \mathcal{M}^{(1)} &= u_1,\\
\mathcal{M}^{(2)} &= \sum_{\sigma\in\Sigma(2)}\sum_{\beta\in J_2} \mathbf{Q}(\mathcal{B}_2^\beta u_{\sigma(2)}\cdot \mathcal{B}_1^\beta u_{\sigma(1)}),\\
\mathcal{M}^{(3)} &= \sum_{\sigma\in\Sigma(3)}\sum_{\beta\in J_3} \mathbf{Q}(\mathcal{B}_3^\beta u_{\sigma(3)}\cdot \mathcal{C}_1^\beta S_1^\beta(\mathcal{B}_2^\beta u_{\sigma(2)}\cdot \mathcal{B}_1^\beta u_{\sigma(1)})),\\
\mathcal{M}^{(4)} &= \mathbf{Q}\mathcal{F}^{(4)},\quad \mathcal{F}^{(4)} = \sum_{\sigma\in\Sigma(4)}\sum_{\beta\in J_4}(\mathcal{G}_\sigma^{(4),\beta}+\widetilde{\mathcal{G}}_\sigma^{(4),\beta}),
\end{aligned}$$

where $\Sigma(\ell)$ is the set of permutations, that is, bijections $\sigma:\{1,2,\dots,\ell\}\to\{1,2,\dots,\ell\}$, and $J_2,J_3,J_4\subset\mathbb{Z}_+$ are finite sets. Moreover, the operators $\mathcal{C}_j^\beta$ are of the same form as the operators $\mathcal{B}_j^\beta$. The orders $k_j^\beta = ord(\mathcal{B}_j^\beta)$ of the differential operators $\mathcal{B}_j^\beta$ and the orders $\ell_j^\beta = ord(\mathcal{C}_j^\beta)$ satisfy $k_1^\beta+k_2^\beta\le 2$ and $k_3^\beta+\ell_1^\beta\le 2$. Moreover,

$$\begin{aligned}
(63)\qquad \mathcal{G}_\sigma^{(4),\beta} &= \mathcal{B}_4^\beta u_{\sigma(4)}\cdot \mathcal{C}_2^\beta S_2^\beta(\mathcal{B}_3^\beta u_{\sigma(3)}\cdot \mathcal{C}_1^\beta S_1^\beta(\mathcal{B}_2^\beta u_{\sigma(2)}\cdot \mathcal{B}_1^\beta u_{\sigma(1)}))\\
\mathcal{M}_\sigma^{(4),\beta} &= \mathbf{Q}\mathcal{G}_\sigma^{(4),\beta},
\end{aligned}$$

where the orders of the differential operators satisfy $k_4^\beta+\ell_2^\beta\le 2$, $k_3^\beta+\ell_1^\beta\le 2$, $k_2^\beta+k_1^\beta\le 2$, and finally

$$\begin{aligned}
(64)\qquad \widetilde{\mathcal{G}}_\sigma^{(4),\beta} &= \mathcal{C}_2^\beta S_2^\beta(\mathcal{B}_4^\beta u_{\sigma(4)}\cdot \mathcal{B}_3^\beta u_{\sigma(3)})\cdot \mathcal{C}_1^\beta S_1^\beta(\mathcal{B}_2^\beta u_{\sigma(2)}\cdot \mathcal{B}_1^\beta u_{\sigma(1)}),\\
\widetilde{\mathcal{M}}_\sigma^{(4),\beta} &= \mathbf{Q}\widetilde{\mathcal{G}}_\sigma^{(4),\beta},
\end{aligned}$$

where $\ell_1^\beta+\ell_2^\beta\le 2$, $k_4^\beta+k_3^\beta\le 2$, $k_2^\beta+k_1^\beta\le 2$. Note that due to the conditions (60), for $\ell=2,3,4$, $\mathcal{M}_\sigma^{(\ell),\beta}$ and $\widetilde{\mathcal{M}}_\sigma^{(\ell),\beta}$ do not contain terms that involve sources f_j, for example, using (36) we obtain the formula

$$\mathcal{M}^{(2)} = \mathbf{Q}\Big(B_1(u_1)f_2 + B_2(u_2)f_1 + K(u_1,u_2)\Big),$$

where the terms $B_1(u_1)f_2$ and $B_2(u_2)f_2$ vanish due to (60) and K is a bilinear operator.

We make also the observation that when $\vec{S}_\beta = (\mathbf{Q},\mathbf{Q})$, the terms $\mathcal{M}_\sigma^{(4),\beta}$ and $\widetilde{\mathcal{M}}_\sigma^{(4),\beta}$ can be written in the form

$$(65)\qquad \mathcal{M}_\sigma^{(4),\beta} = \mathbf{Q}(A[u_{\sigma(4)},\mathbf{Q}(A[u_{\sigma(3)},\mathbf{Q}(A[u_{\sigma(2)},u_{\sigma(1)}])])]),$$

$$(66)\qquad \widetilde{\mathcal{M}}_\sigma^{(4),\beta} = \mathbf{Q}(A[\mathbf{Q}(A[u_{\sigma(4)},u_{\sigma(3)}]),\mathbf{Q}(A[u_{\sigma(2)},u_{\sigma(1)}])])$$

where $A[V,W]$ is a generic notation (i.e., its exact form can vary even inside the formula) for a 2nd order multilinear operator in V and W having the form

$$(67)\qquad A[V,W] = \sum_{|\alpha|+|\gamma|\le 2} a_{\alpha\gamma}(x)(\partial_x^\alpha V(x))\cdot(\partial_x^\gamma W(x)).$$

We use in particular two such bilinear forms that are given for $V = (v_{jk}, \phi)$, and $W = (w^{jk}, \phi')$, by

$$A_1[V, W] = -\widehat{g}^{jb} v_{ab} \widehat{g}^{ak} \partial_j \partial_k w_{pq}, \quad A_2[V, W] = A_1[W, V]. \tag{68}$$

By considering the terms that we obtain by substituting formulas (34) in (35), we see that the case when the quadratic forms A appearing in the formulas (65) and (66) have second order derivatives, that is, when the terms with $|\gamma| = 2$ or $|\alpha| = 2$ in (67) are non-zero, can happen only when A is either the bilinear form A_1 or A_2.

3.6.2. *On the singular support of the non-linear interaction of three waves.* Below, we use for a pair $(x, \xi) \in L^+M_0$ the notation

$$(x(h), \xi(h)) = (\gamma_{x,\xi}(h), \dot{\gamma}_{x,\xi}(h)). \tag{69}$$

Let us next consider four light-like future pointing directions (x_j, ξ_j), $j = 1, 2, 3, 4$, and use below the notation

$$(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4.$$

We will consider the case when we send pieces of spherical waves propagating on surfaces $K(x_j, \xi_j; t_0, s_0)$, $t_0, s_0 > 0$, cf. (55), and these waves interact. Let us use for (x_j, ξ_j) the notation (69) and assume that $x_k(t_0) \notin K(x_j, \xi_j; t_0, s_0)$ for $k \neq j$, see Fig. 9.

Next we consider the 3-interactions of the waves, see Fig. 9 on analogous considerations in $(1 + 2)$ dimensional Lorentz space. For $1 < j_1 < j_2 < j_3 \leq 4$, let $K_{j_p} = K(x_{j_p}, \xi_{j_p}; t_0, s_0)$, $p = 1, 2, 3$. We define

$$\mathcal{X}(j_1, j_2, j_3; t_0, s_0) = \bigcup_{z \in K_{j_1} \cap K_{j_2} \cap K_{j_3}} (N_z K_{j_1} + N_z K_{j_2} + N_z K_{j_3}) \cap L_z^+ M_0. \tag{70}$$

Note that $K_{123} = K_1 \cap K_2 \cap K_3$ is a space like curve and $N_z^* K_{123} = N_z^* K_1 + N_z^* K_2 + N_z^* K_3$. Moreover, we define $\mathcal{Y}(j_1, j_2, j_3; t_0, s_0)$ to be the set of all $y \in M_0$ such that there are $z \in K_{j_1} \cap K_{j_2} \cap K_{j_3}$, $\zeta \in \mathcal{X}(j_1, j_2, j_3; t_0, s_0)$, and $t \geq 0$ such that $\gamma_{z,\zeta}(t) = y$. We use below the 3-interaction sets (See Figs. 10 and 11)

$$\begin{aligned}
\mathcal{Y}((\vec{x}, \vec{\xi}); t_0, s_0) &= \bigcup_{1 \leq j_1 < j_2 < j_3 \leq 4} \mathcal{Y}(j_1, j_2, j_3; t_0, s_0), \\
\mathcal{Y}((\vec{x}, \vec{\xi}); t_0) &= \bigcap_{s_0 > 0} \mathcal{Y}((\vec{x}, \vec{\xi}); t_0, s_0) \subset M_0, \\
\mathcal{X}((\vec{x}, \vec{\xi}); t_0, s_0) &= \bigcup_{1 \leq j_1 < j_2 < j_3 \leq 4} \mathcal{X}(j_1, j_2, j_3; t_0, s_0), \\
\mathcal{X}((\vec{x}, \vec{\xi}); t_0) &= \bigcap_{s_0 > 0} \mathcal{X}((\vec{x}, \vec{\xi}); t_0, s_0) \subset TM_0.
\end{aligned} \tag{71}$$

For instance in Minkowski space, when three plane waves (which singular supports are hyper-planes) collide, the intersections of the hyperplanes is a 1-dimensional space-like line K_{123} in the 4-dimensional space-time. This corresponds to a point moving continuously in time. Roughly speaking, the point seem to move at a higher speed than light (i.e. it appears like a tachyonic point-like object) and produces a shock wave type of singularity that moves on the set $\mathcal{Y}((\vec{x},\vec{\xi});t_0,s_0)$ in the space-time.

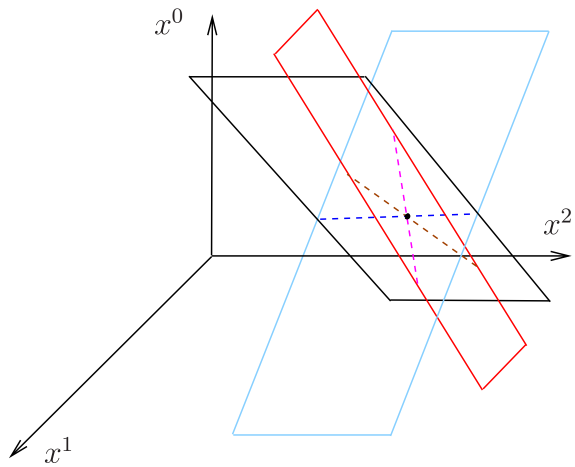

FIGURE 9. A schematic figure where the space-time is represented as the 3-dimensional set $\mathbb{R}^{2+1}$. In the figure 3 pieces or plane waves have singularities on strips of hyperplanes (in fact planes) K_1, K_2, K_3, colored by light blue, red, and black. These planes have intersections, and in the figure the sets $K_{12} = K_1 \cap K_2$, $K_{23} = K_2 \cap K_3$, and $K_{13} = K_1 \cap K_3$ are shown as dashed lines with dark blue, magenta, and brown colors. These dashed lines intersect at a point $\{q\} = K_{123} = K_1 \cap K_2 \cap K_3$.

3.6.3. *Gaussian beams.* Our aim is to consider interactions of 4 waves to produce a new source, and to this end we use test sources that produce gaussian beams.

Let $y \in \widehat{U}$ and $\eta \in T_yM$ be a future pointing light-like vector. We choose a complex function $p \in C^\infty(M_0)$ such that $\operatorname{Im} p(x) \geq 0$ and $\operatorname{Im} p(x)$ vanishes only at y, $p(y) = 0$, $d(\operatorname{Re} p)|_y = \eta^\sharp$, $d(\operatorname{Im} p)|_y = 0$ and the Hessian of $\operatorname{Im} p$ at y is positive definite. To simplify notations, we use below also complex sources and waves. The physical linearized waves can be obtained by taking the real part of the corresponding complex wave. We use a large parameter τ and define a test source

$$F_\tau(x) = \tau^{-1} \exp(i\tau p(x))h(x) \tag{72}$$

where h is section on $\operatorname{sym}(\Omega^2 M) \times \mathbb{R}^L$ supported in a small neighborhood W of y. The construction of $p(x)$ and F_τ is discussed later.

We consider both the usual causal solutions and the solutions for which time is reversed, that is, we use the anti-causal parametrix $\mathbf{Q}^* =$

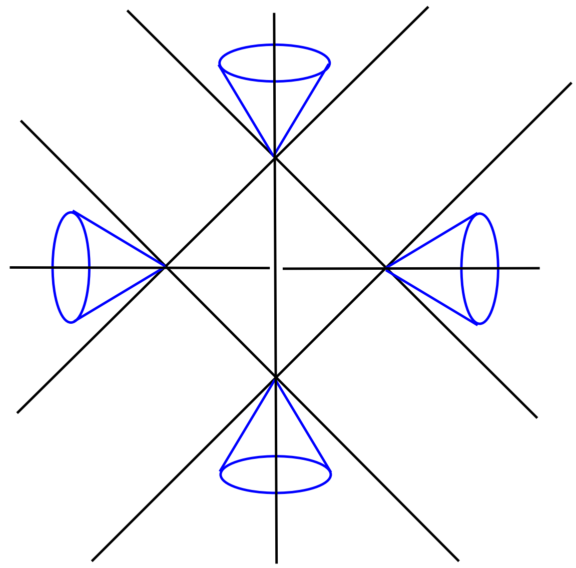

FIGURE 10. In the section 3.6.2 we consider four colliding pieces of spherical waves. In the figure we consider the Minkowski space $\mathbb{R}^{1+3}$ and the figure corresponds to a "time-slice" $\{T_1\} \times \mathbb{R}^3$. We assume that the spherical waves are sent so far away that the waves look like pieces of plane waves. The plane waves $u_j \in \mathcal{I}(K_j)$, $j = 1, 2, 3, 4$, are conormal distributions that are solutions of the linear wave equation and their singular supports are the sets K_j, that are pieces of 3-dimensional planes in the space-time. The sets K_j are not shown in the figure. The 2-wave interaction wave $\mathcal{M}^{(2)}$ is singular on the set $\cup_{j \neq k} K_j \cap K_j$. There are 6 intersection sets $K_j \cap K_j$ that are shown as black line segments. Note that these lines have 4 intersection points, that is, the vertical and the horizontal black lines do not intersect in $\{T_1\} \times \mathbb{R}^3$. The four intersection points of the black lines are the sets $(\{T_1\} \times \mathbb{R}^3) \cap (K_j \cap K_j \cap K_n)$. These points correspond to points moving in time (i.e., they are curves in the space-time) that produce singularities of the 3-interaction wave $\mathcal{M}^{(3)}$. The points seem to move faster than the speed of the light (similarly, as a shadow of a far away object may seem to move faster than the speed of the light). Such point sources produce "shock waves", and due to this, $\mathcal{M}^{(3)}$ is singular on the sets $\mathcal{Y}((\vec{x}, \vec{\xi}), t_0, s_0)$ defined in formulas (65)-(66). The set $(\{T_1\} \times \mathbb{R}^3) \cap \mathcal{Y}((\vec{x}, \vec{\xi}), t_0, s_0)$ is the union of the four blue cones shown in the figure.

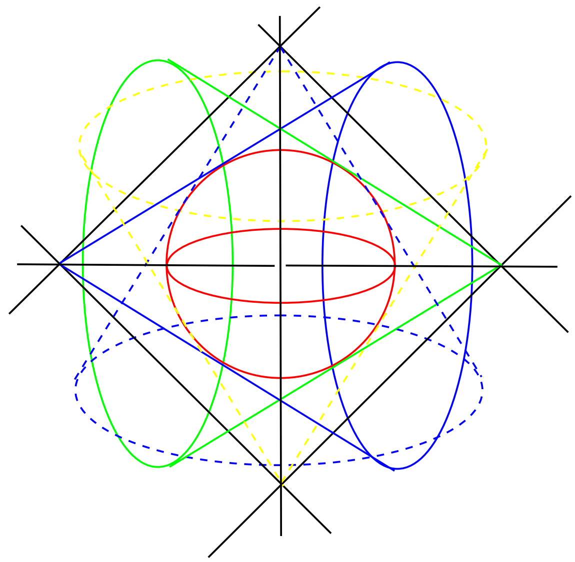

FIGURE 11. The same situation that was described in Fig. 9 is shown at a later time, that is, the figure shows the time-slice $\{T_2\} \times \mathbb{R}^3$ with $T_2 > T_1$, when the parameter s_0 is quite large. The four pieces of the spherical waves have now collided and produced a point source in the space-time at a point $q \in K_1 \cap K_2 \cap K_3 \cap K_4$, that produces the singularities of the 4-interaction wave $\mathcal{M}^{(4)}$. In the figure $T_1 < t < T_2$, where q has the time coordinate t. The four cones in the figure, shown with solid blue and green curves and dashed blue and yellow curves are the intersection of the time-slice $\{T_2\} \times \mathbb{R}^3$ and the set $\mathcal{Y}((\vec{x}, \vec{\xi}), t_0, s_0)$. Inside the cones the red sphere is the set $\mathcal{L}^+(q) \cap (\{T_2\} \times \mathbb{R}^3)$ that corresponds to the spherical wave produced by the point source at q.

$\mathbf{Q}^*_{\widehat{g}}$ instead of the usual causal parametrix $\mathbf{Q} = (\square_{\widehat{g}} + V(x, D))^{-1}$. The wave $u_\tau = \mathbf{Q}^* F_\tau$. It satisfies by [75]

$$\|u_\tau - u_\tau^N\|_{C^k(J(p^-,p^+))} \le C_N \tau^{-n_{N,k}} \tag{73}$$

where $n_{N,k} \to \infty$ as $N \to \infty$ and u_τ^N is a formal Gaussian beam [75] of order N having the form

$$u_\tau^N(x) = \exp(i\tau\varphi(x)) \left(\sum_{n=0}^{N} U_n(x) \tau^{-n} \right), \tag{74}$$

where $\varphi(x) = A(x) + iB(x)$ and $A(x)$ and $B(x)$ are real functions, $B(x) \ge 0$, and $B(x)$ vanishes only on $\gamma_{y,\eta}(\mathbb{R}) \cap J^-_{\widehat{g}}(W)$, and for $z = \gamma_{y,\eta}(t)$, and $\zeta = \dot{\gamma}^\flat_{y,\eta}(t)$, $t < 0$ we have $dA|_z = \zeta$, $dB|_z = 0$, and the Hessian of B at z restricted to the orthocomplement of ζ (with

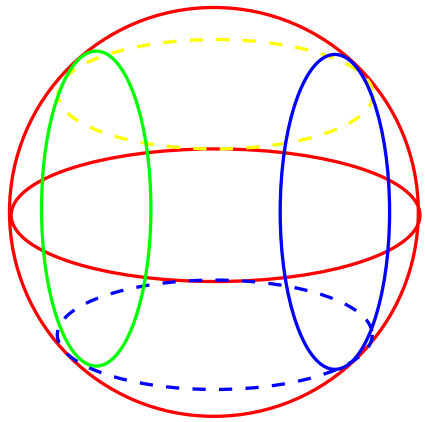

FIGURE 12. The same situation that was described in Fig. 11, that is, the figure shows in the time-slice $\{T_2\} \times \mathbb{R}^3$ the singularities produced by four colliding spherical waves, when the parameter s_0 is very small. In this case the truncated cones degenerate to circles.

respect to $\widehat{g}^+$) is positive definite. Above, functions h and U_n can be chosen to be smooth and supported in any neighborhood V of y and any neighborhood of $\gamma_{y,\eta}((-\infty,0])$.

The source function F_τ can be constructed in local coordinates using the asymptotic representation of the gaussian beam, namely, by considering first

$$F_\tau^{(series)}(x) = c\tau^{-1/2}\int_{\mathbb{R}} e^{-\tau s^2}\phi(s) f_\tau(x;s)ds,$$

where $f_\tau(x;s) = \square_{\widehat{g}}(H(s-x^0)u_\tau(x))$, $\phi \in C_0^\infty(\mathbb{R})$ has value 1 in some neighborhood of zero, and $H(s)$ is the Heaviside function, and writing

$$e^{-i\tau p(x)}F_\tau^{(series)}(x) = e^{-i\tau p(x)}F^{(1)}(x)\tau^{-1} + e^{-i\tau p(x)}F^{(2)}(x)\tau^{-2} + O(\tau^{-3}).$$

Then, F_τ can be defined by $F_\tau = F^{(1)}(x)\tau^{-1}$. Indeed, we see that

$$\begin{aligned} \square_{\widehat{g}}^{-1} F_\tau^{(series)} &= \tau^{-1}\Phi_\tau(x)\, u_\tau(x), \\ \Phi_\tau(x) &= c\tau^{1/2}\int_{\mathbb{R}} e^{-\tau s^2}\phi(s)H(s-x^0)ds \\ &= \begin{cases} O(\tau^{-N}), & \text{for } x_0 < 0, \\ 1 + O(\tau^{-N}), & \text{for } x_0 > 0. \end{cases} \end{aligned}$$

Moreover, by writing $\square_{\widehat{g}} = \widehat{g}^{jk}\partial_j\partial_k + \widehat{\Gamma}^j\partial_j$, $\widehat{\Gamma}^j = \widehat{g}^{pq}\widehat{\Gamma}^j_{pq}$ we see that

$$\begin{aligned} f_\tau(x;s) &= \widehat{g}^{00}u_\tau(x)\delta'(s-x^0) \\ &\quad -2\sum_{j=1}^3 \widehat{g}^{j0}(\partial_j u_\tau(x))\,\delta(s-x^0) - \widehat{\Gamma}^0 u_\tau(x)\,\delta(x^0-s). \end{aligned}$$

Let us use normal coordinates centered at the point y so that $\Gamma^j(0)=0$ and $\widehat g^{jk}(0)$ is the Minkowski metric. Then we define $p(x)=(x^0)^2+\varphi(x)$ and see that the leading order term of $F_\tau^{(series)}$ is given by

$$F_\tau=c\tau^{-1/2}e^{-\tau p(x)}\left(2x^0\phi(x^0)U_0(x)+2i(\partial_0\varphi(x))U_0(x)\phi(x^0)\right),$$

where $\partial_0\varphi|_y$ does not vanish.

3.6.4. *Indicator function for singularities produced by interactions.* Let $y\in U$ and $\eta\in T_{x_0}M$ be a future pointing light-like vector. We will next make a test to see if $(y,\eta^\flat)\in \mathrm{WF}\,(\mathcal{M}^{(\ell)})$ with $\ell\le 4$.

Using the functions $\mathcal{M}^{(\ell)}$ defined in (62) with the pieces of plane waves $u_j\in\mathcal{I}(\Lambda(x_j,\xi_j;t_0,s_0))$, $j\le 4$, and the source F_τ in (72), we define indicator functions

$$\Theta_\tau^{(\ell)}=\langle F_\tau,\mathcal{M}^{(\ell)}\rangle_{L^2(U)},\quad \ell=1,2,3,4, \tag{75}$$

We can write $\Theta_\tau^{(\ell)}$ is a sum of terms $T_{\tau,\sigma}^{(\ell),\beta}$ and $\widetilde T_{\tau,\sigma}^{(\ell),\beta}$, where $\beta\in\mathbb{Z}_+$ are just numbers indexing terms, and $\sigma:\{1,2,\dots,\ell\}\to\{1,2,\dots,\ell\}$ is in the set of permutations of ℓ indexes,

$$\Theta_\tau^{(\ell)}=\sum_{\beta\in J_\ell}\sum_{\sigma\in\Sigma(\ell)}(T_{\tau,\sigma}^{(\ell),\beta}+\widetilde T_{\tau,\sigma}^{(\ell),\beta}).$$

To define the terms $T_{\tau,\sigma}^{(\ell),\beta}$ and $\widetilde T_{\tau,\sigma}^{(\ell),\beta}$ appearing above, we use generic notations where we drop the index β, that is, we denote $\mathcal{B}_j=\mathcal{B}_j^\beta$ and $S_j=S_j^\beta$. Then $T_{\tau,\sigma}^{(2),\beta}$ are terms of the form

$$\begin{aligned} T_{\tau,\sigma}^{(2),\beta} &= \langle F_\tau,\mathbf{Q}(\mathcal{B}_2u_{\sigma(2)}\cdot\mathcal{B}_1u_{\sigma(1)})\rangle_{L^2(M_0)}\\ &= \langle \mathbf{Q}^*F_\tau,\mathcal{B}_2u_{\sigma(2)}\cdot\mathcal{B}_1u_{\sigma(1)}\rangle_{L^2(M_0)}\\ &= \langle u_\tau,\mathcal{B}_2u_{\sigma(2)}\cdot\mathcal{B}_1u_{\sigma(1)}\rangle_{L^2(M_0)}, \end{aligned} \tag{76}$$

and $\widetilde T_{\tau,\sigma}^{(2),\beta}=0$. Moreover, $T_\tau^{(3),\beta}$ are of the form

$$\begin{aligned} T_{\tau,\sigma}^{(3),\beta} &= \langle F_\tau,\mathbf{Q}(\mathcal{B}_3u_{\sigma(3)}\cdot\mathcal{C}_1S_1(\mathcal{B}_2u_{\sigma(2)}\cdot\mathcal{B}_1u_{\sigma(1)}))\rangle_{L^2(M_0)}\\ &= \langle u_\tau,\mathcal{B}_3u_{\sigma(3)}\cdot\mathcal{C}_1S_1(\mathcal{B}_2u_{\sigma(2)}\cdot\mathcal{B}_1u_{\sigma(1)})\rangle_{L^2(M_0)}, \end{aligned} \tag{77}$$

and $\widetilde T_{\tau,\sigma}^{(3),\beta}=0$. When we consider the 4th order interaction terms $T_{\tau,\sigma}^{(4),\beta}$ we change our notations by commuting S_j and $\mathcal{C}_j$ and using Leibniz rule. For instance, if $\mathcal{C}_1$ is a first order operator, we write

$$\begin{aligned} \mathcal{C}_1^\beta S_1^\beta(\mathcal{B}_2^\beta u_2\cdot\mathcal{B}_1^\beta u_1) &= [\mathcal{C}_1^\beta,S_1^\beta](\mathcal{B}_2^\beta u_2\cdot\mathcal{B}_1^\beta u_1)+\\ &\quad +S_1((\mathcal{C}_1^\beta\mathcal{B}_2^\beta u_2)\cdot\mathcal{B}_1^\beta u_1)+S_1(\mathcal{B}_2^\beta u_2\cdot((\mathcal{C}_1^\beta\mathcal{B}_1^\beta u_1)). \end{aligned}$$

Using this we can eliminate the operators $\mathcal{C}_j^\beta$ and increase the order of the differential operators $\mathcal{B}_j^\beta$ and allow S_j^β also be a commutator of $\mathbf{Q}$

and $\mathcal{C}_j^\beta$, that is, below we allow $\mathcal{B}_j^\beta$ and S_j^β to be of the form

$$\begin{aligned} &\mathcal{B}_j^\beta : (v_p)_{p=1}^{10+L} \mapsto (b_p^{r,(j,\beta)}(x)\partial_x^{\vec{k}(\beta,j)} v_r(x))_{p=1}^{10+L}, \text{ and} \\ &S_j^\beta = \mathbf{Q} \text{ or } S_j^\beta = I, \text{ or } S_j^\beta = [\mathbf{Q}, a(x)D^\alpha], \end{aligned} \tag{78}$$

where $|\vec{k}(\beta)| = \sum_{j=1}^4 k(\beta,j) \leq 6$. To simplify notations, we also enumerate again the obtained terms with the indexes β, that is, we consider β as an index running over a non-specified finite set. To simplify notations, we sometimes drop the super-index β below. With these notations,

$$\begin{aligned} T_{\tau,\sigma}^{(4),\beta} &= \langle F_\tau, \mathbf{Q}(\mathcal{B}_4 u_{\sigma(4)} \cdot S_2(\mathcal{B}_3 u_{\sigma(3)} \cdot S_1(\mathcal{B}_2 u_{\sigma(2)} \cdot \mathcal{B}_1 u_{\sigma(1)})))\rangle_{L^2(M_0)} \\ &= \langle \mathbf{Q}^* F_\tau, \mathcal{B}_4 u_{\sigma(4)} \cdot S_2(\mathcal{B}_3 u_{\sigma(3)} \cdot S_1(\mathcal{B}_2 u_{\sigma(2)} \cdot \mathcal{B}_1 u_{\sigma(1)}))\rangle_{L^2(M_0)} \\ &= \langle (\mathcal{B}_4 u_{\sigma(4)}) \cdot u_\tau, S_2(\mathcal{B}_3 u_{\sigma(3)} \cdot S_1(\mathcal{B}_2 u_{\sigma(2)} \cdot \mathcal{B}_1 u_{\sigma(1)}))\rangle_{L^2(M_0)} \\ &= \langle (\mathcal{B}_3 u_{\sigma(3)}) \cdot S_2^*((\mathcal{B}_4 u_{\sigma(4)}) \cdot u^\tau), S_1(\mathcal{B}_2 u_{\sigma(2)} \cdot \mathcal{B}_1 u_{\sigma(1)})\rangle_{L^2(M_0)} \end{aligned} \tag{79}$$

and

$$\begin{aligned} \widetilde{T}_{\tau,\sigma}^{(4),\beta} &= \langle F_\tau, \mathbf{Q}(S_2(\mathcal{B}_4 u_{\sigma(4)} \cdot \mathcal{B}_3 u_{\sigma(3)}) \cdot S_1(\mathcal{B}_2 u_{\sigma(2)} \cdot \mathcal{B}_1 u_{\sigma(1)}))\rangle_{L^2(M_0)} \\ &= \langle \mathbf{Q}^* F_\tau, S_2(\mathcal{B}_4 u_{\sigma(4)} \cdot \mathcal{B}_3 u_{\sigma(3)}) \cdot S_1(\mathcal{B}_2 u_{\sigma(2)} \cdot \mathcal{B}_1 u_{\sigma(1)})\rangle_{L^2(M_0)} \\ &= \langle S_2(\mathcal{B}_4 u_{\sigma(4)} \cdot \mathcal{B}_3 u_{\sigma(3)}) \cdot u_\tau, S_1(\mathcal{B}_2 u_{\sigma(2)} \cdot \mathcal{B}_1 u_{\sigma(1)})\rangle_{L^2(M_0)}. \end{aligned} \tag{80}$$

When σ is the identity, we will omit it in our notations and denote $T_\tau^{(4),\beta} = T_{\tau,id}^{(4),\beta}$, etc.

In particular, the term $\langle F_\tau, \mathbf{Q}(A_2[u_4, \mathbf{Q}(A_2[u_3, \mathbf{Q}(A_2[u_2, u_1])])])\rangle$, where we recall that $A_2[v,w] = \widehat{g}^{np}\widehat{g}^{mq} v_{nm}\partial_p\partial_q w_{jk}$, can be written as a sum of terms of the type $T_{\tau,\sigma}^{(4),\beta}$, we obtain one term that will later cause the leading order asymptotics. This term corresponds to $\sigma = Id$ and the indexes $k_1 = 6$, $k_2 = k_3 = k_2 = 0$, and we enumerate this term to correspond $\beta = \beta_1 := 1$ (we define these indexes used later)

$$\vec{S}_{\beta_1} = (\mathbf{Q}, \mathbf{Q}) \text{ and } k_1^{\beta_1} = 6,\ k_2^{\beta_1} = k_3^{\beta_1} = k_2^{\beta_1} = 0. \tag{81}$$

3.6.5. *Properties of indicator functions on a general manifold.* To consider the properties of the indicator functions related to the sources $f_j \in \mathcal{I}(Y(x_j,\xi_j;t_0,s_0))$, considered as sections of the bundle $\mathcal{B}^K$, where $(x_j,\xi_j) \in L^+M_0$, $x_j \in \widehat{U}$, $j \leq 4$, and the source F_τ determined by $(y,\eta) \in L^+M_0$, $y \in \widehat{U}$ we denote in the following $(x_5,\xi_5) = (y,\eta)$ and continue to use the notation $(\vec{x},\vec{\xi}) = ((x_j,\xi_j))_{j=1}^4$.

Definition 3.3. *We say that the geodesics corresponding to $(\vec{x},\vec{\xi}) = ((x_j,\xi_j))_{j=1}^4$ intersect and the intersection takes place at the point q if there is $q \in M_0$ such that for all $j = 1,2,3,4$, there are $t_j \in (0,\mathbf{t}_j)$, $\mathbf{t}_j = \rho(x_j,\xi_j)$ such that $q = \gamma_{x_j,\xi_j}(t_j)$. In this case that such q exists and is of the form $q = \gamma_{x_5,\xi_5}(t_5)$, $t_5 < 0$, we say that (x_5,ξ_5) comes from the 4-intersection of rays corresponding to $(\vec{x},\vec{\xi}) = ((x_j,\xi_j))_{j=1}^4$.*

Also, we say that q is the intersection point corresponding to $(\vec{x},\vec{\xi})$ and (x_5,ξ_5).

Let Λ_q^+ be the lagrangian manifold

$$\Lambda_q^+ = \{(x,\xi)\in T^*M_0;\ x=\gamma_{q,\zeta}(t),\ \xi^\sharp=\dot\gamma_{q,\zeta}(t),\ \zeta\in L_q^+M_0,\ t>0\}$$

Note that the projection of Λ_q^+ on M_0 is the light cone $\mathcal{L}^+_{\widehat g}(q)\setminus\{q\}$.

Below we say that a function $F(z)$ is a meromorphic function of the variables $z\in\mathbb{R}^m$ if $F(z)=P(z)/Q(z)$ where $P(z)$ and $Q(z)$ are real-analytic functions and $Q(z)$ does not vanish identically. Next we consider $(\vec{x},\vec{\xi})=((x_j,\xi_j))_{j=1}^4$ and $\vartheta_0,t_0>0$ that satisfy (see Fig. 13)

$$\begin{aligned}
(82)\quad &(i)\ x_j\in\widehat U,\ \xi_j\in L^+_{x_j}M_0,\ x_j\not\in J^+_{\widehat g}(x_k),\ \text{for } j,k\le 4,\ j\neq k,\\
&(ii)\ \text{For all } j,k\le 4,\ d_{\widehat g^+}((x_j,\xi_j),(x_k,\xi_k))<\vartheta_0,\\
&(iii)\ \text{There is } y\in\widehat\mu \text{ such that for all } j\le 4,\ d_{\widehat g^+}(y,x_j)<\vartheta_0,\\
&(iv)\ \text{For } j,k\le 4,\ j\neq k, \text{ we have } x_j(t_0)\not\in\gamma_{x_k,\xi_k}(\mathbb{R}_+).
\end{aligned}$$

Above, $(x_j(h),\xi_j(h))$ are defined in (69). We also consider a point $x_6\in U_{\widehat g}$ that satisfies for $(\vec{x},\vec{\xi})$ and t_0 satisfies the condition

$$\begin{aligned}
(83)\qquad &\text{For all } j\le 4,\ x_j^{cut}=\gamma_{x_j(t_0),\xi_j(t_0)}(\mathbf{t}_j)\not\in J^-_{\widehat g}(x_6),\\
&\text{where } \mathbf{t}_j:=\rho(x_j(t_0),\xi_j(t_0)).\\
&\text{Then, we denote } \mathcal{V}((\vec{x},\vec{\xi}),t_0)=M_0\setminus\bigcup_{j=1}^4 J^+_{\widehat g}(\gamma_{x_j(t_0),\xi_j(t_0)}(\mathbf{t}_j)).
\end{aligned}$$

Observe that $\mathcal{V}((\vec{x},\vec{\xi}),t_0)$ is an open neighborhood of $J^-_{\widehat g}(x_6)$.

The condition (83) implies that when we consider the past of x_6, then no geodesics $\gamma_{x_j(t_0),\xi_j(t_0)}$ has conjugate or cut points. Note that two such geodesics $\gamma_{x_j(t_0),\xi_j(t_0)}([0,\infty))$ can intersect only once in $\mathcal{V}((\vec{x},\vec{\xi}),t_0)$.

In the proposition below we will consider four spherical waves $u_j\in\mathcal{I}^{n-1/2}(\Lambda(x_j,\xi_j;t_0,s_0))$, $j=1,2,3,4$, where the order n is low enough, that is $n\le -n_1$. A suitable value of n_1 is $n_1=12$, and the reason for this is that we need to consider fourth order interaction terms and each interaction involves two derivativs.

Proposition 3.4. *Let $(\vec{x},\vec{\xi})=((x_j,\xi_j))_{j=1}^4$ be future pointing light-like vectors and $x_6\in U_{\widehat g}$ satisfing (82)-(83). Let $x_5\in\mathcal{V}((\vec{x},\vec{\xi}),t_0)\cap U_{\widehat g}$ and (x_5,ξ_5) be a future pointing light-like vector such that $x_5\not\in\gamma_{x_j,\xi_j}(\mathbb{R})$ for $j\le 4$, $t_0>0$, and $x_5\not\in\mathcal{Y}=\mathcal{Y}(((x_j,\xi_j))_{j=1}^4;t_0)$, see (55) and (70).*

There exists $n_1\in\mathbb{Z}_+$ such that the following holds: When $s_0>0$ is small enough, the function $\Theta^{(\ell)}_\tau$, see (75), corresponding to the linear waves $u_j\in\mathcal{I}^{n-1/2}(\Lambda(x_j,\xi_j;t_0,s_0))$, $j\le 4$, defined in Lemma 3.1 with $n\le -n_1$, and the source F_τ satisfies the following:

(i) If (x_5,ξ_5) does not come from the 4-intersection of rays corresponding to $(\vec{x},\vec{\xi})$, we have $|\Theta^{(4)}_\tau|\le C_N\tau^{-N}$ for all $N>0$.

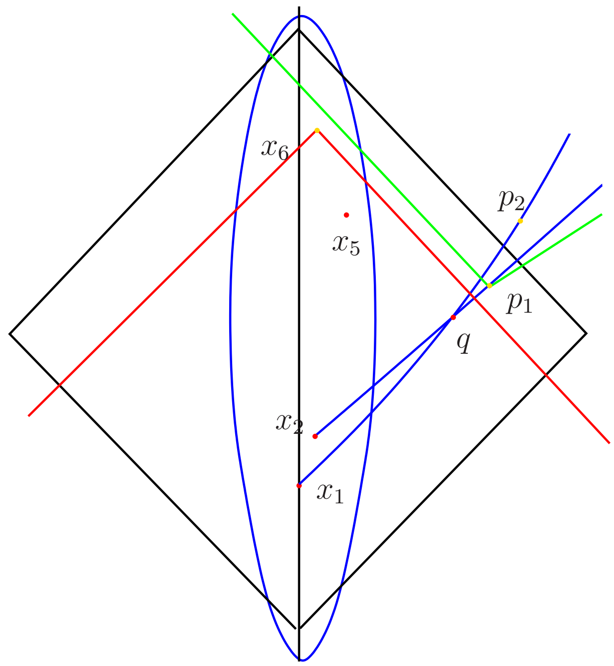

FIGURE 13. A schematic figure where the space-time is represented as the 2-dimensional set $\mathbb{R}^{1+1}$. The figure shows the configuration in formulas (82) and (83). The points x_1 and x_2, marked with red dots, are the points from where we send light-like geodesics $\gamma_{x_j,\xi_j}([t_0,\infty))$, $j=1,2$. The geodesics $\gamma_{x_j,\xi_j}([t_0,\infty))$, $j=3,4$ are not in the figure. We assume the these geodesics intersect at point q that is shown as a red point. The geodesics $\gamma_{x_j,\xi_j}([t_0,\infty))$ have cut points and the first cut points p_j are shown as golden points. The point $x_6\in U_{\widehat g}$ is such that no cut points p_j of geodesics are in the causal past $J^-(x_6)$ of the point x_6 shown with red lines. Observations are done at the point $x_5\in J^-(x_6)\cap U_{\widehat g}$. The set $J^-(x_6)$ has a neighborhood $\mathcal{V}=\mathcal{V}((\vec x,\vec\xi),t_0)$ that is the complement of the set $\bigcup_{j=1}^4 J^+(p_j)$. The boundary of $\mathcal{V}$ is shown with green lines. The the black "diamond" is again the set $J_{\widehat g}(\widehat p^-,\widehat p^+)$ and the black line is the geodesic $\widehat\mu$. The domain inside the blue curve is the set $U_{\widehat g}$ where the sources are supported and the observations are done.

(ii) if (x_5,ξ_5) *comes from the 4-intersection of rays corresponding to* $(\vec x,\vec\xi)$ *and* q *is the corresponding intersection point,* $q=\gamma_{x_j,\xi_j}(t_j)$*, then*

$$\Theta^{(4)}_\tau \sim \sum_{k=m}^{\infty} s_k\tau^{-k} \tag{84}$$

as $\tau\to\infty$ where $m=-4n+2$. Here we use $\sim$ to denote that the terms have the same asymptotics up an error $O(\tau^{-N})$ for all $N>0$.

Moreover, let $b_j=(\dot\gamma_{x_j,\xi_j}(t_j))^\flat$ and $\mathbf{b}=(b_j)_{j=1}^5\in(T_q^*M_0)^5$, w_j be the principal symbols of the waves u_j at (q,b_j), and $\mathbf{w}=(w_j)_{j=1}^5$. Then there is a real-analytic function $\mathcal{G}(\mathbf{b},\mathbf{w})$ such that the leading order term in (84) satisfies

$$s_m=\mathcal{G}(\mathbf{b},\mathbf{w}). \tag{85}$$

Proof. Below, to simplify notations, we denote $K_j=K(x_j,\xi_j;t_0,s_0)$ and $K_{123}=K_1\cap K_2\cap K_3$ and $K_{124}=K_1\cap K_2\cap K_4$, etc. We will denote $\Lambda_j=\Lambda(x_j,\xi:j;t_0,s_0)$ to consider also the singularities of K_j related to conjugate points.

Below we will consider separately the case when the following linear independency condition,

(LI) Assume if that $J\subset\{1,2,3,4\}$ and $y\in J^-(x_6)$ are such that for all $j\in J$ we have $\gamma_{x_j,\xi_j}(t_j')=y$ with some $t_j'\geq 0$, then the vectors $\dot\gamma_{x_j,\xi_j}(t_j')$, $j\in J$ are linearly independent.

is valid and the case when (LI) is not valid.

Let us first consider the case when (LI) is valid.

By the definition of $\mathbf{t}_j$, if the intersection $\gamma_{x_5,\xi_5}(\mathbb{R}_-)\cap(\cap_{j=1}^4\gamma_{x_j,\xi_j}((0,\mathbf{t}_j)))$ is non-empty, it can contain only one point. In the case that such a point exists, we denote it by q. When q exists, the intersection of K_j at this point are transversal and we see that when s_0 is small enough, the set $\cap_{j=1}^4K_j$ consists only of the point q. Next we consider so small s_0 that this is true.

We consider two local coordinates $Z:W_0\to\mathbb{R}^4$ and $Y:W_1\to\mathbb{R}^4$ such that $W_0,W_1\subset\mathcal{V}((\vec x,\vec\xi),t_0)$, see definition after (83). We assume that local coordinates are such that $K_j\cap W_0=\{x\in W_0;\ Z^j(x)=0\}$ and $K_j\cap W_1=\{x\in W_1;\ Y^j(x)=0\}$ for $j=1,2,3,4$. In the Fig. 14, W_0 is a neighborhood of z and W_1 is the neighborhood of y. We note that the origin $0=(0,0,0,0)\in\mathbb{R}^4$ does not necessarily belong to the set $Z(W_0)$ or the set $Y(W_1)$, for instance in the case when the four geodesic γ_{x_j,ξ_j} do not intersect. However, this is the case when the four geodesic intersect at the point q, we need to consider the case when W_0 and W_1 are neighborhoods of 0. Note that W_0 and W_1 do not contain any cut points of the geodesic $\gamma_{x_j,\xi_j}([t_0,\infty)$.

We will denote $z^j=Z^j(x)$. We assume that $Y:W_1\to\mathbb{R}^4$ are similar coordinates and denote $y^j=Y^j(x)$. We also denote below $dy^j=dY^j$ and $dz^j=dZ^j$. Let $\Phi_0\in C_0^\infty(W_0)$ and $\Phi_1\in C_0^\infty(W_1)$.

Let us next considering the map $\mathbf{Q}^*:C_0^\infty(W_1)\to C^\infty(W_0)$. By [65], $\mathbf{Q}^*\in I(W_1\times W_0;\Delta'_{TM_0},\Lambda_{\widehat g})$ is an operator with a classical symbol and its canonical relation $\Lambda'_{\mathbf{Q}^*}$ is associated to a union of two intersecting lagrangian manifolds, $\Lambda'_{\mathbf{Q}^*}=\Lambda'_{\widehat g}\cup\Delta_{TM_0}$, intersecting cleanly [65]. Let $\varepsilon_2>\varepsilon_1>0$ and $B_{\varepsilon_1,\varepsilon_2}$ be a pseudodifferential operator on

M_0 which is microlocally smoothing operator (i.e., the full symbol vanishes in local coordinates) outside in the ε_2-neighborhood $\mathcal{V}_2 \subset T^*M_0$ of the set of the light like covectors L^*M_0 and for which $(I - B_{\varepsilon_1,\varepsilon_2})$ is microlocally smoothing operator in the ε_1-neighborhood $\mathcal{V}_2$ of L^*M_0. The neighborhoods here are defined with respect to the Sasaki metric of $(T^*M_0, \widehat{g}^+)$ and $\varepsilon_2, \varepsilon_1$ are chosen later in the proof. Let us decompose the operator $\mathbf{Q}^* = \mathbf{Q}_1^* + \mathbf{Q}_2^*$ where $\mathbf{Q}_1^* = \mathbf{Q}^*(I - B_{\varepsilon_1,\varepsilon_2})$ and $\mathbf{Q}_2^* = \mathbf{Q}^* B_{\varepsilon_1,\varepsilon_2}$. As $\Lambda_{\mathbf{Q}^*} = \Lambda_{\widehat{g}} \cup \Delta'_{TM_0}$, we see that then there is a neighborhood $\mathcal{W}_2 = \mathcal{W}_2(\varepsilon_2)$ of $L^*M_0 \times L^*M_0 \subset (T^*M_0)^2$ such that the Schwartz kernel $\mathbf{Q}_2^*(r, y)$ of the operator $\mathbf{Q}_2^*$ satisfies

$$\mathrm{WF}(\mathbf{Q}_2^*) \subset \mathcal{W}_2. \tag{86}$$

Moreover, $\Lambda_{\mathbf{Q}_1^*} \subset \Delta'_{TM_0}$ implying that $\mathbf{Q}_1^*$ is a pseudodifferential operator with a classical symbol, $\mathbf{Q}_1^* \in I(W_1 \times W_0; \Delta'_{TM_0})$, and $\mathbf{Q}_2^* \in I(W_1 \times W_0; \Delta'_{TM_0}, \Lambda_{\widehat{g}})$ is a Fourier integral operator (FIO) associated to two cleanly intersecting lagrangian manifolds, similarly to $\mathbf{Q}^*$.

