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Full-wave invisibility of active devices at all frequencies

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Abstract: There has recently been considerable interest in the possibility, both theoretical and practical, of invisibility (or "cloaking") from observation by electromagnetic (EM) waves. Here, we prove invisibility with respect to solutions of the Helmholtz and Maxwell's equations, for several constructions of cloaking devices. The basic idea, as in the papers [GLU2, GLU3, Le, PSS1], is to use a singular transformation that pushes isotropic electromagnetic parameters forward into singular, anisotropic ones. We define the notion of finite energy solutions of the Helmholtz and Maxwell's equations for such singular electromagnetic parameters, and study the behavior of the solutions on the entire domain, including the cloaked region and its boundary. We show that, neglecting dispersion, the construction of [GLU3, PSS1] cloaks passive objects, i.e., those without internal currents, at all frequencies k. Due to the singularity of the metric, one needs to work with weak solutions. Analyzing the behavior of such solutions inside the cloaked region, we show that, depending on the chosen construction, there appear new "hidden" boundary conditions at the surface separating the cloaked and uncloaked regions. We also consider the effect on invisibility of active devices inside the cloaked region, interpreted as collections of sources and sinks or internal currents. When these conditions are overdetermined, as happens for Maxwell's equations, generic internal currents prevent the existence of finite energy solutions and invisibility is compromised.

We give two basic constructions for cloaking a region D contained in a domain $\Omega \subset \mathbb{R}^n, n \geq 3$, from detection by measurements made at $\partial\Omega$ of Cauchy data of waves on Ω. These constructions, the *single* and *double coatings*, correspond to surrounding either just the outer boundary ∂D^+ of the cloaked region, or both ∂D^+ and ∂D^-, with metamaterials whose EM material parameters (index of refraction or electric permittivity and magnetic permeability) are conformal to a singular Riemannian metric on Ω. For the single coating construction, invisibility holds for the Helmholtz equation, but fails for Maxwell's equations with generic internal currents. However, invisibility can be restored by modifying the single coating construction, by either inserting a physical surface at ∂D^- or using the double coating. When cloaking an infinite cylinder, invisibility results for Maxwell's equations are valid if the coating material is lined on ∂D^- with a surface satisfying the soft and hard surface (SHS) boundary condition, but in general not without such a lining, even for passive objects.

1. Introduction

There has recently been considerable interest [AE, MN, Le, PSS1, MBW] in the possibility, both theoretical and practical, of a region or object being shielded (or "cloaked") from detection via electromagnetic (EM) waves. [GLU1, §4] established a non-tunnelling result for time-independent Schrödinger operators with highly singular potentials. This can be interpreted as cloaking, at any frequency, with respect to solutions of the Helmholtz equation using a layer of isotropic, negative index of refraction material. [GLU2, GLU3] raised the possibility of passive objects being undetectable, in the context of electrical impedance tomography (EIT). Motivated by consideration of certain degenerating families of Riemannian metrics, families of singular conductivities, i.e., not bounded below or above, were given and rigorous results obtained for the conductivity equation of electrostatics, i.e., for waves of frequency zero. A related example of a complete but noncompact two-dimensional Riemannian manifold with boundary having the same Dirichlet-to-Neumann map as a compact one was given in [LTU].

More recently, there has been exciting work based on the availability of metamaterials which allow fairly arbitrary behavior of EM material parameters. The constructions in [Le] are based on conformal mapping in two dimensions and are justified via change of variables on the exterior of the cloaked region. [PSS1] also proposes a cloaking construction for Maxwell's equations based on a singular transformation of the original space, again observing that, outside the cloaked region, the solutions of the homogeneous Maxwell equations in the original space become solutions of the transformed equations. The transformation used is the same as used previously in [GLU2, GLU3] in the context of Calderón's inverse conductivity problem. The paper [PSS2] contains analysis of cloaking on the level of ray-tracing, while full wave numerical simulations are discussed in [CPSSP]. Striking positive experimental evidence for cloaking from microwaves has recently been reported in [SMJCPSS].

Since the metamaterials proposed to physically implement these constructions need to be fabricated with a given wavelength, or narrow range of wavelengths, in mind, it is natural to consider this problem in the frequency domain. (As in the earlier works, dispersion, i.e., dependence of the EM material parameters on k, which is certainly present for metamaterials, is neglected.)

The question we wish to consider is then whether, at some (or all) frequencies k, these constructions allow cloaking with respect to solutions of the Helmholtz equation or time-harmonic solutions of Maxwell's equations. We prove that this indeed is the case for the constructions of [GLU3, PSS1], as long as the object being cloaked is passive; in fact, for the Helmholtz equation, the object can be an active device in the sense of having sources and sinks. On the other hand, for Maxwell's equations with generic internal currents, invisibility in a physically realistic sense seems highly problematic. We give several ways of augmenting or modifying the original construction so as to obtain invisibility for all internal currents and at all frequencies.