In the case when $p = 1$ we can write $\mathbf{Q}_p^*$ as

$$(\mathbf{Q}_1^* v)(z) = \int_{\mathbb{R}^{4+4}} e^{i\psi_1(z,y,\xi)} q_1(z, y, \xi) v(y)\, dy d\xi, \tag{87}$$

where

$$\psi_1(z, y, \xi) = (y - z) \cdot \xi, \quad \text{for } (z, y, \xi) \in W_1 \times W_0 \times \mathbb{R}^4, \tag{88}$$

and a classical symbol $q_1(z, y, \xi) \in S^{-2}(W_1 \times W_0; \mathbb{R}^4)$, having a real valued principal symbol

$$\widetilde{q}_1(z, y, \xi) = \frac{\chi(z, \xi)}{g_z(\xi, \xi)}$$

where $\chi(z, \xi) \in \mathbb{C}^\infty$ is a cut-off function vanishing in a neighborhood of the set where $g_z(\xi, \xi) = 0$. Note that then $Q_1 - Q_1^* \in \Psi^{-3}(W_1 \times W_0)$.

Furthermore, let us decompose $\mathbf{Q}_1^* = \mathbf{Q}_{1,1}^* + \mathbf{Q}_{1,2}^*$ corresponding to the decomposition $q_1(z, y, \xi) = q_{1,1}(z, y, \xi) + q_{1,2}(z, y, \xi)$ of the symbol, where

$$q_{1,1}(z, y, \xi) = q_1(z, y, \xi)\psi_R(\xi), \quad q_{1,2}(z, y, \xi) = q_1(z, y, \xi)(1 - \psi_R(\xi)), \tag{89}$$

where $\psi_R \in C_0^\infty(\mathbb{R}^4)$ is a cut-off function that is equal to one in a ball $B(R)$ of radius R specified below.

Next we start to consider the terms $T_\tau^{(4),\beta}$ and $\widetilde{T}_\tau^{(4),\beta}$ of the type (79) and (80). In these terms, we can represent the gaussian beam $u_\tau(z)$ in W_1 in the form

$$u_\tau(y) = e^{i\tau\varphi(y)} a_5(y, \tau) \tag{90}$$

where the function φ is a complex phase function having non-negative imaginary part such that $\operatorname{Im}\varphi$, defined on W_1, vanishes exactly on the geodesic $\gamma_{x_5,\xi_5} \cap W_1$. Note that $\gamma_{x_5,\xi_5} \cap W_1$ may be empty. Moreover, $a_5 \in S^0_{clas}(W_1; \mathbb{R})$ is a classical symbol.

Also, for $y=\gamma_{x_5,\xi_5}(t)\in W_1$ we have that $d\varphi(y)=c\dot\gamma_{x_5,\xi_5}(t)^\flat$, with $c\in\mathbb{R}\setminus\{0\}$, is light-like.

We consider first the asymptotics of terms $T_\tau^{(4),\beta}$ and $\widetilde T_\tau^{(4),\beta}$ of the type (79) and (80) where $S_1=S_2=\mathbf{Q}$ and, symbols $a_j(z,\theta_j)$, $j\leq 3$ and $a_j(y,\theta_j)$, $j\in\{4,5\}$ are scalar valued symbols written in the Z and Y coordinates, $\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3$ are multiplication operators with Φ_0, and $\mathcal{B}_4$ is a multiplication operators with Φ_1 and consider section-valued symbols and general operators later.

Let us consider functions $U_j\in\mathcal{I}^{p_j}(K_j)$, $j=1,2,3$, supported in W_0 and $U_4\in\mathcal{I}(K_4)$, supported in W_1, having classical symbols,

$$U_j(x)=\int_{\mathbb{R}}e^{i\theta_jx^j}a_j(x,\theta_j)d\theta_j,\quad a_j\in S^{p_j}_{clas}(W_{k(j)};\mathbb{R}),\tag{91}$$

for all $j=1,2,3,4$ (Note that here the phase function is $\theta_jx^j=\theta_1x^1$ for $j=1$ etc, that is, there is no summing over index j). We may assume that $a_j(x,\theta_j)$ vanish near $\theta_j=0$.

As $x_5\in\mathcal{V}((\vec x,\vec\xi),t_0)\cap U_{\widehat g}$ and $W_0,W_1\subset\mathcal{V}((\vec x,\vec\xi),t_0)$, we see that an example of functions (91) are $U_j(z)=\Phi_{k(j)}(z)u_j(z)$, $j=1,2,3,4$, where $u_j(z)$ are the pieces of the spherical waves. Here and below, $k(j)=0$ for $j=1,2,3$ and $k(4)=1$ and we also denote $k(5)=1$. Note that $p_j=n$ correspond to the case when $U_j\in\mathcal{I}^n(K_j)=\mathcal{I}^{n-1/2}(N^*K_j)$.

Denote $\Lambda_j=N^*K_j$ and $\Lambda_{jk}=N^*(K_j\cap K_k)$. By [28, Lem. 1.2 and 1.3], the pointwise product $U_2\cdot U_1\in\mathcal{I}(\Lambda_1,\Lambda_{12})+\mathcal{I}(\Lambda_2,\Lambda_{12})$ and thus by [28, Prop. 2.2], $\mathbf{Q}(U_2\cdot U_1)\in\mathcal{I}(\Lambda_1,\Lambda_{12})+\mathcal{I}(\Lambda_2,\Lambda_{12})$ and it can be written as

$$\mathbf{Q}(U_2\cdot U_1)=\int_{\mathbb{R}^2}e^{i(\theta_1z^1+\theta_2z^2)}c_1(z,\theta_1,\theta_2)d\theta_1d\theta_2.\tag{92}$$

Note that here $c_1(z,\theta_1,\theta_2)$ is sum of product type symbols, see (26). As $N^*(K_1\cap K_2)\setminus(N^*K_1\cup N^*K_2)$ consist of vectors which are non-characteristic for the wave operator, that is, the wave operator is elliptic in a neighborhood of this subset of the cotangent bundle, the principal symbol $\widetilde c_1$ of c_1 on $N^*(K_1\cap K_2)\setminus(N^*K_1\cup N^*K_2)$ is given by

$$\begin{aligned}&\widetilde c_1(z,\theta_1,\theta_2)\sim s(z,\theta_1,\theta_2)a_1(z,\theta_1)a_2(z,\theta_2),\\&s(z,\theta_1,\theta_2)=1/g(\theta_1b^{(1)}+\theta_2b^{(2)},\theta_1b^{(1)}+\theta_2b^{(2)})=1/(2g(\theta_1b^{(1)},\theta_2b^{(2)})).\end{aligned}\tag{93}$$

Note that $s(z,\theta_1,\theta_2)$ is a smooth function on $N^*(K_1\cap K_2)\setminus(N^*K_1\cup N^*K_2)$ and homogeneous of order (-2) in $\theta=(\theta_1,\theta_2)$. Here, we use $\sim$ to denote that the symbols have the same principal symbol. Let us next make computations in the case when $a_j(z,\theta_j)\in C^\infty(\mathbb{R}^4\times\mathbb{R})$ is positively homogeneous for $|\theta_j|>1$, that is, we have $a_j(z,s)=a_j'(z)s^{p_j}$, where $p_j\in\mathbb{N}$ and $|s|>1$. We consider $T_\tau^{(4),\beta}=\sum_{p=1}^2T_{\tau,p}^{(4),\beta}$ where $T_{\tau,p}^{(4),\beta}$ is defined as $T_\tau^{(4),\beta}$ by replacing the term $\mathbf{Q}^*(U_4\cdot u_\tau)$ by $\mathbf{Q}_p^*(U_4\cdot u_\tau)$.

Let us now consider the case $p=2$ and choose the parameters ε_1 and ε_2 that determine the decomposition $\mathbf{Q}^*=\mathbf{Q}_1^*+\mathbf{Q}_2^*$. First, we observe that for $p = 2$ we can write using Z and Y coordinates

$$
\begin{aligned}
(94)\qquad T^{(4),\beta}_{\tau,2} &= \tau^4 \int_{\mathbb{R}^{12}} e^{i\tau\Psi_2(z,y,\theta)} c_1(z,\tau\theta_1,\tau\theta_2)\cdot \\
&\quad \cdot a_3(z,\tau\theta_3)\mathbf{Q}^*_2(z,y) a_4(y,\tau\theta_4) a_5(y,\tau)\, d\theta_1 d\theta_2 d\theta_3 d\theta_4 dy dz, \\
\Psi_2(z,y,\theta) &= \theta_1 z^1 + \theta_2 z^2 + \theta_3 z^3 + \theta_4 y^4 + \varphi(y).
\end{aligned}
$$

Denote $\theta = (\theta_1,\theta_2,\theta_3,\theta_4) \in \mathbb{R}^4$. Consider the case when (z,y,θ) is a critical point of Ψ_2 satisfying $\operatorname{Im}\varphi(y) = 0$. Then we have $\theta' = (\theta_1,\theta_2,\theta_3) = 0$ and $z' = (z^1,z^2,z^3) = 0$, $y^4 = 0$, $d_y\varphi(y) = (0,0,0,-\theta_4)$, implying that $y \in K_4$ and $(y, d_y\varphi(y)) \in N^*K_4$. Since $\operatorname{Im}\varphi(y) = 0$, we have that $y = \gamma_{x_5,\xi_5}(t_0)$ with some $t_0 \in \mathbb{R}_-$. As we have $\dot\gamma_{x_5,\xi_5}(t_0)^\flat = d_y\varphi(y) \in N^*_y K_4$, we obtain $\gamma_{x_5,\xi_5}([t_0,0]) \subset K_4$. However, this is not possible by our assumption $x_5 \notin \cup_{j=1}^4 K(x_j,\xi_j;s_0)$ when s_0 is small enough. Thus the phase function $\Psi_2(z,y,\theta)$ has no critical points satisfying $\operatorname{Im}\varphi(y) = 0$.

When the orders p_j of the symbols a_j are small enough, the integrals in the θ variable in (94) are convergent in the classical sense. Next we use properties of wave front set to compute the asymptotics of an oscillatory integrals and to this end we introduce the function

$$
(95)\qquad \widetilde{\mathbf{Q}}^*_2(z,y,\theta) = \mathbf{Q}^*_2(z,y),
$$

that is, consider $\mathbf{Q}^*_2(z,y)$ as a constant function in θ. Below, denote $\psi_4(y,\theta_4) = \theta_4 y^4$ and $r = d\varphi(y)$. Note that then $d_{\theta_4}\psi_4 = y^4$ and $d_y\psi_4 = (0,0,0,\theta_4)$. Then in $W_1 \times W_0 \times \mathbb{R}^4$

$$
\begin{aligned}
d_{z,y,\theta}\Psi_2 &= (\theta_1,\theta_2,\theta_3,0; r + d_y\psi_4(y,\theta_4), z^1, z^2, z^3, d_{\theta_4}\psi_4(y,\theta_4)) \\
&= (\theta_1,\theta_2,\theta_3,0; d\varphi(y) + (0,0,0,\theta_4), z^1, z^2, z^3, y^4)
\end{aligned}
$$

and we see that if $((z,y,\theta), d_{z,y,\theta}\Psi_2) \in \mathrm{WF}(\widetilde{\mathbf{Q}}^*_2)$ and $\operatorname{Im}\varphi(y) = 0$, we have $(z^1,z^2,z^3) = 0$, $y^4 = d_{\theta_4}\psi_4(y,\theta_4) = 0$ and $y \in \gamma_{x_5,\xi_5}$. Thus $z \in K_{123}$ and $y \in \gamma_{x_5,\xi_5} \cap K_4$.

Let us use the following notations

$$
\begin{aligned}
(96)\qquad & z \in K_{123}, \quad \omega_\theta := (\theta_1,\theta_2,\theta_2,0) = \sum_{j=1}^3 \theta_j dz^j \in T^*_z M_0, \\
& y \in K_4 \cap \gamma_{x_5,\xi_5}, \quad (y,w) := (y, d_y\psi_4(y,\theta_4)) \in N^*K_4, \\
& r = d\varphi(y) = r_j dy^j \in T^*_y M_0, \quad \kappa := r + w.
\end{aligned}
$$

Then, y and θ_4 satisfy $y^4 = d_{\theta_4}\psi_4(y,\theta_4) = 0$ and $w = (0,0,0,\theta_4)$.

Note that by definition of the Y coordinates w is a light-like covector. By definition of the Z coordinates, $\omega_\theta \in N^*K_1 + N^*K_2 + N^*K_3 = N^*K_{123}$.

Let us first consider what happens if $\kappa = r + w = d\varphi(y) + (0,0,0,\theta_4)$ is light-like. In this case, all vectors κ, w, and r are light-like and satisfy $\kappa = r + w$. This is possible only if $r \parallel w$, i.e., r and w are parallel, see [81, Cor. 1.1.5]. Thus $r+w$ is light-like if and only if r and w are parallel.

Consider next the case when $(x,y,\theta)\in W_1\times W_0\times \mathbb{R}^4$ is such that $((x,y,\theta),d_{z,y,\theta}\Psi_2)\in \mathrm{WF}(\widetilde{\mathbf{Q}}_2^*)$ and $\mathrm{Im}\,\varphi(y)=0$. Using the above notations (96), we obtain $d_{z,y,\theta}\Psi_2=(\omega_\theta,r+w;(0,0,0,d_{\theta_4}\psi_4(y,\theta_4)))=(\omega_\theta,d\varphi(y)+(0,0,0,\theta_4);(0,0,0,y^4))$, where $y^4=d_{\theta_4}\psi_4(y,\theta_4)=0$, and thus we have

$$((z,\omega_\theta),(y,r+w))\in \mathrm{WF}(\mathbf{Q}_2^*)=\Lambda_{\mathbf{Q}_2^*}.$$

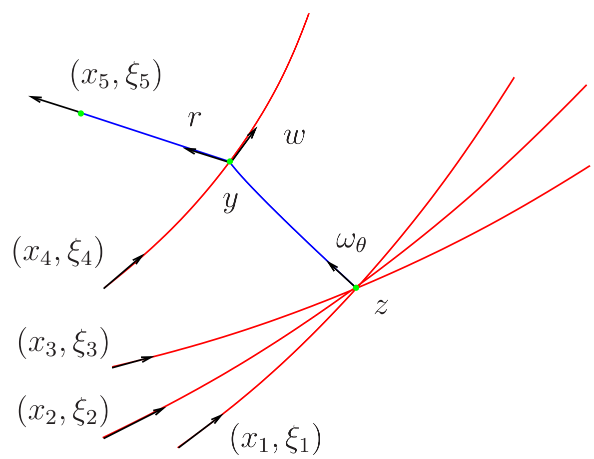

FIGURE 14. A schematic figure where the space-time is represented as the 3-dimensional set $\mathbb{R}^{1+1}$. In the figure we consider the case A1 where three geodesics intersect at z and the waves propagating near these geodesics interact and create a wave that hits the fourth geodesic at the point y, the produced singularities are detected by the gaussian beam source at the point x_5. Note that z and y can be conjugate points on the geodesic connecting them. In the case A2 the points y and z are the same.

As $\Lambda_{\mathbf{Q}_2^*}\subset \Lambda_{\widehat g}\cup\Delta'_{TM_0}$, this implies that one of the following conditions are valid:

(A1) $((z,\omega_\theta),(y,r+w))\in\Lambda_{\widehat g}$,

or

(A2) $((z,\omega_\theta),(y,r+w))\in\Delta'_{TM_0}$.

Let γ_0 be the geodesic with $\gamma_0(0)=z$, $\dot\gamma_0(0)=\omega_\theta^\sharp$. Then (A1) and (A2) are equivalent to the following conditions:

(A1) There is $t_0\in\mathbb{R}$ such that $(\gamma_0(t_0),\dot\gamma_0(t_0)^\flat)=(y,\kappa)$ and, the vector κ is light-like,

or

(A2) $z=y$ and $\kappa=-\omega_\theta$.

Consider next the case when (A1) is valid. As κ is light-like, r and w are parallel. Then, as $(\gamma_{x_5,\xi_5}(t_1),\dot\gamma_{x_5,\xi_5}(t_1)^\flat)=(y,r)$ we see that γ_0 is a continuation of the geodesic γ_{x_5,ξ_5}, that is, for some t_2 we have $(\gamma_{x_5,\xi_5}(t_2),\dot\gamma_{x_5,\xi_5}(t_2))=(z,\omega_\theta)\in N^*K_{123}$. This implies that $x_5\in\mathcal{Y}$ that is not possible by our assumptions. Hence (A1) is not possible.

Consider next the case when (A2) is valid. If we then would also have that $r\parallel w$ then r is parallel to $\kappa=-\omega_\theta\in N^*K_{123}$, and as $(\gamma_{x_5,\xi_5}(t_1),\dot\gamma_{x_5,\xi_5}(t_1)^\flat)=(y,r)$ we would have $x_5\in\mathcal{Y}$. As this is not possible by our assumptions, we see that r and w are not parallel. This implies that $\omega_\theta=-\kappa$ is not light-like.

For any given $(\vec x,\vec\xi)$ and (x_5,ξ_5) there exists $\varepsilon_2>0$ so that $(\{(y,\omega_\theta)\}\times T^*M)\cap\mathcal{W}_2(\varepsilon_2)=\emptyset$, see (86), and thus

$$(\{(y,\omega_\theta)\}\times T^*M)\cap \mathrm{WF}(\mathbf{Q}_2^*)'=\emptyset. \tag{97}$$

Next we assume that $\varepsilon_2>0$ and also $\varepsilon_1\in(0,\varepsilon_2)$ are chosen so that (97) is valid. Then there are no (z,y,θ) such that $((z,y,\theta),d\Psi_2(z,y,\theta))\in \mathrm{WF}(\mathbf{Q}_2^*)$ and $\mathrm{Im}\,\varphi(y)=0$. Thus by Corollary 1.4 in [20] or [75, Lem. 4.1] yields $T^{(4),\beta}_{\tau,2}=O(\tau^{-N})$ for all $N>0$. Alternatively, one can use the complex version of [22, Prop. 1.3.2], obtained using combining the proof of [22, Prop. 1.3.2] and the method of stationary phase with a complex phase, see [40, Thm. 7.7.17].

Thus to analyze the asymptotics of $T^{(4),\beta}_{\tau}$ we need to consider only $T^{(4),\beta}_{\tau,1}$. Next, we analyze the case when U_4 is a conormal distribution and has the form (91).

Let us thus consider the case $p=1$. Now

$$U_4(y)\cdot u_\tau(y) \;=\; \int_{\mathbb{R}^1} e^{i\theta_4 y^4+i\tau\varphi(y)}a_4(y,\theta_4)a_5(y,\tau)\,d\theta_4. \tag{98}$$

Denoting by $\psi_1(z,t,\xi)=(y-z)\cdot\xi$ the phase function of the pseudo-differential operator $\mathbf{Q}_1^*$ we obtain by (87)

$$\begin{aligned}(\mathbf{Q}_1^*(U_4\cdot u_\tau))(z) \;&=\; \int_{\mathbb{R}^9} e^{i(\psi_1(z,y,\xi)+\theta_4y^4+\tau\varphi(y))}q_1(z,y,\xi)\cdot\\ &\qquad\cdot a_4(y,\theta_4)a_5(y,\tau)\,d\theta_4\,dyd\xi.\end{aligned} \tag{99}$$

Then $T^{(4),\beta}_{\tau,1}=T^{(4),\beta}_{\tau,1,1}+T^{(4),\beta}_{\tau,1,2}$, cf. (89), where

$$\begin{aligned}T^{(4),\beta}_{\tau,1,k}&=\int_{\mathbb{R}^{16}} e^{i(\theta_1z^1+\theta_2z^2+\theta_3z^3+\psi_1(z,y,\xi)+\theta_4y^4+\tau\varphi(y))}c_1(z,\theta_1,\theta_2)\cdot\\ &\quad\cdot a_3(z,\theta_3)q_{1,k}(z,y,\xi)a_4(y,\theta_4)a_5(y,\tau)\,d\theta_1d\theta_2d\theta_3d\theta_4dydzd\xi,\end{aligned} \tag{100}$$

or

$$\begin{aligned}T^{(4),\beta}_{\tau,1,k}&=\tau^8\int_{\mathbb{R}^{16}} e^{i\tau(\theta_1z^1+\theta_2z^2+\theta_3z^3+\psi_1(z,y,\xi)+\theta_4y^4+\varphi(y))}c_1(z,\tau\theta_1,\tau\theta_2)\cdot\\ &\quad\cdot a_3(z,\tau\theta_3)q_{1,k}(z,y,\tau\xi)a_4(y,\tau\theta_4)a_5(y,\tau)\,d\theta_1d\theta_2d\theta_3d\theta_4dydzd\xi.\end{aligned} \tag{101}$$

Let (z,θ,y,ξ) be a critical point of the phase function

(102) $\Psi_3(z,\theta,y,\xi)=\theta_1 z^1+\theta_2 z^2+\theta_3 z^3+(y-z)\cdot\xi+\theta_4 y^4+\varphi(y).$

Then

$$\begin{aligned}
&\partial_{\theta_j}\Psi_3=0,\ j=1,2,3\quad\text{yield}\quad z\in K_{123},\\
&\partial_{\theta_4}\Psi_3=0\quad\text{yields}\quad y\in K_4,\\
&\partial_z\Psi_3=0\quad\text{yields}\quad \xi=\omega_\theta,\\
&\partial_\xi\Psi_3=0\quad\text{yields}\quad y=z,\\
&\partial_y\Psi_3=0\quad\text{yields}\quad \xi=-\partial_y\varphi(y)-w.
\end{aligned}\tag{103}$$

The critical points we need to consider for the asymptotics satisfy also

(104)

$$\operatorname{Im}\varphi(y)=0,\quad\text{so that } y\in\gamma_{x_5,\xi_5},\ \operatorname{Im}d\varphi(y)=0,\ \operatorname{Re}d\varphi(y)\in L^{*,+}_yM_0.$$

Next we analyze the terms $T^{(4),\beta}_{\tau,1,k}$ starting with $k=2$. Observe that the 3rd and 5th equations in (103) imply that at the critical points $\xi=(\partial_{y_1}\varphi(y),\partial_{y_2}\varphi(y),\partial_{y_3}\varphi(y),0)$. Thus the critical points are bounded in the ξ variable. Let us now fix the parameter R determining $\psi_R(\xi)$ in (89) so that ξ-components of the critical points are in a ball $B(R)\subset\mathbb{R}^4$. Using the identity $e^{\Psi_3}=|\xi|^{-2}(\nabla_z-\omega_\theta)^2e^{\Psi_3}$ where $\omega_\theta=(\theta_1,\theta_1,\theta_1,0)$ we can include the operator $|\xi|^{-2}(\nabla_z-\omega_\theta)^2$ in the integral (101) with $k=2$ and integrate by parts. Doing this two times we can show that this oscillatory integral (101) with $k=2$ becomes an integral of a Lebesgue-integrable function. Then, by using method of stationary phase and the fact that $\psi_R(\xi)$ vanishes at all critical points of Ψ_3 where $\operatorname{Im}\Psi_3$ vanishes, we see that $T^{(4),\beta}_{\tau,1,2}=O(\tau^{-n})$ for all $n>0$.

Above, we have shown that the term $T^{(4),\beta}_{\tau,1,1}$ has the same asymptotics as $T^{(4),\beta}_{\tau}$. Next we analyze this term. Let (z,θ,y,ξ) be a critical point of $\Psi_3(z,\theta,y,\xi)$ such that y satisfies (104). Let us next use the same notations (96) which we used above. Then (103) and (104) imply

$$z=y\in\gamma_{x_5,\xi_5}\cap\bigcap_{j=1}^4K_j,\ \xi=\omega_\theta=-r-w.\tag{105}$$

Note that in this case all the four geodesics γ_{x_j,ξ_j} intersect at the point q and by our assumptions, $r=d\varphi(y)$ is such a co-vector that in the Y-coordinates $r=(r_j)_{j=1}^4$ with $r_j\neq0$ for all $j=1,2,3,4$. In particular, this shows that the existence of the critical point of $\Psi_3(z,\theta,y,\xi)$ implies that there exists an intersection point of γ_{x_5,ξ_5} and $\bigcap_{j=1}^4K_j$. Equations (105) imply also that

$$r=\sum_{j=1}^4 r_j dy^j=-\omega_\theta-w=-\sum_{j=1}^3\theta_j dz^j-\theta_4dy^4.$$

To consider the case when $y = z$, let us assume for a while that that $W_0 = W_1$ and that the Y-coordinates and Z-coordinates coincide, that is, $Y(x) = Z(x)$. Then the covectors $dz^j = dZ^j$ and $dy^j = dY^j$ coincide for $j = 1, 2, 3, 4$. Then we have

$$r_j = -\theta_j, \quad \text{i.e., } \theta := \theta_j dz^j = -r = r_j dy^j \in T^*_y M_0. \tag{106}$$

Let us apply the method of stationary phase to $T^{(4),\beta}_{\tau,1,1}$ as $\tau \to \infty$. Note that as $c_1(z, \theta_1, \theta_2)$ is a product type symbol, we need to use the fact $\theta_1 \neq 0$ and $\theta_2 \neq 0$ for the critical points as we have by (106) and the fact that $r = d\varphi(y)|_y \notin N^*K_{234} \cup N^*K_{134}$ as $x_5 \notin \mathcal{Y}$ and assuming that s_0 is small enough.

In local Y and Z coordinates where $z = y = (0, 0, 0, 0)$ we can use the method of stationary phase, similarly to proofs of [33, Thm. 1.11 and 3.4], to compute the asymptotics of (101) with $k = 1$. Let us explain this computation in detail. To this end, let us start with some preparatory considerations.

Let $\phi_1(z, y, \theta, \xi)$ be a smooth bounded function in $C^\infty(W_0 \times W_1 \times \mathbb{R}^4 \times \mathbb{R}^4)$ that is homogeneous of degree zero in the (θ, ξ) variables in the set $\{|(\theta, \xi)| > R_0\}$ with some $R_0 > 0$. Assume that $\phi_1(z, y, \theta, \xi)$ is equal to one in a conic neighborhood, with respect to (θ, ξ), of the points where some of the θ_j or ξ_k variable is zero. Note that in this set the positively homogeneous functions $a_j(z, \theta_j/|\theta_j|)$ may be non-smooth. Let $\Sigma \subset W_0 \times W_1 \times \mathbb{R}^4 \times \mathbb{R}^4$ be a conic neighborhood of the critical points of the phase function $\Psi_3(y, z, \theta, \xi)$. Also, assume that the function $\phi_1(y, z, \theta, \xi)$ vanishes in the intersection of Σ and the set $\{|(\theta, \xi)| > R_0\}$. Let

$$\Psi_{(\tau)}(z, y, \theta, \xi) = \theta_1 z^1 + \theta_2 z^2 + \theta_3 z^3 + \psi_1(z, y, \xi) + \theta_4 y^4 + \tau\varphi(y)$$

be the phase function appearing in (100). Note that Σ contains the critical points of $\Psi_{(\tau)}$ for all $\tau > 0$. Let

$$L_\tau = \frac{\phi_1(z, y, \tau^{-1}\theta, \tau^{-1}\xi)}{|d_{z,y,\theta,\xi}\Psi_{(\tau)}(z, y, \theta, \xi)|^2}\,(d_{z,y,\theta,\xi}\overline{\Psi_{(\tau)}(z, y, \theta, \xi)})\cdot d_{z,y,\theta,\xi},$$

so that

$$L_\tau \exp(\Psi_{(\tau)}) = \phi_1(z, y, \tau^{-1}\theta, \tau^{-1}\xi)\exp(\Psi_{(\tau)}). \tag{107}$$

Note that if $\operatorname{Im}\varphi(y) = 0$, then $y \in \gamma_{x_5,\xi_5}$ and hence $d\varphi(y)$ does not vanish and that when τ is large enough, the function $\Psi_{(\tau)}$ has no critical points in the support of ϕ_1. Using these we see that $\gamma_{x_5,\xi_5} \cap W_1$ has a neighborhood $V_1 \subset W_1$ where $|d\varphi(y)| > C_0 > 0$ and there are $C_1, C_2, C_3 > 0$ so that if $\tau > C_1$, $y \in V_1$, and $(z, y, \theta, \xi) \in \operatorname{supp}(\phi_1)$ then

$$|d_{z,y,\theta,\xi}\Psi_{(\tau)}(z, y, \theta, \xi)|^{-1} \leq \frac{C_2}{\tau - C_3}. \tag{108}$$

After these preparatory steps, we are ready to compute the asymptotics of $T^{(4),\beta}_{\tau,1,1}$. To this end, we first transform the integrals in (101) to an integral of a Lebesgue integrable function by using integration by parts of $|\xi|^{-2}(\nabla_z-\omega_\theta)^2$ as explained above. Then we decompose $T^{(4),\beta}_{\tau,1,1}$ into three terms $T^{(4),\beta}_{\tau,1,1}=I_1+I_2+I_3$. To obtain the first term I_1 we include the factor $(1-\phi_1(z,y,\theta,\xi))$ in the integral (101) with $k=1$. The integral I_1 can then be computed using the method of stationary phase similarly to the proof of [33, Thm. 3.4]. Let $\chi_1\in C^\infty(W_1)$ be a function that is supported in V_1 and vanishes on γ_{x_5,ξ_5}. The terms I_2 and I_3 are obtained by including the factor $\phi_1(z,y,\tau^{-1}\theta,\tau^{-1}\xi)\chi_1(y)$ and $\phi_1(z,y,\tau^{-1}\theta,\tau^{-1}\xi)(1-\chi_1(y))$ in the integral (100) with $k=1$, respectively. (Equivalently, the terms I_2 and I_3 are obtained by including the factor $\phi_1(z,y,\theta,\xi)\chi_1(y)$ and $\phi_1(z,y,\theta,\xi)(1-\chi_1(y))$ in the integral (101) with $k=1$, respectively.) Using integration by parts in integral (100) and inequalities (107) and (108), we see that that $I_2=O(\tau^{-N})$ for all $N>0$. Moreover, the fact that $\operatorname{Im}\varphi(y)>c_1>0$ in $W_1\cap V_1$ implies that $I_3=O(\tau^{-N})$ for all $N>0$.

Combining the above we obtain the asymptotics

$$T^{(4),\beta}_{\tau}\sim\tau^{4+4-16/2-2+\rho-2}\sum_{k=0}^{\infty}c_k\tau^{-k}=\tau^{-4+\rho}\sum_{k=0}^{\infty}c_k\tau^{-k}, \tag{109}$$

$$c_0=h(z^{(0)})c_1(0,-r_1,-r_2)\widetilde{a}_3(0,-r_3)\widetilde{q}_1(0,0,-(r_1,r_2,r_3,0))\widetilde{a}_4(0,-r_4)\widetilde{a}_5(0,1),$$

where $\rho=\sum_{j=1}^5 p_j$ and $\widetilde{a}_j$ is the principal symbol of a_j etc. The factor $h(z^{(0)})$ is non-vanishing and is determined by the determinant of the Hessian of the phase function φ at q. A direct computation shows that $\det(\mathrm{Hess}_{z,y,\theta,\xi}\Psi_3(z^{(0)},y^{(0)},\theta^{(0)},\xi^{(0)})=1$. Above, $(z^{(0)},y^{(0)},\theta^{(0)},\xi^{(0)})$ is the critical point satisfying (103) and (104), where in the local coordinates $(z^{(0)},y^{(0)})=(0,0)$ and $h(z^{(0)})$ is constant times powers of values of the cut-off functions Φ_0 and Φ_1 at zero. Recall that we considered above the case when $\mathcal{B}_j$ are multiplication operators with these cut-off functions. The term c_k depends on the derivatives of the symbols a_j and q_1 of order less or equal to $2k$ at the critical point. If $\Psi_3(z,\theta,y,\xi)$ has no critical points, that is, q is not an intersection point, we obtain the asymptotics $T^{(4),\beta}_{\tau,1,1}=O(\tau^{-N})$ for all $N>0$.

For future reference we note that if we use the method of stationary phase in the last integral of (101) only in the integrals with respect to z and ξ, yielding that at the critical point we have $y=z$ and

$\xi = \omega_\beta(\theta) = (\theta_1, \theta_2, \theta_3, 0)$, we see that $T^{(4),\beta}_{\tau,1,1}$ can be written as

$$
\begin{aligned}
(110)\quad T^{(4),\beta}_{\tau,1,1} &= c\tau^4 \int_{\mathbb{R}^8} e^{i(\theta_1 y^1+\theta_2 y^2+\theta_3 y^3+\theta_4 y^4)+i\tau\varphi(y)} c_1(y,\theta_1,\theta_2)\cdot \\
&\cdot a_3(y,\theta_3) q_{1,1}(y,y,\omega_\beta(\theta)) a_4(y,\theta_4) a_5(y,\tau)\, d\theta_1 d\theta_2 d\theta_3 d\theta_4 dy \\
&= c\tau^8 \int_{\mathbb{R}^8} e^{i\tau(\theta_1 y^1+\theta_2 y^2+\theta_3 y^3+\theta_4 y^4+\varphi(y))} c_1(y,\tau\theta_1,\tau\theta_2)\cdot \\
&\cdot a_3(y,\tau\theta_3) q_{1,1}(y,y,\tau\omega_\beta(\theta)) a_4(y,\tau\theta_4) a_5(y,\tau)\, d\theta_1 d\theta_2 d\theta_3 d\theta_4 dy.
\end{aligned}
$$

Next, we consider the terms $\widetilde T^{(4),\beta}_\tau$ of the type (80). Such term is an integral of the product of u_τ and two other factors $\mathbf{Q}(U_2\cdot U_1)$ and $\mathbf{Q}(U_4\cdot U_3)$. As the last two factors can be written in the form (92), one can see using the method of stationary phase that $\widetilde T^{(4),\beta}_\tau$ has similar asymptotics to $T^{(4),\beta}_\tau$ as $\tau\to\infty$, with the leading order coefficient $\widetilde c_0 = \widetilde h(z^{(0)})\widetilde c_1(0,-r_1,-r_2)\widetilde c_2(0,-r_3,-r_4)$ where $\widetilde c_2$ is given as in (93) with symbols $\widetilde a_3$ and $\widetilde a_4$, and moreover, $\widetilde h(z^{(0)})$ is a constant times powers of values of the cut-off functions Φ_0 and Φ_1 at zero.

This proves the claim in the special case where u_j are conormal distributions supported in the coordinate neighborhoods $W_{k(j)}$, a_j are positively homogeneous scalar valued symbols, $S_j = \mathbf{Q}$, and $\mathcal{B}_j$ are multiplication functions with smooth cut-off functions.

By using a suitable partition of unity and summing the results of the above computations, similar results to the above follows when a_j are general classical symbols that are $\mathcal{B}$-valued and the waves u_j are supported on $J^+_{\widehat g}(\mathrm{supp}\,(\mathbf{f}_j))$. Also, S_j can be replaced by operators of type (78) and $\mathcal{B}_j$ can replaced by differential operator without other essential changes expect that the highest order power of τ changes. Then, in the asymptotics of terms $T^{(4),\beta}_\tau$ the function $h(z^{(0)})$ in (109) is a section in dual bundle $(\mathcal{B}_L)^4$. The coefficients of $h(z^{(0)})$ in local coordinates are polynomials of $\widehat g^{jk}$, $\widehat g_{jk}$, $\widehat\phi_\ell$, and their derivatives at $z^{(0)}$. Similar representation is obtained for the asymptotics of terms $\widetilde T^{(4),\beta}_\tau$.

As we integrated by parts two times the operator $(\nabla_z-\omega_\theta)^2$ and the total order of of $\mathcal{B}_j$ is less or equal to 6, we see that it is enough to assume above that the symbols $a_j(z,\theta_j)$ are of order (-12) or less. The leading order asymptotics come from the term where the sum of orders of $\mathcal{B}_j$ is 6 and $p_j = n$ for $j=1,2,3,4$, $p_5=0$ so that $m = -4-\rho+6 = 4n+2$. We also see that the terms containing permutation $\sigma=\sigma_\beta$ of the indexes of the spherical waves can be analyzed analogously. This proves (84).

Making the above computations explicitly, we obtain an explicit formula for the leading order coefficient s_m in (84) in terms of $\mathbf{b}$ and $\mathbf{w}$, multiplied with the power $(-1/2)$ of the determinant of the Hessian of the phase function Ψ_3 at the critical point q in the Z-coordinates determined by $(\vec x,\vec\xi)$ and (x_5,ξ_5). This show that s_m coincides with some real-analytic function $G(\mathbf{b},\mathbf{w})$ multiplied by a non-vanishing function $\mathcal{R}(p,(\vec{x},\vec{\xi}),(x_5,\xi_5))$, corresponding to the power of the Hessian, that depends on the phase function φ in the Z coordinates. This proves the claim in the case when the linear independency condition (LI) is valid.

Next, consider the case when the linear independency condition (LI) is not valid. Again, by the definition of $\mathbf{t}_j$, if the intersection $\gamma_{x_5,\xi_5}(\mathbb{R}_-)\cap(\cap_{j=1}^4\gamma_{x_j,\xi_j}((0,\mathbf{t}_j)))$ is non-empty, it can contain only one point. In the case that such a point exists, we denote it by q.

When (LI) is not valid, we have that the linear space $\mathrm{span}(b_j;\ j=1,2,3,4)\subset T_q^*M_0$ has dimension 3 or less. We use the facts that for $w\in I(\Lambda_1,\Lambda_2)$ we have $\mathrm{WF}(w)\subset\Lambda_1\cup\Lambda_2$ and the fact, see [22, Thm. 1.3.6]

$$\mathrm{WF}(v\cdot w)\subset \mathrm{WF}(v)\cup\mathrm{WF}(w)\cup\{(x,\xi+\eta);\ (x,\xi)\in\mathrm{WF}(v),\ (x,\eta)\in\mathrm{WF}(w)\}.$$

Let us next consider the terms corresponding to the permutation $\sigma=Id$. The above facts imply that $\widetilde{\mathcal{G}}^{(4),\beta}$ in (62) satisfies

$$\mathrm{WF}(\widetilde{\mathcal{G}}^{(4),\beta})\cap T_q^*M_0\subset\mathcal{Z}_{s_0}:=\mathcal{X}_{s_0}\cup\bigcup_{1\le j\le 4}N^*K_j\cup\bigcup_{1\le j<k\le 4}N^*K_{jk},$$

where $\mathcal{X}_{s_0}=\mathcal{X}((\vec{x},\vec{\xi});t_0,s_0)$. Also, for

$$w_{123}=\mathcal{B}_3^\beta u_3\cdot\mathcal{C}_1^\beta S_1^\beta(\mathcal{B}_2^\beta u_2\cdot\mathcal{B}_1^\beta u_1),$$

appearing in (63), we have $\mathrm{WF}(w_{123})\subset\mathcal{Z}_{s_0}$ and thus using Hörmander's theorem [42, Thm. 26.1.1], we see that $\mathrm{WF}(S_2^\beta(w_{123}))\subset\Lambda^{(3)}$, where $\Lambda^{(3)}$ is the flowout of $\mathcal{Z}_{s_0}$ in the canonical relation of $\mathbf{Q}$. Then

$$\pi(\Lambda^{(3)})\subset\mathcal{Y}_{s_0}\cup\bigcup_{1\le j\le 4}K_j\cup\bigcup_{1\le j<k\le 4}K_{jk},$$

where $\mathcal{Y}_{s_0}=\mathcal{Y}((\vec{x},\vec{\xi});t_0,s_0)$ and $\pi:T^*M_0\to M_0$ is the projection to the base point.

Observe that $E=\mathrm{span}(b_j;\ j=1,2,3,4)\subset T_q^*M_0$ has dimension 3 or less, $\Lambda^{(3)}\cap T_q^*M_0\subset E$ and $\mathrm{WF}(u_4)\cap T_q^*M_0\subset E$. Thus, $\mathcal{G}^{(4),\beta}=\mathcal{B}_4^\beta u_4\cdot C_2^\beta S_2^\beta(w_{123})$ satisfies $\mathrm{WF}(\mathcal{G}^{(4),\beta})\cap T_q^*M_0\subset E$. Now, $E\subset\mathcal{Z}_{s_0}$. By our assumption, $(q,b_5)\notin\mathcal{X}((\vec{x},\vec{\xi});t_0)$, and thus, cf. (62), we see that

$$\langle u_\tau,\mathbf{Q}(\sum_{\beta\le n_1}\mathcal{G}^{(4),\beta}+\widetilde{\mathcal{G}}^{(4),\beta})\rangle=O(\tau^{-N})$$

for all $N>0$ when s_0 is small enough. The terms where the permutation σ is not the identity can be analyzed similarly. This proves the claim in the case when the linear independency condition (LI) is not valid. □

Proposition 3.5. *Let the assumptions of Proposition 3.4 be valid. Moreover, assume that (x_5,ξ_5) comes from the 4-intersection of rays corresponding to $(\vec{x},\vec{\xi})$ and q is the corresponding intersection point.*

Then the point x_5 has a neighborhood V so that $\mathcal{M}^4$ in V satisfies $\mathcal{M}^4|_V \in \mathcal{I}(V;\Lambda_q^+)$.

Proof. Let us fix (x_j,ξ_j), $j\leq 4$, s_0, and the waves $u_j\in \mathcal{I}(K_j)$, $j\leq 4$.

Let us consider the same condition (LI) that was used in the proof of Prop. 3.4. First we observe that if (LI) is not valid, we see using the proof of Prop. 3.4, (see the end of the proof where the case when (LI) is not valid is considered) that $\mathcal{M}^{(4)}$ is C^∞ smooth in $\mathcal{V}\setminus(\mathcal{Y}\cup\bigcup_{j=1}^4 K_j)$. Thus to prove the claim of the proposition we can assume that (LI) is valid.

Let us decompose $\mathcal{F}^{(4)}$, given by (62) and (63)-(64) as $\mathcal{F}^{(4)}=\mathcal{F}^{(4)}_1+\mathcal{F}^{(4)}_2$ where $\mathcal{F}^{(4)}_p$ is defined similarly to $\mathcal{F}^{(4)}$ in (62) and (63)-(64) by modifying these formulas so that the operator S_1^β is replaced by $S_{1,p}^\beta$, where $S_{1,p}^\beta=\mathbf{Q}_p$, when $S_1^\beta=\mathbf{Q}$, and $S_{1,p}^\beta=(2-p)I$, when $S_1^\beta=I$. Here, the operators $\mathbf{Q}_p$ are defined as above using the parameters ε_2 and ε_1 defined below.

Using formulae (62), (91), (92), and (99) we see that near near q in the Y coordinates $\mathcal{M}^{(4)}_1=\mathbf{Q}\mathcal{F}^{(4)}_1$ can be calculated using that

$$\mathcal{F}^{(4)}_1(y)=\int_{\mathbb{R}^4} e^{iy^j\theta_j}b(y,\theta)\,d\theta, \tag{111}$$

where K_j in local coordinates is given by $\{y^j=0\}$ and $b(y,\theta)$ is a finite sum of terms that are products of some of the following terms: at most one product type symbol $c_l(y,\theta_j,\theta_k)\in S(W_0;\mathbb{R}\times(\mathbb{R}\setminus\{0\}))$ (they appear in the terms (79)-(80) where the S_j^β operators are $\mathbf{Q}$ and do not appear if these operators are the identity), and one ore more term which is either the symbols $a_j(y,\theta_j)\in S^n(W_0;\mathbb{R})$, or the functions $q_1(y,y,\omega_\beta(\theta))$, cf. (110), where $\omega_\beta(\theta)$ is equal to some of the vectors $(\theta_1,\theta_2,\theta_3,0)$, $(\theta_1,\theta_2,0,\theta_4)$, $(\theta_1,0,\theta_3,\theta_4)$, or $(0,\theta_1,\theta_3,\theta_4)$, depending on the permutation σ.

Let us consider next the source F_τ is determined by the functions (p,h) in (72). Then using the method of stationary phase gives the asymptotics, c.f. (110),

$$\langle u_\tau,\mathcal{F}^{(4)}_1\rangle\sim\tau^8\int_{\mathbb{R}^8} e^{i\tau(\varphi(y)+y^j\theta_j)}(a_5(y,\tau),b(y,\tau\theta))_{\widehat G}d\theta dy\sim\sum_{k=m}^\infty s_k(p,h)\tau^{-k}$$

where $\widehat G$ is a Riemannian metric of the fiber of $\mathcal{B}^L$ at y, that is isomorphic to $\mathbb{R}^{10+L}$, and the critical point of the phase function is $y=0$ and $\theta=-d\varphi(0)$. As we saw above, we have that when $\varepsilon_2>0$ is small enough then for $p=2$ we have $\langle u_\tau,\mathcal{F}^{(4)}_p\rangle=O(\tau^{-N})$ for all $N>0$.

Let us choose sufficiently small $\varepsilon_3>0$ and choose a function $\chi(\theta)\in C^\infty(\mathbb{R}^4)$ that vanishes in a ε_3-neighborhood (in the $\widehat g^+$ metric) of $\mathcal{A}_q$,

$$\mathcal{A}_q \;:=\; N_q^*K_{123}\cup N_q^*K_{134}\cup N_q^*K_{124}\cup N_q^*K_{234} \tag{112}$$

and is equal to 1 outside the $(2\varepsilon_3)$-neighborhood of this set.

Let $\phi \in C_0^\infty(W_1)$ be a function that is one near q. Also, let

$$b_0(y,\theta) = \phi(y)\chi(\theta)b(y,\theta)$$

be a classical symbol, $p = \sum_{j=1}^4 p_j$, and let $\mathcal{F}^{(4),0}(y) \in \mathcal{I}^{p-4}(q)$ be the conormal distribution that is given by the formula (111) with $b(y,\theta)$ being replaced by $b_0(y,\theta)$.

When ε_3 is small enough (depending on the point x_5), we see that F_τ is determined by functions (p,h) and the corresponding gaussian beams u_τ propagating on the geodesic $\gamma_{x_5,\xi_5}(\mathbb{R})$ such that the geodesic passes through $x_5 \in V$, we have

$$\langle u_\tau, \mathcal{F}^{(4),0}\rangle \sim \sum_{k=m}^{\infty} s_k(p,h)\tau^{-k},$$

that is, we have $\langle u_\tau, \mathcal{F}^{(4),0}\rangle - \langle u_\tau, \mathcal{F}^{(4)}\rangle = O(\tau^{-N}$ for all N. When $\gamma_{x_5,\xi_5}(\mathbb{R})$ does not pass through q, we have that $\langle u_\tau, \mathcal{F}^{(4)}\rangle$ and $\langle u_\tau, \mathcal{F}^{(4),0}\rangle$ are both of order $O(\tau^{-N})$ for all $N > 0$.

Let $V \subset \mathcal{V}((\vec{x},\vec{\xi}),t_0) \setminus \bigcup_{j=1}^4 \gamma_{x_j,\xi_j}([0,\infty))$, see (83), be an open set. By varying the source F_τ, defined in (72), we see, by multiplying the solution with a smooth cut of function and using Corollary 1.4 in [20] in local coordinates, or [64], we see that the function $\mathcal{M}^4 - \mathbf{Q}\mathcal{F}^{(4),0}$ has no wave front set in $T^*(V)$ and it is thus C^∞-smooth function in V.

As by [28], $\mathbf{Q} : \mathcal{I}^{p-4}(\{q\}) \to \mathcal{I}^{p-4-3/2,-1/2}(N^*(\{q\}),\Lambda_q^+)$, the above implies that

$$\mathcal{M}^4|_{V\setminus\mathcal{Y}} \in \mathcal{I}^{p-4-3/2}(V\setminus\mathcal{Y};\Lambda_q^+), \tag{113}$$

where $\mathcal{Y} = \mathcal{Y}((\vec{x},\vec{\xi}),t_0,s_0)$. When x_5 if fixed, choosing s_0 to be small enough, we obtain the claim. □

Next we will show that the function $\mathcal{G}$ is not identically vanishing.

3.6.6. *WKB computations and the indicator functions in the Minkowski space.* To show that the function (85) is not identically vanishing, we will consider waves in Minkowski space.

In this section $\widehat{g}_{jk} = \text{diag}\,(-1,1,1,1)$ denotes the metric in the standard coordinates of the Minkowski space $\mathbb{R}^4$. Below we call the principal symbols of the linearized waves the polarizations to emphasize their physical meaning. To show that $\mathcal{G}(\mathbf{b},\mathbf{w})$ is non-vanishing, recall that $\mathbf{w} = (w_j)_{j=1}^5$ where for $j \le 4$ the polarizations $w_j = (v_j, v_j')$, represented as a pair of metric and scalar field polarizations, the metric part of the polarization v_j has to satisfy 4 linear conditions. Because of this, below we study the case when all polarizations of the matter fields at q vanish, that is, $v_j' = 0$ for all j, and v_j satisfies 4 conditions. In this case, in Minkowski space the function $\mathcal{G}(v,0,\mathbf{b})$ can be analyzed by assuming that there are no matter fields, which we do next. Later we return to the case of general polarizations. Next, we denote the g-components amplitudes by $v_j = v^{(j)}$.

Instead of the indicator function $\Theta^{(4)}(\mathbf{v},0,\mathbf{b})$ given in (84) parametrized by $(\mathbf{v},0,\mathbf{b})$ we will consider the function parametrized by the variables $(\mathbf{v},\mathbf{b})$, where $\mathbf{b}=(b^{(j)})_{j=1}^5$, $b^{(j)}\in\mathbb{R}^4$, are as before but $\mathbf{v}=(v^{(j)})_{j=1}^5$ are the amplitudes of the linear waves taking values in symmetric 4×4 matrices. In addition, we assume that the waves $u_j(x)$, $j=1,2,3,4$, solving the linear wave equation in the Minkowski space, are of the form

$$u_j(x)=v^{(j)}\left(b_p^{(j)}x^p\right)_+^a,\quad t_+^a=|t|^aH(t),$$

where $b_p^{(j)}dx^p$, $p=1,2,3,4$ are four linearly independent light-like co-vectors of $\mathbb{R}^4$, $a>0$ and $v^{(j)}$ are constant 4×4 matrices. We also assume that $b^{(5)}$ is not in the linear span of any three vectors $b^{(j)}$, $j=1,2,3,4$. In the following, we denote $b^{(j)}\cdot x:=b_p^{(j)}x^p$.

Let us next consider the wave produced by interaction of two plane wave type solutions in the Minkowski space.

Next, we will consider an operator $\mathbf{Q}_0$, which is an algebraic inverse of $\square_{\widehat g}$ in Minkowski space, that we will define for certain products of Heaviside functions and polynomials. Let $b^{(1)}=(1,p^1,p^2,p^3)$ and $b^{(2)}=(1,q^1,q^2,q^3)$ be light like vectors. We use the notations

$$v^{a_1,a_2}(x;b^{(1)},b^{(2)})=(b^{(1)}\cdot x)_+^{a_1}\cdot(b^{(2)}\cdot x)_+^{a_2}$$

and define

$$\mathbf{Q}_0(v^{a_1,a_2}(x;b^{(1)},b^{(2)}))=\frac{1}{2(a_1+1)(a_2+1)\,\widehat g(b^{(1)},b^{(2)})}v^{a_1+1,a_2+1}(x;b^{(1)},b^{(2)}).$$

Then $\square_{\widehat g}(\mathbf{Q}_0(v^{a_1,a_2}(x;b^{(1)},b^{(2)})))=v^{a_1,a_2}(x;b^{(1)},b^{(2)})$. Note that the function $v^{a_1,a_2}(x;b^{(1)},b^{(2)})$ is a product of two plane waves and it is supported in the causal future of the space-like set K_{12}. This implies that

$$\mathbf{Q}_0(v^{a_1,a_2}(x;b^{(1)},b^{(2)}))=\mathbf{Q}(v^{a_1,a_2}(x;b^{(1)},b^{(2)}))$$

where $\mathbf{Q}$ is the causal inverse of the wave operator $\square$ in the Minkowski space.

Similarly to the above we denote

$$v_\tau^{a,0}(x;b^{(4)},b^{(5)})=u_4(x)\,u_0^\tau(x),\quad u_4(x)=(b^{(4)}\cdot x)_+^a,\quad u_0^\tau(x)=e^{i\tau\, b^{(5)}\cdot x}.$$

Then $\square_{\widehat g}(v_\tau^{a,0}(x;b^{(4)},b^{(5)}))=2a\,\widehat g(b^{(4)},b^{(5)})i\tau\, v_\tau^{a-1,0}(x;b^{(4)},b^{(5)})$ and hence we define

(114)

$$\mathbf{Q}_0(v_\tau^{a,0}(x;b^{(4)},b^{(5)}))=\frac{1}{2i(a+1)\,\widehat g(b^{(4)},b^{(5)})\tau}v_\tau^{a+1,0}(x;b^{(4)},b^{(5)}).$$

Later, we will consider the relation between $\mathbf{Q}_0$ and the causal inverse $\mathbf{Q}$.

Next we prove that the indicator function $\mathcal{G}(v,\mathbf{b})$ in (85) does not vanish identically by showing that it coincides with the *formal* indicator function $\mathcal{G}^{(\mathbf{m})}(v,\mathbf{b})$, which is a real-analytic function that does not vanish identically

Below, let $x=(x^0,x^1,x^3,x^4)$ be the standard coordintes in the Minkowski space and let $z=(z^j)_{j=1}^4$ be light-like coordinates $z^j=b^{(z)}\cdot x$. We denote $x=X(z)$ and $z=Z(x)$. Also, let P be a vector such that $b^{(5)}\cdot x=P\cdot z$.

Let $h\in C_0^\infty(\mathbb{R}^4)$ be a function that has value 1 in a neigbhorhood of $x=0$, $T_0>1$ be such that supp (h) is contained in the set $\{x;\ x^0<T_0\}$.

Let $\chi=\chi(x^0)\in C^\infty(\mathbb{R})$ be zero for $x^0>T_0+1$ and one for $x^0<T_0$.

We define the (Minkowski) indicator function (c.f. (85))

$$\mathcal{G}^{(\mathbf{m})}(v,\mathbf{b})=\lim_{\tau\to\infty}\tau^m(\sum_{\beta\leq n_1}\sum_{\sigma\in\Sigma(4)}T_{\tau,\sigma}^{(\mathbf{m}),\beta}+\widetilde{T}_{\tau,\sigma}^{(\mathbf{m}),\beta}),$$

where the super-index $(\mathbf{m})$ refers to the word "Minkowski". Above, $\sigma:\{1,2,3,4\}\to\{1,2,3,4\}$ runs over all permutations of indexes of the waves u_j, where $\mathbf{b}=(b^{(1)},b^{(2)},\dots,b^{(5)})$ and $T_{\tau,\sigma}^{(\mathbf{m}),\beta}$ and $\widetilde{T}_{\tau,\sigma}^{(\mathbf{m}),\beta}$ are counterparts of the functions $T_\tau^{(4),\beta}$ and $\widetilde{T}_\tau^{(4),\beta}$, see (79)-(80), obtained by replacing the pieces of the spherical waves and the gaussian beam by plane waves by replacing the parametrix $\mathbf{Q}$ with a formal parametrix $\mathbf{Q}_0$ and including in the obtained formula a smooth cut off function $h\in C_0^\infty(M)$ which is one near the intersection point of the waves, and permutating indexes, that is,

$$T_{\tau,\sigma}^{(\mathbf{m}),\beta}=\langle S_2^0(u^\tau\cdot\mathcal{B}_4u_{\sigma(4)}),h\cdot\mathcal{B}_3u_{\sigma(3)}\cdot S_1^0(\mathcal{B}_2u_{\sigma(2)}\cdot\mathcal{B}_1u_{\sigma(1)})\rangle_{L^2(\mathbb{R}^4)},\tag{115}$$

$$\widetilde{T}_{\tau,\sigma}^{(\mathbf{m}),\beta}=\langle u^\tau,h\cdot S_2^0(\mathcal{B}_4u_{\sigma(4)}\cdot\mathcal{B}_3u_{\sigma(3)})\cdot S_1^0(\mathcal{B}_2u_{\sigma(2)}\cdot\mathcal{B}_1u_{\sigma(1)})\rangle_{L^2(\mathbb{R}^4)},\tag{116}$$

where $u_j=v_{(j)}(b^{(j)}\cdot x)_+^a$, $j=1,2,3,4$ and

$$u^\tau(x)=\chi(x^0)v_{(5)}\exp(i\tau b^{(5)}\cdot x).$$

Moreover, $\mathcal{B}_j=\mathcal{B}_{j,\beta}$ and finally, $S_j^0=S_{j,\beta}^0\in\{\mathbf{Q}_0,I\}$. We note that here that the algebraic inverse $\mathbf{Q}_0$ is used to replace both the causal parametrix $\mathbf{Q}$ and the anti-causal parametrix $\mathbf{Q}^*$, and the commutator terms do not appear at all.

Let us now consider the orders of the differential operators appearing above. The orders $k_j=ord(\mathcal{B}_j^\beta)$ of the differential operators $\mathcal{B}_j^\beta$, defined in (78), depend on $\vec{S}_\beta^0=(S_{1,\beta}^0,S_{2,\beta}^0)$ as follows: When β is such that $\vec{S}_\beta^0=(\mathbf{Q}_0,\mathbf{Q}_0)$, for the terms $T_{\tau,\sigma}^{(\mathbf{m}),\beta}$ we have

$$k_1+k_2+k_3+k_4\leq 6,\quad k_3+k_4\leq 4,\quad k_4\leq 2\tag{117}$$

and for the terms $\widetilde{T}_{\tau,\sigma}^{(\mathbf{m}),\beta}$ we have

$$k_1+k_2+k_3+k_4\leq 6,\quad k_1+k_2\leq 4,\quad k_3+k_4\leq 4.\tag{118}$$

When β is such that $\vec{S}^0_\beta = (I, Q_0)$ we have for terms $T^{(\mathbf{m}),\beta}_{\tau,\sigma}$

$$k_1 + k_2 + k_3 + k_4 \le 4, \quad k_4 \le 2, \tag{119}$$

and for terms $\widetilde{T}^{(\mathbf{m}),\beta}_{\tau,\sigma}$ we have

$$k_1 + k_2 + k_3 + k_4 \le 4, \quad k_1 + k_2 \le 2. \tag{120}$$

When β is such that $\vec{S}^0_\beta = (\mathbf{Q}_0, I)$, both for the terms $T^{(\mathbf{m}),\beta}_{\tau,\sigma}$ and $\widetilde{T}^{(\mathbf{m}),\beta}_{\tau,\sigma}$ we have

$$k_1 + k_2 + k_3 + k_4 \le 4, \quad k_3 + k_4 \le 2. \tag{121}$$

Finally, when β is such that $\vec{S}^0_\beta = (I, I)$, for the terms $T^{(\mathbf{m}),\beta}_{\tau,\sigma}$ and $\widetilde{T}^{(\mathbf{m}),\beta}_{\tau,\sigma}$ we have $k_1 + k_2 + k_3 + k_4 \le 2$.

Lemma 3.6. *When $b^{(j)}$, $j = 1, 2, 3, 4$ are linearly independent light-like co-vectors and light-like co-vector $b^{(5)}$ is not in the linear span of any three vectors $b^{(j)}$, $j = 1, 2, 3, 4$ we have $\mathcal{G}(\mathbf{w}, \mathbf{b}) = \mathcal{G}^{(\mathbf{m})}(\mathbf{v}, \mathbf{b})$ when $w_{(j)} = (v_{(j)}, 0) \in \mathbb{R}^{10} \times \mathbb{R}^L$.*

Proof. Let us start by considering the relation of $\mathbf{Q}_0$ with the causal inverse $\mathbf{Q}$ in (114). Let

$$\begin{aligned} w_{\tau,0} &= \mathbf{Q}_0(v^{a,0}_\tau(\,\cdot\,; b^{(4)}, b^{(5)})) \\ &= \int_{\mathbb{R}} e^{i\theta_4 z^4 + i\tau P\cdot z} a_4(z, \theta_4) d\theta_4, \\ w_\tau &= \mathbf{Q}^*(J), \\ J &= u_4 \cdot (\chi \cdot u^\tau_0), \end{aligned}$$

where

$$\begin{aligned} J(z) &= \chi(X^0(z))\,\Box w_{\tau,0} \\ &= \int_{\mathbb{R}} e^{i\theta_4 z^4 + i\tau P\cdot z} (\tau b_1(z, \theta_4) + b_2(z, \theta_4)) a_5(z, \tau) d\theta_4, \end{aligned}$$

where $a_5(z, \tau) = 1$. Then

$$\begin{aligned} &\Box(w_\tau - \chi w_{\tau,0}) = J_1, \quad \text{for } x \in \mathbb{R}^4, \\ &J_1 = [\Box, \chi] w_{\tau,0} \\ &\quad = \int_{\mathbb{R}} e^{i\theta_4 z^4 + i\tau P\cdot z} (\tau b_3(z, \theta_4) + b_4(z, \theta_4)) d\theta_4, \end{aligned}$$

where $b_3(z, \theta_4)$ and $b_4(z, \theta_4)$ are supported in the domain $T_0 < X^0(z) < T_0 + 1$ and $w_\tau - \chi w_{\tau,0}$ is supported in domain $X^0(z) < T_0 + 1$. Thus

$$w_\tau = \chi w_{\tau,0} + \mathbf{Q}^* J_1.$$

Here, we can write

$$
\begin{aligned}
(122)\qquad & J_1(z) = u_4^{(1)}(z)u^{\tau,(1)}(z) + u_4^{(2)}(z)u^{\tau,(2)}(z), \quad \text{where} \\
& u_4^{(1)}(z) = \int_{\mathbb{R}} e^{i\theta_4 z^4} b_3(z,\theta_4)d\theta_4, \\
& u^{\tau,(1)}(z) = \tau u^\tau(z), \\
& u_4^{(2)}(z) = \int_{\mathbb{R}} e^{i\theta_4 z^4} b_4(z,\theta_4)d\theta_4, \\
& u^{\tau,(2)}(z) = u^\tau(z).
\end{aligned}
$$

Let us now substitute this in to the above microlocal computations done in the proof of Prop. 3.4.

Recall that $b^{(j)}$, $j = 1,2,3,4$, are four linearly independent co-vectors. This means that a condition analogous to (LI) in the proof of Prop. 3.4 is satisfied, and that $b^{(5)}$ is not in the space spanned by any of three of the co-vectors $b^{(j)}$, $j = 1,2,3,4$. Also, observe that hyperplanes $K_j = \{x \in \mathbb{R}^4;\ b^{(j)} \cdot x = 0\}$ intersect at origin of $\mathbb{R}^4$. Thus, we see that the arguments in the proof of Prop. 3.4 are valid mutatis mutandis if the phase function of the gaussian beam $\varphi(x)$ is replaced by the phase function of the plane wave, $b^{(5)} \cdot x$, and the geodesic γ_{x_5,ξ_5}, on which the gaussian beam propagates, is replaced by the whole space $\mathbb{R}^4$. In particular, as $b^{(5)}$ is not in the space spanned by any of three of those co-vectors, the case (A1) in the proof of Prop. 3.4 cannot occur. In particular, we see that the leading order asymptotics of the terms $T^\beta_{\tau,\sigma}$ and $\widetilde{T}^\beta_{\tau,\sigma}$ does not change as these asymptotics are obtained using the method of stationary phase for the integral (110) and the other analogous integrals at the critical point $z = 0$. In other words, we can replace the gaussian beam by a plane wave in our considerations similar to those in the proof of Prop. 3.4.

Using (122) and the fact that $b_3(z,\theta_4)$ and $b_4(z,\theta_4)$ vanish near $z = 0$, we see that if u_4 and u^τ are replaced by $u_4^{(j)}$ and $u^{\tau,(j)}$, respectively, where $j \in \{1,2\}$ and we can do similar computations based on the method of stationary phase as are done in the proof of Proposition 3.4. Then both terms $T^\beta_{\tau,\sigma}$ and $\widetilde{T}^\beta_{\tau,\sigma}$ have asymptotics $O(\tau^{-N})$ for all $N > 0$ as $\tau \to \infty$. In other words, in the proof of Prop. 3.4 the term $w_\tau = \mathbf{Q}^*(u_4 u^\tau)$ can be replaced by $\chi w_{\tau,0}$ without changing the leading order asymptotics. This shows that $\mathcal{G}(\mathbf{w},\mathbf{b}) = \mathcal{G}^{(\mathbf{m})}(\mathbf{v},\mathbf{b})$, where $w_{(j)} = (v_{(j)}, 0) \in \mathbb{R}^{10} \times \mathbb{R}^L$. $\square$

Proposition 3.7. *Let X be the set of $(\mathbf{b}, v^{(2)}, v^{(3)}, v^{(4)})$, where $\mathbf{b}$ is a 5-tuple of light-like covectors $\mathbf{b} = (b^{(1)}, b^{(2)}, b^{(3)}, b^{(4)}, b^{(5)})$ and $v^{(j)} \in \mathbb{R}^{10}$, $j = 2,3,4$ are the polarizations that satisfy the equation (54) with respect to $b^{(j)}$, i.e., the divergence condition for the principal symbols. Also, let $a \in \mathbb{Z}_+$ be large enough. Then for $(\mathbf{b}, v^{(2)}, v^{(3)}, v^{(4)})$ in a* *generic (i.e. open and dense) subset of X there exist linearly independent vectors $v_q^{(5)}$, $q=1,2,3,4,5,6$, so that if $v^{(5)}\in span(\{v_p^{(5)};\ p=1,2,3,4,5,6\})$ is non-zero, then there exists a vector $v^{(1)}$ for which the pair $(b^{(1)},v^{(1)})$ satisfies the equation (54) and $\mathcal{G}^{(\mathbf{m})}(\mathbf{v},\mathbf{b})\neq 0$ with $\mathbf{v}=(v^{(1)},v^{(2)},v^{(3)},v^{(4)},v^{(5)})$.*

Proof. To show that the coefficient $\mathcal{G}^{(\mathbf{m})}(\mathbf{v},\mathbf{b})$ of the leading order term in the asymptotics is non-zero, we consider a special case when the direction vectors of the intersecting plane waves in the Minkowski space are the linearly independent light-like vectors of the form

$$b^{(5)}=(1,1,0,0),\quad b^{(j)}=(1,1-\frac{1}{2}\rho_j^2,\rho_j+O(\rho_j^3),\rho_j^3),\quad j=1,2,3,4,$$

where $\rho_j>0$ are small parameters for which

$$\|b^{(5)}-b^{(j)}\|_{(\mathbb{R}^4,\widehat{g}^+)}=\rho_j(1+o(\rho_j)),\quad j=1,2,3,4. \tag{123}$$

With an appropriate choice of $O(\rho_k^3)$ above, the vectors $b^{(k)}$, $k\leq 5$ are light-like and

$$\begin{aligned}\widehat{g}(b^{(5)},b^{(j)}) &= -1+(1-\frac{1}{2}\rho_j^2)=-\frac{1}{2}\rho_j^2,\\ \widehat{g}(b^{(k)},b^{(j)}) &= -\frac{1}{2}\rho_k^2-\frac{1}{2}\rho_j^2+O(\rho_k\rho_j).\end{aligned}$$

Below, we denote $\omega_{kj}=\widehat{g}(b^{(k)},b^{(j)})$. We consider the case when the orders of ρ_j are (Note here the "unordered" numbering 4-2-3-1)

$$\rho_4=\rho_2^{100},\ \rho_2=\rho_3^{100},\ \text{and}\ \rho_3=\rho_1^{100}. \tag{124}$$

Note that when ρ_1 is small enough, $b^{(5)}$ is not a linear combination of any three vectors $b^{(j)}$, $j=1,2,3,4$.

The coefficient $\mathcal{G}^{(\mathbf{m})}$ of the leading order asymptotics is computed by analyzing the leading order terms of all 4th order interaction terms, similar to those given in (115) and (116). We will start by analysing the most important terms $T_\tau^{(\mathbf{m}),\beta}$ of the type (115) when β is such that $\vec S_\beta=(\mathbf{Q}_0,\mathbf{Q}_0)$. When $k_j=k_j^\beta$ is the order of $\mathcal{B}_j$, and we denote $\vec k_\beta=(k_1^\beta,k_1^\beta,k_3^\beta,k_4^\beta)$, we see that

$$\begin{aligned}(125)\ T_\tau^{(\mathbf{m}),\beta}&=\langle \mathbf{Q}_0(\mathcal{B}_4u_4\cdot u^\tau),h\cdot\mathcal{B}_3u_3\cdot\mathbf{Q}_0(\mathcal{B}_2u_2\cdot\mathcal{B}_1u_1)\rangle\\ &=C\frac{\mathcal{P}_\beta}{\omega_{45}\tau}\frac{1}{\omega_{12}}\langle v_\tau^{a-k_4+1,0}(\,\cdot\,;b^{(4)},b^{(5)}),h\cdot u_3\cdot v^{a-k_2+1,a-k_1+1}(\,\cdot\,;b^{(2)},b^{(1)})\rangle\\ &=C\frac{\mathcal{P}_\beta}{\omega_{45}\tau}\frac{1}{\omega_{12}}\int_{\mathbb{R}^4}(b^{(4)}\cdot x)_+^{a-k_4+1}e^{i\tau(b^{(5)}\cdot x)}h(x)(b^{(3)}\cdot x)_+^{a-k_3}\cdot\\ &\qquad\cdot(b^{(2)}\cdot x)_+^{a-k_2+1}(b^{(1)}\cdot x)_+^{a-k_1+1}\,dx,\end{aligned}$$

where $\mathcal{P}=\mathcal{P}_\beta$ is a polarization factor involving the coefficients of $\mathcal{B}_j$, the directions $b^{(j)}$, and the polarization $v_{(j)}$. Moreover, $C=C_a$ is a generic constant depending on a and β but not on $b^{(j)}$ or $v_{(j)}$.

We will analyze the polarization factors later, but as a sidetrack, let us already now explain the nature of the polarization term when $\beta = \beta_1$, see (81). Observe that this term appear only when we analyze the term $\langle F_\tau, \mathbf{Q}(A[u_4, \mathbf{Q}(A[u_3, \mathbf{Q}(A[u_2, u_1])])])\rangle$ where all operators $A[v, w]$ are of the type $A_2[v, w] = \widehat{g}^{np}\widehat{g}^{mq}v_{nm}\partial_p\partial_q w_{jk}$, cf. (65) and (66). Due to this, we have the polarization factor

$$\mathcal{P}_{\beta_1} = (v_{(4)}^{rs}b_r^{(1)}b_s^{(1)})(v_{(3)}^{pq}b_p^{(1)}b_q^{(1)})(v_{(2)}^{nm}b_n^{(1)}b_m^{(1)})\mathcal{D}, \tag{126}$$

where $v_{(\ell)}^{nm} = \widehat{g}^{nj}\widehat{g}^{mk}v_{jk}^{(\ell)}$ and

$$\mathcal{D} = \widehat{g}_{nj}\widehat{g}_{mk}v_{(5)}^{nm}v_{(1)}^{jk}. \tag{127}$$

We will postpone the analysis of the polarization factors $\mathcal{P}_\beta$ in $T_\tau^{(\mathbf{m}),\beta}$ with $\beta \neq \beta_1$ later.

Let us now return back to the computation (125). We next use in $\mathbb{R}^4$ the coordinates $y = (y^1, y^2, y^3, y^4)^t$ where $y^j = b_k^{(j)}x^k$, i.e., and let $A \in \mathbb{R}^{4\times 4}$ be the matrix for which $y = A^{-1}x$. Let $\mathbf{p} = (A^{-1})^t b^{(5)}$. In the y-coordinates, $b^{(j)} = dy^j$ for $j \leq 4$ and $b^{(5)} = \sum_{j=1}^4 \mathbf{p}_j dy^j$ and

$$\mathbf{p}_j = \widehat{g}(b^{(5)}, dy^j) = \widehat{g}(b^{(5)}, b^{(j)}) = \omega_{j5} = -\frac{1}{2}\rho_j^2.$$

Then $b^{(5)} \cdot x = \mathbf{p} \cdot y$. We use the notation $\mathbf{p}_j = \omega_{j5} = -\frac{1}{2}\rho_j^2$, that is, we denote the same object with several symbols, to clarify the steps we do in the computations.

Then $\det(A) = 8\rho_1^{-3}\rho_2^{-2}\rho_3^{-1}(1 + O(\rho_1))$ and

$$T_\tau^{(\mathbf{m}),\beta} = \frac{C\mathcal{P}_\beta}{\omega_{45}\tau}\frac{\det(A)}{\omega_{12}}\int_{(\mathbb{R}_+)^4} e^{i\tau\mathbf{p}\cdot y}h(Ay)y_4^{a-k_4+1}y_3^{a-k_3}y_2^{a-k_2+1}y_1^{a-k_1+1}\,dy.$$

Using repeated integration by parts we see that

(128)

$$T_\tau^{(\mathbf{m}),\beta} \;=\; C\det(A)\,\mathcal{P}_\beta\frac{(i\tau)^{-(12+4a-|\vec{k}_\beta|)}(1 + O(\tau^{-1}))}{\rho_4^{2(a-k_4+1+2)}\rho_3^{2(a-k_3+1)}\rho_2^{2(a-k_2+2)}\rho_1^{2(a-k_1+1+2)}}.$$

Note that here and below $O(\tau^{-1})$ may depend also on ρ_j, that is, we have $|O(\tau^{-1})| \leq C(\rho_1, \rho_2, \rho_3, \rho_4)\tau^{-1}$.

To show that $\mathcal{G}^{(\mathbf{m})}(\mathbf{v}, \mathbf{b})$ is non-vanishing we need to estimate $\mathcal{P}_{\beta_1}$ from below. In doing this we encounter the difficulty that $\mathcal{P}_{\beta_1}$ can go to zero, and moreover, simple computations show that as the pairs $(b^{(j)}, v^{(j)})$ satisfies the divergence condtion (54) we have $v_{(r)}^{ns}b_n^{(j)}b_s^{(j)} = O(\rho_r + \rho_j)$. However, to show that $\mathcal{G}^{(\mathbf{m})}(\mathbf{v}, \mathbf{b})$ is non-vanishing for some $\mathbf{v}$ we consider a particular choice of polarizations $v^{(r)}$, namely

$$v_{mk}^{(r)} = b_m^{(r)}b_k^{(r)}, \quad \text{for } r = 2, 3, 4, \text{ but not for } r = 1, 5 \tag{129}$$

so that for $r=2,3,4$, we have

$$\widehat g^{nm}b^{(r)}_n v^{(r)}_{mk}=0,\quad \widehat g^{mk}v^{(r)}_{mk}=0,\quad \widehat g^{nm}b^{(r)}_n v^{(r)}_{mk}-\frac12(\widehat g^{mk}v^{(r)}_{mk})b^{(r)}_k=0.$$

Note that for this choice of $v^{(r)}$ the linearized divergence conditions hold. Moreover, for this choice of $v^{(r)}$ we see that for $\rho_j\le\rho_r^{100}$

$$v^{ns}_{(r)}b^{(j)}_n b^{(j)}_s=\widehat g(b^{(r)},b^{(j)})\,\widehat g(b^{(r)},b^{(j)})=\rho_r^4+O(\rho_r^5). \tag{130}$$

In particular, when $\beta=\beta_1$, so that $k_{\beta_1}=(6,0,0,0)$ and the polarizations are given by (129), we have

$$\mathcal P_{\beta_1}=(\mathcal D+O(\rho_1))\rho_1^4\cdot\rho_1^4\cdot\rho_1^4,$$

where $\mathcal D$ is the inner product of $v^{(1)}$ and $v^{(5)}$ given in (127). Then the term $T^{(\mathbf m),\beta_1}_\tau$, which later turns out to have the strongest asymptotics in our considerations, has the asymptotics

$$T^{(\mathbf m),\beta_1}_\tau=\mathcal L_\tau,\quad\text{where} \tag{131}$$
$$\mathcal L_\tau=C\det(A)(i\tau)^{-(6+4a)}(1+O(\tau^{-1}))\vec\rho^{\,-2(\vec a+\vec 1)}\rho_4^{-4}\rho_2^{-2}\rho_3^0\rho_1^{20}\mathcal D,$$

where $\vec\rho=(\rho_1,\rho_2,\rho_3,\rho_4)$, $\vec a=(a,a,a,a)$, and $\vec 1=(1,1,1,1)$. To compare different terms, we express ρ_j in powers of ρ_1 as explained in formula (124), that is, we write $\rho_4^{n_4}\rho_2^{n_2}\rho_3^{n_3}\rho_1^{n_1}=\rho_1^m$ with $m=100^3n_4+100^2n_2+100n_3+n_1$. In particular, we will below write

$$\mathcal L_\tau=C_{\beta_1}(\vec\rho)\,\tau^{n_0}(1+O(\tau^{-1}))\quad\text{as }\tau\to\infty\text{ for each fixed }\vec\varepsilon,\text{ and}$$
$$C_{\beta_1}(\vec\varepsilon)=c'_{\beta_1}\,\rho_1^{m_0}(1+o(\rho_1))\quad\text{as }\rho_1\to0.$$

Below we will show that c'_{β_1} does not vanish for generic $(\vec x,\vec\xi)$ and (x_5,ξ_5) and polarizations $\mathbf v$. We will consider below $\beta\ne\beta_1$ and show that also these terms have the asymptotics

$$T^{(\mathbf m),\beta}_\tau=C_\beta(\vec\rho)\,\tau^{n}(1+O(\tau^{-1}))\quad\text{as }\tau\to\infty\text{ for each fixed }\vec\varepsilon,\text{ and}$$
$$C_\beta(\vec\varepsilon)=c'_\beta\,\rho_1^{m}(1+o(\rho_1))\quad\text{as }\rho_1\to0.$$

When we have that either $n\le n_0$ and $m<m_0$, or $n<n_0$, we say that $T^{(\mathbf m),\beta}_\tau$ has weaker asymptotics than $T^{(\mathbf m),\beta_1}_\tau$ and denote $T^{(\mathbf m),\beta}_\tau\prec\mathcal L_\tau$.

As we consider here the asymptotic of five small parameters τ^{-1} and ε_i, $i=1,2,3,4$, and compare in which order we make them tend to 0, let us explain the above ordering in detail. Above, we have chosen the order: first τ^{-1}, then ε_4, ε_2, ε_3 and finally, ε_1. In correspondence with this choice we can introduce an ordering on all monomials $c\tau^{-c_\tau}\varepsilon_4^{n_4}\varepsilon_3^{n_3}\varepsilon_2^{n_2}\varepsilon_1^{n_1}$. Namely, we say that

$$C'\tau^{-n'_\tau}\varepsilon_4^{n'_4}\varepsilon_3^{n'_3}\varepsilon_2^{n'_2}\varepsilon_1^{n'_1}\prec C\tau^{-n_\tau}\varepsilon_4^{n_4}\varepsilon_3^{n_3}\varepsilon_2^{n_2}\varepsilon_1^{n_1} \tag{132}$$

if $C\ne0$ and one of the following holds

(i) if $n_\tau<n'_\tau$;
(ii) if $n_\tau=n'_\tau$ but $n_4<n'_4$;
(iii) if $n_\tau=n'_\tau$ and $n_4=n'_4$ but $n_2<n'_2$;

(iv) if $n_\tau = n'_\tau$ and $n_4 = n'_4$, $n_2 = n'_2$ but $n_3 < n'_3$;
(v) if $n_\tau = n'_\tau$ and $n_4 = n'_4$, $n_3 = n'_3$, $n_2 = n'_2$ but $n_1 < n'_1$.

Note that this ordering is not a partial ordering.

Then, we can analyze terms $T^{(\mathbf{m}),\beta}_{\tau,\sigma}$ and $\widetilde T^{(\mathbf{m}),\beta}_{\tau,\sigma}$ in the formula for

$$\Theta^{(4)}_\tau = \Theta^{(4)}_{\tau,\vec\varepsilon} = \sum_{\beta\in J_\ell}\sum_{\sigma\in\Sigma(\ell)}\left(T^{(\mathbf{m}),\beta}_{\tau,\sigma} + \widetilde T^{(\mathbf{m}),\beta}_{\tau,\sigma}\right). \tag{133}$$

Note that here the terms, in which the permutation σ is either the identical permutation id or the permutation $\sigma_0 = (2,1,3,4)$, are the same.

Remark 3.3. We can find the leading order asymptotics of the strongest terms in the decomposition (133) using the following algorithm. First, let us multiply $\Theta^{(4)}_{\tau,\vec\varepsilon}$ by $\tau^{\widehat n_\tau}$, where $\widehat n_\tau = \min_\beta n_\tau(\beta)$. Taking then $\tau\to\infty$ will give non-zero contribution from only those terms $T^{(\mathbf{m}),\beta}_{\tau,\sigma}$ and $\widetilde T^{(\mathbf{m}),\beta}_{\tau,\sigma}$ where $n_\tau(\beta) = \widehat n_\tau$. This corresponds to step (i) above. Multiplying next by $\varepsilon_4^{-\widehat n_4}$, where $\widehat n_4 = \min_\beta n_4(\beta)$ under the condition that $n_\tau(\beta) = \widehat n_\tau$ and taking $\varepsilon_4\to 0$ corresponds to selecting terms $T^{(\mathbf{m}),\beta}_{\tau,\sigma}$ with $n_\tau(\beta) = \widehat n_\tau$ and $n_4(\beta) = \widehat n_4$ and terms $\widetilde T^{(\mathbf{m}),\beta}_{\tau,\sigma}$ with $\widetilde n_\tau(\beta) = \widehat n_\tau$ and $\widetilde n_4(\beta) = \widehat n_4$. This corresponds to step (ii). Continuing this process we obtain a scalar value that gives the leading order asymptotics of the strongest terms in the decomposition (133).

The next results tells what are the strongest terms in (133).

Proposition 3.8. *In (133), the strongest term are* $T^{(\mathbf{m}),\beta_1}_\tau = T^{(\mathbf{m}),\beta_1}_{\tau,id}$ *and* $T^{(\mathbf{m}),\beta_1}_{\tau,\sigma_0}$ *in the sense that for all* $(\beta,\sigma)\notin\{(\beta_1,id),(\beta_1,\sigma_0)\}$ *we have* $T^{(\mathbf{m}),\beta}_{\tau,\sigma} \prec T^{(\mathbf{m}),\beta_1}_{\tau,id}$.

Proof. When $\vec S_\beta = (S_1,S_2) = (\mathbf{Q}_0,\mathbf{Q}_0)$, similar computations to the above ones yield

$$\begin{aligned}\widetilde T^{(\mathbf{m}),\beta}_\tau &= \langle u^\tau, h\cdot \mathbf{Q}_0(\mathcal{B}_4u_4\cdot\mathcal{B}_3u_3)\cdot\mathbf{Q}_0(\mathcal{B}_2u_2\cdot\mathcal{B}_1u_1)\rangle\\ &= C\det(A)\mathcal{P}_\beta\frac{(i\tau)^{-(12+4a-|\vec k_\beta|)}(1+O(\tau^{-1}))}{\rho_4^{2(a-k_4+2)}\rho_3^{2(a-k_3+1+2)}\rho_2^{2(a-k_2+2)}\rho_1^{2(a-k_1+1+2)}}.\end{aligned}$$

Let us next consider the case when $\vec S_\beta = (S_1,S_2) = (I,\mathbf{Q}_0)$. Again, the computations similar to the above ones show that

$$\begin{aligned}T^{(\mathbf{m}),\beta}_\tau &= \langle \mathbf{Q}_0(u^\tau\cdot\mathcal{B}_4u_4), h\cdot\mathcal{B}_3u_3\cdot I(\mathcal{B}_2u_2\cdot\mathcal{B}_1u_1)\rangle\\ &= i\,C\mathcal{P}_\beta\det(A)\frac{(i\tau)^{-(10+4a-|\vec k_\beta|)}(1+O(\tau^{-1}))}{\rho_4^{2(a-k_4+1+2)}\rho_3^{2(a-k_3+1)}\rho_2^{2(a-k_2+1)}\rho_1^{2(a-k_1+1)}}\end{aligned}$$

and

$$\begin{aligned}\widetilde T^{(\mathbf{m}),\beta}_\tau &= \langle u^\tau, h\cdot \mathbf{Q}_0(\mathcal{B}_4 u_4\cdot \mathcal{B}_3\, u_3)\cdot I(\mathcal{B}_2\, u_2\cdot \mathcal{B}_1\, u_1)\rangle\\ &= \mathcal{P}_\beta C\det(A)\frac{(i\tau)^{-(10+4a-|\vec k_\beta|)}(1+O(\tau^{-1}))}{\rho_4^{2(a-k_4+2)}\rho_3^{2(a-k_3+1+2)}\rho_2^{2(a-k_2+1)}\rho_1^{2(a-k_1+1)}}.\end{aligned}$$

When $\vec S_\beta=(S_1,S_2)=(\mathbf{Q}_0,I)$ we have $\widetilde T^{(\mathbf{m}),\beta}_\tau=T^{(\mathbf{m}),\beta}_\tau$ and

$$\begin{aligned}T^{(\mathbf{m}),\beta}_\tau &= \langle I(u^\tau\cdot \mathcal{B}_4 u_4), h\cdot \mathcal{B}_3 u_3\cdot \mathbf{Q}_0(\mathcal{B}_2 u_2\cdot \mathcal{B}_1 u_1)\rangle,\\ &= i\mathcal{P}_\beta \det(A) C\frac{(i\tau)^{-(10+4a-|\vec k_\beta|)}(1-O(\tau^{-1}))}{\rho_4^{2(a-k_4+1)}\rho_3^{2(a-k_3+1)}\rho_2^{2(a-k_2+2)}\rho_1^{2(a-k_1+1+2)}},\end{aligned}$$

and finally when $\vec S_\beta=(S_1,S_2)=(I,I)$

$$\begin{aligned}\widetilde T^{(\mathbf{m}),\beta}_\tau &= \langle u^\tau, h\cdot I(\mathcal{B}_4 u_4\cdot \mathcal{B}_3 u_3)\cdot I(\mathcal{B}_2 u_2\cdot \mathcal{B}_1 u_1)\rangle\\ &= \mathcal{P}_\beta C_a\det(A)\frac{(i\tau)^{-(8+4a-|\vec k_\beta|)}(1+O(\tau^{-1}))}{\rho_4^{2(a-k_4+1)}\rho_3^{2(a-k_3+1)}\rho_2^{2(a-k_2+1)}\rho_1^{2(a-k_1+1)}}.\end{aligned}$$

Next we consider all β such that $\vec S^\beta=(\mathbf{Q}_0,\mathbf{Q}_0)$ but $\beta\neq\beta_1$. Then

$$\widetilde T^{(\mathbf{m}),\beta}_\tau = C\det(A)(i\tau)^{-(6+4a)}(1+O(\tau^{-1}))\vec\rho^{\,-2(\vec a+\vec 1)+2\vec k_\beta}\rho_4^{-2}\rho_2^{-2}\rho_3^{-4}\rho_1^{-4}\cdot \mathcal{P}_\beta$$

where $\vec k_\beta$ is as in (117). Note that for $\beta=\beta_1$ we have $\mathcal{P}_{\beta_1}=(\mathcal{D}+O(\rho_1))\rho_1^4\cdot \rho_1^4\cdot \rho_1^4$ while $\beta\neq\beta_1$ we just use an estimate $\mathcal{P}_\beta=O(1)$. Then we see that $\widetilde T^{(\mathbf{m}),\beta}_\tau\prec \mathcal{L}_\tau$.

When β is such that $\vec S^\beta=(\mathbf{Q}_0,I)$, we see that

$$\begin{aligned}T^{(\mathbf{m}),\beta}_\tau &= C\det(A)(i\tau)^{-(10+4a-|\vec k_\beta|)}(1+O(\tau^{-1}))\vec\rho^{\,-2(\vec a+\vec 1)+2\vec k_\beta}\rho_4^{0}\rho_2^{-2}\rho_3^{0}\rho_1^{-4}\mathcal{P}_\beta,\\ \widetilde T^{(\mathbf{m}),\beta}_\tau &= C\det(A)(i\tau)^{-(10+4a-|\vec k_\beta|)}(1+O(\tau^{-1}))\vec\rho^{\,-2(\vec a+\vec 1)+2\vec k_\beta}\rho_4^{-2}\rho_2^{0}\rho_3^{-4}\rho_1^{0}\mathcal{P}_\beta\end{aligned}$$

where $\mathcal{P}_\beta=O(1)$ and $\vec k_\beta$ is as in (121) and hence $T^{(\mathbf{m}),\beta}_\tau\prec\mathcal{L}_\tau$ and $\widetilde T^{(\mathbf{m}),\beta}_\tau\prec\mathcal{L}_\tau$.

When β is such that $\vec S^\beta=(I,\mathbf{Q}_0)$ or $\vec S^\beta=(I,I)$, using inequalities of the type (119) and (120) in appropriate cases, we see that $T^{(\mathbf{m}),\beta}_\tau\prec\mathcal{L}_\tau$ and $\widetilde T^{(\mathbf{m}),\beta}_\tau\prec\mathcal{L}_\tau$.

The above shows that all terms $T^{(\mathbf{m}),\beta}_\tau$ and $\widetilde T^{(\mathbf{m}),\beta}_\tau$ with maximal allowed k.s have asymptotics with the same power of τ but their $\vec\rho$ asymptotics vary, and when the asymptotic orders of ρ_j are given as explained in after (124), there is only one term, namely $\mathcal{L}_\tau=T^{(\mathbf{m}),\beta_1}_\tau$, that has the strongest order asymptotics given in (131).

Next we analyze the effect of the permutation $\sigma:\{1,2,3,4\}\to\{1,2,3,4\}$ of the indexes j of the waves u_j. We assume below that the permutation σ is not the identity map.

Recall that in the computation (125) there appears a term $\omega_{45}^{-1}\sim\rho_4^{-2}$. As this term does not appear in the computations of the terms $\widetilde T^{(\mathbf{m}),\beta}_{\tau,\sigma}$, we see that $\widetilde T^{(\mathbf{m}),\beta}_{\tau,\sigma}\prec\mathcal{L}_\tau$. Similarly, if σ is such that $\sigma(4)\neq 4$, the term ω_{45}^{-1} does not appear in the computation of $T_{\tau,\sigma}^{(\mathbf{m}),\beta}$ and hence $T_{\tau,\sigma}^{(\mathbf{m}),\beta_1} \prec \mathcal{L}_\tau$. Next we consider the permutations for which $\sigma(4) = 4$.

Next we consider σ that is either $\sigma = (3,2,1,4)$ or $\sigma = (2,3,1,4)$. These terms are very similar and thus we analyze the case when $\sigma = (3,2,1,4)$. First we consider the case when $\beta = \beta_2$ is such that $\vec{S}^{\beta_2} = (\mathbf{Q}_0, \mathbf{Q}_0)$, $\vec{k}_{\beta_2} = (2,0,4,0)$ This term appears in the analysis of the term $A^{(1)}[u_{\sigma(4)}, \mathbf{Q}(A^{(2)}[u_{\sigma(3)}, \mathbf{Q}(A^{(3)}[u_{\sigma(2)}, u_{\sigma(1)}])])]$ when $(A^{(1)}, A^{(2)}, A^{(3)}) = (A_2, A_1, A_2)$, see (68). By a permutation of the indexes in (125) we obtain the formula

$$
\begin{aligned}
(134)\qquad T_{\tau,\sigma}^{(\mathbf{m}),\beta_2} &= c_1' \det(A)(i\tau)^{-(6+4a)}(1+O(\tau^{-1}))\vec{\rho}^{\,-2(\vec{a}+\vec{1})} \\
&\quad \cdot (\omega_{45}\omega_{32})^{-1}\rho_4^{2(k_4-1)}\rho_1^{2k_3}\rho_2^{2(k_2-1)}\rho_3^{2(k_1-1)}\mathcal{P}_{\beta_2}, \\
\widetilde{\mathcal{P}}_{\beta_2} &= (v_{(4)}^{pq} b_p^{(1)} b_q^{(1)})(v_{(3)}^{rs} b_r^{(1)} b_s^{(1)})(v_{(2)}^{nm} b_n^{(3)} b_m^{(3)})\mathcal{D}.
\end{aligned}
$$

Hence, in the case when we use the polarizations (129), we obtain

$$
T_{\tau,\sigma}^{(\mathbf{m}),\beta_2} = c_1(i\tau)^{-(6+4a)}(1+O(\tau^{-1}))\vec{\rho}^{\,-2(\vec{a}+\vec{1})}\rho_4^{-4}\rho_2^{-2}\rho_3^{0+4}\rho_1^{6+8}\mathcal{D}.
$$

Comparing the power of ρ_3 in the above expression, we see that in this case $T_{\tau,\sigma}^{(\mathbf{m}),\beta_2} \prec \mathcal{L}_\tau$. When $\sigma = (3,2,1,4)$, we see in a straightforward way also for other β for which $\vec{S}^\beta = (\mathbf{Q}_0, \mathbf{Q}_0)$, that $T_{\tau,\sigma}^{(\mathbf{m}),\beta} \prec \mathcal{L}_\tau$.

When $\sigma = (1,3,2,4)$, we see that for all β with $|\vec{k}_\beta| = 6$,

$$
\begin{aligned}
T_{\tau,\sigma}^{(\mathbf{m}),\beta} &= C\det(A)(i\tau)^{-(6+4a)}(1+O(\frac{1}{\tau}))\vec{\rho}^{\,-2(\vec{a}-\vec{k}+\vec{1})}\omega_{45}^{-1}\omega_{13}^{-1}\rho_4^{-2}\rho_2^{0}\rho_3^{-2}\rho_1^{-2}\mathcal{P}_\beta \\
&= C\det(A)(i\tau)^{-(6+4a)}(1+O(\frac{1}{\tau}))\vec{\rho}^{\,-2(\vec{a}-\vec{k}+\vec{1})}\rho_4^{-4}\rho_2^{0}\rho_3^{-2}\rho_1^{-4}\mathcal{P}_\beta.
\end{aligned}
$$

Here, $\mathcal{P}_\beta = O(1)$. Thus when $\sigma = (1,3,2,4)$, by comparing the powers of ρ_2 we see that $T_{\tau,\sigma}^{(\mathbf{m}),\beta} \prec \mathcal{L}_\tau$. The same holds in the case when $|\vec{k}_\beta| < 6$. The case when $\sigma = (3,1,2,4)$ is similar to $\sigma = (1,3,2,4)$. This proves Proposition 3.8 □

Summarizing; we have analyzed the terms $T_{\tau,\sigma}^{(\mathbf{m}),\beta}$ corresponding to any β and all σ except $\sigma = \sigma_0 = (2,1,3,4)$. Clearly, the sum $\sum_\beta T_{\tau,\sigma_0}^{(\mathbf{m}),\beta}$ is equal to the sum $\sum_\beta T_{\tau,id}^{(\mathbf{m}),\beta}$. Thus, when the asymptotic orders of ρ_j are given in (124) and the polarizations satisfy (129), we have

$$
\begin{aligned}
\mathcal{G}^{(\mathbf{m})}(v,\mathbf{b}) &= \lim_{\tau\to\infty}\sum_{\beta,\sigma}\frac{T_{\tau,\sigma}^{(\mathbf{m}),\beta}}{(i\tau)^{(6+4a)}} = \lim_{\tau\to\infty}\frac{2T_\tau^{(\mathbf{m}),\beta_1}(1+O(\rho_1))}{(i\tau)^{(6+4a)}} \\
(135)\qquad &= 2c_1\det(A)(1+O(\rho_1))\,\vec{\rho}^{\,-2(\vec{a}+\vec{1})}\rho_4^{-4}\rho_2^{-2}\rho_3^{0}\rho_1^{20}\mathcal{D}.
\end{aligned}
$$

Notice that here ρ_j depend only on $b^{(k)}$, $k = 1,2,3,4,5$.

Let $Y = \mathrm{sym}(\mathbb{R}^{4\times 4})$ and consider the quadratic form $B : (v,w) \mapsto \widehat{g}_{nj}\widehat{g}_{mk}v^{nm}w^{jk}$ as a inner product in Y. Then $\mathcal{D} = B(v^{(5)}, v^{(1)})$.

Let $L(b^{(j)})$ denote the subspace of dimension 6 of the symmetric matrices $v \in Y$ that satisfy equation (54) with covector $b^{(j)}$.

Let $\mathcal{L}$ be the real analytic manifold consisting of $\eta = (\mathbf{b}, \underline{v}, V^{(1)}, V^{(5)})$, where $\mathbf{b} = (b^{(1)}, b^{(2)}, b^{(3)}, b^{(4)}, b^{(5)})$ is a sequence of light-like vectors and $\underline{v} = (v^{(2)}, v^{(3)}, v^{(4)})$ satisfy $v_{(j)} \in L(b^{(j)})$ for all $j = 2,3,4$, and $V^{(1)} = (v_p^{(1)})_{p=1}^6$ be basis of $L(b^{(1)})$ and $V^{(5)} = (v_p^{(5)})_{p=1}^6$ be vectors in Y such that $B(v_p^{(5)}, v_q^{(1)}) = \delta_{pq}$ for $p \le q$.

We define for $\eta \in \mathcal{L}$

$$\kappa(\eta) := \det\Big(\mathcal{G}^{(\mathbf{m})}(\mathbf{v}_{(p,q)}, \mathbf{b})\Big)_{p,q=1}^6, \text{ where } \mathbf{v}_{(p,q)} = (v_p^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}, v_q^{(5)}).$$

Then $\kappa(\eta)$ is a real-analytic function on $\mathcal{L}$.

Let us next consider linearly independent light-like vectors, $\widehat{\mathbf{b}} = (\widehat{b}^{(1)}, \widehat{b}^{(2)}, \widehat{b}^{(3)}, \widehat{b}^{(4)}, \widehat{b}^{(5)})$ satisfying (123) with $\vec{\rho}$ given in (124) with some small $\rho_1 > 0$ and let the polarizations $\widehat{\underline{v}} = (\widehat{v}^{(2)}, \widehat{v}^{(3)}, \widehat{v}^{(4)})$ be such that $\widehat{v}^{(j)} \in L(b^{(j)})$, $j = 2,3,4$, are those given by (129), and $\widehat{V}^{(1)} = (\widehat{v}_p^{(1)})_{p=1}^6$ be a basis of $L(b^{(1)})$. Let $\widehat{V}^{(5)} = (\widehat{v}_p^{(5)})_{p=1}^6$ be vectors in Y such that $B(\widehat{v}_p^{(5)}, \widehat{v}_q^{(1)}) = \delta_{pq}$ for $p \le q$. When $\rho_1 > 0$ is small enough, formula (135) yields that $\kappa(\widehat{\eta}) \neq 0$ for $\widehat{\eta} = (\widehat{\mathbf{b}}, \widehat{\underline{v}}, \widehat{V}^{(1)}, \widehat{V}^{(5)})$. As $\kappa(\eta)$ is a real-analytic function on $\mathcal{L}$, we see that $\kappa(\eta)$ is non-vanishing on a generic subset of the component of $\mathcal{L}$ containing $\widehat{\eta}$. Note that for any $\mathbf{b}$ there is $\eta = (\mathbf{b}, \underline{v}, V^{(1)}, V^{(5)})$ that is in this component.

Consider next $\eta = (\mathbf{b}, \underline{v}, V^{(1)}, V^{(5)})$ that is in the component of $\mathcal{L}$ containing $\widehat{\eta}$. As $v^{(5)} \mapsto \mathcal{G}^{(\mathbf{m})}(\underline{v}, \mathbf{b})$ is linear and thus $\kappa(\eta)$ can be considered as an alternative 6-multilinear form of $V^{(5)}$. Thus if $\widetilde{v}^{(5)} = \sum_{p=1}^6 a_p v_p^{(5)} \neq 0$ is such that $G^{(\mathbf{m})}(v^{(1)}, \underline{v}, \widetilde{v}^{(5)}, \mathbf{b}) = 0$ for all $v^{(1)} \in L(b^{(1)})$, we see that $\kappa(\eta) = 0$. As the image of a open and dense set in the projection $(\mathbf{b}, \underline{v}, V^{(1)}, V^{(5)}) \mapsto (\mathbf{b}, \underline{v})$ is open and dense, we conclude that for an open and dense set of pairs $(\mathbf{b}, \underline{v})$ there is $V^{(5)}$ so that for all $\widetilde{v}^{(5)} \in \mathrm{span}(V^{(5)})$ there is $v^{(1)} \in L(b^{(1)})$ such that $G^{(\mathbf{m})}(v^{(1)}, \underline{v}, \widetilde{v}^{(5)}, \mathbf{b}) \neq 0$. □

4. Observations in normal coordinates

4.1. **Detection of singularities.** Above we have considered the singularities of the metric g in the wave gauge coordinates, that is, we have used the coordinates of manifold M_0 where the metric g solves the $\widehat{g}$-reduced Einstein equations. As the wave gauge coordinates may also be non-smooth, we do not know if the observed singularities are caused by the metric or the coordinates. Because of this, we next consider the metric in normal coordinates.

Let $(g^{\vec{\varepsilon}}, \phi^{\vec{\varepsilon}})$ be the solution of the $\widehat{g}$-reduced Einstein equation (10) with the source $\mathbf{f}_{\vec{\varepsilon}}$ given in (59). We emphasize that $g^{\vec{\varepsilon}}$ is the metric in the $\widehat{g}$-wave gauge coordinates.

Let $(z, \eta) \in \mathcal{U}_{(z_0,\eta_0)}(\widehat{h})$ and denote by $\mu_{\vec{\varepsilon}}([-1,1]) = \mu_{g^{\vec{\varepsilon}},z,\eta}([-1,1])$ the freely falling observers, i.e. time-like geodesic, on $(M_0, g^{\vec{\varepsilon}})$ having the same initial data as $\mu_{\widehat{g},z,\eta}$, see Sect. 1.1. When $\widetilde{r} \in (-1,1]$, we call the family $\mu_{\vec{\varepsilon}}([-1,\widetilde{r}])$, where $\vec{\varepsilon} = (\varepsilon_j)_{j=1}^4$, $\varepsilon_j \geq 0$ the observation geodesics and note that $\vec{\varepsilon}$ may be here the zero-vector. For given (z,η), let us choose $(Z_j(-1))_{j=1}^4$ to be linearly independent vectors at $\mu_{\widehat{g},z,\eta}(-1)$ such that $Z_1(-1) = \dot{\mu}_{\widehat{g},z,\eta}(-1)$. For $s \in [-1,1]$, let $Z_{j,\vec{\varepsilon}}(s)$ be the parallel translation of $Z_j(-1)$ along $\mu_{\vec{\varepsilon}}$. Also, assume that $\mu_{\vec{\varepsilon}}([-1,1]) \subset U_{\vec{\varepsilon}}$ for $|\vec{\varepsilon}|$ small enough and denote $p_{\vec{\varepsilon}} = \mu_{\vec{\varepsilon}}(\widetilde{r})$ where $\widetilde{r} < 1$.

Let then $\Psi_{\vec{\varepsilon}}$ denote normal coordinates of $(M_0, g^{\vec{\varepsilon}})$ defined using the center $p_{\vec{\varepsilon}}$ and the frame $Z_{j,\vec{\varepsilon}}$, $j = 1,2,3,4$. Below, we denote $\mu_0 = \mu_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}$ and $Z_{j,0} = Z_{j,\vec{\varepsilon}}|_{\vec{\varepsilon}=0}$. We say that these normal coordinates are associated to the observation geodesics $\mu_{\vec{\varepsilon}}$, see Fig. 15.

Next we consider the metric $g^{\vec{\varepsilon}}$ in the normal coordinates and study when $\partial^4_{\vec{\varepsilon}}(\Psi_{\vec{\varepsilon}})_* g^{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}$ is smooth. Below, we denote $g_{\vec{\varepsilon}} = g^{\vec{\varepsilon}}$ and $U_{\vec{\varepsilon}} = U_{g^{\vec{\varepsilon}}}$. Recall that $\widehat{U} = U_{\widehat{g}}$. In the next Lemma, we consider observations in normal coordinates, see Fig. 15.

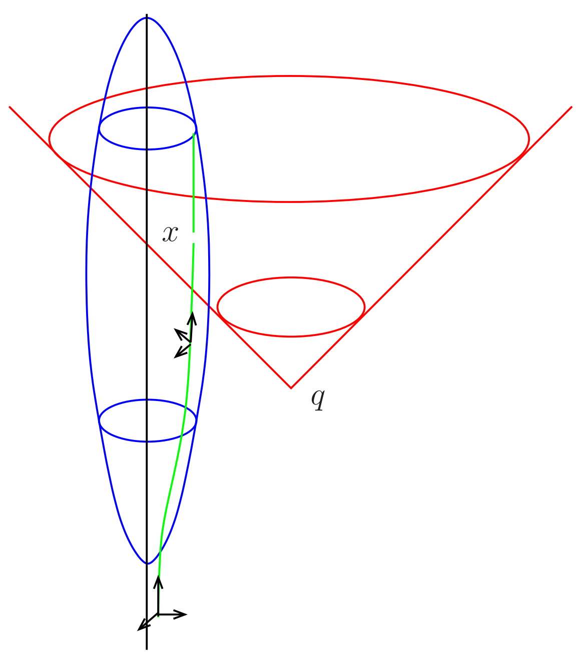

FIGURE 15. A schematic figure where the space-time is represented as the 3-dimensional set $\mathbb{R}^{1+2}$. The future light cone $\mathcal{L}^+_{\widehat{g}}(q)$ corresponding to the point q is shown as a red cone. The green curve is the geodesic $\mu_0 = \mu_{\widehat{g},z,\eta}$. This geodesic intersect the future light cone $\mathcal{L}^+_{\widehat{g}}(q)$ at the point x. The black vectors are the frame (Z_j) that is obtained using parallel translation along the geodesic μ_0. Near the intersection point x we use the normal coordinates centered at x and associated to the frame obtained via parallel translation.

Lemma 4.1. *Let $u^{\vec\varepsilon} = (g^{\vec\varepsilon}, \phi^{\vec\varepsilon})$ be the solution of the reduced Einstein equations (10) with the source $\mathbf{f}_{\vec\varepsilon}$ given in (59), and $\mu_{\vec\varepsilon}([-1,\widetilde r])$, $-1 < \widetilde r < 1$ be the observation geodesics in $U_{\vec\varepsilon}$. Let us consider at the point $p_{\vec\varepsilon} = \mu_{\vec\varepsilon}(\widetilde r)$ the frame $Z_{j,\vec\varepsilon} = Z_{j,\vec\varepsilon}(\widetilde r)$ and let $\Psi_{\vec\varepsilon} : W_{\vec\varepsilon} \to \Psi_{\vec\varepsilon}(W_{\vec\varepsilon}) \subset \mathbb{R}^4$ be the normal coordinates centered at $p_{\vec\varepsilon}$ and associated to the frame $(Z_{j,\vec\varepsilon})_{j=1}^4$. Denote $W = W_0$.*

Let $S \subset \widehat U$ be a smooth 3-dimensional surface such that $p_0 = p_{\vec\varepsilon}|_{\vec\varepsilon=0} \in S$ and

$$g^{(\alpha)} = \partial^\alpha_{\vec\varepsilon} g^{\vec\varepsilon}|_{\vec\varepsilon=0}, \quad \phi^{(\alpha)}_\ell = \partial^\alpha_{\vec\varepsilon} \phi^{\vec\varepsilon}_\ell|_{\vec\varepsilon=0}, \quad \text{for } |\alpha| \le 4,\ \alpha \in \{0,1\}^4,$$

and assume that $g^{(\alpha)}$ and $\phi^{(\alpha)}_l$ are in $C^\infty(W)$ for $|\alpha| \le 3$ and $g^{(\alpha_0)}_{pq}|_W \in \mathcal{I}^{m_0}(W\cap S)$ and $\phi^{(\alpha_0)}_l|_W \in \mathcal{I}^{m_0}(W \cap S)$ for $\alpha_0 = (1,1,1,1)$.

(i) Assume that $S\cap W$ is empty. Then $\partial^4_{\vec\varepsilon}((\Psi_{\vec\varepsilon})_* g_{\vec\varepsilon})|_{\vec\varepsilon=0}$ and $\partial^4_{\vec\varepsilon}((\Psi_{\vec\varepsilon})_* \phi^{\vec\varepsilon}_\ell)|_{\vec\varepsilon=0}$ are C^∞-smooth in $\Psi_0(W)$.

(ii) Assume that $\mu_0([-1,1])$ intersets S transversally at p_0. Consider the conditions

(a) There is a 2-contravariant tensor field v that is a smooth section of $TW \otimes TW$ such that $v(x) \in T_xS \otimes T_xS$ for $x \in S$ and the principal symbol of $\langle v, g^{(\alpha_0)}\rangle \in \mathcal{I}^{m_0}(W; W\cap S)$ is non-vanishing at p_0.

(b) The principal symbol of $\phi^{(\alpha_0)}_\ell$ is non-vanishing at p_0 for some $\ell = 1,2,\dots,L$.

If (a) or (b) holds, then either $\partial^4_{\vec\varepsilon}((\Psi_{\vec\varepsilon})_* g_{\vec\varepsilon})|_{\vec\varepsilon=0}$ or $\partial^4_{\vec\varepsilon}((\Psi_{\vec\varepsilon})_* \phi^{\vec\varepsilon}_\ell)|_{\vec\varepsilon=0}$ is not C^∞-smooth in $\Psi_0(W)$.

Proof. (i) Using the metric $g_{\vec\varepsilon}$ in the $\widehat g$-wave gauge coordinates we can compute the $\Psi_{\vec\varepsilon}$-coordinates and thus find $(\Psi_{\vec\varepsilon})_* g_{\vec\varepsilon}$. As $g^{(\alpha)}$, and thus $\partial^\alpha_{\vec\varepsilon}\Psi_{\vec\varepsilon}|_{\vec\varepsilon=0}$ for $|\alpha| \le 3$ and $g^{(\alpha_0)}$ and $\partial^{\alpha_0}_{\vec\varepsilon}\Psi_{\vec\varepsilon}|_{\vec\varepsilon=0}$ are smooth in $\Psi_0(W)$, we see that $\partial^4_{\vec\varepsilon}(\Psi_{\vec\varepsilon})_* g_{\vec\varepsilon}|_{\vec\varepsilon=0}$ is smooth in $\Psi_0(W)$. This proves (i).

(ii) Let $g^{\vec\varepsilon}$ denote the metric in the $\widehat g$ wave gauge coordinates on M_0. Denote $\gamma_{\vec\varepsilon}(t) = \mu_{\vec\varepsilon}(t + \widetilde r)$. Let $X : W_0 \to V_0 \subset \mathbb{R}^4$, $X(y) = (X^j(y))_{j=1}^4$ be such local coordinates in W_0 that $X(p_0) = 0$ and $X(S\cap W_0) = \{(x^1,x^2,x^3,x^4) \in V_0;\ x^1 = 0\}$ and $y(t) = X(\gamma_0(t)) = (t,0,0,0)$. Note that the coordinates X are independent of $\vec\varepsilon$.

Let $\widehat Y_j(t)$ be $\widehat g$-parallel vector fields on $\gamma_0(t)$ such that $\widehat Y_j(0) = \partial/\partial X^j$ are the coordinate vector fields corresponding to the coordinates X. Let $b^k_j \in \mathbb{R}$ be such that $\widehat Y_j(0) = b^k_j Z_{0,k}(\widetilde r)$ and define $Y^{\vec\varepsilon}_j(t) = b^k_j Z_{\vec\varepsilon,k}(t+\widetilde r)$.

To do computations in local coordinates, let us denote

$$\widetilde g^{(\alpha)} = \partial^\alpha_{\vec\varepsilon}(X_* g^{\vec\varepsilon})|_{\vec\varepsilon=0}, \quad \widetilde\phi^{(\alpha)}_\ell = \partial^\alpha_{\vec\varepsilon}(X_*\phi^{\vec\varepsilon}_\ell)|_{\vec\varepsilon=0}, \quad \text{for } |\alpha| \le 4,\ \alpha \in \{0,1\}^4.$$

We note that as the X-coordinates are independent of $\vec\varepsilon$, we have $\partial^\alpha_{\vec\varepsilon}(X_* g^{\vec\varepsilon})|_{\vec\varepsilon=0} = X_*(\partial^\alpha_{\vec\varepsilon}(g^{\vec\varepsilon})|_{\vec\varepsilon=0})$. In the local X coordinates, let $v(x) = v^{pq}(x)\frac{\partial}{\partial x^p}\frac{\partial}{\partial x^q}$ be such that $v^{pq}(x) = 0$ if $(p,q) \notin \{2,3,4\}^2$ and $x \in X(S\cap W_0)$.

Let $R^{\vec\varepsilon}$ be the curvature tensor of $g^{\vec\varepsilon}$ and define the functions

$$\begin{aligned} h^{\vec\varepsilon}_{mk}(t) &= g^{\vec\varepsilon}(R^{\vec\varepsilon}(\dot\gamma_{\vec\varepsilon}(t),Y^{\vec\varepsilon}_m(t))\dot\gamma_{\vec\varepsilon}(t),Y^{\vec\varepsilon}_k(t)),\\ J_v(t) &= \partial^4_{\vec\varepsilon}(\widehat v^{mq}\,h^{\vec\varepsilon}_{mq}(t))|_{\vec\varepsilon=0}, \end{aligned}$$

where $\widehat v^{mq}=v^{mq}(0)\in\mathbb{R}^{4\times 4}$. Here we can consider $\widehat v^{mq}$ as a constant matrix or alternatively, a tensor field whose representation in the X coordinates is given by a constant matrix.

Observe that $J_v(t)$ is a function defined on the geodesic $\gamma_0(t)$, $t\in I$ that is parametrized along arc length and thus it has a coordinate invariant definition. As all $\vec\varepsilon$ derivatives of order 3 or less of $g_{\vec\varepsilon}$ are smooth, we see that if $\partial^4_{\vec\varepsilon}((\Psi_{\vec\varepsilon})_*g_{\vec\varepsilon})|_{\vec\varepsilon=0}$ would be smooth near $0\in\mathbb{R}^4$, then also $\partial^4_{\vec\varepsilon}((\Psi_{\vec\varepsilon})_*R^{\vec\varepsilon})|_{\vec\varepsilon=0}$ and $\partial^4_{\vec\varepsilon}((\Psi_{\vec\varepsilon})_*\dot\gamma_{\vec\varepsilon})|_{\vec\varepsilon=0}$ and $\partial^4_{\vec\varepsilon}((\Psi_{\vec\varepsilon})_*Y^{\vec\varepsilon}_k)|_{\vec\varepsilon=0}$ would be smooth, and thus also the function $J_v(t)$ would be smooth near $t=0$. Hence, to show that the $\partial^4_{\vec\varepsilon}$-derivatives of the metric tensor in the normal coordinates are not smooth, it is enough to show that for some values of $\widehat v^{mq}$ the function $J_v(t)$ is non-smooth at $t=0$.

The curvature tensor and thus $h^{\vec\varepsilon}_{mk}$ can be written in the X coordiantes as a sum of terms which are products of the x-derivatives of $g_{\vec\varepsilon}$ up to order 2, its inverse matrix $g^{-1}_{\vec\varepsilon}$, evaluated at $\gamma_{\vec\varepsilon}(t)$, and the vector fields $\dot\gamma_{\vec\varepsilon}(t)$ and $Y^{\vec\varepsilon}_k(t)$. When we apply the product rule in the differentation and the chain rule (when we compute e.g. $\partial_{\vec\varepsilon}(g^{jk}_{\vec\varepsilon}(\gamma^{\vec\varepsilon}(t)))$, the first derivative ∂_{ε_1} operates either to $\partial^\beta_x g_{\vec\varepsilon}$, with $|\beta|\le 2$, or $\dot\gamma_{\vec\varepsilon}(t)$, or $Y^{\vec\varepsilon}_k(t)$, or due to chain rule, it produces the x-derivatives of $\widehat g$ multiplied by $\partial_{\varepsilon_1}\gamma_{\vec\varepsilon}(t)$. As in W all ε-derivatives of the metric tensor $g_{\vec\varepsilon}$ up to order 3 are smooth, we see that all other ε-derivatives, namely ∂_{ε_j}, $j=2,3,4$ have to operate on the same term on which the ∂_{ε_1} derivative operated or otherwise, the produced term is C^∞-smooth in the X-coordinates. Thus we need to consider only terms where all four ε-derivatives operate on the same term.

Below, $\Gamma_{\vec\varepsilon}$ are the connection coefficients corresponding to $g_{\vec\varepsilon}$. We will work in the X coordinates and denote $\widetilde R(x)=\partial^4_{\vec\varepsilon}X_*(R_{\vec\varepsilon})(x)|_{\vec\varepsilon=0}$, and $\widetilde\Gamma^j_{nk}$ and $\widetilde\gamma^j$ are the analogous 4th order ε-derivatives, and denote $\widetilde g=\widetilde g^{(\alpha_0)}$. For simplicity we also denote $X_*\widehat g$ and $X_*\widehat\phi_l$ by $\widehat g$ and $\widehat\phi_l$, respectively.

We analyze the functions of $t\in I=(-t_1,t_1)$, e.g., $a(t)$, where $t_1>0$ is small. We say that $a(t)$ is of order n if $a(\,\cdot\,)\in\mathcal{I}^n(\{0\})$.

When $a(t)$ solves an ordinary differential equation (ODE) of the type $\partial_t a(t)+K(t)a(t)=b(t)$, $a(0)=a_0$ where $K(t),b(t)\in\mathcal{I}^n(\{0\})$, we say that $a(t)$ solves an ODE involving $K(t)$. When $n<-1$, this implies, due to [28] and bootstrap arguments, that then $a\in\mathcal{I}^{n-1}(\{0\})$, i.e., a is one order smoother than b and K.

When t_0 is small enough, $S\subset M_0$ intersects $\gamma_0=\gamma_{\vec\varepsilon}|_{\vec\varepsilon=0}$ only at the point p_0, and the intersection there is transversal. Then, we see that the restrictions of conormal distributions in $\mathcal{I}(M_0;S)$ on γ_0

are conormal distributions associated to the submanifold $\{p_0\} \subset \gamma_0$. Thus by the assumptions of the theorem, $(\partial_x^\beta \widetilde g_{jk})(\gamma(t)) \in \mathcal{I}^{m_0+\beta_1}(\{0\})$ when $\beta = (\beta_1,\beta_2,\beta_3,\beta_4)$. As $\widehat g$ and $\widehat \phi_l$ are C^∞-smooth in W_0 and the geodesic γ_0 intersects S transversally, we see that $(\widetilde\gamma(t),\partial_t\widetilde\gamma(t)) = \partial_{\vec\varepsilon}^4(\gamma^{\vec\varepsilon}(t),\partial_t\gamma^{\vec\varepsilon}(t))|_{\vec\varepsilon=0}$ and $\widetilde Y_k = \partial_{\vec\varepsilon}^4 Y_k^{\vec\varepsilon}|_{\vec\varepsilon=0}$ in the X-coordinates are solutions of ODEs (the latter one obtained by differentiating, with respect to $\vec\varepsilon$, the equation $\nabla^{\vec\varepsilon}_{\dot\gamma^{\vec\varepsilon}} Y_k^{\vec\varepsilon}=0$) with coefficients depending on the Christoffel symbols, i.e., on the derivatives $\partial_{\vec\varepsilon}^4(\partial_j g^{\vec\varepsilon}_{pq})|_{\vec\varepsilon=0} \in \mathcal{I}^{m_0+1}(\{0\})$. Thus $(\widetilde\gamma(t),\partial_t\widetilde\gamma(t))$ and $\widetilde Y_k$ are in $\mathcal{I}^{m_0}(\{0\})$.

As the curvature $R_{\vec\varepsilon}$ depends on the 2nd order derivatives of the metric, the above analysis shows that $\widetilde R|_{\gamma_0(t)} \in \mathcal{I}^{m_0+2}(\{0\})$. Thus in the X coordinates $\partial_{\vec\varepsilon}^4(h^{\vec\varepsilon}_{mk}(t))|_{\vec\varepsilon=0} \in \mathcal{I}^{m_0+2}(\{0\})$ can be written as

$$\begin{aligned}
\partial_{\vec\varepsilon}^4(h^{\vec\varepsilon}_{mk}(t))|_{\vec\varepsilon=0} &= \widehat g(\widetilde R(\dot\gamma_0(t),\widehat Y_m(t))\dot\gamma_0(t),\widehat Y_k(t)) + \text{smoother terms}\\
&= \widehat g_{kq}\widetilde R^q_{11m} + \text{smoother terms}\\
&= \widehat g_{qk}\Big(\frac{\partial}{\partial x^1}\widetilde\Gamma^q_{1m} - \frac{\partial}{\partial x^m}\widetilde\Gamma^q_{11}\Big) + \text{smoother terms}\\
&= \frac12\Big(\frac{\partial}{\partial x^1}\Big(\frac{\partial\widetilde g_{km}}{\partial x^1} + \frac{\partial\widetilde g_{k1}}{\partial x^m} - \frac{\partial\widetilde g_{1m}}{\partial x^k}\Big)\\
&\quad -\frac{\partial}{\partial x^m}\Big(\frac{\partial\widetilde g_{k1}}{\partial x^1} + \frac{\partial\widetilde g_{k1}}{\partial x^1} - \frac{\partial\widetilde g_{11}}{\partial x^k}\Big)\Big) + \text{smoother terms},
\end{aligned}$$

where all "smoother terms" are in $\mathcal{I}^{m_0+1}(\{0\})$.

Later, we will use the fact that $\partial/\partial x^1$ raises the order of the singularity by one.

Consider next the case (a). Assume next that for given $(k,m) \in \{2,3,4\}^2$, the principal symbol of $\widetilde g^{(\alpha_0)}_{km}$ is non-vanishing at $0 = X(p_0)$. Let v be such a tensor field that $v^{mk}(0) = v^{km}(0) \neq 0$ and $v^{in}(0) = 0$ when $(i,n) \notin \{(k,m),(m,k)\}$. Then the above yields (in the formula below, we do not sum over k,m)

$$J_v(t) = e_{km}v^{km}\partial_{\vec\varepsilon}^4(h^{\vec\varepsilon}_{mk}(t))|_{\vec\varepsilon=0} = \frac{e_{km}}{2}\widehat v^{km}\Big(\frac{\partial}{\partial x^1}\frac{\partial\widetilde g_{km}}{\partial x^1}\Big) + \text{smoother terms},$$

where $e_{km} = 2-\delta_{km}$. Thus the principal symbol of $J_v(t)$ in $\mathcal{I}^{m_0}(\{0\})$ is non-vanishing and $J_v(t)$ is not a smooth function. Thus in this case $\partial_{\vec\varepsilon}^4((\Psi_{\vec\varepsilon})_*g_{\vec\varepsilon})|_{\vec\varepsilon=0}$ is not smooth.

Next, we consider the case (b). Assume that there is ℓ such that the principal symbol of the field $\widetilde\phi_\ell$ is non-vanishing. As $\partial_t\widetilde\gamma(t)) \in \mathcal{I}^{m_0}(\{0\})$, we see that $\widetilde\gamma(t)) \in \mathcal{I}^{m_0+1}(\{0\})$, Then as $\phi^{\vec\varepsilon}_\ell$ are scalar fields,

$$\begin{aligned}
j_\ell(t) &= \partial_{\vec\varepsilon}^4\Big(\phi^{\vec\varepsilon}_\ell(\gamma_{\vec\varepsilon}(t)))\Big)\Big|_{\vec\varepsilon=0}\\
&= \widetilde\phi_\ell(\widehat\gamma(t)) + \widetilde\gamma^j(t)\frac{\partial\widehat\phi_\ell}{\partial x^j}(\widehat\gamma(t)) + \text{smoother terms},
\end{aligned} \tag{136}$$

where $j_\ell \in \mathcal{I}^{m_0}(\{0\})$ and the smoother terms are in $\mathcal{I}^{m_0+1}(\{0\})$. Again, if both $\partial_{\vec\varepsilon}^4((\Psi_{\vec\varepsilon})_* g_{\vec\varepsilon})|_{\vec\varepsilon=0}$ and $\partial_{\vec\varepsilon}^4((\Psi_{\vec\varepsilon})_* \phi_\ell^{\vec\varepsilon})|_{\vec\varepsilon=0}$ are smooth, we see that $j_\ell(t)$ is smooth, too. Thus to show the claim it is enough to show that $j_\ell(t)$ is not smooth at $t=0$.

In (136), $\partial_1 \widetilde\phi_\ell$ has the order (m_0+1). As we saw above, $\widetilde\gamma(t)$ and $\partial_t\widetilde\gamma(t)$ have the order m_0. Thus $j_\ell(t)$ is not smooth which proves the claim. □

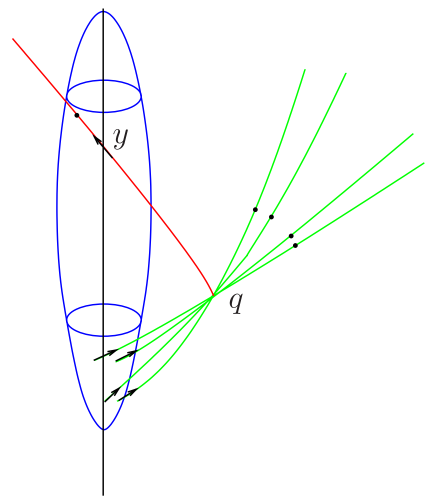

FIGURE 16. A schematic figure where the space-time is represented as the 3-dimensional set $\mathbb{R}^{1+2}$. The light-like geodesic emanating from the point q is shown as a red curve. The point q is the intersection of light-like geodesics corresponding to the starting points and directions $(\vec x, \vec\xi) = ((x_j,\xi_j))_{j=1}^4$. A light like geodesic starting from q passes through the point y and has the direction η at y. The black points are the first conjugate points on the geodesics $\gamma_{x_j(t_0),\xi_j(t_0)}([0,\infty))$, $j=1,2,3,4$, and $\gamma_{q,\zeta}([0,\infty))$. The figure shows the case when the interaction condition (I) is satisfied for $y\in\widehat U$ with light-like vectors $(\vec x,\vec\xi)$.

Next we use the above result to detect singularities in normal coordinates. We will consider the condition that an intersection point q exists and the light cone of q intersects y: We say that the *interaction condition* (I) is satisfied for $y\in\widehat U$ with light-like vectors $(\vec x,\vec\xi)=((x_j,\xi_j))_{j=1}^4$ and $t_0\ge 0$, if

(I) There exists $q\in\bigcap_{j=1}^4 \widehat\gamma_{x_j(t_0),\xi_j(t_0)}((0,\mathbf{t}_j))$, $\mathbf{t}_j=\rho(x_j(t_0),\xi_j(t_0))$, $\zeta\in L_q^+(M_0,\widehat g)$ and $t\ge 0$ such that $y=\widehat\gamma_{q,\zeta}(t)$.

where $(x_j(h),\xi_j(h))$ are given in (69) and the function ρ is defined in (20), see Fig. 16. When (I) holds, we sometimes say that it holds for y with parameters q, ζ, t, t_0, and $\eta=\partial_t\widehat\gamma_{q,\zeta}(t)$.

Let $\mathcal{W}_j(s)=\mathcal{W}_j(s;x_j,\xi_j)$ be the s-neighborhood of (x_j,ξ_j) in TM_0 in the Sasaki metric corresponding to $\widehat g^+$.

As earlier, let $\mu_{\vec\varepsilon}([-1,1])$ be a family of observation geodesics in $(U_{\vec\varepsilon},g_{\vec\varepsilon})$, determined by a geodesic $\mu_0([-1,1])\subset U_{\widehat g}$ of $(M_0,\widehat g)$ and $\Psi_{\vec\varepsilon}:W_{\vec\varepsilon}=B_{g^+_{\vec\varepsilon}}(p_{\vec\varepsilon},R_1)\to\mathbb{R}^4$ be the normal coordinates at the point $p_{\vec\varepsilon}=\mu_{\vec\varepsilon}(\widetilde r)$, $-1\le\widetilde r<1$ associated to the frame $(Z_{j,\vec\varepsilon}(\widetilde r))_{j=1}^4$ obtained by the parallel translation along the geodesic $\mu_{\vec\varepsilon}$.

Next we investigate when some solution $u^{\vec\varepsilon}$ corresponding to $(\vec x,\vec\xi)$ has observable singularities in the normal coordinates $\Psi_{\vec\varepsilon}$ determined by the observation geodesics $\mu_{\vec\varepsilon}([-1,1])$, that have the center $y=\mu_{\vec\varepsilon}(\widetilde r)$ and the frame obtained by parallel translation along $\mu_{\vec\varepsilon}([-1,1])$. Below, we denote $\mu_{\vec\varepsilon}=\mu_0$ when $\vec\varepsilon=\vec 0$ and say that $\Psi_{\vec\varepsilon}$ are the normal coordinates associated to $\mu_{\vec\varepsilon}([-1,1])$ and y.

Using such normal coordinates, we define that point $y\in\widehat U$, satisfy the singularity *detection condition* (D) with light-like directions $(\vec x,\vec\xi)$ and $t_0,\widehat s>0$ if

(D) For any $s,s_0\in(0,\widehat s)$ there are $(x_j',\xi_j')\in\mathcal{W}_j(s;x_j,\xi_j)$, $j=1,2,3,4$, and $\mathbf{f}_j\in\mathcal{I}_S{}^{n-3/2}(Y((x_j',\xi_j');t_0,s_0))$, see (57) and (59), and a family of observation geodesics $\mu_{\vec\varepsilon}([-1,1])$ with $y=\mu_0(\widetilde r)$, such that when $u_{\vec\varepsilon}$ of is the solution of (10) with the source $\mathbf{f}_{\vec\varepsilon}=\sum_{j=1}^4\varepsilon_j\mathbf{f}_j$ and $\Psi_{\vec\varepsilon}$ are the the normal coordinates corresponding to the center $\mu_{\vec\varepsilon}(\widetilde r)$ and the frame obtained by parallel translation along $\mu_{\vec\varepsilon}([-1,1])$, then the function $\partial^4_{\vec\varepsilon}((\Psi_{\vec\varepsilon})_*u_{\vec\varepsilon})|_{\vec\varepsilon=0}$ is not C^∞-smooth in any neighborhood of $0=\Psi_0(y)$.

Below we use the $\mathcal{Y}((\vec x,\vec\xi);t_0,\widehat s)$ set defined in (71).

Lemma 4.2. *Let $(\vec x,\vec\xi)$, and $\mathbf{t}_j$ with $j=1,2,3,4$, $t_0>0$, and $x_6\in\widehat U$ satisfy (82)-(83). Let $t_0,\widehat s>0$ and assume that $y\in\mathcal{V}((\vec x,\vec\xi),t_0)\cap U_{\widehat g}$ satisfies $y\notin\mathcal{Y}((\vec x,\vec\xi);t_0,\widehat s)\cup\bigcup_{j=1}^4\gamma_{x_j,\xi_j}(\mathbb{R})$. Then*

(i) If y does not satisfy condition (I) with $(\vec x,\vec\xi)$ and t_0 then y does not satisfy the condition (D) with $(\vec x,\vec\xi)$ and $t_0,\widehat s>0$.

(ii) Assume $y\in\widehat U$ satisfies the condition (I) with $(\vec x,\vec\xi)$ and t_0 and parameters q,ζ, and $0<t<\rho(q,\zeta)$. Then y satisfies condition (D) with $(\vec x,\vec\xi)$, t_0, and any sufficiently small $\widehat s>0$.

Proof. (i) If $y\notin\mathcal{Y}((\vec x,\vec\xi);t_0,\widehat s)\cup\bigcup_{j=1}^4\gamma_{x_j,\xi_j}(\mathbb{R})$, the same condition holds also for $(\vec x',\vec\xi')$ close to $(\vec x,\vec\xi)$. Thus Prop. 3.4, Prop. 3.5, and Lemma 4.1 imply that (i) holds.

(ii) Let $\mu_{\vec\varepsilon}([-1,1])$ be a family of observation geodesics such that $y=\mu_0(\widetilde r)$ and $\Psi_{\vec\varepsilon}$ be the normal coordinates associated to $\mu_{\vec\varepsilon}([-1,1])$ and y.

Let $R_1>0$ be such that $B_{\widehat g^+}(y,R_1)\subset \mathcal{V}((\vec x,\vec\xi),t_0)\cap \widehat U$, see (83). Our aim is next to show that there is a source $\mathbf{f}_{\vec\varepsilon}$ so that $\partial_{\vec\varepsilon}^4((\Psi_{\vec\varepsilon})_*u_{\vec\varepsilon})|_{\vec\varepsilon=0}$ is not C^∞-smooth in $B_{\widehat g^+}(y,R_1)$ for any $R_1>0$. By making R_1 smaller if necessary, we see that when $\widehat s>0$ is small enough, we have that if p is a cut point of $\gamma_{x_j,\xi_j}([t_0,\infty))$ for some $j\le 4$, then $B_{\widehat g^+}(y,R_1)\cap J^+_{\widehat g}(p)=\emptyset$ and $B_{\widehat g^+}(y,R_1)\cap \mathcal{Y}((\vec x,\vec \xi);t_0,\widehat s)\cup \bigcup_{j=1}^4\gamma_{x_j,\xi_j}(\mathbb{R})=\emptyset$.

Let $\eta=\partial_t\widehat\gamma_{q,\zeta}(t)$ and denote $(y,\eta)=(x_5,\xi_5)$. Let $t_j\in\mathbb{R}$ be such that $\widehat\gamma_{x_j,\xi_j}(t_j)=q$ and denote $b_j=\partial_t\widehat\gamma_{x_j,\xi_j}(t_j)$, $j=1,2,3,4,5$. Note that then $b_5=\zeta$, $x_5=y$, and $t_5=t$.

By Propositions 3.4 and 3.7, arbitrarily near to $b_j\in L^+_qM_0$ there are $b_j'\in L^+_qM_0$ and polarizations $v^{(j)}\in\mathbb{R}^{10+L}$, $j=2,3,4$, i.e., principal symbols at (q,b_j'), and linearly independent polarizations $v^{(5)}_p\in\mathbb{R}^{10+L}$, $p=1,2,3,4,5,6$, and $v^{(1)}_r\in\mathbb{R}^{10+L}$, $r=1,2,3,4,5,6$, having the following properties:

(a) All $j\le 4$, $v^{(j)}$, $j=2,3,4$, and $v^{(1)}_r$, $r=1,2,\dots,6$ are such that their metric components (i.e., g-components of (g,ϕ)) satisfy the divergence conditions for the symbols (54) with the covector ξ being b_j' and b_1', respectively.

(b) If $v^{(5)}\in X_5=\text{span}(\{v^{(5)}_p;\ p=1,2,3,\dots,6\})\setminus\{0\}$ then there exists a vector $v^{(1)}\in X_1=\text{span}(\{v^{(1)}_r;\ r=1,2,3,\dots,6\})$ such that for $\mathbf{v}=(v^{(1)},v^{(2)},v^{(3)},v^{(4)},v^{(5)})$ and $\mathbf{b}'=(b_j')_{j=1}^5$ we have $\mathcal{G}(\mathbf{v},\mathbf{b}')\neq 0$.

Let $s\in(0,\widehat s)$ and $x_j'=\widehat\gamma_{q,b_j'}(-t_j)$ and $\xi_j'=\partial_t\widehat\gamma_{q,b_j'}(-t_j)$, $j=1,2,\dots,4$, and $\xi_5'=\partial_t\widehat\gamma_{q,b_5'}(t_5)$. As the function ρ is lower semi-continuous, we can assume that the b_j' are above chosen to be so close to b_j that $t_j>\rho(q,b_j')$, $(x_j',\xi_j')\in\mathcal{W}_j(s;x_j,\xi_j)$ and $x_5'\in\mathcal{V}((\vec x,\vec\xi),t_0)\cap U_{\widehat g}$, see (83). We denote $(\vec x',\vec\xi')=((x_j',\xi_j'))_{j=1}^4$.

As $\rho(q,\zeta)>t$, and the function ρ is lower semi-continuous, we can also assume that b_5' is so close to ζ that $\rho(q,b_5')>t$ and $x_5'\in B_{\widehat g^+}(y,R_1)$. By assuming that $V\subset B_{\widehat g^+}(y,R_1)$ is a sufficiently small neighborhood of x_5', we have that $S:=\mathcal{L}^+_{\widehat g}(q)\cap V$ is a smooth 3-submanifold.

Next, consider the parametrix $\mathbf{Q}^*_{\widehat g}$ corresponding to the linear wave equation with a reversed causality and the gaussian beam $u_\tau=\mathbf{Q}^*_{\widehat g}F_\tau$, produced by a source F_τ and function h given in (72). When $h(x_5')=\overline{w}$ and w is the principal symbol (i.e. the polarization) of u_τ at (q,b_5'), we can use the techniques of [44, 75], see also [4, 43] to obtain an analogous result to Lemma 3.2 for the propagation of singularities along the geodesic $\widehat\gamma_{q,b_5'}([0,-t_5])$, and see that $w=(R_{(5)})^*\overline{w}$, where $R_{(5)}$ is a bijective linear map similar to map $R_{(5)}(q,b_5';x_5',\xi_5')$ considered in the formula (56), that is obtained by solving a system of linear ordinary differential equations along the geodesic connecting q to x_5'. Let us also denote $R_{(1)}=R(q,b_1';x_1',\xi_1')$ and recall that the map $R_{(1)}$ is bijective, too.

Consider next $\mathbf{b}'$ and $w^{(2)},w^{(3)},w^{(4)}$ as parameters and let

$$\begin{aligned}\mathcal{W}_5 &= \{(R_{(1)})^{-1}w^{(1)},(R^*_{(5)})^{-1}w^{(5)})\ ;\ w^{(5)}\in X_5,\ w^{(1)}\in X_1,\\ &\qquad \text{and } \mathcal{G}(\mathbf{w},\mathbf{b}')\neq 0 \text{ for } \mathbf{w}=(w^{(1)},w^{(2)},w^{(3)},w^{(4)},w^{(5)})\}.\end{aligned}$$

Let $W_5=\pi_2(\mathcal{W}_5)\cup\{0\}$, where $\pi_2:(\overline{w}_1,\overline{w}_5)\mapsto\overline{w}_1$. When $\overline{w}^{(5)}\in W_5$ is non-zero, there is $\overline{w}^{(1)}\in X_1$ such that $(\overline{w}^{(1)},\overline{w}^{(5)})\in\mathcal{W}_5$. Then, by Lemma 3.2, there are $\mathbf{f}_j\in\mathcal{I}_S{}^{n-3/2}(Y((x_j',\xi_j');t_0,s_0))$ that have the principal symbols $\overline{w}^{(j)}=R(q,b_j';x_j',\xi_j')^{-1}w^{(j)}$ at (x_j',ξ_j'), $j\leq 4$. Moreover, let $u_{\vec\varepsilon}=(g^{\vec\varepsilon},\phi^{\vec\varepsilon})$ be the solution corresponding to $\mathbf{f}_{\vec\varepsilon}=\sum_{j=1}^4\varepsilon_j\mathbf{f}_j$ and $u_\tau=\mathbf{Q}^*_{\widehat g}F_\tau$ is a past propagating gaussian beam sent in the direction $(x_5',-\xi_5')$, defined in (72) with functions F_τ and $h(x)$ such that $h(x_5')=\overline{w}^{(5)}$. Then we see for $\mathcal{M}^{(4)}=\partial^4_{\vec\varepsilon}u_{\vec\varepsilon}|_{\vec\varepsilon=0}$ that the inner product $\langle F_\tau,\mathcal{M}^{(4)}\rangle_{L^2(\widehat U)}$ is not of the order $O(\tau^{-N})$ for all $N>0$. Next, let us continue $\overline{w}^{(5)}$ from the point x_5' to a smooth section of $\mathcal{B}^L$. Then the above implies that the function $x\mapsto\langle\overline{w}^{(5)}(x),\mathcal{M}^{(4)}(x)\rangle_{\mathcal{B}^L}$ is not smooth in any neighborhood of x_5'.

Roughly speaking, this means that in the wave gauge coordinates $\mathcal{M}^{(4)}$ has wave front set at (x_5',ξ_5') with a polarization that is not perpendicular to $\overline{w}^{(5)}$.

Let us consider a family of observation geodesics $\mu'_{\vec\varepsilon}([0,1])$ such that $p_{\vec\varepsilon}=\mu'_{\vec\varepsilon}(\widetilde r)$ satisfies $p_0=x_5'$ and that for some $r_-,r_+\in(-1,1)$ satisfying $r_-<\widetilde r<r_+$, the curve $\widehat\gamma(r)=\mu_0(r)$, $r\in[r_-,r_+]$, is a causal geodesic that intersects S only at x_5', the intersection is transversal, and $\widehat\gamma\subset V$. Let $\Psi_{\vec\varepsilon}$ be the normal coordinates at $p_{\vec\varepsilon}$ associated to the observation geodesics $\mu'_{\vec\varepsilon}([-1,1])$ centered at the point $\mu'_{\vec\varepsilon}(\widetilde r)$.

Let $\mathcal{X}=\text{symm}(T_{x_5'}M_0\otimes T_{x_5'}M_0)+\mathbb{R}^L$ be the linear space of dimension $(10+L)$. By the property (b) above, $W_5\subset\mathcal{X}$ is a linear subspace containing X_5 that has dimension 6 and the dimension of $V_5=\text{symm}(T_{x_5'}S\otimes T_{x_5'}S)+\mathbb{R}^L\subset X$ is $(6+L)$. Thus the dimension of $\mathcal{V}_5\cap W_5$ is at least two. In particular, it contains a non-zero vector.

Next, let $\overline{w}^{(5)}\in(W_5\cap V_5)\setminus\{0\}$ and $\overline{w}^{(1)}\in(R_{(1)})^{-1}X_1$ be such that $(\overline{w}^{(1)},\overline{w}^{(5)})\in\mathcal{W}_5$. By Lemma 4.1, we then see that $\partial^4_{\vec\varepsilon}((\Psi_{\vec\varepsilon})_*u_{\vec\varepsilon})|_{\vec\varepsilon=0}$ is not C^∞-smooth in any neighborhood of $\Psi_0(x_5')$. This means that there are sources $\mathbf{f}_j\in\mathcal{I}_S{}^{n-3/2}(Y((x_j',\xi_j');t_0,s_0))$ with polarizations $(\overline{w}^{(1)},\overline{w}^{(2)},\overline{w}^{(3)},\overline{w}^{(4)})$ that cause singularities in $\partial^4_{\vec\varepsilon}((\Psi_{\vec\varepsilon})_*u_{\vec\varepsilon})|_{\vec\varepsilon=0}$ near $\Psi_0(x_5')$, that is, the singularities that one can observe in the normal coordinates. As R_1 is arbitrarily small so that x_5' can be assumed to be in an arbitrary neighborhood of y, we see using the above and the normal coordinates associated a family of observation geodesics $\mu_{\vec\varepsilon}([-1,1])$ and $y=\mu_0(\widetilde r)$ that condition (D) is valid for y. This proves (ii). $\square$

Above we considered the solution $u_{\vec{\varepsilon}}$ and a source $\mathbf{f}_{\vec{\varepsilon}}$ satisyfing the conditions (60) in the wave guide coordinates. However, we do not know the wave guide coordinates in the set (U_g, g) and thus thus we do not know which element of the data set $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$, see (15), corresponds to the source $\mathbf{f}_{\vec{\varepsilon}}$. This problem is solved in the following lemma.

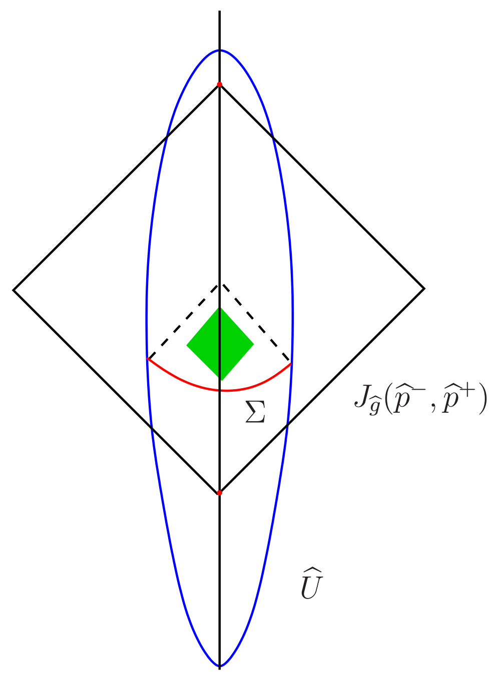

FIGURE 17. A schematic figure where the space-time is represented as the 3-dimensional set $\mathbb{R}^{1+1}$. The figure shows the objects used in the proof of Lemma 4.3. Figure displays the subsets of $(M_0, \widehat{g})$. The green diamond in the figure is the set V where the source is supported. The dashed line shows the set from which all causal curves intersect the surface $\Sigma \subset \widehat{U}$.

Lemma 4.3. *Assume that we are given* $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$ *where* $\varepsilon > 0$ *is small enough. Let* $0 < r_3 < r_2 < r_1$*, where* r_1 *is the parameter used to define* W_g *and* $s_- + r_1 < s_1 < s_+$*. When* $\varepsilon_1, \varepsilon_2 > 0$ *are small enough the following holds:*

(i) Assume that we are given (U', g', ϕ', F') *such that the equivalence class* $[(U', g', \phi', F')]$ *is in* $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$*, and moreover, we have* $\mathcal{N}^{(16)}_{\widehat{g}}(g') < \varepsilon_1$*,* $\mathcal{N}^{(16)}(F') < \varepsilon_1$*, and*

$$K' \subset I_{g'}(\mu_{g'}(s_1 - r_2), \mu_{g'}(s_1)) \quad \text{where } K' = supp\,(F'). \tag{137}$$

Then we can determine a source $F \in C^\infty_0(W_{\widehat{g}})$ *such that* $(U', g', \phi', F') \in [(U_g, g|_{U_g}, \phi|_{U_g}, F|_{U_g})]$*, where* (g, ϕ) *is the solution of the* $\widehat{g}$*-reduced Einstein equations (10) with the source* F*.*

(ii) Let $K \subset \widehat{U}$ be a compact set such that

$$K \subset I_{\widehat{g}}(\mu_{\widehat{g}}(s_1 - r_3), \mu_{\widehat{g}}(s_1)). \tag{138}$$

When $F \in C_0^\infty(W_{\widehat{g}})$ satisfies supp$(F) \subset K$ and $\mathcal{N}^{(16)}(F) < \varepsilon_2$, we can find the element $[(U_g, g|_{U_g}, \phi|_{U_g}, F|_{U_g})]$ in $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$, where (g, ϕ) is the solution of the $\widehat{g}$-reduced Einstein equations (10) with the source F. Moreover, we can find $(\Psi_\mu)_* g$ and $(\Psi_\mu)_* \phi$ where Ψ_μ are normal coordinates associated to a given geodesic $\mu = \mu_{g,z,\eta}([-1, 1]) \subset U_g$, that is, these normal coordinates are centered at the end point of μ and are associated to the frame obtained by parallel translation along μ.

Proof. (i) As $[(U', g', \phi', F')] \in \mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$, there exists a source F on M_0 and a solution (g, ϕ) of the $\widehat{g}$-reduced Einstein equations (10) on M_0 with the source F such that $(U_g, g|_{U_g}, \phi|_{U_g}, F|_{U_g}) \in [(U', g', \phi', F')]$.

By definition, $(U_g, g|_{U_g}, \phi|_{U_g}, F|_{U_g}) \in [(U', g', \phi', F')]$. This implies that there exists a diffeomorphic isometry $f : (U', g') \to (U_g, g)$. Let us next denote the causal domain in (U', g') by $I^+_{g'}(p')$ and $I^-_{g'}(p')$ etc., where $p' \in U'$. Let $V' = I_{g'}(\mu_{g'}(s_1 - r_2), \mu_{g'}(s_1))$ and denote $V = f(V')$, see Fig. 17. For clarity, we denote the causal domains in (M, g) by $I^+_g(p)$ and $I^-_g(p)$, etc., where $p \in M$.

Let $\Sigma' \subset U'$ be a smooth space-like 3-dimensional surface such that $J^+_{g'}(K') \cap \Sigma' = \emptyset$ and for all $x \in \mathrm{cl}(V')$ any non-extendable past-directed causal curve in (U', g') starting from x intersects Σ'. Then the space-like surface $\Sigma = f(\Sigma') \subset \widehat{U}$ is such that $(J^-_{g'}(\Sigma'), g')$ is isometric to $(J^-_{U_g, g}(\Sigma), g)$ and also to $(J^-_{\widehat{g}}(\Sigma), \widehat{g})$. Next we identify these sets as well Σ' and Σ. Then the restriction $f_1 = f|_{J^-_{g'}(V') \cap J^+_{g'}(\Sigma)}$, that is, the isometry $f_1 : (J^-_{g'}(V') \cap J^+_{g'}(\Sigma), g') \to (J^-_g(V) \cap J^+_g(\Sigma), \mathrm{g})$, is the identity map in a neighborhood of Σ.

Assume next that ε_1 and r_2 are so small that $f(J^-_{g'}(V') \cap J^+_{g'}(\Sigma)) \subset \widehat{U}$. Considering the particular case when (U', g', ϕ', F') is equal to (U_g, g, ϕ, F) and f_1 is the identity map, we see using the harmonicity condition (155) that f_1 is also a $(g', \widehat{g})$-wave map $f_1 : J^-_{g'}(V') \cap J^+_{g'}(\Sigma) \to \widehat{U}$ that is an identity map near Σ. Thus, as the given data contain the pairs (U', g') and $(\widehat{U}, \widehat{g})$ and we have fixed above the surface Σ, we can determine f_1 by solving a Cauchy problem for the wave map equation. Hence, we can find $F = (f_1)_* F'$ in $f_1(J^-_{g'}(V') \cap J^+_{g'}(\Sigma))$. As F vanishes outside this this set, we can find F in $\widehat{U}$ by extending the obtained function by zero. This proves (i).

(ii) Assume that $\varepsilon_2 \in (0, \varepsilon)$ is so small that the Einstein equation (10) have solution (g, ϕ) for all $F \in \mathbb{H}$, where

$$\mathbb{H} = \{F \in C_0^\infty(W_{\widehat{g}});\ K = \mathrm{supp}\,(F) \text{ satisfies (138) and } \mathcal{N}^{(16)}_{\widehat{g}}(F) < \varepsilon_2\}.$$

When $\varepsilon_2 > 0$ and $r_3 > 0$ are small enough, for any $F \in \mathbb{H}$ is such that the unique solution (g, ϕ) and F satisfy $\mathcal{N}^{(16)}_{\widehat{g}}(g) < \varepsilon_1$ and (137). Then, there is a unique equivalence class $[(U_g, g|_{U_g}, \phi|_{U_g}, F|_{U_g})] \in \mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$ and thus the map $\mathcal{M} : \mathbb{H} \to \mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$ is injective. Observe that for given $[(U_g, g|_{U_g}, \phi|_{U_g}, F|_{U_g})] \in \mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$ we can verify if it belongs in $\mathbb{H}$. By (i) we can construct the inverse of the map $\mathcal{M} : \mathbb{H} \to \mathcal{M}(\mathbb{H})$, and thus we can construct also the map $\mathcal{M} : \mathbb{H} \to \mathcal{M}(\mathbb{H})$. Hence, when $F \in \mathbb{H}$, we can determine $[(U_g, g|_{U_g}, \phi|_{U_g}, F|_{U_g})]$ and represent g and ϕ in the normal coordinates given in the claim. This yields (ii). $\square$

By Lemma 4.3, for the smooth sources $\mathbf{f}_{\vec{\varepsilon}}$ satisfying conditions (60) and $|\vec{\varepsilon}|$ small enough we can find $(\Psi_{\vec{\varepsilon}})_* u_{\vec{\varepsilon}}$ where $\Psi_{\vec{\varepsilon}}$ are the normal coordinates associated to an observation geodesic on $(M_0, g_{\vec{\varepsilon}})$, where $u_{\vec{\varepsilon}} = (g_{\vec{\varepsilon}}, \phi_{\vec{\varepsilon}})$. The non-smooth sources F, given on $\widehat{U}$ e.g. in the Fermi coordinates of $(\widehat{U}, \widehat{g})$, for which $\mathcal{N}^{(16)}(F) < \varepsilon$, can clearly been approximated by smooth sources. Thus when $(\vec{x}, \vec{\xi})$ and the observation geodesic $\mu_0([-1,1])$ are given as well as the data set $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$, for the sources $\mathbf{f}_{\vec{\varepsilon}}$ with $\mathbf{f}_j \in \mathcal{I}_S{}^{n-3/2}(Y((x_j', \xi_j'); t_0, s_0))$, where $(-n)$ is large enough and $(x_j', \xi_j') \in \mathcal{W}(s; x_j, \xi_j)$, we can compute the derivatives $\partial_{\vec{\varepsilon}}^4((\Psi_{\vec{\varepsilon}})_* u_{\vec{\varepsilon}})|_{\vec{\varepsilon}=0}$. Hence, using the observation geodesics $\mu_{\vec{\varepsilon}}([-1,1])$ and $\mu_0(\widetilde{r}) = y$ we can check if the condition (D) is valid for y with the given $(\vec{x}, \vec{\xi})$, s_0 and t_0 or not.

5. Determination of light observation sets for Einstein equations

In this section we use only the metric $\widehat{g}$ and denote often $\widehat{g} = g$, $\widehat{\gamma} = \gamma$, $U = U_{\widehat{g}}$, etc. Below, let $\mathrm{cl}(A)$ denote the closure of the set A.

Our next aim is to handle the technical problem that in the set $\mathcal{Y}((\vec{x}, \vec{\xi}))$, see (71) we have not analyzed if we observe singularities or not. To/aim this end, we define next the sets $\mathcal{S}_{reg}((\vec{x}, \vec{\xi}), t_0)$ of points near which we observe singularities in a 3-dimensional set.

Definition 5.1. *Let $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4$ be a collection of light-like vectors with $x_j \in U_g$ and $t_0 > 0$. We define to $\mathcal{S}((\vec{x}, \vec{\xi}), t_0)$ be the set of those $y \in U_g$ that satisfies the property (D) with $(\vec{x}, \vec{\xi})$ and t_0 and some $\widehat{s} > 0$.*

Moreover, let $\mathcal{S}_{reg}((\vec{x}, \vec{\xi}), t_0)$ be the set of the points $y_0 \in \mathcal{S}((\vec{x}, \vec{\xi}), t_0)$ having a neighborhood $W \subset U_g$ such that the intersection $W \cap \mathcal{S}((\vec{x}, \vec{\xi}), t_0)$ is a non-empty smooth 3-dimensional submanifold. We denote

$$\mathcal{S}^{cl}(\vec{x}, \vec{\xi}), t_0) = cl(\mathcal{S}_{reg}((\vec{x}, \vec{\xi}), t_0)) \cap U_g.$$

Note that the data $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$ determines the sets $\mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$. Our goal is to show that $\mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$ coincides with the intersection of the light cone $\mathcal{L}_g^+(q)$ and U_g where q is the intersection point of the geodesics corresponding to $(\vec{x}, \vec{\xi})$, see Fig. 15 and 16.

Let us next motivate the analysis we do below: We will consider how to create an artificial point source using interaction of spherical waves propagating along light-like geodesics $\gamma_{x_j,\xi_j}(\mathbb{R}_+)$ where $(\vec x,\vec \xi)=((x_j,\xi_j))_{j=1}^4$ are perturbations of a light-like (y,ζ), $y\in\widehat\mu$. We will use the fact that for all $q\in J(p^-,p^+)\setminus\widehat\mu$ there is a light-like geodesic $\gamma_{y,\zeta}([0,t])$ from $y=\widehat\mu(f_{\widehat\mu}^-(q))$ to q with $t\le\rho(y,\zeta)$. We will next show that when we choose (x_j,ξ_j) to be suitable perturbations of $\partial_t\widehat\gamma_{y,\zeta}(t_0)$, $t_0>0$, it is possible that all geodesics $\widehat\gamma_{x_j,\xi_j}(\mathbb{R}_+)$ intersect at q before their first cut points, that is, $\widehat\gamma_{x_j,\xi_j}(r_j)=q$, $r_j<\rho(x_j,\xi_j)$. We note that we can not analyze the interaction of the waves if the geodesics intersect after the cut points as then the spherical waves can have caustics. Such interactions of wave caustics can, in principle, cause propagating singularities. Thus the sets $\mathcal{S}^{cl}((\vec x,\vec\xi),t_0)$ contain singularities propagating along the light cone $\mathcal{L}_g^+(q)$ and in addition that they may contain singularities produced by caustics that we do not know how to analyze (that could be called "messy waves"). Fortunately, near an open and dense set of geodesics $\mu_{z,\eta}$ the nice singularities propagating along the light cone $\mathcal{L}_g^+(q)$ arrive before the "messy waves". This is the reason why we consider below the first observed singularities on geodesics $\mu_{z,\eta}$. Let us now return to the rigorous analysis.

Below in this section we fix t_0 to have the value $t_0=4\kappa_1$, cf. Lemma 2.10. Recall the notation that

$$(x(t_0),\xi(t_0))=(\gamma_{x,\xi}(t_0),\dot\gamma_{x,\xi}(t_0)),$$
$$(\vec x(t_0),\vec\xi(t_0))=((x_j(t_0),\xi_j(t_0)))_{j=1}^4.$$

Lemma 5.2. *Let $\vartheta>0$ be arbitrary, $q\in J(p^-,p^+)\setminus\widehat\mu$ and let $y=\widehat\mu(f_{\widehat\mu}^-(q))$ and $\zeta\in L_y^+M_0$, $\|\zeta\|_{g^+}=1$ be such that $\gamma_{y,\zeta}([0,r_1])$, $r_1>t_0=4\kappa_1$ is a longest causal (in fact, light-like) geodesic connecting y to q. Then there exists a set $\mathcal{G}$ of 4-tuples of light-like vectors $(\vec x,\vec\xi)=((x_j,\xi_j))_{j=1}^4$ such that the points $x_j(t_0)=\gamma_{x_j,\xi_j}(t_0)$ and the directions $\xi_j(t_0)=\dot\gamma_{x_j,\xi_j}(t_0)$ have the following properties:*

(i) *$x_j\in U_g$, $x_j\notin J^+(x_k)$ for $j\neq k$,*
(ii) *$d_{g^+}((x_l,\xi_l),(y,\zeta))<\vartheta$ for $l\le 4$,*
(iii) *$q=\gamma_{x_j,\xi_j}(r_j)$ and $\rho(x_j(t_0),\xi_j(t_0))+t_0>r_j$,*
(iv) *when $(\vec x,\vec\xi)$ run through the set $\mathcal{G}$, the directions $(\dot\gamma_{x_j,\xi_j}(r_j))_{j=1}^4$ form an open set in $(L_q^+M_0)^4$.*

In addition, $\mathcal{G}$ contains elements $(\vec x,\vec\xi)$ for which $(x_1,\xi_1)=(y,\zeta)$.

Proof. Let $\eta=\dot\gamma_{y,\zeta}(r_1)\in L_q^+M_0$. Let us choose light-like directions $\eta_j\in T_qM_0$, $j=1,2,3,4$, close to η so that η_j and η_k are not parallel for $j\neq k$. In particular, it is possible (but not necessary) that $\eta_1=\eta$. Let $\mathbf{t}:M\to\mathbb{R}$ be a time-function on M that can be used to identify M and $\mathbb{R}\times N$. Moreover, let us choose $T\in(\mathbf{t}(y),\mathbf{t}(\gamma_{y,\zeta}(r_0-t_0)))$ and for $j=1,2,3,4$, let $s_j>0$ be such that $\mathbf{t}(\gamma_{q,\eta_j}(-s_j))=T$. Choosing first T to be sufficiently close $\mathbf{t}(y)$ and then all η_j, $j = 1, 2, 3, 4$ to be sufficiently close to η and defining $x_j = \gamma_{q,\eta_j}(-s_j)$ and $\xi_j = \dot\gamma_{q,\eta_j}(-s_j)$ we obtain the pairs (x_j, ξ_j) satisfying the properties stated in the claim. As vectors η_j can be varied in sufficiently small open sets so that the properties stated in claim stay valid, we obtain that claim concerning the open set of light-like directions.

The last claim follows from the fact that η_1 may be equal to η and $T = \mathbf{t}(y)$. □

Next we analyze the set $\mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0) = \text{cl}\,(\mathcal{S}_{reg}((\vec{x}, \vec{\xi}), t_0))$. We recall that the set $\mathbf{e}_U(q)$ is the points on $\mu_{z,\eta}$ on which the light cone $\mathcal{L}^+_g(q)$ is observed at the earliest time, see Def. 2.4.

Lemma 5.3. *Let $(\vec{x}, \vec{\xi})$, and $\mathbf{t}_j = \rho(x_j(t_0), \xi_j(t_0))$ with $j = 1, 2, 3, 4$, $t_0 = 4\kappa_1$, and $x_6 \in U_g$ and satisfy (82)-(83) and assume that ϑ_0 in (82) and Lemma 2.10 is so small that for all $j \le 4$, $x_j \in I_g(\mu_g(s - r_1), \mu_g(s)) = I^+_g(\mu_g(s - r_1)) \cap I^-_g(\mu_g(s))$ with some $s \in (s_- + r_1, s_+)$, where r_1 appears in Lemma 4.3. Let $\mathcal{V}((\vec{x}, \vec{\xi}), t_0)$ be the set defined in (83) and consider $y \in \mathcal{V}((\vec{x}, \vec{\xi}), t_0) \cap U_g$. Then*

(i) Recall that if $y = \gamma_{q,\zeta}(t)$ with $t \le \rho(q, \zeta)$, then $y \in \mathbf{e}_U(q)$. Assume that y satisfies the condition (I) with $(\vec{x}, \vec{\xi})$ and t_0 with parameters q, ζ, and t such that $0 \le t \le \rho(q, \zeta)$, that is, $y \in \mathbf{e}_U(q)$. Then $y \in \mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$.

(ii) Assume y does not satisfy condition (I) with $(\vec{x}, \vec{\xi})$ and t_0. Then y has a neighborhood that does not intersect $\mathcal{S}_{reg}((\vec{x}, \vec{\xi}), t_0)$. In particular, $y \notin \mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$.

(iii) If $\mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0) \cap \mathcal{V}((\vec{x}, \vec{\xi}), t_0) \neq \emptyset$ then the geodesics corresponding to $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4$ intersect (see Def. 3.3) and there is a unique point $q \in \mathcal{V}((\vec{x}, \vec{\xi}), t_0)$ where the the intersection takes place.

The point q considered in Lemma 5.3 (iii) is the earliest point in the set $\cap_{j=1}^4 \gamma_{x_j,\xi_j}([t_0, \infty))$ and we denote it by $q = Q((\vec{x}, \vec{\xi}), t_0)$. If such intersection point does not exists, we define $Q((\vec{x}, \vec{\xi}), t_0)$ to be the empty set.

Proof. (i) Clearly, if $t = 0$ so that $q \in U_g$, we have $q \in \mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$.

Assume next that $0 < t < \rho(q, \zeta)$. Let $r_2 > 0$ and $V(y, r_2)$ be the set of points $y' \in B_{g^+}(y, r_2)$ such that there exists a family of observation geodesics $\mu_{\vec{\varepsilon}}([-1, 1])$ and $y' = \mu_{\vec{\varepsilon}}(\widetilde{r})$ satisfying condition (D). Let $\Gamma = \exp^g_q(\mathcal{X}((\vec{x}, \vec{\xi}), t_0)$, see (71), be the set of points on the light cone on which singularities caused by 3-interactions may appear. Then the closure of $\mathcal{L}^+_g(q) \setminus \Gamma$ is $\mathcal{L}^+_g(q)$. As $t < \rho(q, \zeta)$, we see that when $r_2 > 0$ is small enough, $B_{g^+}(y, r_2) \cap \mathcal{L}^+_g(q)$ is a smooth 3-submanifold having non-empty intersection with any neighborhood of y and $B_{g^+}(y, r_2) \cap \Gamma$ is a submanifold of dimension 2. Moreover, we see using Lemma 4.2

that for $r_2 > 0$ small enough that $B_{g^+}(y,r_2) \cap (\mathcal{L}_g^+(q) \setminus \Gamma) \subset V(y,r_2)$ and $V(y,r_2) \subset \mathcal{L}_g^+(q)$. Thus $(\mathcal{L}_g^+(q) \setminus \Gamma) \cap U_g \subset S_{reg} \subset \mathcal{L}_g^+(q) \cap U_g$, where the complement of Γ in $\mathcal{L}_g^+(q)$ is dense in $\mathcal{L}_g^+(q)$. The claim (i) follows from this in the case when $t < \rho(q,\zeta)$.

Assume next that $y = \gamma_{q,\zeta}(t)$ where ζ is light-like vector and $t = \rho(q,\zeta)$. Let $t_j < t$ be such that $t_j \to t$ as $j \to \infty$. Then the $t_j < \rho(q,\zeta)$ and the above yields $y_j = \gamma_{q,\zeta}(t_j) \in \mathcal{S}^{cl}((\vec{x},\vec{\xi}),t_0)$. As $y_j \to y$ as $j \to \infty$ we see that $y \in \mathcal{S}^{cl}((\vec{x},\vec{\xi}),t_0)$. This proves (i).

(ii) It follows from the assumption that either there are no intersection point q or that $y \notin \mathcal{L}_g^+(q)$. If $y \notin A = \Gamma \cup \bigcup_{j=1}^4 \gamma_{x_j,\xi_j}([t_0,\infty))$, it follows from Lemma 4.2 that y has a neighborhood V that does not intersect the set $S((\vec{x},\vec{\xi}),t_0)$. If $y \in A$, we see, using the fact that y does not satisfy condition (I) with $(\vec{x},\vec{\xi})$ and t_0, that y has a neighborhood V such that $V \setminus A$ does not intersect $S((\vec{x},\vec{\xi}),t_0)$. The Hausdorff dimension of the set $A \cap V$ is less or equal to 2. Thus $V \cap S_{reg}((\vec{x},\vec{\xi}),t_0) = \emptyset$. This proves (ii).

(iii) Using the conditions posed for $(\vec{x},\vec{\xi})$, t_0 and $\mathbf{t}_j$ with $j = 1,2,3,4$, and x_6 we see that geodesics $\gamma_{x_j,\xi_j}([t_0,\infty)$ can intersect only once in $\mathcal{V}((\vec{x},\vec{\xi}),t_0)$. Moreover, if there is no such intersection, the above shows that $\mathcal{S}^{cl}((\vec{x},\vec{\xi}),t_0) \cap \mathcal{V}((\vec{x},\vec{\xi}),t_0) = \emptyset$, which proves the claim (iii). □

Recall that $\mathcal{D}(\widehat{g},\widehat{\phi},\varepsilon)$ determines the sets $\mathcal{S}^{cl}((\vec{x},\vec{\xi}),t_0)$. Below, denote

$$\mathcal{S}^{cl}{}_{z,\eta}((\vec{x},\vec{\xi}),t_0) = e_{z,\eta}(\mathcal{S}^{cl}((\vec{x},\vec{\xi}),t_0)),$$

(see (18) and Def. 5.1).

Next we summarize the above results in a form that is convenient below.

Lemma 5.4. *Let* $(\vec{x},\vec{\xi})$, *and* $\mathbf{t}_j = \rho(x_j(t_0),\xi_j(t_0))$ *with* $j = 1,2,3,4$, $t_0 = 4\kappa_1$, *and* $x_6 \in U_g$ *and conditions (82)-(83) are some satisfied with some sufficiently small* ϑ_0.

Let $(z,\eta) \in \mathcal{U}_{z_0,\eta_0}$. *Then we have*

(i) If $Q((\vec{x},\vec{\xi}),t_0) = \emptyset$ *or* $Q((\vec{x},\vec{\xi}),t_0) \notin J^-(x_6)$ *then*

$$\mathcal{S}^{cl}{}_{z,\eta}((\vec{x},\vec{\xi}),t_0) \cap J^-(x_6) = \emptyset, \tag{139}$$

(ii) If $Q((\vec{x},\vec{\xi}),t_0) \neq \emptyset$ *and* $q = Q((\vec{x},\vec{\xi}),t_0) \in J^-(x_6)$ *then*

$$\mathcal{S}^{cl}{}_{z,\eta}((\vec{x},\vec{\xi}),t_0) \cap J^-(x_6) = \begin{cases} \{y_{z,\eta}\}, & \text{if } y_{z,\eta} \in J^-(x_6), \\ \emptyset, & \text{if } y_{z,\eta} \notin J^-(x_6), \end{cases} \tag{140}$$

where $y_{z,\eta} = \mu_{z,\eta}(f^+_{\mu(z,\eta)}(q))$, *i.e.,* $\mathbf{e}_{z,\eta}(q) = \{y_{z,\eta}\}$.

Note that this lemma combined with Theorem 1.2 prove Theorem 1.4 in the case when there are no cut points on the manifold (M,g). Later we consider general manifolds with cut points.

Proof. The claim (i) follows by applying Lemma 5.3 (ii) for all points $y \in J^-(x_6) \cap U_g$.

Let us next consider the claim (ii). Denote $\mathcal{V} = \mathcal{V}((\vec{x}, \vec{\xi}), t_0)$. Let $q = Q((\vec{x}, \vec{\xi}), t_0)$. Then $q \in J^-(x_6) \subset \mathcal{V}$.

Let $y \in U_g \cap \mathcal{V}$. If $y \notin \mathcal{L}_g(q)$, Lemma 5.3 yields $y \notin \mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$. On the other hand, if $y \in \mathbf{e}_U(q)$, Lemma 5.3 yields that $y \in \mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$. Thus $\mathbf{e}_U(q) \cap \mathcal{V} \subset \mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0) \subset \mathcal{L}_g(q)$ yielding

$$\mathbf{e}_{z,\eta}(q) \cap \mathcal{V} = e_{z,\eta}(\mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)) \cap \mathcal{V}. \tag{141}$$

If $y_{z,\eta} \in J^-(x_6)$ then $J^-(x_6) \subset \mathcal{V}$ implies that $\mathbf{e}_{z,\eta}(q) \cap \mathcal{V} = \{y_{z,\eta}\}$. On the other hand, if $y_{z,\eta} \notin J^-(x_6)$, then $e_{z,\eta}(F) \cap J^-(x_6) = \mathbf{e}_{z,\eta}(q) \cap J^-(x_6) = \emptyset$. This yields (ii). □

Below, let $t_0 = 4\kappa_1$, cf. Lemma 2.10 and $\mathcal{K}_{t_0} \subset \widehat{U}$ be the set of points $x = \gamma_{y,\zeta}(r)$ where $y = \widehat{\mu}(s)$, $s \in [s^-, s^+]$, $\zeta \in L^+_y M_0$ satisfies $g^+(\zeta, \zeta) = 1$ and $r \in [0, 2t_0)$. Recall that $\mathcal{U}_{z_0,\eta_0} = \mathcal{U}_{z_0,\eta_0}(\widehat{h})$ was defined using the parameter $\widehat{h}$. We see that if $t_0' > 0$ and $\widehat{h}' \in (0, \widehat{h})$ are small enough and $(z, \eta) \in \mathcal{U}_{z_0,\eta_0}(\widehat{h}')$, then the longest geodesic from $x \in \overline{\mathcal{K}}_{t_0'}$ to the point $\mathbf{e}_{z,\eta}(x)$ is contained in $\mathcal{U}_g$, and hence we can determine the point $\mathbf{e}_{z,\eta}(x)$ for such x and (z, η). Let us replace the parameters $\widehat{h}$ and t_0 by $\widehat{h}'$ and t_0', correspondingly in our considerations below. Then we may assume that in addition to the data given in the original formulation of the problem, we are given also the set $\mathbf{e}_U(\mathcal{K}_{t_0})$. Next we do this.

For technical reasons, we will next replace U by $V = U \cap I^-(\widehat{p}^+)$ and consider the sets $\mathbf{e}_V(q)$, $q \in J^+(\widehat{p}^-) \cap I^-(\widehat{p}^+)$.

Next we consider step-by-step construction of the set $\mathbf{e}_V(J^+(\widehat{p}^-) \cap I^-(\widehat{p}^+))$ by constructing $\mathbf{e}_V(J^+(y) \cap I^-(\widehat{p}^+))$ with $y = \widehat{\mu}(s)$ and decreasing s in small steps. The difficulty we encounter here is that we do not know how the spherical waves propagating along geodesics interact after the geodesics have a cut point.

Our aim is construct sets $\mathcal{Z}_k = \mathbf{e}_V(I^-(\widehat{p}^+) \cap J^+(y_k)) \setminus \mathbf{e}_V(I^-(\widehat{p}^+) \cap J^+(y_{k-1}))$, with $\mathcal{Z}_0 = \emptyset$ and $y_k = \widehat{\mu}(s_k)$, $s_k < s_{k-1}$, $s_0, \dots, s_K \in [s_-, s_+]$ with $s_K = s_-$ and s_0 being close to s_+. The union of $\mathcal{Z}_k$, $k = 1, 2, \dots, K$ is the set $\mathbf{e}_V(J^+(\widehat{p}^-) \cap I^-(\widehat{p}^+))$. The idea of this construction is to choose s_j so that when $\mathcal{Z}_{k-1}$ is constructed we obtain the sets $\mathcal{Z}_k$ as a union of sets $\mathbf{e}_V(\gamma_{y,\zeta}([0, r(y, \zeta)))$ where $y \in \widehat{\mu}([s_k, s_{k-1}))$, ζ is light-like, and $r(y, \zeta)$ is chosen so that the geodesic $\gamma_{y,\zeta}([t_0, r(y, \zeta)))$ does not contain cut points and does not intersect $J^+(y_{k-1})$ but still lies in $I^-(\widehat{p}^+)$.

The construction is the following:

Let below $\kappa_1, \kappa_2, \kappa_3$ be constants given in Lemma 2.10. Let $s_0 \in [s_-, s_+]$ be so close to s_+ that $J^+(\widehat{\mu}(s_0)) \cap I^-(\widehat{p}^+) \subset \mathcal{K}_{t_0}$. Then the given data determines $\mathbf{e}_V(J^+(\widehat{\mu}(s_0)) \cap I^-(\widehat{p}^+))$. Moreover, let $s_k < s_{k-1}$,

$s_0, \dots, s_K \in [s_-, s_+]$ be such that $s_{j+1} > s_j - \kappa_3$ and $s_K = s_-$, and denote $y_k = \widehat{\mu}(s_k)$, see Fig. 18 for the points y_1 and y_2.

Above, $\mathbf{e}_V(J^+(\widehat{\mu}(s_0)) \cap I^-(\widehat{p}^+))$ was determined from the data. Next we use induction: We consider $s_1 \in (s_-, s_+)$ and assume we are given $\mathbf{e}_V(J^+(y_1) \cap I^-(\widehat{p}^+))$ with $y_1 = \widehat{\mu}(s_1)$. Let us then consider $s_2 \in (s_1 - \kappa_3, s_1)$. Our next aim is to find the light observation points $\mathbf{e}_V(J^+(y_2) \cap I^-(\widehat{p}^+))$ with $y_2 = \widehat{\mu}(s_2)$. To this end we need to make the following definitions (see Fig. 18, 19, and 20).

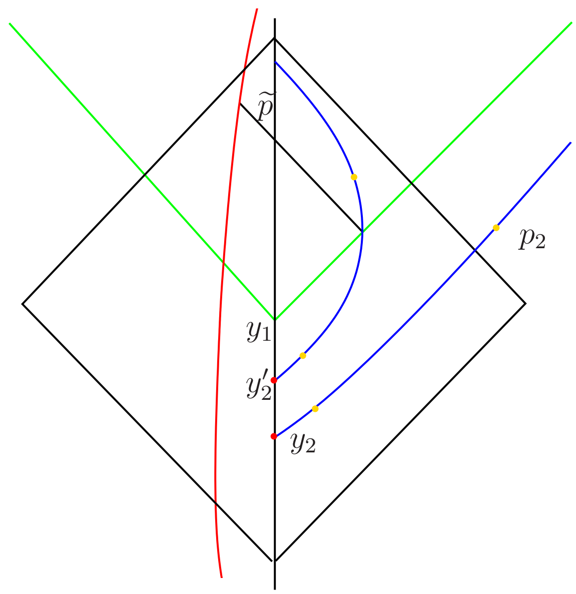

FIGURE 18. A schematic figure where the space-time is represented as the 2-dimensional set $\mathbb{R}^{1+1}$. In section 5 we consider geodesic $\gamma_{y_2,\zeta_2}([0,\infty))$ sent from $y_2 = \widehat{\mu}(s_2)$. We consider two cases corresponding to geodesics $\gamma_{y_2,\zeta_2}([0,\infty))$ and $\gamma_{y_2',\zeta_2'}([0,\infty))$. For $t_0 > 0$, the first cut point p_2, denoted by a golden point in the figure, of the geodesic $\gamma_{y_2,\zeta_2}([t_0,\infty))$ is either outside the set $J_g(\widehat{p}^-, \widehat{p}^+)$ or is in the set $J^+(y_1)$, denoted by the green boundary, where $y_1 = \widehat{\mu}(s_1)$. At the point $\widetilde{p} = \mu_{z,\eta}((\mathbb{S}(y,\zeta,z,\eta,s_1)))$ we observe the first time on the geodesic $\mu_{z,\eta} \cap I^-(\widehat{p})$ that the geodesic $\gamma_{y_2,\zeta_2}([0,\infty))$ has entered in the set $J^+(y_1)$. The red curve is $\mu_{z,\eta}$

Let

$$m(z,\eta) = \sup\{s \le 1;\ \mu_{z,\eta}(s) \in I^-(\widehat{p}^+)\}, \quad (z,\eta) \in \mathcal{U}_{z_0,\eta_0}. \tag{142}$$

Note that we can determine $m(z,\eta)$ using U_g.

Definition 5.5. *Let $s_- \le s_2 < s_1 \le s_+$ satisfy $s_1 < s_2 + \kappa_3$, let $s \in [s_2, s_1)$, $y_j = \widehat{\mu}(s_j)$, $j = 1, 2$, $y = \widehat{\mu}(s)$ and $\zeta \in L^+_y U$, $\|\zeta\|_{g^+} =$*

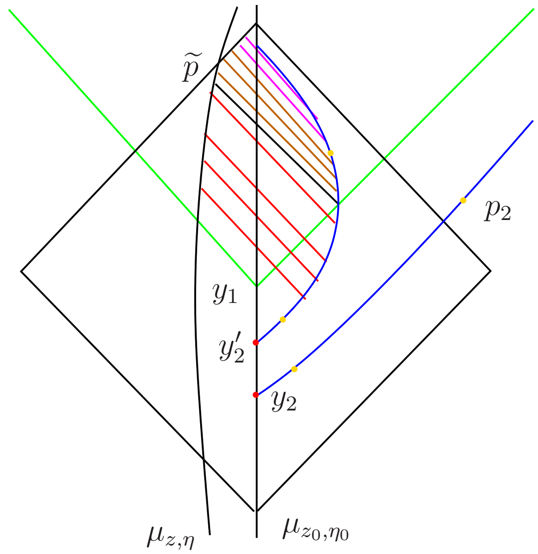

FIGURE 19. A schematic figure where the space-time is represented as the 2-dimensional set $\mathbb{R}^{1+1}$. We consider the geodesics emanating from a point $x(r) = \gamma_{(y_2',\zeta_2')}(r)$, When r is smallest value for which $x(r) \in J^+(y_1)$, a light-like geodesic (black line segment) emanating from $x(r)$ is observed at the point $\widetilde{p} \in \mu_{z,\eta} \cap I^-(\widehat{p}^+)$. Then $\widetilde{p} = \mu_{z,\eta}(\mathbb{S}(y,\zeta,z,\eta,s_1))$. When r is small enough so that $x(r) \notin J^+(y_1)$, the light-geodesics (red line segments) can be observed on $\mu_{z,\eta}$ in the set $J^-(\widetilde{p})$. Moreover, when r is such that $x(r) \in J^+(y_1)$, the light-geodesics can be observed at $\mu_{z,\eta}$ in the set $J^+(\widetilde{p})$. The golden point is the cut point on $\gamma_{y_2',\zeta_2'}([t_0,\infty))$ and the singularities on the light-like geodesics starting before this point (brown line segments) can be analyzed, but after the cut point the singularities on the light-like geodesics (magenta line segments) are not analyzed in this paper.

1. *Moreover, recall that here* $g = \widehat{g}$ *and let* $(z,\eta) \in \mathcal{W}_{z_0,\eta_0}$ *and* $\mu = \mu(z,\eta) = \mu(g,z,\eta) = \mu_{g,z,\eta}$ *and recall that* $\widehat{\mu} = \mu(g,z_0,\eta_0)$.
When $\gamma_{y,\zeta}(\mathbb{R}_+)$ *intersects* $J^+(\widehat{\mu}(s_1)) \cap I^-(\widehat{p}^+)$ *we define*

$$\mathbb{S}(y,\zeta,z,\eta,s_1) = \min(m(z,\eta), f^+_{\mu(z,\eta)}(q_0)), \tag{143}$$

where $q_0 = \gamma_{y,\zeta}(r_0)$ *and* $r_0 > 0$ *is the smallest* $r \geq 0$ *such that* $\gamma_{y,\zeta}(r) \in J^+(\widehat{\mu}(s_1))$. *In the case when* $\gamma_{y,\zeta}(\mathbb{R}_+)$ *does not intersect* $J^+(\widehat{\mu}(s_1)) \cap I^-(\widehat{p}^+)$, *we define* $\mathbb{S}(y,\zeta,z,\eta,s_1) = m(z,\eta)$.

In other words, if $\mathbb{S}(y,\zeta,z,\eta,s_1) < m(z,\eta)$, then it is the first time when $\gamma_{y,\zeta}(t)$ is observed on $\mu(z,\eta)$ to enter the set $J^+(\widehat{\mu}(s_1)) \cap I^-(\widehat{p}^+)$.

Definition 5.6. *Let* $s \in [s_2,s_1)$, $y = \widehat{\mu}(s)$ *and* $\zeta \in L^+_y U$, $\|\zeta\|_{g^+} = 1$.

For $\vartheta \in (0,\vartheta_0)$, let $\mathcal{R}^{(0)}_\vartheta(y,\zeta)$ be the set of $(\vec x,\vec \xi) = ((x_j,\xi_j))_{j=1}^4$ such that

(i) $(x_j,\xi_j) \in L^+U_g$, $j = 1,2,3,4$,
(ii) $(x_1,\xi_1) = (y,\zeta)$ *and* (x_k,ξ_k), $k = 2,3,4$ *are in ϑ-neighborhood of (y,ζ) in (TM_0,g^+) and $x_l \notin J^+_g(x_j)$ for $j,l = 1,2,3,4$,*

Also, we define $\mathcal{R}_\vartheta(y,\zeta,z,\eta,s_1)$ to be the set of 4-tuples compatible with earlier observations near $\mu_{z,\eta}$, that is, the set $\mathcal{R}_\vartheta(y,\zeta,z,\eta,s_1)$ consist of all such $(\vec x,\vec \xi) \in \mathcal{R}^{(0)}_\vartheta(y,\zeta)$ for which we have

(iii) *there is an open neighborhood $\mathcal{W}_0 \subset \mathcal{U}_{z_0,\eta_0}$ of (z,η) such that*

$$\mathcal{S}^{cl}{}_{z_1,\eta_1}((\vec x,\vec \xi),t_0) \in \mathbf{e}_{z_1,\eta_1}(J^+(y_1)\cap I^-(\widehat p^+)), \quad \text{for all } (z_1,\eta_1) \in \mathcal{W}_0,$$

and

$$\mathcal{S}^{cl}{}_{z_1,\eta_1}((\vec x,\vec \xi),t_0) \in I^-(\widehat p^+), \quad \text{for all } (z_1,\eta_1) \in \mathcal{W}_0.$$

We denote $\mathcal{R}^{(0)}(y,\zeta) = \mathcal{R}^{(0)}_{\vartheta_0}(y,\zeta)$ and $\mathcal{R}(y,\zeta,z,\eta,s_1) = \mathcal{R}_{\vartheta_0}(y,\zeta,z,\eta,s_1)$.

Below, let $\kappa_4 = \min(\kappa_2,\kappa_0)$ cf. Lemma 2.10. Let $R_0(y,\zeta,s_1) \in [0,\infty]$ be the smallest value for which $q_0 = \gamma_{y,\zeta}(r_0) \in \partial J^+(\widehat\mu(s_1))$, or $R_0(y,\zeta,s_1) = \infty$ if no such value of r exists. By Lemma 2.10 (iv), $\gamma_{y,\zeta}([0,t_0])$ does not intersect $J^+(y_1)\cap I^-(\widehat p^+)$. Thus, if $\gamma_{y,\zeta}([0,\infty))$ intersects $J^+(y_1)\cap I^-(\widehat p^+)$ if and only if $\gamma_{y,\zeta}([t_0,\infty))$ intersects it.

Lemma 5.7. *Assume that $y = \widehat\mu(s)$, $y_1 = \widehat\mu(s_1)$ with $-1 < s_1 - \kappa_3 \le s < s_1 < s^+$, $\zeta \in L^+_yM$, $\|\zeta\|_{g^+} = 1$. Then for all $\delta > 0$ there is $\vartheta_1(y,\zeta,s_1,\delta) > 0$ such that if $0 < \vartheta < \vartheta_1(y,\zeta,s_1,\delta)$ and $(\vec x,\vec \xi) \in \mathcal{R}^{(0)}_\vartheta(y,\zeta)$ and for some $j = 1,2,3,4$, we have $\mathbf{t}_j = \rho(x_j(t_0),\xi_j(t_0)) < \mathcal{T}(x_j(t_0),\xi_j(t_0))$, then the cut point $p_j = \gamma_{x_j(t_0),\xi_j(t_0)}(\mathbf{t}_j)$ of the geodesics $\gamma_{x_j(t_0),\xi_j(t_0)}([0,\infty))$ satisfies either*

(i) $p_j \notin J^-(\widehat p^+)$,

or

(ii) it holds that $r_0 = R_0(y,\zeta,s_1) < \infty$, $q_0 = \gamma_{y,\zeta}(r_0)$ satisfies $q_0 \in J^-(\widehat\mu(s_{+2}))$ and $r_1 = r_0+\kappa_4 < t_0 + \rho(y(t_0),\zeta(t_0))$ and $q_1 = \gamma_{y,\zeta}(r_1)$ satisfy $q_1 \in I^-(\widehat\mu(s_{+3}))$, cf. (1) and text above it, and $f^+_{\mu(z,\eta)}(p_j) > f^+_{\mu(z,\eta)}(q_1) - \delta$.

Proof. We start with a Lipschitz estimate that we will need below. Consider $r > 0$ such that $\gamma_{y,\zeta}(r) \in J^-(\widehat\mu(s_{+3}))$ and let $(\vec x,\vec \xi) \in \mathcal{R}^{(0)}(y,\zeta)$. Using the Lipschitz estimate (24) for the geodesic flow in $J(\widehat\mu(-1),\widehat\mu(1))$, we see that there are $L_0 = L_0(y,\zeta,r) > 0$ and $\vartheta_2(y,\zeta,r) > 0$ such that if $0 < \vartheta < \vartheta_2(y,\zeta,r)$ and $d_{g^+}((x_j,\xi_j),(y,\zeta)) < \vartheta$ then

$$d_{g^+}(\gamma_{x_j,\xi_j}(r),\gamma_{y,\zeta}(r)) < L_0\vartheta. \tag{144}$$

Next, let $t_2 = \rho(y,\zeta)$, $t_3 = \rho(y(t_0),\zeta(t_0))+t_0$, and $(\vec{x},\vec{\xi}) \in \mathcal{R}^{(0)}(y,\zeta)$. Let $q_2 = \gamma_{y,\zeta}(t_2)$ and $q_3 = \gamma_{y,\zeta}(t_3)$. By Lemma 2.10 (i)-(ii), we see that if $t_2 \geq R_1+\kappa_0$ then $\rho(x_j(t_0),\xi_j(t_0))+t_0 > R_1$ and hence $p_j \notin J^-(\widehat{\mu}(s_{+2}))$. Thus it is enough to consider the case when $t_2 < R_1 + \kappa_0$

We see using Lemma 2.10 (ii) that $t_3 \geq t_2 + 3\kappa_2$. Note that $q_3 = \gamma_{y(t_0),\zeta(t_0)}(\rho(y(t_0),\zeta(t_0)))$ is the first cut point on $\gamma_{y(t_0),\zeta(t_0)}([0,\infty))$. If this cut point satisfies $q_3 \in J^-(\widehat{\mu}(s_{+2}))$, then by Lemma 2.10 (iii), we have $f^-_{\widehat{\mu}}(q_3) > f^-_{\widehat{\mu}}(q_2) + 3\kappa_3 \geq s_2 + 3\kappa_3$, implying $q_3 \in I^+(y_1)$ as $s_1 < s_2 + 2\kappa_3$. Thus the first cut point q_3 on $\gamma_{y(t_0),\zeta(t_0)}([0,\infty))$ satisfies (see Fig. 18)

$$q_3 \in I^+(y_1) \quad \text{or} \quad q_3 \notin J^-(\widehat{\mu}(s_{+2})). \tag{145}$$

Below, we consider two cases separately:

In case (a): $q_3 \notin J^-(\widehat{\mu}(s_{+2}))$.

In case (b): $q_3 \in J^-(\widehat{\mu}(s_{+2}))$.

Assume next the alternative (b) is valid.

Then by (145) we have $q_3 \in I^+(y_1)$ and hence $\gamma_{y,\zeta}([0,\infty))$ intersects $J^+(y_1) \cap J^-(\widehat{\mu}(s_{+2}))$. There exists the smallest number $r_0 > 0$ such that $\gamma_{y,\zeta}(r_0) \in \partial J^+(y_1)$. We define $q_0 = \gamma_{y,\zeta}(r_0) \in \partial J^+(y_1)$. Note that then $q_0 \in J^-(\widehat{\mu}(s_{+2}))$.

Let $r_1 = r_0 + \kappa_4$. Then by Lemma 2.10 (i), we have $q_1 = \gamma_{y,\zeta}(r_1) \in I^-(\widehat{\mu}(s_{+3}))$.

Using Lemma 2.10 (iii) for the point q_0, we see that as $f^-_{\widehat{\mu}}(q_0) = s_1 < s + 2\kappa_3$, we have $r_0 < \rho(y,\zeta) + \kappa_2$. On the other hand, by Lemma 2.10 (ii), $\rho(x_j(t_0),\xi_j(t_0))+t_0 > \rho(y,\zeta)+3\kappa_2$, and hence $\rho(x_j(t_0),\xi_j(t_0))+t_0 > r_0 + 2\kappa_2 \geq r_1 + \kappa_2$.

Let $S^* = f^+_{\mu(z,\eta)}(q_1)$ and $\delta > 0$. As the functions $(x,\xi) \mapsto \gamma_{x,\xi}(r_1)$ and $q \mapsto f^+_{\mu(z,\eta)}(q)$ are continuous and $f^+_{\mu(z,\eta)}(q_1) > S^* - \delta$, we see that when there is $\vartheta_1(y,\zeta,s_1,r_1,\delta)$ such that if $0 < \vartheta < \vartheta_1(y,\zeta,s_1,r_1,\delta)$ then $q_j' = \gamma_{x_j,\xi_j}(r_1)$ satisfies $f^+_{\mu(z,\eta)}(q_j') > S^* - \delta$. As $q_j' < p_j$, we have $f^+_{\mu(z,\eta)}(p_j) > S^* - \delta$.

Let us next consider the case when the alternative (a) is valid, that is, $q_3 \notin J^-(\widehat{\mu}(s_{+2}))$.

Let $r_4 = r_4(y,\zeta,s_2) > 0$ be the smallest number such that $\gamma_{y,\zeta}(r_4) \in \partial J^-(\widehat{\mu}(s_{+2}))$ and define $q_4 = \gamma_{y,\zeta}(r_4)$. Then using this and the definition of q_4, we have $t_3 = \rho(y(t_0),\zeta(t_0)) + t_0 > r_4$. As $(x,\xi) \mapsto \rho(x,\xi)$ is a lower-semicontinuous function, there exists $\vartheta_3(y,\zeta,s_1) > 0$ such that if $0 < \vartheta < \vartheta_3(y,\zeta,s_1)$ then $\rho(x_j(t_0),\xi_j(t_0)) + t_0 > r_4$.

Observe that $q_4 \notin J(\widehat{\mu}(s_{-2}),\widehat{p}^+) = J^+(\widehat{\mu}(s_{-2})) \cap J^-(\widehat{p}^+)$ and thus $h_2 := d_{g^+}(q_4, J(\widehat{\mu}(s_{-2}),\widehat{p}^+)) > 0$. Using (144) we see that there exists $\vartheta_1(y,\zeta,s_1) \in (0,\vartheta_3(y,\zeta,s_1))$ such that if $0 < \vartheta < \vartheta_1(y,\zeta,s_1)$ and $(\vec{x},\vec{\xi}) \in \mathcal{R}^{(0)}_\vartheta(y,\zeta)$, then $x_j \in J^+(\widehat{\mu}(s_{-2}))$ and

$$d_{g^+}(\gamma_{x_j,\xi_j}(r_4), q_4) < \frac{1}{2}h_2,$$

and hence

$$d_{g^+}(\gamma_{x_j,\xi_j}(r_4), J(\widehat{\mu}(s_{-2})), \widehat{p}^+)) > \frac{1}{2}h_2.$$

Assume next that $0 < \vartheta < \vartheta_1(y,\zeta,r_4)$. Then, as $x_j \in J^+(\widehat{\mu}(s_{-2}))$, we have $\gamma_{x_j,\xi_j}(r_4) \in J^+(\widehat{\mu}(s_{-2}))$ and hence $\gamma_{x_j,\xi_j}(r_4) \notin J^-(\widehat{p}^+)$. Then $\rho(x_j(t_0),\xi_j(t_0)) + t_0 > r_4$ and $\gamma_{x_j,\xi_j}(r_4) \notin J^-(\widehat{p}^+)$ imply that $p_j = \gamma_{x_j(t_0),\xi_j(t_0)}(\rho(x_j(t_0),\xi_j(t_0))) \notin J^-(\widehat{p}^+)$. This proves the claim. □

Next our aim is show that we can determine the function $\mathbb{S}(y,\zeta,z,\eta,s_1)$. To this end we have to take care of the difficulty that the geodesic $\gamma_{y,\zeta}$ can exit U_g, later return to it and intersects the geodesic $\widehat{\mu}$ or a geodesic $\mu_{z,\eta}$. This happen for instance in the Lorentzian manifold $\mathbb{R}\times\mathbb{S}^3$. First we consider the case when the geodesic $\gamma_{y,\zeta}$ does not intersect e.g. a geodesic $\mu_{z,\eta}$.

Lemma 5.8. *Let $s_- \le s_2 \le s < s_1 \le s_+$ satisfy $s_1 < s_2 + \kappa_3$, let $y_j = \widehat{\mu}(s_j)$, $j = 1,2$, $y = \widehat{\mu}(s)$ and $\zeta \in L^+_yU$, $\|\zeta\|_{g^+} = 1$. Assume that $(z,\eta) \in \mathcal{U}_{z_0,\eta_0}$ is such that $\gamma_{y,\zeta}([t_0,\infty))$ does not intersect $\mu_{z,\eta}([-1,1])$.*

There is $\vartheta_3(y,\zeta,s_1,z,\eta) > 0$ such that if $0 < \vartheta < \vartheta_3(y,\zeta,s_1,z,\eta)$ and $(\vec{x},\vec{\xi}) \in \mathcal{R}^{(0)}_\vartheta(y,\zeta)$ the following holds:

If $\rho(x_j(t_0),\xi_j(t_0)) < \mathcal{T}(x_j(t_0),\xi_j(t_0))$ for some $j = 1,2,3,4$, then the cut point $p_j = \gamma_{x_j(t_0),\xi_j(t_0)}(\rho(x_j(t_0),\xi_j(t_0)))$ satisfies either $p_j \notin J^-(\widehat{p}^+)$ or $f^+_{\mu(z,\eta)}(p_j) > \mathbb{S}(y,\zeta,z,\eta,s_1)$.

Proof. We denote below $S = \mathbb{S}(y,\zeta,z,\eta,s_1)$. The fact that $\gamma_{y,\zeta}([t_0,\infty))$ does not intersect $\mu_{z,\eta}([-1,1])$ implies that when $\vartheta > 0$ is small enough, we have that $(\vec{x},\vec{\xi}) \in \mathcal{R}^{(0)}_\vartheta(y,\zeta)$ that $\gamma_{x_j,\xi_j}([t_0,\infty))$ does not intersect $\mu_{z,\eta}([-1,1])$. Assume next that ϑ is so small that this is valid.

Using a short-cut argument that if $t_2 > t_1$ then for all $j = 1,2,3,4$,

$$f^+_{\mu(z,\eta)}(\gamma_{x_j,\xi_j}(t_2)) > f^+_{\mu(z,\eta)}(\gamma_{x_j,\xi_j}(t_1)). \tag{146}$$

Assume that $\rho(x_j(t_0),\xi_j(t_0)) < \mathcal{T}(x_j(t_0),\xi_j(t_0))$ for some $j = 1,2,3,4$. We can assume that $p_j \in J^-(\widehat{p}^+)$ as otherwise the claim is proven.

By Lemma 5.7 (ii), then $r_0 = R(y,\zeta,s_1) < \infty$, $\gamma_{y,\zeta}([t_0,\infty))$ intersects $J^+(y_1) \cap I^-(\widehat{\mu}(s_{+2})))$, and $q_0 = \gamma_{y,\zeta}(r_0) \in \partial J^+(y_1)$. Let $r_1 = r_0 + \kappa_4$ so that $q_1 = \gamma_{y,\zeta}(r_1) \in I^-(\widehat{\mu}(s_{+3}))$. Using (146) we see the $S_1 = f^+_{\mu(z,\eta)}(q_1) > S$, and define $\delta = (S_1 - S)/2 > 0$. By Lemma 5.7, when $0 < \vartheta < \vartheta_1(y,\zeta,s_1,\delta)$, we have $\rho(x_j(t_0),\xi_j(t_0)) + t_0 > r_1$ and $f^+_{\mu(z,\eta)}(p_j) > S_1 - \delta > S$. □

Definition 5.9. *Let $T(y,\zeta,z,\eta,s_1)$ be the infimum of $s \in [-1,m(z,\eta)]$ for which for every $\vartheta \in (0,\vartheta_0)$ there exists $(\vec{x},\vec{\xi}) \in \mathcal{R}_\vartheta(y,\zeta,z,\eta,s_1)$ such that $\mu_{z,\eta}(s) = e_{z,\eta}(\mathcal{S}^{cl}((\vec{x},\vec{\xi}),t_0))$ if such values of s exist, and otherwise, let $T(y,\zeta,z,\eta,s_1) = m(z,\eta)$.*

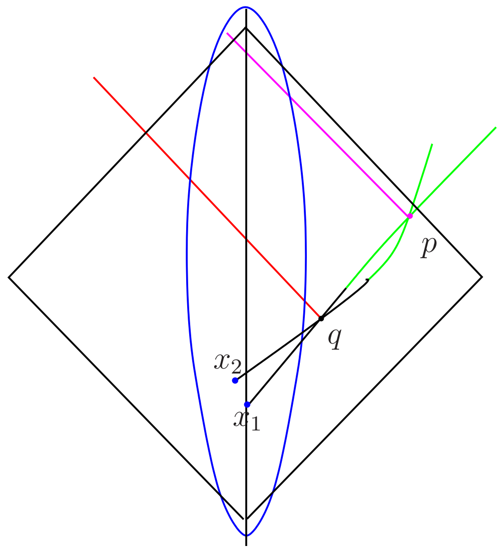

FIGURE 20. A schematic figure where the space-time is represented as the 2-dimensional set $\mathbb{R}^{1+1}$. In section 5 we consider geodesics $\gamma_{x_j,\xi_j}([0,\infty))$, $j = 1,2,3,4$, that all intersect for the first time at a point q. In the figure we consider only geodesics $\gamma_{x_j,\xi_j}([0,\infty))$, $j = 1,2$. When we consider geodesics $\gamma_{x_j,\xi_j}([t_0,\infty))$, $j = 1,2$ with $t_0 > 0$, they may have cut points at $x_j^{cut} = \gamma_{x_j,\xi_j}(\mathbf{t}_j)$. In the figure $\gamma_{x_j,\xi_j}([0,\mathbf{t}_j))$ are colored by black and the geodesics $\gamma_{x_j,\xi_j}([\mathbf{t}_j,\infty))$ are colored by green. In the figure the geodesics intersect at the point q before the cut point and for the second time at p after the cut point. We can analyze the singularities caused by the spherical waves that interact at q but not the interaction of waves after the cut points of the geodesics. It may be that e.g. the intersection at p causes new singularities to appear and we observe those in $\widehat{U}$. As we cannot ianalyze these singularities, we consider these singularities as "messy waves". However, the "nice" singularities caused by the interaction at q propagate along the future light cone of the point q arrive to $\widehat{U}$ so that on a dense and open subset of geodesics $\mu_{g,z,\eta}$ these "nice" singularities are observed before the "messy waves". Due to this we consider the first singularities observed on geodesics $\mu_{g,z,\eta}$.

Lemma 5.10. *Let* $s_- \le s_2 \le s < s_1 \le s_+$ *satisfy* $s_1 < s_2 + \kappa_3$, *let* $y_j = \widehat{\mu}(s_j)$, $j = 1,2$, $y = \widehat{\mu}(s)$ *and* $\zeta \in L_y^+ U$, $\|\zeta\|_{g^+} = 1$, *and* $(z,\eta) \in \mathcal{U}_{z_0,\eta_0}$.

Then

(i) We always have $T(y,\zeta,z,\eta,s_1)\ge \mathbb{S}(y,\zeta,z,\eta,s_1)$.

(ii) Assume that $(z,\eta)\in \mathcal{U}_{z_0,\eta_0}$ *is such that* $\gamma_{y,\zeta}([t_0,\infty))$ *does not intersect* $\mu_{z,\eta}([-1,1])$. *Then* $T(y,\zeta,z,\eta,s_1)=\mathbb{S}(y,\zeta,z,\eta,s_1)$.

(iii) For all $(z,\eta)\in \mathcal{U}_{z_0,\eta_0}$ *we have*

$$\mathbb{S}(y,\zeta,z,\eta,s_1)=\min(m(z,\eta),\liminf_{(z_1,\eta_1)\to(z,\eta)} T(y,\zeta,z_1,\eta_1,s_1)).$$

Proof. In the proof we continue to use the notations given in Def. 5.6 and denote $T=T(y,\zeta,z,\eta,s_1)$. The objects used in the proof are shown in Figs. 19 and 20.

(i) Let $s_1^+<\mathbb{S}(y,\zeta,z,\eta,s_1)$ and $x_6=\mu_{z,\eta}(s_1^+)$. Note that as then $s_1^+<m(z,\eta)$. As $\mu_{z,\eta}(m(z,\eta))\in J^-(\widehat p^+)$, we have $x_6\ll \widehat p^+$.

Let $\delta\in(0,\mathbb{S}(y,\zeta,z,\eta,s_1)-s_1^+)$. Assume that $(\vec x,\vec\xi)\in \mathcal{R}^{(0)}_\vartheta(y,\zeta)$ with some $0<\vartheta<\vartheta_1(y,\zeta,s_1,r_1,\delta)$, where $\vartheta_1(y,\zeta,s_1,r_1,\delta)$ is defined in Lemma 5.7.

Assume that for some $j=1,2,3,4$ it holds that $\rho(x_j(t_0),\xi_j(t_0))<\mathcal{T}(x_j(t_0),\xi_j(t_0))$ and consider the cut point $p_j=\gamma_{x_j(t_0),\xi_j(t_0)}(\rho(x_j(t_0),\xi_j(t_0)))$. By Lemma 5.7, p_j either satisfies $p_j\not\in J^-(\widehat p^+)$, or alternatively, $r_0=R(y,\zeta,s_1)<\infty$ and $q_0=\gamma_{y,\zeta}(r_0)\in\partial J^+(y_1)$ and for $r_1=r_0+\kappa_4$ we have $q_1=\gamma_{y,\zeta}(r_1)\in I^-(\widehat\mu(s_{+3}))$ and

$$f^+_{\mu(z,\eta)}(q_1)\ge f^+_{\mu(z,\eta)}(q_0)=\mathbb{S}(y,\zeta,z,\eta,s_1)$$

and

$$f^+_{\mu(z,\eta)}(p_j)\ge f^+_{\mu(z,\eta)}(q_1)-\delta\ge \mathbb{S}(y,\zeta,z,\eta,s_1)-\delta>s_1^+.$$

Notice that in the latter case $f^+_{\mu(z,\eta)}(p_j)>s_1^+$.

Thus in all cases the cut points p_j of geodesics $\gamma_{x_j(t_0),\xi_j(t_0)}$ satisfy $p_j\not\in J^-(x_6)$. Assume below that ϑ is so small enough that the above holds.

Next, assume that the geodesics corresponding to $(\vec x,\vec\xi)\in\mathcal{R}^{(0)}_\vartheta(y,\zeta)$ intersect at some point $q\in J^-(x_6)$. Then $\mathbf{e}_{z,\eta}(q)\cap J^-(x_6)=\{\mu_{z,\eta}(\widetilde s)\}$ with $\widetilde s=f^+_{\mu(z,\eta)}(q)\le s_1^+$. Hence Lemma 5.4 yields that ${\mathcal{S}^{cl}}_{z,\eta}((\vec x,\vec\xi),t_0)\cap J^-(x_6)$ is equal to $\{\mu_{z,\eta}(\widetilde s)\}$. However, for all $q'\in\gamma_{y,\zeta}\cap(J^+(y_1)\cap I^-(\widehat p^+))$, we have $\mathbf{e}_{z,\eta}(q')=\{\mu_{z,\eta}(s')\}$ with $s'\ge\mathbb{S}(y,\zeta,z,\eta,s_1)$. As $\widetilde s\le s_1^+<\mathbb{S}(y,\zeta,z,\eta,s_1)$, we see that the condition (iii) in Def. 5.6 can not be satisfied for $(\vec x,\vec\xi)$ and hence $(\vec x,\vec\xi)\not\in\mathcal{R}_\vartheta(y,\zeta,z,\eta,s_1)$.

Assume next that $(\vec x,\vec\xi)\in\mathcal{R}_\vartheta(y,\zeta,z,\eta,s_1)$. The above yields that either the geodesics corresponding to $(\vec x,\vec\xi)$ do not intersect, that is, $Q((\vec x,\vec\xi),t_0)=\emptyset$ or the intersection point $q=Q((\vec x,\vec\xi),t_0)$ is not in $J^-(x_6)$. Then we see using Lemma 5.4 that $\mathcal{S}^{cl}((\vec x,\vec\xi),t_0)\cap J^-(x_6)=\emptyset$. As $\mathcal{S}^{cl}((\vec x,\vec\xi),t_0)$ is closed, we have either $e_{z,\eta}(\mathcal{S}^{cl}((\vec x,\vec\xi),t_0))=\emptyset$ or $\mu_{z,\eta}(s_1^+)=x_6\ll e_{z,\eta}(\mathcal{S}^{cl}((\vec x,\vec\xi),t_0))$. By definition of T, this gives that $T\ge s_1^+$. As above $s_1^+<\mathbb{S}(y,\zeta,z,\eta,s_1)$ was arbitrary, this yields $T\ge\mathbb{S}(y,\zeta,z,\eta,s_1)$.

(ii) Denote below $S=\mathbb{S}(y,\zeta,z,\eta,s_1)$. If $S=m(z,\eta)$ and the claim holds trivially. Thus, let us assume below that $S<m(z,\eta)$. By definition (143), this implies that $r_0=R(y,\zeta,s_1)<\infty$ and $q_0=\gamma_{y,\zeta}(r_0)\in\partial J^+(y_1)$ is such that $f^+_{\mu(z,\eta)}(q_0)=S$.

By Lemma 5.8, if $\rho(x_j(t_0),\xi_j(t_0))<\mathcal{T}(x_j(t_0),\xi_j(t_0))$ for some $j=1,2,3,4$, and ϑ is small enough, the cut point $p_j=\gamma_{x_j(t_0),\xi_j(t_0)}(\rho(x_j(t_0),\xi_j(t_0)))$ satisfies either $p_j\not\in J^-(\widehat p^+)$ or $f^+_{\mu(z,\eta)}(p_j)>S$.

As $S<m(z,\eta)$ and above holds for all $j\in\{1,2,3,4\}$, we see that there is $s_*\in(S,m(z,\eta))$ such that $p_j=\gamma_{x_j(t_0),\xi_j(t_0)}(\rho(x_j(t_0),\xi_j(t_0)))\not\in J^-(\mu(s_*))$. We observe that then $(\vec x(t_0),\vec\xi(t_0))$ satisfies conditions (82)-(83) with $x_6=\mu(s_*)$ and $q_0\in I^-(x_6)$.

Let $s'\in(S,s_*)$. Note that then $\mu_{z,\mu}(s')\in I^-(x_6)$. As $f^+_{\mu(z,\eta)}(q_0)=S$, the functions $(z_1,\eta_1)\mapsto f^+_{\mu(z_1,\eta_1)}(q_0)$ and $(z_1,\eta_1)\mapsto\mu_{z_1,\eta_1}(s')$ are continuous, we see that (z,η) has a neighborhood $\mathcal{W}_1\subset\mathcal{U}_{z_0,\eta_0}$ such that for all $(z_1,\eta_1)\in\mathcal{W}_1$ we have $f^+_{\mu(z_1,\eta_1)}(q_0)<s'$ and $\mu_{z_1,\eta_1}(s')\subset I^-(x_6)$. In particular then $\mathbf{e}_{z_1,\eta_1}(q_0)=\{\mu_{z_1,\eta_1}(f^+_{\mu(z_1,\eta_1)}(q_0))\}\subset I^-(x_6)$. Then by Lemma 5.4, ${\mathcal{S}^{cl}}_{z_1',\eta_1'}((\vec x,\vec\xi),t_0))\cap J^-(x_6)$ coincides with the set $\mathbf{e}_{z_1',\eta_1'}(q_0)\cap J^-(x_6)$ for all $(z_1',\eta_1')\in\mathcal{U}_{z_0,\eta_0}$. This and the above imply that ${\mathcal{S}^{cl}}_{z_1,\eta_1}(((\vec x,\vec\xi),t_0)=\mathbf{e}_{z_1,\eta_1}(q_0)$ when $(z_1,\eta_1)\in\mathcal{W}_1$. As $q_0\in J^+(y_1)\cap I^-(\widehat p^+)$, we have $(\vec x,\vec\xi)=((x_j,\xi_j))_{j=1}^4\in\mathcal{R}_\vartheta(y,\zeta,z,\eta,s_1)$.

The above shows that there is $(\vec x,\vec\xi)\in\mathcal{R}_\vartheta(y,\zeta,z,\eta,s_1)$ such that the geodesics corresponding to $(\vec x,\vec\xi)$ intersect in q_0 and $\mu_{z,\eta}(s)={\mathcal{S}^{cl}}_{z,\eta}(((\vec x(t_0),\vec\xi(t_0)),t_0)$ with $s=S=f^+_{\mu(z,\eta)}(q_0)$.

Hence $T(y,\zeta,z,\eta,s_1)\le S=\mathbb{S}(y,\zeta,z,\eta,s_1)$ and the claim (ii) follows from (i).

(iii) Using standard results of differential topology, we see that there is an open and dense set $\mathcal{W}\subset\mathcal{U}_{z_0,\eta_0}$ such that if $(z,\eta)\in\mathcal{W}$ then $\mu_{z,\eta}([-1,1])$ does not intersect $\gamma_{y,\zeta}([t_0,\infty))$.

Let $q_0=\gamma_{y,\zeta}(r_0)\in\partial J^+(y_1)$ be the point defined in the proof of (ii). By Lemma 2.3 (v), the function $(z,\eta)\mapsto f^+_{z,\eta}(q_0)$ is continuous and as $M_0\setminus I^-(\widehat p^+)$ is closed, similarly to the first part of the proof of Lemma 2.3 (v), we see that the function $(z,\eta)\mapsto m(z,\eta)$ is lower-semicontinuous. As $\mathbb{S}(y,\zeta,z,\eta,s_1)=\min(m(z,\eta),f^+_{z,\eta}(q_0))$ the claim (iii) follows easily from (ii). □

Next we reconstruct $\mathbf{e}_V(q)$ when q runs over a geodesic segment.

Lemma 5.11. *Let $s_-\le s_2\le s<s_1\le s_+$ satisfy $s_1<s_2+\kappa_3$, let $y_j=\widehat\mu(s_j)$, $j=1,2$, $y=\widehat\mu(s)$, and $\zeta\in L^+_yU$, $\|\zeta\|_{g^+}=1$. When we are given the data set $\mathcal{D}(\widehat g,\widehat\phi,\varepsilon)$, we can determine the collection $\{\mathbf{e}_V(q);\ q\in(\gamma_{y,\zeta}([t_0,\infty))\cap I^-(\widehat p^+))\setminus J^+(y_1)\}$, where $V=U\cap I^-(\widehat p^+)$.*

Proof. In the proof, we consider y, ζ, s_1, and t_0 as fixed parameters and do not always indicate the dependency on the other parameters on those.

Let us denote by $\mathcal{W} = \mathcal{W}(y,\zeta) \subset \mathcal{U}_{z_0,\eta_0}$ the open and dense set of those $(z,\eta) \in \mathcal{U}_{z_0,\eta_0}$ for which $\mu_{z,\eta}([-1,1]) \cap \gamma_{y,\zeta}([t_0,\infty)) = \emptyset$.

Next, let $(z,\eta) \in \mathcal{W}$ and denote $S = \mathbb{S}(y,\zeta,z,\eta,s_1)$. Let $0 < \vartheta < \vartheta_3(y,\zeta,s_1,z,\eta)$ and consider $(\vec{x},\vec{\xi}) \in \mathcal{R}^{(0)}_\vartheta(y,\zeta)$.

We define $x' = x'(z,\eta) := \mu_{z,\eta}(\mathbb{S}(y,\zeta,z,\eta,s_1))$. Then $x' \in \partial J^+(q_0)$, $x' \in J^-(\widehat{p}^+)$.

As $0 < \vartheta < \vartheta_3(y,\zeta,s_1,z,\eta)$, using Lemma 5.8 we see that if $\rho(x_j(t_0),\xi_j(t_0)) < \mathcal{T}(x_j(t_0),\xi_j(t_0))$ for some $j = 1,2,3,4$, then $p_j = \gamma_{x_j(t_0),\xi_j(t_0)}(\rho(x_j(t_0),\xi_j(t_0)))$ either satisfies $p_j \notin J^-(\widehat{p}^+)$ or $f^+_{\mu(z,\eta)}(p_j) > S$. In both cases the cut points p_j of geodesics $\gamma_{x_j(t_0),\xi_j(t_0)}$ satisfy $p_j \notin J^-(x')$.

Recall that for $(\vec{x},\vec{\xi})$ we denote the first intersection point of the geodesics γ_{x_j,ξ_j} by $q = Q(\vec{x},\vec{\xi})$ if such intersection point exists and otherwise we define $Q(\vec{x},\vec{\xi}) = \emptyset$.

With the above definitions all cut points satisfy $p_j \notin J^-(x')$ for all $j \le 4$. By definition of x' and Lemma 5.4, we see using x' as the point x_6, that if the geodesic corresponding to $(\vec{x},\vec{\xi})$ intersect at some point $q = Q(\vec{x},\vec{\xi}) \ll x'$ we have $\mathcal{S}^{cl}{}_{z,\eta}((\vec{x},\vec{\xi}),t_0) = \mathbf{e}_{z,\eta}(q) \ll x'$, and otherwise, $\mathcal{S}^{cl}{}_{z,\eta}((\vec{x},\vec{\xi}),t_0) \cap I^-(x') = \emptyset$.

Consider next a general $(z,\eta) \in \mathcal{U}_{z_0,\eta_0}$, that is, we do not anymore require that $(z,\eta) \in \mathcal{W}(y,\eta)$. We say that a sequence $((\vec{x}^{(\ell)},\vec{\xi}^{(\ell)}))_{\ell=1}^\infty$ is a $\mathcal{A}_{z,\eta}(y,\zeta)$ sequence and denote $((\vec{x}^{(\ell)},\vec{\xi}^{(\ell)}))_{\ell=1}^\infty \in \mathcal{A}_{z,\eta}(y,\zeta)$, if $(\vec{x}^{(\ell)},\vec{\xi}^{(\ell)}) \in \mathcal{R}^{(0)}_{\vartheta(\ell)}(y,\zeta)$ with $\vartheta(\ell) = 1/\ell$ and there is $\ell_0 > 0$ such that either there is $p \in \mu_{z,\eta}$ such that for all $\ell \ge \ell_0$

$$\mathcal{S}^{cl}{}_{z,\eta}((\vec{x}^{(\ell)},\vec{\xi}^{(\ell)}),t_0) \cap I^-(x'(z,\eta)) = \{p\}$$

or alternatively, for all $\ell \ge \ell_0$

$$\mathcal{S}^{cl}{}_{z,\eta}((\vec{x}^{(\ell)},\vec{\xi}^{(\ell)}),t_0) \cap I^-(x'(z,\eta)) = \emptyset.$$

Note that as $x'(z,\eta) = \mu_{z,\eta}(\mathbb{S}(y,\zeta,z,\eta,s_1)) \le \widehat{p}^+$, we have $I^-(x'(z,\eta)) \subset I^-(\widehat{p}^+)$ and we can therefore have

$$\begin{aligned}&\mathcal{S}^{cl}{}_{z,\eta}((\vec{x}^{(\ell)},\vec{\xi}^{(\ell)}),t_0) \cap I^-(x'(z,\eta))\\ &= \mathcal{S}^{cl}{}_{z,\eta}((\vec{x}^{(\ell)},\vec{\xi}^{(\ell)}),t_0) \cap \mu_{z,\eta}((-1,\mathbb{S}(y,\zeta,z,\eta,s_1))).\end{aligned}$$

We denote then

$$\mathbf{p}_{z,\eta}(((\vec{x}^{(\ell)},\vec{\xi}^{(\ell)}))_{\ell=1}^\infty) = \mathcal{S}^{cl}{}_{z,\eta}((\vec{x}^{(\ell_1)},\vec{\xi}^{(\ell_1)}),t_0) \cap I^-(x'(z,\eta))$$

where ℓ_1 above is chosen so that the right hand side does not change if ℓ_1 is replaced with any larger value. Note that we can use any $\ell_1 \ge \ell_0$. Note that here $\mathbf{p}_{z,\eta}(((\vec{x}^{(\ell)},\vec{\xi}^{(\ell)}))_{\ell=1}^\infty)$ is in fact a set-valued function; its value can either be one point, or an empty set.

We say also that $((\vec{x}^{(\ell)},\vec{\xi}^{(\ell)}))_{\ell=1}^\infty$ is a $\mathcal{A}(y,\zeta)$ sequence if there is open and dense set $\mathcal{W}'\subset\mathcal{U}_{z_0,\eta_0}$ and a non-empty open set $\mathcal{W}_0'\subset\mathcal{W}'$ such that $((\vec{x}^{(\ell)},\vec{\xi}^{(\ell)}))_{\ell=1}^\infty$ is a $\mathcal{A}_{z,\eta}(y,\zeta)$ sequence for all $(z,\eta)\in\mathcal{W}'$ and the set $\mathbf{p}_{z,\eta}(((\vec{x}^{(\ell)},\vec{\xi}^{(\ell)}))_{\ell=1}^\infty)$ is non-empty for all $(z,\eta)\in\mathcal{W}_0'$.

Next, let us denote the $\mathcal{A}(y,\zeta)$ sequences by $\mathcal{X}=((\vec{x}^{(\ell)},\vec{\xi}^{(\ell)}))_{\ell=1}^\infty$.

Observe that $\mathcal{D}(\widehat{g},\widehat{\phi},\varepsilon)$ determines the points $x'(z,\eta)$, the sets $\mathcal{A}(y,\zeta)$, and $\mathbf{p}_{z,\eta}(\mathcal{X})$ for $\mathcal{X}\in\mathcal{A}(y,\zeta)$.

Let $(z,\eta)\in\mathcal{W}$ and consider a $\mathcal{A}_{z,\eta}(y,\zeta)$ sequence $\mathcal{X}=((\vec{x}^{(\ell)},\vec{\xi}^{(\ell)}))_{\ell=1}^\infty$. Below, we use $x_6=x'(z,\eta)$. We consider two cases separately:

Case (a): Assume that for all ℓ large enough the geodesics corresponding to $(\vec{x}^{(\ell)},\vec{\xi}^{(\ell)})$ intersect at a point $q_\ell\in(I^-(\widehat{p}^+)\cap I^-(x'(z,\eta)))\setminus J^+(y_1)$. As $\mathbf{e}_{z,\eta}(q_\ell)\ll x_6$, Lemma 5.4 implies $\mathbf{e}_{z,\eta}(q_\ell)={\mathcal{S}^{cl}}_{z,\eta}((\vec{x}^{(\ell)},\vec{\xi}^{(\ell)}),t_0)$ for ℓ is large enough. By definition of $\mathcal{X}$ we have $\mathbf{e}_{z,\eta}(q_\ell)=\mathbf{p}_{z,\eta}(\mathcal{X})$ for all ℓ large enough. Recall that when $\mathbf{e}_{z,\eta}(q_\ell)$ is given for all $(z,\eta)\in\mathcal{W}_0'$ we can determine by Lemma 2.6 the point q_ℓ uniquely. Thus we see that the points q_ℓ have to coincide when ℓ is large enough. Next we denote the point q_ℓ, when ℓ is large enough, by q.

Case (b): Assume that there are arbitrarily large ℓ such that the intersection point q_ℓ of the geodesics corresponding to $(\vec{x}^{(\ell)},\vec{\xi}^{(\ell)})$ is not in $(I^-(\widehat{p}^+)\cap I^-(x'(z,\eta)))\setminus J^+(y_1)$ or it does not exists. Then, when ℓ is large enough, then the set ${\mathcal{S}^{cl}}_{z,\eta}(\vec{x}^{(\ell)},\vec{\xi}^{(\ell)}),t_0)$ does not intersect $I^-(\widehat{p}^+)\cap I^-(x'(z,\eta))$.

By definition, for all $\mathcal{X}=((\vec{x}^{(\ell)},\vec{\xi}^{(\ell)}))_{\ell=1}^\infty\in\mathcal{A}(y,\zeta)$ there is a non-empty open set $\mathcal{W}_0'$ such that for large enough ℓ the intersection of ${\mathcal{S}^{cl}}_{z,\eta}(\vec{x}^{(\ell)},\vec{\xi}^{(\ell)}),t_0)$ and $I^-(\widehat{p}^+)\cap I^-(x'(z,\eta))$ is non-empty for all $(z,\eta)\in\mathcal{W}_0'$. Thus, as $\mathcal{W}_0'\cap\mathcal{W}$ is non-empty, we see that the case (b) is not possible. Thus for all $\mathcal{X}\in\mathcal{A}(y,\zeta)$ the case (a) has to hold and we have a well-defined intersection point $q=Q(\vec{x}^{(\ell)},\vec{\xi}^{(\ell)})$ where the geodesics corresponding to $(\vec{x}^{(\ell)},\vec{\xi}^{(\ell)})$ intersect when ℓ is large enough. Below we denote it by $Q(\mathcal{X})=q$. This point has to be on the geodesic $\gamma_{y,\eta}$, and by the above considerations, we see that it has to in the set $\gamma_{y,\zeta}([t_0,\infty))\cap(I^-(\widehat{p}^+)\setminus J^+(y_1))$. On the other hand, by Lemma 5.2, for all $q\in\gamma_{y,\zeta}([t_0,\infty))\cap(I^-(\widehat{p}^+)\setminus J^+(y_1))$ and all $\vartheta>0$ there is $(\vec{x},\vec{\xi})\in\mathcal{R}_\vartheta(y,\zeta)$ such that the geodesics corresponding to $(\vec{x},\vec{\xi})$ intersect at q. As $q\in I^-(\widehat{p}^+)$, we see that there is an open set $\mathcal{W}_0'\subset\mathcal{W}$ such that for all $(z,\eta)\in\mathcal{W}_0'$ we have $\mu_{z,\eta}((-1,m(z,\eta)))\cap\mathcal{L}_g^+(q)\neq\emptyset$ and thus $f^+_{\mu(z,\eta)}(q)<m(z,\eta)$. Combining these observations we conclude that the set $\{Q(\mathcal{X});\ \mathcal{X}\in\mathcal{A}(y,\zeta)\}$ coincides with the set $\gamma_{y,\zeta}([t_0,\infty))\cap(I^-(\widehat{p}^+)\setminus J^+(y_1))$.

When $\mathcal{W}'$ is equal to the set $\mathcal{W}$ we have

$$\begin{aligned}(147)\qquad &\mathbf{p}_{z,\eta}(\mathcal{X})\cap(I^-(\widehat{p}^+)\cap I^-(x'(z,\eta)))=\\ &\mathbf{e}_{z,\eta}(Q(\mathcal{X}))\cap(I^-(\widehat{p}^+)\cap I^-(x'(z,\eta)))\ \text{for all}\ (z,\eta)\in\mathcal{W}'.\end{aligned}$$

Note that then the equation (147) is valid in particular when $\mathcal{W}' = \mathcal{W}$. Unfortunately, we have not above determined the set $\mathcal{W}$ and thus can not assumed it to be known. However, for any open and dense set $\mathcal{W}'$ the intersection of $\mathcal{W}' \cap \mathcal{W}$ is an open and dense subset of $\mathcal{W}$ and thus

$$\{\mathbf{p}_{z,\eta}(\mathcal{X});\ (z,\eta) \in \mathcal{W}' \cap \mathcal{W}\} = \bigcup_{(z,\eta)\in\mathcal{W}'\cap\mathcal{W}} \mu_{z,\eta}((-1, m(z,\eta))) \cap \mathbf{e}_V(q)$$

is a dense subset of $\mathbf{e}_V(q) \cap I^-(\widehat{p}^+)$. Hence for all open and dense sets $\mathcal{W}'$,

$$\mathbf{e}_V(q) \cap I^-(\widehat{p}^+) \subset \mathrm{cl}(\{\mathbf{p}_{z,\eta}(\mathcal{X});\ (y,\eta) \in \mathcal{W}'\}) \cap I^-(\widehat{p}^+)$$

and the equality holds when $\mathcal{W}'$ is the set $\mathcal{W}$. Using this we see that if we take the intersection of the all sets $\mathrm{cl}(\{\mathbf{p}_{z,\eta}(\mathcal{X});\ (y,\eta) \in \mathcal{W}'\}) \cap I^-(\widehat{p}^+)$ where $\mathcal{W}' \subset \mathcal{U}_{z_0,\eta_0}$ is an open and dense subset, we obtain the set $\mathbf{e}_V(q)$ for $q = Q(\mathcal{X})$, where $V = U_g \cap I^-(\widehat{p}^+)$.

Doing this construction for all for $\mathcal{X} \in \mathcal{A}(y,\zeta)$, we determine the set $\mathbf{e}_V((\gamma_{y,\zeta}([t_0,\infty)) \cap I^-(\widehat{p}^+)) \setminus J^+(y_1))$. □

The above shows that the given data $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$ determine the collection $\{\mathbf{e}_V(q);\ q \in \gamma_{y,\zeta}([t_0,\infty)) \cap (I^-(\widehat{p}^+) \setminus J^+(y_1))\}$ for all $y = \widehat{\mu}(s)$ and $\zeta \in L^+_y M_0$, $\|\zeta\|_{g^+} = 1$, where $s \in [s_1 - \kappa_3, s_1)$. Taking the union of all such collections and $\mathbf{e}_V(J^+(\widehat{\mu}(s_1)) \cap I^-(\widehat{p}^+))$ and $\mathbf{e}_V(\mathcal{K}_{t_0} \cap J^+(\widehat{\mu}(s)))$, we obtain the set $\mathbf{e}_V(J^+(\widehat{\mu}(s)) \cap I^-(\widehat{p}^+))$. Iterating this construction for $s_2, s_3, \dots, s_K$, with $s_{k+1} \in (s_k - \kappa_3, s_k)$, we can find $\mathbf{e}_V(I(\widehat{\mu}(s'), \widehat{p}^+))$ for all $s_- < s' < s_+$.

Let now for $s_- < s' < s'' < s_+$. Observe that $\mathbf{e}_V(q) \in \mathbf{e}_V(J^+(\widehat{\mu}(s)) \cap I^-(\widehat{p}^+))$ satisfies $\mathbf{e}_V(q) \in \mathbf{e}_V(I(\widehat{\mu}(s'), \widehat{\mu}(s'')))$ if and only if $\mathbf{e}_{z_0,\eta_0}(q) \ll \widehat{\mu}(s'')$. Thus we can find the sets $\mathbf{e}_V(I(\widehat{\mu}(s'), \widehat{\mu}(s'')))$ for all $s_- < s' < s'' < s_+$.

By Theorem 1.2 we can reconstruct the manifold $I(\widehat{\mu}(s'), \widehat{\mu}(s''))$ for all $s_- < s' < s'' < s_+$. Glueing these constructions together we obtain $I(\widehat{p}^-, \widehat{p}^+)$ and the conformal structure on it. This proves Theorem 1.4. □

Proof. (of Thm. 1.5) As noted in Appendix B, by assuming that Condition A is valid and by making the parameter $\widehat{h}$ determining U_g smaller, there are adaptive source function, or $\mathcal{S}$-functions, given in the formula (170) in Appendix B with $L \geq 5$ and $K = L \cdot (L!) + 1$. for which the Assumption S is valid. Next we consider these $\mathcal{S}$-functions.

Let us denote by (g', ϕ') the solutions of (7) with some sources $(\mathcal{F}_1, \mathcal{F}_2)$ supported compactly in $U_{g'}$. We want to find out when the observations $(U_{g'}, g'|_{U_{g'}}, \phi'|_{U_{g'}}, \mathcal{F}_1, \mathcal{F}_2)$, representing an equivalence class in $\mathcal{D}^{alt}(\widehat{g}, \widehat{\phi}, \varepsilon)$ that satisfy the conservation law $\nabla^g_j(\mathbf{T}^{jk}(g,\phi) + \mathcal{F}_1^{jk}) = 0$, are equivalent to the observations of some solution (g,ϕ) of the model (10) when the $\mathcal{S}$-functions are given by the formula (170) in Appendix

B. To do this, assume that we are given the restrictions of $\widehat{g}$ and the solution of (7), denoted g', ϕ' in $U_{g'}$ and $(\mathcal{F}_1, \mathcal{F}_2)$. Then we can compute

$$R' := \mathcal{F}_1,$$
$$S_\ell = (\mathcal{F}_2)_\ell, \quad \ell = 1, 2, \dots, L,$$
$$Z' := \sum_{\ell=1}^{L} S_\ell \phi'_\ell$$

and find $P' := R' - Z'g'$. After we have found these functions, we test if equations (165) and (166) in the Appendix B hold for $g = g'$, $\phi = \phi'$, $P = P'$, $Z = Z'$, $R = R'$, $Q_{L+1} = Z$ and some functions $(Q_\ell)_{\ell=1}^L$ that are compactly supported in $U_g = U_{g'}$. If this is the case the functions g, ϕ in U_g and (P, Q), supported in U_g, are restrictions of the solutions of (10) with the $\mathcal{S}$-functions given by the formula (170) in Appendix B with some P and Q.

Clearly, all solutions (10) with the $\mathcal{S}$-functions (170) correspond to the solutions of (7) with some $(\mathcal{F}_1, \mathcal{F}_2)$,

Summarizing, for a given $(U_{g'}, g'|_{U_{g'}}, \phi'|_{U_{g'}}, \mathcal{F}_1, \mathcal{F}_2)$ that represents an equivalence class in $\mathcal{D}^{alt}(\widehat{g}, \widehat{\phi}, \varepsilon)$ and satisfies the conservation law $\nabla^g_j(\mathbf{T}^{jk}(g, \phi) + \mathcal{F}_1^{jk}) = 0$, we can find out if there exists an element $(U_{g'}, g'|_{U_{g'}}, \phi'|_{U_{g'}}, F) \in \mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$ with some F supported in $U_{g'}$, and if so, we can find this element. Hence, when we are given the collection $\mathcal{D}^{alt}(\widehat{g}, \widehat{\phi}, \varepsilon)$, we can choose from it all elements that correspond to some element of $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$, and thus we can find $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$. The claim follows then from Theorem 1.4. □

Proof. (of Corollary 1.6) In the above proof of Theorem 1.4, we used the assumption that $\widehat{Q} = 0$ and $\widehat{P} = 0$ to obtain equations (62). In the setting of Cor. 1.6 where the background source fields $\widehat{Q}$ and $\widehat{P}$ are not zero, we need to assume in the computations related to sources (59) that there are neighborhoods V_j of the geodesics γ_j that satisfy $\text{supp}\,(\mathbf{f}_j) \subset V_j$, the linearized waves $u_j = \mathbf{Q}_{\widehat{g}}\mathbf{f}_j$ satisfy $\text{singsupp}(u_j) \subset V_j$, and $V_i \cap V_j \cap (\text{supp}\,(\widehat{Q}) \cup \text{supp}\,(\widehat{P})) = \emptyset$ for $i \neq j$. To this end, we have first consider measurements for the linearized waves and check for given $(\vec{x}, \vec{\xi})$ that no two geodesic $\gamma_{x_j,\xi_j}(\mathbb{R}_+)$ do intersect at $U_{\widehat{g}}$ and restrict all considerations for such $(\vec{x}, \vec{\xi})$. Notice that such $(\vec{x}, \vec{\xi})$ form an open and dense set in $(TU_{\widehat{g}})^4$. If then the width $\widehat{s}$ of the used spherical waves is chosen to be small enough, we see that condition $V_i \cap V_j \cap (\text{supp}\,(\widehat{Q}) \cup \text{supp}\,(\widehat{P})) = \emptyset$ is satisfied.

The above restriction causes only minor modifications in the above proof and thus, mutatis mutandis, we see that we can determine the conformal type of the metric in all relatively compact subsets $I_{\widehat{g}}(\widehat{\mu}(s'), \widehat{\mu}(s'')) \setminus (\text{supp}\,(\widehat{Q}) \cup \text{supp}\,(\widehat{P}))$, $s_- < s' < s'' < s_+$, of $I_{\widehat{g}}(\widehat{p}^-, \widehat{p}^+) \setminus (\text{supp}\,(\widehat{Q}) \cup \text{supp}\,(\widehat{P}))$. By glueing these manifolds and $U_{\widehat{g}}$ together, we find the conformal type of the metric in $I_{\widehat g}(\widehat p^-,\widehat p^+)$. After this the claim follows from Corollary 1.3.

□

5.1. **A discussion on an application for a dark matter related example.** The determination of the conformal class of the Lorentzian metric considered above can be done also for a model that is related to dark matter and energy [73]. We note that by reconstructing the conformal class of the metric tensor g in area of space that contain dark matter but not usual "observable matter" tells how the "dark matter" would change the path of light rays that would travel in this area, even the path of light rays on which we can not do direct measurements.

Let us consider a model where the fields ϕ_ℓ, $\ell\le L-1$ can be observed and correspond to "usual" matter. The field ϕ_L could correspond to "dark" matter. We write $\phi=(\phi',\phi_L)$, where $\phi'=(\phi_\ell)_{\ell=1}^{L-1}$ and $Q=(Q',Q_L)$, where $Q'=(Q_\ell)_{\ell=1}^{L-1}$.

Moreover, we assume that in the model (10) the adaptive source functions $\mathcal{S}_\ell(\phi,\nabla^g\phi,Q,\nabla^gQ,P,\nabla^gP,g)$ are such that

$$\begin{aligned}(148)\quad &\mathcal{S}_\ell(\phi,\nabla^g\phi,Q,\nabla^gQ,P,\nabla^gP,g)=\widetilde{\mathcal{S}}_\ell(\phi',\nabla^g\phi',Q',\nabla^gQ',P,\nabla^gP,g),\\ &\quad \text{for } \ell\le L-1,\\ &\mathcal{S}_L(\phi,\nabla^g\phi,Q,\nabla^gQ,P,\nabla^gP,g)=0.\end{aligned}$$

We consider the model (10) assuming that $Q_L=0$ and that we can observe only the g and ϕ' components of the waves in U_g. Moreover, we assume that the Assumption B is valid with permutations σ for which $L\notin\{\sigma(j);\ j=1,2,3,4,5\}$. Analyzing the wave equation in the $\widehat g$ wave map coordinates, we see that the ϕ_L component of linearized waves $(\dot g,\dot\phi)$ satisfies

$$(149)\,(\widehat g^{jk}\partial_j\partial_k-\widehat g^{pq}\widehat\Gamma^j_{pq}\partial_j+m)\dot\phi_L=-(\dot g^{jk}\partial_j\partial_k+\dot g^{pq}\widehat\Gamma^j_{pq}\partial_j)\widehat\phi_L.$$

Thus, using the notations introduced earlier in the paper, we consider $(\vec x,\vec\xi)$ such that $\gamma_{x_j,\xi_j}([t_0,\infty))$ intersect at a point q and that there is a light-like geodesic from q to the point $y\in U_{\widehat g}$. Assume that $K_j\subset\mathcal{L}^+_{\widehat g}(x_j)$ are such that $\gamma_{x_j,\xi_j}([t_0,\infty))\subset K_j$ and consider linearized waves $\dot u^{(j)}=(\dot g^{(j)},\dot\phi^{(j)})\in\mathcal{I}(K_j)$, $j=1,2,3,4$. We showed earlier that in a generic case the interaction of the waves at q can be observed in normal coordinates at y at least in two polarizations, that is, in two components of $\dot u^{(j)}$. If we do not observe the ϕ_L component of the waves, then the interaction of the waves can be observed in normal coordinates at y at least in one polarization. We also showed that there are some principal symbols of the waves $\dot u^{(j)}$ at q that produce such observable singularities and these principal symbols are such that the ϕ-components of the principal symbols are zero. In particular, the ϕ_L-component the principal symbols of $\dot u^{(j)}$ is zero at $(q,\eta)\in N^*K_j$. As the linearized equation (149) contains no derivatives of $\dot g$, we see then that the ϕ_L-component the principal symbols of $\dot u^{(j)}$ is zero also at $(x_j,\xi_j^\flat)$ that is on the same bicharacteristic as (q,η). A linearized wave $\dot u^{(j)}$ with such principal symbol can be produced with sources Q for which $Q_L=0$, that is, without source terms in the "dark" matter component. Because of the above considerations, we see that with a slight modification of the proof, we can find the conformal type of the metric.

Appendix A: Reduced Einstein equation

In this section we review known results on Einstein equations and wave maps.

A.1. Summary of the used notations. Let us recall some definitions given in Introduction, in the Subsection 1.3.4. Let $(M,\widehat g)$ be a C^∞-smooth globally hyperbolic Lorentzian manifold and $\widetilde g$ be a C^∞-smooth globally hyperbolic metric on M such that $\widehat g<\widetilde g$.

Recall that there is an isometry $\Phi:(M,\widetilde g)\to(\mathbb{R}\times N,\widetilde h)$, where N is a 3-dimensional manifold and the metric $\widetilde h$ can be written as $\widetilde h=-\beta(t,y)dt^2+\overline h(t,y)$ where $\beta:\mathbb{R}\times N\to(0,\infty)$ is a smooth function and $\overline h(t,\cdot\,)$ is a Riemannian metric on N depending smoothly on $t\in\mathbb{R}$. As in the main text identify these isometric manifolds and denote $M=\mathbb{R}\times N$. Also, for $t\in\mathbb{R}$, recall that $M(t)=(-\infty,t)\times N$. We use parameters $t_1>t_0>0$ and denote $M_j=M(t_j)$, $j\in\{0,1\}$. We use the time-like geodesic $\widehat\mu=\mu_{\widehat g}$, $\mu_{\widehat g}:[-1,1]\to M_0$ on $(M_0,\widehat g)$ and the set $\mathcal{K}_j:=J^+_{\widehat g}(\widehat p^-)\cap M_j$ with $\widehat p^-=\widehat\mu(s_-)\in(0,t_0)\times N$, $s_-\in(-1,1)$ and recall that $J^+_{\widehat g}(\widehat p^-)\cap M_j$ is compact and there exists $\varepsilon_0>0$ such that if $\|g-\widehat g\|_{C^0_b(M_1;\widehat g^+)}<\varepsilon_0$, then $g|_{\mathcal{K}_1}<\widetilde g|_{\mathcal{K}_1}$, and in particular, we have $J^+_g(p)\cap M_1\subset\mathcal{K}_1$ for all $p\in\mathcal{K}_1$.

Let us use local coordinates on M_1 and denote by $\nabla_k=\nabla_{X_k}$ the covariant derivative with respect to the metric g to the direction $X_k=\frac{\partial}{\partial x^p}$ and by $\widehat\nabla_k=\widehat\nabla_{X_k}$ the covariant derivative with respect to the metric $\widehat g$ to the direction X_k.

A.2. Reduced Ricci and Einstein tensors. Following [26] we recall that

$$\mathrm{Ric}_{\mu\nu}(g)\ =\ \mathrm{Ric}^{(h)}_{\mu\nu}(g)+\frac12\big(g_{\mu q}\frac{\partial\Gamma^q}{\partial x^\nu}+g_{\nu q}\frac{\partial\Gamma^q}{\partial x^\mu}\big)\tag{150}$$

where $\Gamma^q=g^{mn}\Gamma^q_{mn}$,

$$\mathrm{Ric}^{(h)}_{\mu\nu}(g)=-\frac12 g^{pq}\frac{\partial^2 g_{\mu\nu}}{\partial x^p\partial x^q}+P_{\mu\nu},\tag{151}$$

$$P_{\mu\nu}=g^{ab}g_{ps}\Gamma^p_{\mu b}\Gamma^s_{\nu a}+\frac12\big(\frac{\partial g_{\mu\nu}}{\partial x^a}\Gamma^a+g_{\nu l}\Gamma^l_{ab}g^{aq}g^{bd}\frac{\partial g_{qd}}{\partial x^\mu}+g_{\mu l}\Gamma^l_{ab}g^{aq}g^{bd}\frac{\partial g_{qd}}{\partial x^\nu}\big).$$

Note that $P_{\mu\nu}$ is a polynomial of g_{jk} and g^{jk} and first derivatives of g_{jk}. The harmonic Einstein tensor is

$$\mathrm{Ein}^{(h)}_{jk}(g) = \mathrm{Ric}^{(h)}_{jk}(g) - \frac{1}{2} g^{pq}\mathrm{Ric}^{(h)}_{pq}(g)\, g_{jk}. \tag{152}$$

The harmonic Einstein tensor is extensively used to study Einstein equations in local coordinates where one can use the Minkowski space $\mathbb{R}^4$ as the background space. To do global constructions with a background space $(M,\widehat g)$ one uses the reduced Einstein tensor. The $\widehat g$-reduced Einstein tensor $\mathrm{Ein}_{\widehat g}(g)$ and the $\widehat g$-reduced Ricci tensor $\mathrm{Ric}_{\widehat g}(g)$ are given by

$$(\mathrm{Ein}_{\widehat g}(g))_{pq} = (\mathrm{Ric}_{\widehat g}(g))_{pq} - \frac{1}{2}(g^{jk}(\mathrm{Ric}_{\widehat g} g)_{jk})g_{pq}, \tag{153}$$

$$(\mathrm{Ric}_{\widehat g}(g))_{pq} = \mathrm{Ric}_{pq} g - \frac{1}{2}(g_{pn}\widehat\nabla_q \widehat F^n + g_{qn}\widehat\nabla_p \widehat F^n) \tag{154}$$

where $\widehat F^n$ are the harmonicity functions given by

$$\widehat F^n = \Gamma^n - \widehat\Gamma^n, \quad \text{where } \Gamma^n = g^{jk}\Gamma^n_{jk}, \quad \widehat\Gamma^n = g^{jk}\widehat\Gamma^n_{jk}, \tag{155}$$

where Γ^n_{jk} and $\widehat\Gamma^n_{jk}$ are the Christoffel symbols for g and $\widehat g$, correspondingly. Note that $\widehat\Gamma^n$ depends also on g^{jk}. As $\Gamma^n_{jk} - \widehat\Gamma^n_{jk}$ is the difference of two connection coefficients, it is a tensor. Thus $\widehat F^n$ is tensor (actually, a vector field), implying that both $(\mathrm{Ric}_{\widehat g}(g))_{jk}$ and $(\mathrm{Ein}_{\widehat g}(g))_{jk}$ are 2-covariant tensors. Observe that the $\widehat g$-reduced Einstein tensor is sum of the harmonic Einstein tensor and a term that is a zeroth order in g,

$$(\mathrm{Ein}_{\widehat g}(g))_{\mu\nu} = \mathrm{Ein}^{(h)}_{\mu\nu}(g) + \frac{1}{2}(g_{\mu q}\frac{\partial \widehat\Gamma^q}{\partial x^\nu} + g_{\nu q}\frac{\partial \widehat\Gamma^q}{\partial x^\mu}). \tag{156}$$

A.3. Wave maps and reduced Einstein equations. Let us consider the manifold $M_1 = (-\infty, t_1) \times N$ with a C^m-smooth metric g', $m \geq 8$, which is a perturbation of the metric $\widehat g$ and satisfies the Einstein equation

$$\mathrm{Ein}(g') = T' \quad \text{on } M_1, \tag{157}$$

or equivalently,

$$\mathrm{Ric}(g') = \rho', \quad \rho'_{jk} = T'_{jk} - \frac{1}{2}((g')^{nm}T'_{nm})g'_{jk} \quad \text{on } M_1.$$

Assume also that $g' = \widehat g$ in the domain A, where $A = M_1 \setminus \mathcal{K}_1$ and $\|g' - \widehat g\|_{C^2_b(M_1,\widehat g^+)} < \varepsilon_0$, so that (M_1, g') is globally hyperbolic. Note that then $T' = \widehat T$ in the set A and that the metric g' coincides with $\widehat g$ in particular in the set $M^- = \mathbb{R}_- \times N$

We recall next the considerations of [13]. Let us consider the Cauchy problem for the wave map $f : (M_1, g') \to (M, \widehat g)$, namely

$$\Box_{g',\widehat g} f = 0 \quad \text{in } M_1, \tag{158}$$

$$f = Id, \quad \text{in } \mathbb{R}_- \times N, \tag{159}$$

where $M_1=(-\infty,t_1)\times N\subset M$. In (158), $\square_{g',\widehat g}f=g'\cdot\widehat\nabla^2 f$ is the wave map operator, where $\widehat\nabla$ is the covariant derivative of a map $(M_1,g')\to(M,\widehat g)$, see [13, formula (7.32)]. In local coordinates $X:V\to\mathbb{R}^4$ of $V\subset M_1$, denoted $X(z)=(x^j(z))_{j=1}^4$ and $Y:W\to\mathbb{R}^4$ of $W\subset M$, denoted $Y(z)=(y^A(z))_{A=1}^4$, the wave map $f:M_1\to M$ has representation $Y(f(X^{-1}(x)))=(f^A(x))_{A=1}^4$ and the wave map operator in equation (158) is given by

$$(160)\qquad (\square_{g',\widehat g}f)^A(x)=(g')^{jk}(x)\Big(\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^k}f^A(x)-\Gamma'^n_{jk}(x)\frac{\partial}{\partial x^n}f^A(x)+\widehat\Gamma^A_{BC}(f(x))\,\frac{\partial}{\partial x^j}f^B(x)\,\frac{\partial}{\partial x^k}f^C(x)\Big)$$

where $\widehat\Gamma^A_{BC}$ denotes the Christoffel symbols of metric $\widehat g$ and Γ'^j_{kl} are the Christoffel symbols of metric g'. When (158) is satisfied, we say that f is wave map with respect to the pair $(g',\widehat g)$. The important property of the wave maps is that, if f is wave map with respect to the pair $(g',\widehat g)$ and $g=f_*g'$ then, as follows from (160), the identity map $Id:x\mapsto x$ is a wave map with respect to the pair $(g,\widehat g)$ and, the wave map equation for the identity map is equivalent to (cf. [13, p. 162])

$$(161)\qquad \Gamma^n=\widehat\Gamma^n,\quad \text{where } \Gamma^n=g^{jk}\Gamma^n_{jk},\quad \widehat\Gamma^n=g^{jk}\widehat\Gamma^n_{jk}$$

where the Christoffel symbols $\widehat\Gamma^n_{jk}$ of the metric $\widehat g$ are smooth functions.

As $g=g'$ outside a compact set $\mathcal{K}_1\subset(0,t_1)\times N$, we see that this Cauchy problem is equivalent to the same equation restricted to the set $(-\infty,t_1)\times B_0$, where $B_0\subset N$ is such an open relatively compact set that $\mathcal{K}_1\subset(0,t_1]\times B_0$ with the boundary condition $f=Id$ on $(0,t_1]\times\partial B_0$. Then using results of [38], that can be applied for equations on manifold as is done in Appendix C, and combined with the Sobolev embedding theorem, we see[3] that there is $\varepsilon_1>0$ such that if $\|g'-\widehat g\|_{C^m_b(M_1;\widehat g^+)}<\varepsilon_1$, then there is a map $f:M_1\to M$ satisfying the Cauchy problem (158)-(159) and the solution depends continuously, in $C^{m-5}_b([0,t_1]\times N,g^+)$, on the metric g'. Moreover, by the uniqueness of the wave map, we have $f|_{M_1\setminus\mathcal{K}_1}=id$ so that $f(\mathcal{K}_1)\cap M_0\subset\mathcal{K}_0$.

As the inverse function of the wave map f depends continuously, in $C^{m-5}_b([0,t_1]\times N,g^+)$, on the metric g' we can also assume that ε_1 is so small that $M_0\subset f(M_1)$.

Denote next $g:=f_*g'$, $T:=f_*T'$, and $\rho:=f_*\rho'$ and define $\widehat\rho=\widehat T-\frac12(\mathrm{Tr}\,\widehat T)\widehat g$. Then g is C^{m-6}-smooth and the equation (157) implies

$$(162)\qquad \mathrm{Ein}(g)=T\quad\text{on } M_0.$$

[3] See also: Thm. 4.2 in App. III of of [13], and its proof for the estimates for the time on which the solution exists.

As f is a wave map, g satisfies (161) and thus by the definition of the reduced Einstein tensor, (153), we have

$$\mathrm{Ein}_{pq}(g) = (\mathrm{Ein}_{\widehat{g}}(g))_{pq} \quad \text{on } M_0.$$

This and (162) yield the $\widehat{g}$-reduced Einstein equation

$$(\mathrm{Ein}_{\widehat{g}}(g))_{pq} = T_{pq} \quad \text{on } M_0. \tag{163}$$

This equation is useful for our considerations as it is a quasilinear, hyperbolic equation on M_0. Recall that g coincides with $\widehat{g}$ in $M_0 \setminus \mathcal{K}_0$. The unique solvability of this Cauchy problem is studied in e.g. [13, Thm. 4.6 and 4.13] and [38] and in Appendix C below.

A.4. Relation of the reduced Einstein equations and for the original Einstein equation. The metric g which solves the $\widehat{g}$-reduced Einstein equation $\mathrm{Ein}_{\widehat{g}}(g) = T$ is a solution of the original Einstein equation $\mathrm{Ein}(g) = T$ if the harmonicity functions $\widehat{F}^n$ vanish identically. Next we recall the result that the harmonicity functions vanish on M_0 when

$$\begin{aligned} &(\mathrm{Ein}_{\widehat{g}}(g))_{jk} = T_{jk}, \quad \text{on } M_0,\\ &\nabla_p T^{pq} = 0, \quad \text{on } M_0,\\ &g = \widehat{g}, \quad \text{on } M_0 \setminus \mathcal{K}_0. \end{aligned} \tag{164}$$

To see this, let us denote $\mathrm{Ein}_{jk}(g) = S_{jk}$, $S^{jk} = g^{jn}g^{km}S_{nm}$, and $T^{jk} = g^{jn}g^{km}T_{nm}$. Following the standard arguments, see [13], we see from (153) that in local coordinates

$$S_{jk} - (\mathrm{Ein}_{\widehat{g}}(g))_{jk} = \frac{1}{2}(g_{jn}\widehat{\nabla}_k \widehat{F}^n + g_{kn}\widehat{\nabla}_j \widehat{F}^n - g_{jk}\widehat{\nabla}_n \widehat{F}^n).$$

Using equations (164), the Bianchi identity $\nabla_p S^{pq} = 0$, and the basic property of Lorentzian connection, $\nabla_k g^{nm} = 0$, we obtain

$$\begin{aligned} 0 &= 2\nabla_p(S^{pq} - T^{pq})\\ &= \nabla_p(g^{qk}\widehat{\nabla}_k F^p + g^{pm}\widehat{\nabla}_m \widehat{F}^q - g^{pq}\widehat{\nabla}_n \widehat{F}^n)\\ &= g^{pm}\nabla_p \widehat{\nabla}_m \widehat{F}^q + (g^{qk}\nabla_p \widehat{\nabla}_k \widehat{F}^p - g^{qp}\nabla_p \widehat{\nabla}_n \widehat{F}^n)\\ &= g^{pm}\nabla_p \widehat{\nabla}_m \widehat{F}^q + W^q(\widehat{F}) \end{aligned}$$

where $\widehat{F} = (\widehat{F}^q)_{q=1}^4$ and the operator

$$W : (\widehat{F}^q)_{q=1}^4 \mapsto (g^{qk}(\nabla_p \widehat{\nabla}_k \widehat{F}^p - \nabla_k \widehat{\nabla}_p \widehat{F}^p))_{q=1}^4$$

is a linear first order differential operator which coefficients are polynomial functions of $\widehat{g}_{jk}$, $\widehat{g}^{jk}$, g_{jk}, g^{jk} and their first derivatives.

Thus the harmonicity functions $\widehat{F}^q$ satisfy on M_0 the hyperbolic initial value problem

$$\begin{aligned} &g^{pm}\nabla_p \widehat{\nabla}_m \widehat{F}^q + W^q(\widehat{F}) = 0, \quad \text{on } M_0,\\ &\widehat{F}^q = 0, \quad \text{on } M_0 \setminus \mathcal{K}_0, \end{aligned}$$

and as this initial Cauchy problem is uniquely solved by [13, Thm. 4.6 and 4.13] or [38], we see that $\widehat{F}^q = 0$ on M_0. Thus equations (164) yield that Einstein equation $\mathrm{Ein}(g) = T$ holds on M_0.

Appendix B: An example satisfying Assumption S

Next we give an example of functions $\mathcal{S}_\ell(\phi, \nabla\phi, Q', \nabla^g P, Q_K, \nabla Q_K, g)$ in the model (10) for which Assumption S is valid.

Let $L \geq 5$, g be a C^2-smooth metric and $\phi = (\phi_\ell)_{\ell=1}^L$ be C^2-smooth functions on $\widehat{U} \subset M$. Let us fix a symmetric (0,2)-tensor P and a scalar function Z that are C^2-smooth and compactly supported in $\widehat{U}$. Let $[P_{jk}(x)]_{j,k=1}^4$ be the coefficients of P in local coordinates.

To obtain adaptive source functions satisfying the assumption S, let us start implications of the conservation law (11). To this end, consider C^2-smooth functions $S_\ell(x)$ on $\widehat{U}$. The conservation law (11) gives for all $j = 1, 2, 3, 4$ equations (see [13, Sect. 6.4.1])

$$\begin{aligned}
0 &= \frac{1}{2}\nabla^g_p(g^{pk}T_{jk}) \\
&= \sum_{\ell=1}^L (g^{pk}\nabla^g_p\partial_k\phi_\ell)\,\partial_j\phi_\ell - (m_\ell\phi_\ell\partial_p\phi_\ell)\delta^p_j + \frac{1}{2}\nabla^g_p(g^{pk}g_{jk}S_\ell\phi_\ell + g^{pk}P_{jk}) \\
&= \sum_{\ell=1}^L (g^{pk}\nabla^g_p\partial_k\phi_\ell - m_\ell\phi_\ell)\,\partial_j\phi_\ell + \frac{1}{2}\nabla^g_p(g^{pk}g_{jk}S_\ell\phi_\ell + g^{pk}P_{jk}) \\
&= \sum_{\ell=1}^L S_\ell\,\partial_j\phi_\ell + \frac{1}{2}\nabla^g_p(g^{pk}g_{jk}S_\ell\phi_\ell) + \frac{1}{2}g^{pk}\nabla^g_p P_{jk} \\
&= \left(\sum_{\ell=1}^L S_\ell\,\partial_j\phi_\ell\right) + \frac{1}{2}\partial_j\left(\sum_{\ell=1}^L S_\ell\phi_\ell\right) + \frac{1}{2}g^{pk}\nabla^g_p P_{jk}.
\end{aligned}$$

Recall that S_ℓ should satisfy

$$\sum_{\ell=1}^L S_\ell\phi_\ell = Z. \tag{165}$$

Then, the conservation law (11) holds if we have

$$\sum_{\ell=1}^L S_\ell\,\partial_j\phi_\ell = -\frac{1}{2}g^{pk}\nabla^g_p R_{jk}, \quad R_{jk} = (P_{jk} + g_{jk}Z), \tag{166}$$

for $j = 1, 2, 3, 4$.

Equations (165) and (166) give together five point-wise equations for the functions $S_1, \dots, S_L$.

Next we denote the set of permutations $\sigma : \{1, 2, \dots, L\} \to \{1, 2, \dots, L\}$ by $\Sigma(L)$. Next we assume Condition A, that is, that at any $x \in \mathrm{cl}(\widehat{U})$

there is a permutation $\sigma:\{1,2,\dots,L\}\to\{1,2,\dots,L\}$ such that the 5×5 matrix $(B^\sigma_{jk}(\widehat\phi(x),\nabla\widehat\phi(x)))_{j,k\le 5}$ is invertible, where

$$(B^\sigma_{jk}(\phi(x),\nabla\phi(x)))_{j,k\le 5}=\begin{pmatrix}(\,\partial_j\phi_{\sigma(\ell)}(x))_{j\le 4,\ \ell\le 5}\\ (\phi_{\sigma(\ell)}(x))_{\ell\le 5}\end{pmatrix}.$$

Let $V_\sigma\subset \mathrm{cl}(\widehat U)$ be the set where $(B^\sigma_{jk}(\widehat\phi(x),\nabla\widehat\phi(x)))_{j,k\le 5}$ is invertible.

Below, let us use $K=L\cdot(L!)+1$ and identify the set index set $\{1,2,\dots,K-1\}$ with the the set $\Sigma(L)\times\{1,2,\dots,L\}$. We consider a $\mathbb{R}^K$ valued function $Q(x)=(Q'(x),Q_K(x))$, where

$$Q'=(Q_{\sigma,\ell})_{\ell\in\{1,2,\dots,L\},\ \sigma\in\Sigma(L)}.$$

Note that we have introduced the following renumbering: identify the set index set $\{1,2,\dots,\widetilde K-1\}$ with the the set $\Sigma(L)\times\{1,2,\dots,L\}$ using a bijective map $j\mapsto(\sigma(j),l_j)$.

Also, below $R_{jk}=P_{jk}+g_{jk}Z$ and we set

$$Q_K=Z. \tag{167}$$

Our next aim is to consider first a fixed permutation σ and point $x\in V_\sigma$, and construct scalar functions $\mathcal{S}_{\sigma,\ell}(Q',Q_K,R,g,\phi)$, $\ell=1,2,\dots,L$ that satisfy

$$\sum_{\ell=1}^5\mathcal{S}_{\sigma,\ell}(Q',Q_K,R,g,\phi)\,\partial_j\phi_{\sigma(\ell)}=-\frac12 g^{pk}\nabla^g_pR_{jk}-\sum_{\ell=6}^L Q_{\sigma,\ell}\,\partial_j\phi_{\sigma(\ell)}, \tag{168}$$

$$\sum_{\ell=1}^5\mathcal{S}_{\sigma,\ell}(Q',Q_K,R,g,\phi)\,\phi_{\sigma(\ell)}=Q_K-\sum_{\ell=6}^L Q_{\sigma,\ell}\phi_{\sigma(\ell)}. \tag{169}$$

Recall that for $x\in V_\sigma$ the matrix $\mathcal{B}^\sigma(\phi,\nabla\phi)=(B^\sigma_{jk}(\phi(x),\nabla\phi(x)))_{j,k=1}^5$ is invertible.

Let $(Y_\sigma(\phi,\nabla\phi))(x)=(B^\sigma(\phi,\nabla\phi))^{-1}$ for $x\in V_\sigma$, and zero for $x\notin V_\sigma$. Then we define $\mathcal{S}_{\sigma,\ell}=\mathcal{S}_{\sigma,\ell}(Q',Q_K,R,g,\phi)$, $\ell=1,2,\dots,L$, to be

$$\begin{aligned}(S_{\sigma,\ell})_{\ell\le 5}&=Y(\phi,\nabla\phi)\begin{pmatrix}(-\frac12 g^{pk}\nabla^g_pR_{jk}-\sum_{\ell=6}^L Q_{\sigma,\ell}\,\partial_j\phi_{\sigma(\ell)})_{j\le 4}\\ Q_K-\sum_{\ell=6}^L Q_{\sigma,\ell}\phi_{\sigma(\ell)}\end{pmatrix}\\ &=Y(\phi,\nabla\phi)\begin{pmatrix}(-\frac12 g^{pk}\nabla^g_pR_{jk}\\ Q_K\end{pmatrix}+Q_{\sigma,\ell},\\ (S_{\sigma,\ell})_{\ell\ge 6}&=(Q_{\sigma,\ell})_{\ell\ge 6}.\end{aligned} \tag{170}$$

Observe that $Q_{\widetilde\sigma,\ell}$, with $\widetilde\sigma\neq\sigma$, do not appear in the formula (170).

Above, Z and P were fixed. Let us now choose Q' to be arbitrary C^1-functions. Recall that $R_{jk}=P_{jk}+g_{jk}Z$. Then, we see that if we denote (note that we later will change the meaning of symbols S_ℓ)

$$\begin{aligned}Q_K&=Z,\\ S_{\sigma,\ell}&=\mathcal{S}_{\sigma,\ell}(\phi,\nabla\phi,Q',Q_K,\nabla Q_K,\nabla^gP,g),\end{aligned}$$

we see that the equations (165) and (166) are satisfied in $x \in V_\sigma$ when functions S_ℓ are replaced by $S_{\sigma,\ell}$. Moreover, we see that the derivative of $\mathcal{S}_\sigma = (\mathcal{S}_{\sigma,\ell})_{\ell=1}^L$ with respect to (Q', Q_K, R), that is,

$$(171) \qquad D_{Q',Q_K,R}\mathcal{S}_\sigma(\widehat{\phi}, \widehat{\nabla}\widehat{\phi}, Q', Q_K, R, \widehat{g}) : \mathbb{R}^{K+4} \to \mathbb{R}^L$$

is surjective.

Let us next combine the above constructions that were done for a single σ. Let $\psi_\sigma \in C^\infty(\mathrm{cl}(\widehat{U}))$ be the partition of unity such that $\mathrm{supp}\,(\psi_\sigma) \subset V_\sigma$ and $\sum_{\sigma\in\Sigma(L)} \psi_\sigma(x) = 1$ in $\mathrm{cl}(\widehat{U})$.

We define

$$\mathcal{S}_\ell(\phi, \nabla\phi, Q', Q_K, \nabla Q_K, \nabla^g P, g) = \sum_{\sigma\in\Sigma(L)} \psi_\sigma(x)\, \mathcal{S}_{\sigma,\ell}(\phi, \nabla\phi, Q', Q_K, \nabla Q_K, \nabla^g P, g),$$

Let us next denote

$$\begin{aligned} Q_K &= Z, \\ S_\ell &= \mathcal{S}_\ell(\phi, \nabla\phi, Q', Q_K, \nabla Q_K, \nabla^g P, g). \end{aligned}$$

Then we see, using the partition of unity, that the functions S_ℓ satisfy the equations (165) and (166) for all $x \in \widehat{U}$.

Using the fact that $Q_{\widetilde{\sigma},\ell}$, with $\widetilde{\sigma} \neq \sigma$, do not appear in the formula (170) and that derivatives (171) are surjective, we see that in the derivative of $\mathcal{S} = (\mathcal{S}_\ell)_{\ell=1}^L$, c.f. Assumption S, with respect to (Q', Q_K, R), that is,

$$D_{Q',Q_K,R}\mathcal{S}(\widehat{\phi}, \widehat{\nabla}\widehat{\phi}, Q', Q_K, R, \widehat{g}) : \mathbb{R}^{K+4} \to \mathbb{R}^L$$

is surjective[4] at all $x \in \mathrm{cl}(U_{\widehat{g}})$. Hence (iii) in the Assumption S is valid.

Appendix C: Stability and existence of the direct problem. Let us start by explaining how we can choose a C^∞-smooth metric $\widetilde{g}$ such that $\widehat{g} < \widetilde{g}$ and $(M, \widetilde{g})$ is globally hyperbolic: When $v(x)$ the eigenvector corresponding to the negative eigenvalue of $\widehat{g}(x)$, we can choose a smooth, strictly positive function $\eta : M \to \mathbb{R}_+$ such that

[4] We make the following note related the case considered in the main part of the paper when $Q, P, R \in \mathcal{I}^m(Y)$, where Y is 2-dimensional sufrace, and we need to use the principal symbol $\mathbf{r}$ of R as an independent variable: Let us consider also a point $x_0 \in M_0$ and η be a light-like covector choose coordinates so that $g = \mathrm{diag}\,(-1,1,1,1)$ and $\eta = (1,1,0,0)$. When $c = (c^k)_{k=1}^4 \in \mathbb{R}^4$ and $P^{jk} = C^{jk}(x\cdot\eta)_+^a$, where C^{jk} is such a symmetric matrix that $C^{11} = c_1$, $C^{12} = C^{21} = \frac{1}{2}c_2$, $C^{13} = C^{31} = c_3$, and $C^{14} = C^{41} = c_4$ and other C^{jk} are zeros. Then we have

$$\nabla_j^g(C^{jk}(x\cdot\eta)_+^a) \;=\; \eta_j P^{jk} = (C^{1k} + C^{2k})(x\cdot\eta)_+^{a-1} = c_k(x\cdot\eta)_+^{a-1}.$$

As we can always choose coordinates that $\widehat{g}$ and η have at a given point the above forms, we can obtain arbitrary vector $\mathbf{r}$, as principal symbol of R, by considering $Q', Q_{L+1}, P \in \mathcal{I}^m(Y)$, where Y is 2-submanifold, with principal symbols of $\widetilde{\mathbf{p}}$ and $\widetilde{\mathbf{z}}$ satisfing equations corresponding to equations $\widehat{g}^{jk}\widehat{\nabla}_j(\mathbf{p}_{kn} + \mathbf{z}\widehat{g}_{kn}) \in \mathcal{I}^m(Y)$, and the sub-principal symbols of $\mathbf{p}_{kn}$ and $\mathbf{z}$ vary arbitrarily.

$\widetilde g' := \widehat g - \eta v\otimes v < \widetilde g$. Then $(M,\widetilde g')$ is globally hyperbolic, $\widetilde g'$ is smooth and $\widehat g < \widetilde g'$. Thus we can replace $\widetilde g$ by the smooth metric $\widetilde g'$ having the same properties that are required for $\widetilde g$.

Let us now return to consider existence and stability of the solutions of the Einstein-scalar field equations. Let $t = \mathbf{t}(x)$ be local time so that there is a diffeomorphism $\Psi : M \to \mathbb{R}\times N$, $\Psi(x) = (\mathbf{t}(x), Y(x))$, and $S(T) = \{x \in M_0;\ \mathbf{t}(x) = T\}$, $T \in \mathbb{R}$ are Cauchy surfaces. Let $t_0 > 0$. Next we identify M and $\mathbb{R}\times N$ via the map Ψ and just denote $M = \mathbb{R}\times N$. Let us denote $M(t) = (-\infty, t)\times N$, and $M_0 = M(t_0)$. By [5, Cor. A.5.4] the set $\mathcal{K} = J^+_{\widetilde g}(\widehat p^-)\cap M_0$, where $\widehat p^0 \in M_0$, is compact. Let $N_1, N_2 \subset N$ be such open relatively compact sets with smooth boundary that $N_1 \subset N_2$ and $Y(J^+_{\widetilde g}(\widehat p^0)\cap M_0) \subset N_1$.

To simplify citations to existing literature, let us define $\widetilde N$ to be a compact manifold without boundary such that N_2 can be considered as a subset of $\widetilde N$. Using a construction based on a suitable partition of unity, the Hopf double of the manifold N_2, and the Seeley extension of the metric tensor, we can endow $\widetilde M = (-\infty, t_0)\times \widetilde N$ with a smooth Lorentzian metric $\widehat g^e$ (index e is for "extended") so that $\{t\}\times \widetilde N$ are Cauchy surfaces of $\widetilde N$ and that $\widehat g$ and $\widehat g^e$ coincide in the set $\Psi^{-1}((-\infty, t_0)\times N_1)$ that contains the set $J^+_{\widetilde g}(\widehat p^0)\cap M_0$. We extend the metric $\widetilde g$ to a (possibly non-smooth) globally hyperbolic metric $\widetilde g^e$ on $\widetilde M_0 = (-\infty, t_0)\times \widetilde N)$ such that $\widehat g^e < \widetilde g^e$.

To simplify notations below we denote $\widehat g^e = \widehat g$ and $\widetilde g^e = \widetilde g$ on the whole $\widetilde M_0$. Our aim is to prove the estimate (32).

Let us denote by $t = \mathbf{t}(x)$ the local time. Recall that when (g,ϕ) is a solution of the scalar field-Einstein equation, we denote $u = (g - \widehat g, \phi - \widehat\phi)$. We will consider the equation for u, and to emphasize that the metric depends on u, we denote $g = g(u)$ and assume below that both the metric g is dominated by $\widetilde g$, that is, $g < \widetilde g$. We use the pairs $\mathbf{u}(t) = (u(t,\cdot), \partial_t u(t,\cdot)) \in H^1(\widetilde N)\times L^2(\widetilde N)$ and the notations $\mathbf{v}(t) = (v(t), \partial_t v(t))$ etc. Let us consider a generalization of the system (30) of the form

$$\begin{aligned}(172)\qquad &\square_{g(u)}u + V(x,D)u + H(u,\partial u) = R(x,u,\partial u)F + K, \quad x\in \widetilde M_0,\\ &\mathrm{supp}\,(u)\subset \mathcal{K},\end{aligned}$$

where $\square_{g(u)}$ is the Lorentzian Laplace operator operating on the sections of the bundle $\mathcal{B}^L$ on M_0 and $\mathrm{supp}\,(F)\cup \mathrm{supp}\,(K)\subset \mathcal{K}$. Note that above $u = (g-\widehat g, \phi - \widehat\phi)$ and $g(u) = g$. Also, $F\mapsto R(x,u,\partial u)F$ is a linear first order differential operator which coefficients at x are depending smoothly on $u(x)$, $\partial_j u(x)$ and the derivatives of $(\widehat g, \widehat\phi)$ at x and

$$V(x,D) = V^j(x)\partial_j + V(x)$$

is a linear first order differential operator which coefficients at x are depending smoothly on the derivatives of $(\widehat{g},\widehat{\phi})$ at x, and finally, $H(u,\partial u)$ is a polynomial of $u(x)$ and $\partial_j u(x)$ which coefficients at x are depending smoothly on the derivatives of $(\widehat{g},\widehat{\phi})$ such that $\partial_v^\alpha\partial_w^\beta H(v,w)|_{v=0,w=0}=0$ for $|\alpha|+|\beta|\leq 1$. By [80, Lemma 9.7], the equation (172) has at most one solution with given C^2-smooth source functions F and K. Next we consider the existence of u and its dependency on F and K.

Below we use notations, c.f. (30) and (41)

$$\mathcal{R}(\mathbf{u},F)=R(x,u(x),\partial u(x))F(x),\quad \mathcal{H}(\mathbf{u})=H(u(x),\partial u(x)).$$

Note that $u=0$, i.e., $g=\widehat{g}$ and $\phi=\widehat{\phi}$ satisfies (172) with $F=0$ and $K=0$. Let us use the same notations as in [38] cf. also [46, section 16], to consider quasilinear wave equation on $[0,t_0]\times\widetilde{N}$. Let $\mathbb{H}^{(s)}(\widetilde{N})=H^s(\widetilde{N})\times H^{s-1}(\widetilde{N})$ and

$$Z=\mathbb{H}^{(1)}(\widetilde{N}),\quad Y=\mathbb{H}^{(k+1)}(\widetilde{N}),\quad X=\mathbb{H}^{(k)}(\widetilde{N}).$$

The norms on these space are defined invariantly using the smooth Riemannian metric $h=\widehat{g}|_{\{0\}\times\widetilde{N}}$ on $\widetilde{N}$. Note that $\mathbb{H}^{(s)}(\widetilde{N})$ are in fact the Sobolev spaces of sections on the bundle $\pi:\mathcal{B}_K\to\widetilde{N}$, where $\mathcal{B}_K$ denotes also the pull back bundle of $\mathcal{B}_K$ on $\widetilde{M}$ in the map $id:\{0\}\times\widetilde{N}\to\widetilde{M}_0$, or on the bundle $\pi:\mathcal{B}_L\to\widetilde{N}$. Below, ∇_h denotes the standard connection of the bundle $\mathcal{B}_K$ or $\mathcal{B}_L$ associated to the metric h.

Let $k\geq 4$ be an even integer. By definition of H and R we see that there are $0<r_0<1$ and $L_1,L_2>0$, all depending on $\widehat{g}$, $\widehat{\phi}$, $\mathcal{K}$, and t_0, such that if $0<r\leq r_0$ and

$$\begin{aligned}(173)\qquad &\|\mathbf{v}\|_{C([0,t_0];\mathbb{H}^{(k+1)}(\widetilde{N}))}\leq r,\quad \|\mathbf{v}'\|_{C([0,t_0];\mathbb{H}^{(k+1)}(\widetilde{N}))}\leq r,\\ &\|F\|_{C([0,t_0];H^{(k+1)}(\widetilde{N}))}\leq r^2,\quad \|K\|_{C([0,t_0];H^{(k+1)}(\widetilde{N}))}\leq r^2\\ &\|F'\|_{C([0,t_0];H^{(k+1)}(\widetilde{N}))}\leq r^2,\quad \|K'\|_{C([0,t_0];H^{(k+1)}(\widetilde{N}))}\leq r^2\end{aligned}$$

then

$$\begin{aligned}(174)\qquad &\|g(\cdot\,;v)^{-1}\|_{C([0,t_0];H^s(\widetilde{N}))}\leq L_1,\\ &\|\mathcal{H}(\mathbf{v})\|_{C([0,t_0];H^{s-1}(\widetilde{N}))}\leq L_2r^2,\quad \|\mathcal{H}(\mathbf{v}')\|_{C([0,t_0];H^{s-1}(\widetilde{N}))}\leq L_2r^2,\\ &\|\mathcal{H}(\mathbf{v})-\mathcal{H}(\mathbf{v}')\|_{C([0,t_0];H^{s-1}(\widetilde{N}))}\leq L_2r\,\|\mathbf{v}-\mathbf{v}'\|_{C([0,t_0];\mathbb{H}^{(s)}(\widetilde{N}))},\\ &\|\mathcal{R}(\mathbf{v}',F')\|_{C([0,t_0];H^{s-1}(\widetilde{N}))}\leq L_2r^2,\quad \|\mathcal{R}(\mathbf{v},F)\|_{C([0,t_0];H^{s-1}(\widetilde{N}))}\leq L_2r^2,\\ &\|\mathcal{R}(\mathbf{v},F)-\mathcal{R}(\mathbf{v}',F')\|_{C([0,t_0];H^{s-1}(\widetilde{N}))}\\ &\leq L_2r\,\|\mathbf{v}-\mathbf{v}'\|_{C([0,t_0];\mathbb{H}^{(s)}(\widetilde{N}))}+L_2\|F-F'\|_{C([0,t_0];H^{s+1}(\widetilde{N}))\cap C^1([0,t_0];H^s(\widetilde{N}))},\end{aligned}$$

for all $s\in[1,k+1]$.

Next we write (172) as a first order system. To this end, let $\mathcal{A}(t,\mathbf{v}):\mathbb{H}^{(s)}(\widetilde{N})\to\mathbb{H}^{(s-1)}(\widetilde{N})$ be the operator $\mathcal{A}(t,\mathbf{v})=\mathcal{A}_0(t,\mathbf{v})+\mathcal{A}_1(t,\mathbf{v})$

where in local coordinates and in the local trivialization of the bundle $\mathcal{B}^L$

$$\mathcal{A}_0(t,\mathbf{v}) = -\begin{pmatrix} 0 & I \\ \frac{1}{g^{00}(v)}\sum_{j,k=1}^3 g^{jk}(v)\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^k} & \frac{1}{g^{00}(v)}\sum_{m=1}^3 g^{0m}(v)\frac{\partial}{\partial x^m}\end{pmatrix}$$

with $g^{jk}(v) = g^{jk}(t,\cdot\,;v)$ is a function on $\widetilde N$ and

$$\mathcal{A}_1(t,\mathbf{v}) = \frac{-1}{g^{00}(v)}\begin{pmatrix} 0 & 0 \\ \sum_{j=1}^3 B^j(v)\frac{\partial}{\partial x^j} & B^0(v)\end{pmatrix}$$

where $B^j(v)$ depend on $v(t,x)$ and its first derivatives, and the connection coefficients (the Christoffel symbols) corresponding to $g(v)$. We denote $\mathcal{S} = (F,K)$ and

$$\begin{aligned} &f_{\mathcal{S}}(t,\mathbf{v}) = (f^1_{\mathcal{S}}(t,\mathbf{v}), f^2_{\mathcal{S}}(t,\mathbf{v})) \in \mathbb{H}^{(k)}(\widetilde N), \quad \text{where} \\ &f^1_{\mathcal{S}}(t,\mathbf{v}) = 0, \quad f^2_{\mathcal{S}}(t,\mathbf{v}) = \mathcal{R}(\mathbf{v},F)(t,\cdot\,) - \mathcal{H}(\mathbf{v})(t,\cdot\,) + K(t,\cdot\,). \end{aligned}$$

Note that when (173) are satisfied with $r < r_0$, inequalities (174) imply that there exists $C_2 > 0$ so that

$$\begin{aligned} &\|f_{\mathcal{S}}(t,\mathbf{v})\|_Y + \|f_{\mathcal{S}'}(t,\mathbf{v}')\|_Y \le C_2 r^2, \\ &\|f_{\mathcal{S}}(t,\mathbf{v}) - f_{\mathcal{S}}(t,\mathbf{v}')\|_Y \le C_2 r\,\|\mathbf{v}-\mathbf{v}'\|_{C([0,t_0];Y)}. \end{aligned} \tag{175}$$

Let $U^{\mathbf{v}}(t,s)$ be the wave propagator corresponding to metric $g(v)$, that is, $U^{\mathbf{v}}(t,s) : \mathbf{h} \mapsto \mathbf{w}$, where $\mathbf{w}(t) = (w(t), \partial_t w(t))$ solves

$$(\square_{g(v)} + V(x,D))w = 0 \quad \text{for } (t,y) \in [s,t_0]\times \widetilde N, \quad \text{with } \mathbf{w}(s,y) = \mathbf{h}.$$

Let $S = (\nabla_h^*\nabla_h + 1)^{k/2} : Y \to Z$ be an isomorphism. As k is an even integer, we see using multiplication estimates for Sobolev spaces, see e.g. [38, Sec. 3.2, point (2)], that there exists $c_1 > 0$ (depending on r_0, L_1, and L^2) so that $\mathcal{A}(t,\mathbf{v})S - S\mathcal{A}(t,\mathbf{v}) = C(t,\mathbf{v})$, where $\|C(t,\mathbf{v})\|_{Y\to Z} \le c_1$ for all $\mathbf{v}$ satisfying (173). This yields that the property (A2) in [38] holds, namely that $S\mathcal{A}(t,\mathbf{v})S^{-1} = \mathcal{A}(t,\mathbf{v}) + B(t,\mathbf{v})$ where $B(t,\mathbf{v})$ extends to a bounded operator in Z for which $\|B(t,\mathbf{v})\|_{Z\to Z} \le c_1$ for all $\mathbf{v}$ satisfying (173). Alternatively, to see the mapping properties of $B(t,\mathbf{v})$ we could use the fact that $B(t,\mathbf{v})$ is a zeroth order pseudodifferential operator with H^k-symbol.

Thus the proof of [38, Lemma 2.6] shows that there is a constant $C_3 > 0$ so that

$$\|U^{\mathbf{v}}(t,s)\|_{Z\to Z} \le C_3 \quad \text{and} \quad \|U^{\mathbf{v}}(t,s)\|_{Y\to Y} \le C_3 \tag{176}$$

for $0 \le s < t \le t_0$. By interpolation of estimates (176), we see also that

$$\|U^{\mathbf{v}}(t,s)\|_{X\to X} \le C_3, \tag{177}$$

for $0 \le s < t \le t_0$.

Let us next modify the reasoning given in [46]: let $r_1 \in (0, r_0)$ be a parameter which value will be chosen later, $C_1 > 0$ and E be the space of functions $\mathbf{u} \in C([0,t_0];X)$ for which

$$\|\mathbf{u}(t)\|_Y \le r_1 \quad \text{and} \tag{178}$$

$$\|\mathbf{u}(t_1) - \mathbf{u}(t_2)\|_X \le C_1|t_1 - t_2| \tag{179}$$

for all $t, t_1, t_2 \in [0, t_0]$. The set E is endowed by the metric of $C([0,t_0];X)$. We note that by [46, Lemma 7.3], a convex Y-bounded, Y-closed set is closed also in X. Similarly, functions $G : [0, t_0] \to X$ satisfying (179) form a closed subspace of $C([0,t_0];X)$. Thus $E \subset X$ is a closed set implying that E is a complete metric space.

Let

$$W = \{(F,K) \in C([0,t_0];H^{k+1}(\widetilde{N}))^2;\ \sup_{t\in[0,t_0]} \|F(t)\|_{H^{k+1}(\widetilde{N})} + \|K(t)\|_{H^{k+1}(\widetilde{N})} < r_1\}.$$

Following [46, p. 44], we see that the solution of equation (172) with the source $\mathcal{S} \in W$ is found as a fixed point, if it exists, of the map $\Phi_\mathcal{S} : E \to C([0,t_0];Y)$ where $\Phi_\mathcal{S}(\mathbf{v}) = \mathbf{u}$ is given by

$$\mathbf{u}(t) = \int_0^t U^\mathbf{v}(t,\widetilde{t}) f_\mathcal{S}(\widetilde{t},\mathbf{v})\, d\widetilde{t}, \quad 0 \le t \le t_0.$$

Below, we denote $\mathbf{u}^\mathbf{v} = \Phi_\mathcal{S}(\mathbf{v})$.

As $\Phi_{\mathcal{S}_0}(0) = 0$ where $\mathcal{S}_0 = (0,0)$, we see using the above and the inequality $\|\cdot\|_X \le \|\cdot\|_Y$ that the function $\mathbf{u}^\mathbf{v}$ satifies

$$\|\mathbf{u}^\mathbf{v}\|_{C([0,t_0];Y)} \le C_3C_2t_0r_1^2,$$
$$\|\mathbf{u}^\mathbf{v}(t_2) - \mathbf{u}^\mathbf{v}(t_1)\|_X \le C_3C_2r_1^2|t_2 - t_1|, \quad t_1, t_2 \in [0, t_0].$$

When $r_1 > 0$ is so small that $C_3C_2(1 + t_0) < r_1^{-1}$ and $C_3C_2r_1^2 < C_1$ we see that $\|\Phi_\mathcal{S}(\mathbf{v})\|_{C([0,t_0];Y)} < r_1$ and $\|\Phi_\mathcal{S}(\mathbf{v})\|_{C^{0,1}([0,t_0];X)} < C_1$. Hence $\Phi_\mathcal{S}(E) \subset E$ and we can consider $\Phi_\mathcal{S}$ as a map $\Phi_\mathcal{S} : E \to E$.

As $k > 1 + \frac{3}{2}$, it follows from Sobolev embedding theorem that $X = \mathbb{H}^{(k)}(\widetilde{N}) \subset C^1(\widetilde{N})^2$. This yields that by [46, Thm. 3], for the original reference, see Theorems III-IV in [45],

$$\begin{aligned}&\|(U^\mathbf{v}(t,s) - U^{\mathbf{v}'}(t,s))\mathbf{h}\|_X \\ &\le C_3\Big(\sup_{t'\in[0,t]} \|\mathcal{A}(t',\mathbf{v}) - \mathcal{A}(t',\mathbf{v}')\|_{Y\to X}\|U^\mathbf{v}(t',0)\mathbf{h}\|_Y\Big) \\ &\le C_3^2\|\mathbf{v} - \mathbf{v}'\|_{C([0,t_0];X)}\|\mathbf{h}\|_Y.\end{aligned}$$

Thus,

$$\begin{aligned}&\|U^\mathbf{v}(t,s)f_\mathcal{S}(s,\mathbf{v}) - U^{\mathbf{v}'}(t,s)f_\mathcal{S}(s,\mathbf{v}')\|_X \\ \le\ &\|(U^\mathbf{v}(t,s) - U^{\mathbf{v}'}(t,s))f_\mathcal{S}(s,\mathbf{v})\|_X + \|U^{\mathbf{v}'}(t,s)(f_\mathcal{S}(s,\mathbf{v}) - f_\mathcal{S}(s,\mathbf{v}'))\|_X \\ \le\ &(1 + C_3)^2C_2r_1^2\|\mathbf{v} - \mathbf{v}'\|_{C([0,t_0];X)}.\end{aligned}$$

This implies that

$$\|\Phi_{\mathcal{S}}(\mathbf{v})-\Phi_{\mathcal{S}}(\mathbf{v}')\|_{C([0,t_0];X)}\leq t_0(1+C_3)^2C_2r_1^2\|\mathbf{v}-\mathbf{v}'\|_{C([0,t_0];X))}.$$

Assume next that $r_1>0$ is so small that we have also

$$t_0(1+C_3)^2C_2r_1^2<\frac{1}{2}.$$

cf. Thm. I in [38] (or (9.15) and (10.3)-(10.5) in [46]). For $\mathcal{S}\in W$ this implies that $\Phi_{\mathcal{S}}:E\to E$ is a contraction with a contraction constant $C_L\leq \frac{1}{2}$, and thus $\Phi_{\mathcal{S}}$ has a unique fixed point $\mathbf{u}$ in the space $E\subset C^{0,1}([0,t_0];X)$.

Moreover, elementary considerations related to fixed point of the map $\Phi_{\mathcal{S}}$ show that $\mathbf{u}$ in $C([0,t_0];X)$ depends in $E\subset C([0,t_0];X)$ Lipschitz-continuously on $\mathcal{S}\in W\subset C([0,t_0];H^{k+1}(\widetilde N))^2$. Indeed, if $\|\mathcal{S}-\mathcal{S}'\|_{C([0,t_0];H^{k+1})^2}<\varepsilon$, we see that

$$\|f_{\mathcal{S}}(t,\mathbf{v})-f_{\mathcal{S}'}(t,\mathbf{v})\|_Y\leq C_2\varepsilon,\quad t\in[0,t_0],\tag{180}$$

and when (175) and (176) are satisfied with $r=r_1$, we have

$$\|\Phi_{\mathcal{S}}(\mathbf{v})-\Phi_{\mathcal{S}'}(\mathbf{v}')\|_{C([0,t_0];Y)}\leq C_3C_2t_0r_1^2.$$

Hence

$$\|\Phi_{\mathcal{S}}(\mathbf{v})-\Phi_{\mathcal{S}'}(\mathbf{v})\|_{C([0,t_0];Y)}\leq t_0C_3C_2\varepsilon.$$

This and standard estimates for fixed points, yield that when ε is small enough the fixed point $\mathbf{u}'$ of the map $\Phi_{\mathcal{S}'}:E\to E$ corresponding to the source $\mathcal{S}'$ and the fixed point $\mathbf{u}$ of the map $\Phi_{\mathcal{S}}:E\to E$ corresponding to the source $\mathcal{S}$ satisfy

$$\|\mathbf{u}-\mathbf{u}'\|_{C([0,t_0];X)}\leq \frac{1}{1-C_L}t_0C_3C_2\varepsilon.\tag{181}$$

Thus the solution $\mathbf{u}$ depends in $C([0,t_0];X)$ Lipschitz continuously on $\mathcal{S}\in C([0,t_0];H^{k+1}(\widetilde N))^2$ (see also [46, Sect. 16], and [80]). In fact, for analogous systems it is possible to show that u is in $C([0,t_0];Y)$, but one can not obtain Lipschitz or Hölder stability for u in the Y-norm, see [46], Remark 7.2.

Finally, we note that the fixed point $\mathbf{u}$ of $\Phi_{\mathcal{S}}$ can be found as a limit $\mathbf{u}=\lim_{n\to\infty}\mathbf{u}_n$ in $C([0,t_0];X)$, where $\mathbf{u}_0=0$ and $\mathbf{u}_n=\Phi_{\mathcal{S}}(\mathbf{u}_{n-1})$. Denote $\mathbf{u}_n=(g_n-\widehat g,\phi_n-\widehat\phi)$. We see that if $\text{supp}\,(\mathbf{u}_{n-1})\subset J_{\widehat g}(\text{supp}\,(\mathcal{S}))$ then also $\text{supp}\,(g_{n-1}-\widehat g)\subset J_{\widehat g}(\text{supp}\,(\mathcal{S}))$. Hence for all $x\in M_0\setminus J_{\widehat g}(\text{supp}\,(\mathcal{S}))$ we see that $J^-_{g_{n-1}}(x)\cap J_{\widehat g}(\text{supp}\,(\mathcal{S}))=\emptyset$. Then, using the definition of the map $\Phi_{\mathcal{S}}$ we see that $\text{supp}\,(\mathbf{u}_n)\subset J_{\widehat g}(\text{supp}\,(\mathcal{S}))$. Using induction we see that this holds for all n and hence we see that the solution $\mathbf{u}$ satisfies

$$\text{supp}\,(\mathbf{u})\subset J_{\widehat g}(\text{supp}\,(\mathcal{S})).\tag{182}$$

Appendix D: An inverse problem for a non-linear wave equation. In this appendix we explain how a problem for a scalar wave equation can be solved with the same techniques that we used for the Einstein equations.

Let (M_j, g_j), $j = 1,2$ be two globally hyperbolic $(1+3)$ dimensional Lorentzian manifolds represented using global smooth time functions as $M_j = \mathbb{R} \times N_j$, $\mu_j = \mu_j([-1,1]) \subset M_j$ be a time-like geodesic and $U_j \subset M_j$ be open, relatively compact neighborhood of $\mu_j([s_-, s_+])$, $-1 < s_- < s_+ < 1$. Let $M_j^0 = (-\infty, T_0) \times N_j$ where $T_0 > 0$ is such that $U_j \subset M_j^0$. Consider the non-linear wave equation

$$\begin{aligned} &\square_{g_j} u(x) + a_j(x)\, u(x)^2 = f(x) \quad \text{on } M_j^0, \\ &\operatorname{supp}(u) \subset J^+_{g_j}(\operatorname{supp}(f)), \end{aligned} \tag{183}$$

where $\operatorname{supp}(f) \subset U_j$,

$$\square_g u = -\sum_{p,q=1}^4 \det(-g(x))^{-1/2} \frac{\partial}{\partial x^p}\left((-\det(g(x))^{1/2} g^{pq}(x) \frac{\partial}{\partial x^q} u(x) \right),$$

$\det(g) = \det((g_{pq}(x))_{p,q=1}^4)$, $f \in C_0^6(U_j)$ is a controllable source, and a_j is a non-vanishing C^∞-smooth function. Our goal is to prove the following result:

Theorem 5.12. *Let (M_j, g_j), $j = 1,2$ be two open, smooth, globally hyperbolic Lorentzian manifolds of dimension $(1+3)$. Let $p_j^+ = \mu_j(s_+), p_j^- = \mu_j(s_-) \in M_j$ the points of a time-like geodesic $\mu_j = \mu_j([-1,1]) \subset M_j$, $-1 < s_- < s_+ < 1$, and let $U_j \subset M_j$ be an open relatively compact neighborhood of $\mu_j([s_-, s_+])$ given in (2). Let $a_j : M_j \to \mathbb{R}$, $j = 1,2$ be C^∞-smooth functions that are non-zero on M_j.*

Let L_{U_j}, $j = 1,2$ be measurement operators defined in an open set $\mathcal{W}_j \subset C_0^6(U_j)$ containing the zero function by setting

$$L_{U_j} : f \mapsto u|_{U_j}, \quad f \in C_0^6(U_j), \tag{184}$$

where u satisfies the wave equation (183) on (M_j^0, g_j).

Assume that there is a diffeomorphic isometry $\Phi : U_1 \to U_2$ so that $\Phi(p_1^-) = p_2^-$ and $\Phi(p_1^+) = p_2^+$ and the measurement maps satisfy

$$((\Phi^{-1})^* \circ L_{U_1} \circ \Phi^*) f = L_{U_2} f$$

for all $f \in \mathcal{W}$ where $\mathcal{W}$ is some neighborhood of the zero function in $C_0^6(U_2)$.

Then there is a diffeomorphism $\Psi : I(p_1^-, p_1^+) \to I(p_2^-, p_2^+)$, and the metric $\Psi^* g_2$ is conformal to g_1 in $I(p_1^-, p_1^+) \subset M_1$, that is, there is $\beta(x)$ such that $g_1(x) = \beta(x)(\Psi^* g_2)(x)$ in $I(p_1^-, p_1^+)$.

We note that the smoothness assumptions assumed above on the functions a and the source f are not optimal. The proof, presented below, is based on using the interaction of singular waves. The techniques used can be modified used to study different non-linearities, such as the equations $\square_{\widehat g} u + a(x)u^3 = f$, $\square_{\widehat g} u + a(x)u_t^2 = f$, or $\square_{g(x,u(x))} u = f$, but these considerations are outside the scope of this paper.

Theorem 5.12 can be applicable for example in the mathematical analysis of non-destructive testing or imaging in non-linear medium e.g, in imaging the non-linearity of the acoustic material parameter inside a given body when it is under large, time-varying, possibly periodic, changes of the external pressure and at the same time the body is probed with small-amplitude fields. Such acoustic measurements are analogous to the recently developed Ultrasound Elastography imaging technique where the interaction of the elastic shear and pressure waves is used for medical imaging, see e.g. [37, 62, 63, 72]. There, the slowly progressing shear wave is imaged using a pressure wave and the image of the shear wave inside the body is used to determine approximately the material parameters. In other words, the changes which the elastic wave causes in the medium are imaged using the interaction of the s-wave and p-wave components of the elastic wave.

Let us also consider some implications of theorem 5.12 for inverse problems for a non-linear equation involving a time-independent metric

$$g(t,y) = -dt^2 + \sum_{\alpha,\beta=1}^{3} h_{\alpha\beta}(y)dy^\alpha dy^\beta, \quad (t,y) \in \mathbb{R} \times N. \tag{185}$$

The metric (185) corresponds to the hyperbolic operator $\partial_t^2 - \Delta_h$, with a time-independent Riemannian metric $h = (h_{\alpha\beta}(y))_{\alpha,\beta=1}^3$, $y \in N$, where N is a 3-dimensional manifold and

$$\Delta_h u(y,t) = \sum_{\alpha,\beta=1}^{3} \frac{\partial}{\partial y^\alpha}\left(h^{\alpha\beta}(y)\frac{\partial}{\partial y^\beta}u(t,y)\right).$$

Corollary 5.13. *Let (M_j, g_j), $M_j = \mathbb{R} \times N_j$, $j = 1,2$ be two open, smooth, globally hyperbolic Lorentzian manifolds of dimension $(1+3)$. Assume that g_j is the product metric of the type (185), $g_j = -dt^2 + h_j(y)$, $j = 1,2$. Let $p_j^+ = \mu_j(s_+), p_j^- = \mu_j(s_-) \in (0,T_0) \times N_j$ be points of a time-like geodesic $\mu_j = \mu_j([-1,1]) \subset M_j$, $-1 < s_- < s_+ < 1$, and to fix the time variable, assume that $\mu_j(s_-) \in \{1\} \times N_j$.*

Let $U_j \subset (0,T_0) \times N_j$ be an open relatively compact neighborhood of $\mu_j([s_-,s_+])$ given in (2). Let $a_j : M_j \to \mathbb{R}$, $j = 1,2$ be C^∞-smooth functions that are non-zero on M_j and $x = (t,y) \in \mathbb{R} \times N$.

For $j = 1,2$, consider the non-linear wave equations

$$\begin{aligned}&(\frac{\partial^2}{\partial t^2} - \Delta_{h_j})u(t,y) + a_j(y,t)(u(t,y))^2 = f(t,y) \quad \text{on } (0,T_0) \times N_j,\\ &\text{supp}\,(u) \subset J^+_{g_j}(\text{supp}\,(f)),\end{aligned} \tag{186}$$

where $f \in C_0^6(U_j)$, $j = 1,2$. Let $L_{U_j} : f \mapsto u|_{U_j}$ be the measurement operator (184) for the wave equation (186) with the Riemannian metric

$h_j(x)$ and the coefficient $a_j(x,t)$ for $j=1,2$, defined in some $C^6_0(U_j)$ neighborhood of the zero function.

Assume that there is a diffeomorphism $\Phi : U_1 \to U_2$ of the form $\Phi(t,y) = (t,\phi(y))$ so that

$$((\Phi^{-1})^* \circ L_{U_1} \circ \Phi^*) f = L_{U_2} f$$

for all $f \in \mathcal{W}$ where $\mathcal{W}$ is some neighborhood of the zero function in $C^6_0(U_2)$.

Then there is a diffeomorphism $\Psi : I^+(p_1^-) \cap I^-(p_1^+) \to I^+(p_2^-) \cap I^-(p_2^+)$ of the form $\Psi(t,y) = (\psi(y),t)$, the metric $\Psi^* g_2$ is isometric to g_1 in $I^+(p_1^-) \cap I^-(p_1^+)$, and $a_1(t,y) = a_2(t,\psi(y))$ in $I^+(p_1^-) \cap I^-(p_1^+)$.

Next we consider the proofs.

Proof. (of Theorem 5.12). We will explain how the proof of Theorem 1.4 for the Einstein equation needs to be modified to obtain the similar result for the non-linear wave equation.

Let $(M,\widehat{g})$ be a smooth globally hyperbolic Lorentzian manifold that we represent using a global smooth time function as $M = (-\infty,\infty)\times N$, and consider $M^0 = (-\infty,T)\times N \subset M$. Assume that the set U, where the sources are supported and where we observe the waves, satisfies $U \subset [0,T]\times N$.

The results of section 3.2 concerning the direct problem for Einstein equations can be modified for the wave equation

$$\begin{aligned} &\square_{\widehat{g}} u + a u^2 = f, \quad \text{in } M^0 = (-\infty,T)\times N,\\ &u|_{(-\infty,0)\times N} = 0, \end{aligned} \tag{187}$$

where $a = a(x)$ is a smooth, non-vanishing function. Here we denote the metric by $\widehat{g}$ to emphasize the fact that it is independent on the solution u. Below, let Q be the causal inverse operator of $\square_{\widehat{g}}$.

When f in $C_0([0,t_0];H^6_0(B)) \cap C^1_0([0,t_0];H^5_0(B))$ is small enough, we see by using [80, Prop. 9.17] and [38, Thm. III], see also (181) in Appendix C, that the equation (187) has a unique solution $u \in C_0([0,t_0];H^5(N)) \cap C^1_0([0,t_0];H^4(N))$. Moreover, we can consider the case when $f = \varepsilon f_0$ where $\varepsilon > 0$ is small. Then, we can write

$$u = \varepsilon w_1 + \varepsilon^2 w_2 + \varepsilon^3 w_3 + \varepsilon^4 w_4 + E_\varepsilon$$

where w_j and the reminder term E_ε satisfy

$$\begin{aligned}
w_1 &= Qf,\\
w_2 &= -Q(a\,w_1\,w_1),\\
w_3 &= -2Q(a\,w_1\,w_2)\\
&= 2Q(a\,w_1\,Q(a\,w_1\,w_1)),\\
w_4 &= -Q(a\,w_2\,w_2) - 2Q(a\,w_1\,w_3)\\
&= -Q(a\,Q(a\,w_1\,w_1)\,Q(a\,w_1\,w_1))\\
&\quad +4Q(a\,w_1\,Q(a\,w_1\,w_2))\\
&= -Q(a\,Q(a\,w_1\,w_1)\,Q(a\,w_1\,w_1))\\
&\quad -4Q(a\,w_1\,Q(a\,w_1\,Q(a\,w_1\,w_1))),\\
\|E_\varepsilon\|&_{C([0,t_0];H_0^4(N))\cap C^1([0,t_0];H_0^3(N))} \leq C\varepsilon^5.
\end{aligned}$$

If we consider sources $f_{\vec\varepsilon}(x) = \sum_{j=1}^4 \varepsilon_j f_{(j)}(x)$, $\vec\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$, and the corresponding solution $u_{\vec\varepsilon}$ of (187), we see that

$$\begin{aligned}
\mathcal{M}^{(4)} &= \partial^4_{\vec\varepsilon} u_{\vec\varepsilon}|_{\vec\varepsilon=0}\\
&= \partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3}\partial_{\varepsilon_4} u_{\vec\varepsilon}|_{\vec\varepsilon=0}\\
&= -\sum_{\sigma\in\Sigma(4)} \Big(Q(a\,Q(a\,u_{(\sigma(1))}\,u_{(\sigma(2))})\,Q(a\,u_{(\sigma(3))}\,u_{(\sigma(4))}))\\
&\qquad +4Q(a\,u_{(\sigma(1))}\,Q(a\,u_{(\sigma(2))}\,Q(a\,u_{(\sigma(3))}\,u_{(\sigma(4))})))\Big),
\end{aligned} \tag{188}$$

where $u_{(j)} = Qf_{(j)}$ and $\Sigma(\ell)$ is the set of permutations of the set $\{1, 2, 3, \ldots, \ell\}$.

The results of Lemma 3.1 can be replaced by the results of [28, Prop. 2.1] as follows. Using the same notations as in Lemma 3.1, let $Y = Y(x_0, \zeta_0; t_0, s_0)$, $K = K(x_0, \zeta_0; t_0, s_0)$, and $\Lambda_1 = \Lambda(x_0, \zeta_0; t_0, s_0)$, and consider a source $f \in \mathcal{I}^{n-3/2}(Y)$. Then $u = Qf$ satisfies $u|_{M_0\setminus Y} \in \mathcal{I}^{n-1/2}(M_0 \setminus Y; \Lambda_1)$. Assume that $(x, \xi), (y, \eta) \in L^+M$ are on the same bicharacteristics of $\square_{\widehat g}$, and $x < y$, that is, $((x, \xi), (y, \eta)) \in \Lambda'_{\widehat g}$. Moreover, assume that $(x, \xi) \in N^*Y$. Let $\widetilde b(x, \xi)$ be the principal symbol of f at (x, ξ) and $\widetilde a(y, \eta)$ be the principal symbol of u at (y, η). Then $\widetilde a(y, \eta)$ depends linearly on $\widetilde f(x, \xi)$ and $\widetilde a(y, \eta)$ vanishes if and only if $\widetilde f(x, \xi)$ vanishes.

Analogously to the Einstein equations, we consider the indicator function

$$\Theta^{(4)}_\tau = \langle F_\tau, \mathcal{M}^{(4)}\rangle_{L^2(U)}, \tag{189}$$

where $\mathcal{M}^{(4)}$ is given by (188) with $u_{(j)} = Qf_{(j)}$, $j = 1, 2, 3, 4$, where $f_{(j)} \in \mathcal{I}^{n-3/2}(Y(x_j, \xi_j; t_0, s_0))$, $n \leq -n_1$, and F_τ is the source producing a gaussian beam Q^*F_τ that propagates to the past along the geodesic $\gamma_{x_5,\xi_5}(\mathbb{R}_-)$, see (72).

Similar results to the ones given in Proposition 3.4 are valid. Let us consider next the case when (x_5,ξ_5) comes from the 4-intersection of rays corresponding to $(\vec x,\vec \xi)=((x_j,\xi_j))_{j=1}^4$ and q is the corresponding intersection point, that is, $q=\gamma_{x_j,\xi_j}(t_j)$ for all $j=1,2,3,4,5$. Then

$$\Theta^{(4)}_\tau\sim \sum_{k=m}^\infty s_k\tau^{-k} \tag{190}$$

as $\tau\to\infty$ where $m=-4n+4$. Moreover, let $b_j=(\dot\gamma_{x_j,\xi_j}(t_j))^\flat$ and $\mathbf{b}=(b_j)_{j=1}^5\in (T_q^*M_0)^5$, w_j be the principal symbols of the waves $u_{(j)}$ at (q,b_j), and $\mathbf{w}=(w_j)_{j=1}^5$. Then we see as in Proposition 3.4 that there is a real-analytic function $\mathcal{G}(\mathbf{b},\mathbf{w})$ such that the leading order term in (84) satisfies

$$s_m=\mathcal{G}(\mathbf{b},\mathbf{w}). \tag{191}$$

The proof of Prop. 3.7 dealing with Einstein equations needs significant changes and we need to prove the following:

Proposition 5.14. *The function $\mathcal{G}(\mathbf{b},\mathbf{w})$ given in (191) for the non-linear wave equation is a non-identically vanishing real-analytic function.*

Proof. Let us use the notations introduced in Prop. 3.7.

As for the Einstein equations, we consider light-like vectors

$$b_5=(1,1,0,0),\quad b_j=(1,1-\frac12\rho_j^2,\rho_j+O(\rho_j^3),\rho_j^3),\quad j=1,2,3,4,$$

in the Minkowski space $\mathbb{R}^{1+3}$, endowed with the standard metric $g=\operatorname{diag}(-1,1,1,1)$, where the terms $O(\rho_k^3)$ are such that the vectors b_j, $j\le 5$, are light-like. Then

$$g(b_5,b_j)=-\frac12\rho_j^2,\quad g(b_k,b_j)=-\frac12\rho_k^2-\frac12\rho_j^2+O(\rho_k\rho_j).$$

Below, we denote $\omega_{kj}=g(b_k,b_j)$. Note that if $\rho_j<\rho_k^4$, we have $\omega_{kj}=-\frac12\rho_k^2+O(\rho_k^3)$.

For the wave equation, we use different parameters ρ_j than for the Einstein equations, and define (so, we use here the "unordered" numbering 4-2-1-3)

$$\rho_4=\rho_2^{100},\ \ \rho_2=\rho_1^{100},\ \text{and}\ \rho_1=\rho_3^{100}. \tag{192}$$

Below in this proof, we denote $\vec\rho\to 0$ when $\rho_3\to 0$ and ρ_4, ρ_2, and ρ_1 are defined using ρ_3 as in (192).

Let us next consider in Minkowski space the coordinates $(x^j)_{j=1}^4$ such that $K_j=\{x^j=0\}$ are light-like hyperplanes and the waves $u_j=u_{(j)}$ that satisfy in the Minkowski space $\square u_j=0$ and can be written as

$$u_j(x)=\int_{\mathbb{R}} e^{ix^j\theta}a_j(x,\theta)\,d\theta,\quad a_j(x,\theta')\in S^n(\mathbb{R}^4;\mathbb{R}\setminus 0),\quad j\le 4,$$

and

$$u^\tau(x)=\chi(x^0)w_{(5)}\exp(i\tau b^{(5)}\cdot x).$$

Note that the singular supports of the waves u_j, $j=1,2,3,4$, intersect then at the point $\cap_{j=1}^4 K_j=\{0\}$. Analogously to the definition (85) we considered for Einstein equations, we define the (Minkowski) indicator function

$$\mathcal{G}^{(\mathbf{m})}(v,\mathbf{b})=\lim_{\tau\to\infty}\tau^m(\sum_{\beta\le n_1}\sum_{\sigma\in\Sigma(4)}T^{(\mathbf{m}),\beta}_{\tau,\sigma}+\widetilde T^{(\mathbf{m}),\beta}_{\tau,\sigma}),$$

where

$$\begin{aligned}T^{(\mathbf{m}),\beta}_{\tau,\sigma}&=\langle Q_0(u^\tau\cdot\alpha u_{\sigma(4)}),h\cdot\alpha u_{\sigma(3)}\cdot Q_0(\alpha u_{\sigma(2)}\cdot u_{\sigma(1)})\rangle,\\ \widetilde T^{(\mathbf{m}),\beta}_{\tau,\sigma}&=\langle u^\tau,h\alpha\, Q_0(\alpha u_{\sigma(4)}\cdot u_{\sigma(3)})\cdot Q_0(\alpha u_{\sigma(2)}\cdot u_{\sigma(1)})\rangle.\end{aligned}$$

As for Einstein equations, we see that when α is equal to the value of the function $a(t,y)$ at the intersection point $q=0$ of the waves, we have $\mathcal{G}^{(\mathbf{m})}(v,\mathbf{b})=\mathcal{G}(v,\mathbf{b})$.

Similarly to the Lemma 3.4 we analyze next the functions

$$\Theta^{(\mathbf{m})}_\tau=\sum_{\beta\in J_\ell}\sum_{\sigma\in\Sigma(4)}(T^{(\mathbf{m}),,\beta}_{\tau,\sigma}+\widetilde T^{(\mathbf{m}),\beta}_{\tau,\sigma}).$$

Here $(\mathbf{m})$ refers to "Minkowski". We denote $T^{(\mathbf{m}),\beta}_\tau=T^{(\mathbf{m}),\beta}_{\tau,id}$ and $\widetilde T^{(\mathbf{m}),\beta}_\tau=\widetilde T^{(\mathbf{m}),\beta}_{\tau,id}$.

Let us first consider the case when the permutation $\sigma=id$. Then, as in the proof of Prop. 3.7, in the case when $\vec S^\beta=(Q_0,Q_0)$, we have

$$\begin{aligned}&T^{(\mathbf{m}),\beta}_\tau\\ &=C_1\det(A)\cdot(i\tau)^m(1+O(\frac{1}{\tau}))\vec\rho^{\,2\vec n}(\omega_{45}\omega_{12})^{-1}\rho_4^{-4}\rho_2^{-4}\rho_1^{-4}\rho_3^{2}\cdot\mathcal{P}\\ &=C_2\det(A)\cdot(i\tau)^m(1+O(\frac{1}{\tau}))\vec\rho^{\,2\vec n}\rho_4^{-4-2}\rho_2^{-4}\rho_1^{-4-2}\rho_3^{-2}\cdot\mathcal{P}\end{aligned}$$

where $\mathcal{P}$ is the product of the principal symbols of the waves u_j at zero, $\vec\rho^{\,2\vec n}=\rho_1^{2n}\rho_2^{2n}\rho_3^{2n}\rho_4^{2n}$, and C_1 and C_2 are non-vanishing. Similarly, a direct computation yields

$$\begin{aligned}&\widetilde T^{(\mathbf{m}),\beta}_\tau\\ &=C_1\det(A)\cdot(i\tau)^n(1+O(\frac{1}{\tau}))\vec\rho^{\,2\vec n}(\omega_{43}\omega_{21})^{-1}\rho_4^{-4}\rho_2^{-4}\rho_1^{-4}\rho_3^{-4}\cdot\mathcal{P}\\ &=C_2\det(A)\cdot(i\tau)^m(1+O(\frac{1}{\tau}))\vec\rho^{\,2\vec n}\rho_4^{-4}\rho_2^{-4}\rho_1^{-4-2}\rho_3^{-4-2}\cdot\mathcal{P},\end{aligned}$$

where again, $\mathcal{P}$ is the product of the principal symbols of the waves u_j at zero and C_1 and C_2 are non-vanishing.

Considering formula (188), we see that for the wave equation we do not need to consider the terms that for the Einstein equations correspond to the cases when $\vec{S}^\beta = (Q_0, I)$, $\vec{S}^\beta = (I, Q_0)$, or $\vec{S}^\beta = (I, I)$ as the corresponding terms do not appear in formula (188).

Let us now consider permutations σ of the indexes $(1,2,3,4)$ and compare the terms

$$L^{(\mathbf{m}),\beta}_\sigma = \lim_{\tau\to\infty} \tau^m T^{(\mathbf{m}),\beta}_{\tau,\sigma},$$
$$\widetilde{L}^{(\mathbf{m}),\beta}_\sigma = \lim_{\tau\to\infty} \tau^m \widetilde{T}^{(\mathbf{m}),\beta}_{\tau,\sigma}.$$

Due to the presence of $\omega_{45}\omega_{12}$ in the above computations, we observe that all the terms $\widetilde{L}^{(\mathbf{m}),\beta}_{\tau,\sigma}/L^{(\mathbf{m}),\beta}_{\tau,id} \to 0$ as $\vec{\rho} \to 0$, see (192). Also, if $\sigma \neq (1,2,3,4)$ and $\sigma \neq \sigma_0 1 = (2,1,3,4)$, we see that $\widetilde{L}^{(\mathbf{m}),\beta}_{\tau,\sigma}/L^{(\mathbf{m}),\beta}_{\tau,id} \to 0$ as $\vec{\rho} \to 0$. Also, we observe that $L^{(\mathbf{m}),\beta}_{\tau,\sigma_1} = L^{(\mathbf{m}),\beta}_{\tau,id}$. Thus we see that the equal terms $L^{(\mathbf{m}),\beta}_{\tau,\sigma_1} = L^{(\mathbf{m}),\beta}_{\tau,id}$ that give the largest contributions as $\vec{\rho} \to 0$ and that when $\mathcal{P} \neq 0$ the sum

$$S(\vec{\rho}, \mathcal{P}) = \sum_{\sigma\in\Sigma(4)} (L^{(\mathbf{m}),\beta}_{\tau,\sigma} + \widetilde{L}^{(\mathbf{m}),\beta}_{\tau,\sigma})$$

is non-zero when $\rho_3 > 0$ is small enough and ρ_4, ρ_2, and ρ_1 are defined using ρ_3 as in (192). As the indicator function is real-analytic, this shows that the indicator function in non-vanishing in a generic set. □

We need also to change the the singularity *detection condition* (D) with light-like directions $(\vec{x}, \vec{\xi})$ as follows: We define that point $y \in \widehat{U}$, satisfies the singularity *detection condition* (D') with light-like directions $(\vec{x}, \vec{\xi})$ and $t_0, \widehat{s} > 0$ if

(D') For any $s, s_0 \in (0, \widehat{s})$ there are $(x_j', \xi_j') \in \mathcal{W}_j(s; x_j, \xi_j)$, $j = 1,2,3,4$, and $f_{(j)} \in \mathcal{I}_S{}^{n-3/2}(Y((x_j', \xi_j'); t_0, s_0))$, and such that if $u_{\vec{\varepsilon}}$ of is the solution of (187) with the source $f_{\vec{\varepsilon}} = \sum_{j=1}^4 \varepsilon_j f_{(j)}$, then the function $\partial^4_{\vec{\varepsilon}} u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}$ is not C^∞-smooth in any neighborhood of y.

When condition (D) is replace by (D'), the considerations in the Sections 4 and 5 show that we can recover the conformal class of the metric. This proves Theorem 5.12. □

Proof. (of Corollary 5.13). Let us denote $W_j = I^+(p_j^-)\cap I^-(p_j^+) \subset M_j$. By Theorem 5.12, there is a map $\Psi : W_1 \to W_2$ such that the product type metrics $g_1 = -dt^2 + h_1(y)$ and $g_2 = -dt^2 + h_2(y)$ are conformal. At the vector field $V = \partial/\partial x^0$ satisfies $\nabla^{g_j} V = 0$, for given $\mu_1(s) = x_0 = (y_0, t_0)$, $s \in [s_-, s_+]$ we can consider all smooth paths $a : [0,1] \to W_1$ that satisfy $a(0) = x_0 = (y_0, t_0)$ and $g_1(\dot{a}(s), V) = 0$. The set of the end points of such paths is equal to the set $W_1 \cap (\{t_0\} \times N_1)$. Considering all such paths on W_1 and the analogous paths on W_2, we see that $\Psi : W_1 \cap (\{t_0\} \times N_1) \to W_2 \cap (\{t_0\} \times N_2)$ is a diffeomorphism. Hence

$\Psi : W_1 \to W_2$ has the form $\Psi(t,y) = (t, \widetilde{\psi}(y,t))$. This means that we can determine uniquely the foliation given by the t-coordinate. As the metric tensors g_1 and g_2 are conformal and their $(0,0)$-components in the (t,y) coordinates satisfy $(g_1)_{00} = -1$ and $(g_2)_{00} = -1$, we conclude that g_1 and g_2 are isometric. Moreover, as g_1 and g_2 are independent of t, we see that there is a diffeomorphism $\Psi : W_1 \to W_2$ of the form $\Psi(t,y) = (t, \psi(y))$ such that $g_1 = \Psi^* g_2$. Note also that if $\pi_2 : (t,y) \mapsto y$, then $h_1 = \psi^* h_2$ on $\pi_2(W_1)$. Thus, we can assume next that the metric tensors g_1 and g_2 are isometric and identify the sets W_1 and W_2 denoting $W = W_1 = W_2$.

As the linearized waves $u_{(j)} = Qf_{(j)}$ depend only on the metric g, using the proof of Theorem 5.12 we see that the indicator functions $\mathcal{G}(\mathbf{b}, \mathbf{w})$ for (U_1, g_1, a_1) and (U_1, g_1, a_1) coincide for all $\mathbf{b}$ and $\mathbf{w}$. This implies that $a_1(t,y)^3 = a_2(t,y)^3$ for all $(t,y) \in W$. Hence $a_1(t,y) = a_2(t,y)$ for all $(t,y) \in W$. □

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