Due to the degeneracy of the equations at the surface of the cloaked region, is important to rigorously consider weak solutions to the Helmholtz and Maxwell's equations on all of the domain, not just the exterior of the cloaked region. We analyze various constructions for cloaking from observation on the level of physically meaningful EM waves, i.e., finite energy distributional solutions of the equations, showing that careful formulation of the problem is necessary both mathematically and for correct understanding of the physical phenomena. It turns out that the cloaking structure imposes hidden boundary conditions on such waves at the surface of the cloak. When these conditions are overdetermined, finite energy solutions typically do not exist. The time-domain physical interpretation of this is not entirely clear, but it seems to indicate an accumulation of energy or blow-up of the fields which would compromise the desired cloaking effect.

As mentioned earlier, the example in [PSS1] turns out to be a special case of one of the constructions from [GLU2, GLU3], which gave, in dimensions $n \geq 3$, counterexamples to uniqueness for Calderón's inverse problem [C] for the conductivity equation. (Such counterexamples have now also been given for $n = 2$ [V, KSVW].) Thus, since the equations of electromagnetism (EM) reduce at frequency $k = 0$ to the conductivity equation with conductivity parameter $\sigma(x)$, namely $\nabla \cdot (\sigma \nabla u) = 0$, for the electrical potential u, the invisibility question has already been answered affirmatively in the case of electrostatics.

The present work addresses the invisibility problem at all frequencies $k \neq 0$. We wish to cloak not just a passive object, but rather an active "device", interpreted as a collection of sources and sinks, or an internal current, within D. Since the boundary value problems in question may fail to have unique solutions (e.g., when $-k^2$ is a Dirichlet eigenvalue on D), it is natural, as in [GLU1], to use the set of Cauchy data at $\partial\Omega$ of all of the solutions, rather than the Dirichlet-to-Neumann operator on $\partial\Omega$, which may not be well-defined.

The basic idea of [GLU3, Le, PSS1] is to form new EM material parameters by pushing forward old ones via a singular mapping F. Solutions of the relevant equations, Helmholtz or Maxwell, with respect to the old parameters then push forward to solutions of the modified equations with respect to the new parameters outside the cloaked region. However, when dealing with a *singular* mapping F, the converse is not in general true. This means that outside D, depending upon the class of solutions considered, there are solutions to the equations with respect to the new parameters which are not the push forwards of solutions to the equations with the old parameters. Furthermore, it is crucial that the solutions be dealt with on *all* of Ω, and not only outside D. Especially when dealing with the cloaking of active devices, this gives rise to the question of what are the proper *transmission* conditions on ∂D, which allow arbitrary internal sources to be made invisible to an external observer. To address these issues rigorously, one needs to make a suitable choice of the class of *weak* solutions (on all of Ω) to the singular equations being considered. For both mathematical and, even more so, physical reasons, the weak solutions that are appropriate to consider seem to be the *locally finite energy* solutions; these belong to the Sobolev space H^1 with respect to the singular volume form $|\widetilde{g}|^{1/2}dx$ on Ω for Helmholtz, and $L^2(\Omega, |\widetilde{g}|^{1/2}dx)$ for Maxwell.

These considerations are absent from [Le, PSS1, PSS2], where the cloaking is justified by appealing to both the transformation of solutions on the exterior of D under smooth mappings F (essentially the chain rule) and the fact, in the high frequency limit, that rays originating in $\Omega \setminus D$ avoid ∂D and do not enter D. As we will show, analysis of the transmission conditions at ∂D shows that the constructions of [PSS1, PSS2], although adequate for cloaking active devices in the absence of polarization, i.e., for Helmholtz, and cloaking passive devices in the presence of polarization, i.e., for Maxwell, fail to admit finite energy solutions to Maxwell when generic active devices are present. Furthermore, analysis of cloaking of an infinite cylinder, which was numerically explored in [CPSSP] and provides a model of the experimental verification of cloaking in [SMJCPSS], shows that even cloaking a passive object may be problematic. Fortunately, it is possible to remedy the situation by augmenting or modifying this construction.

We now describe the results of this paper. For what we call the *single coating*, which is the construction of [GLU3], and apparently that intended in [PSS1], we establish invisibility with respect to the Helmholtz equation at all frequencies $k \neq 0$. In fact, one can not only cloak a passive object in a region $D \subset\subset \Omega$, containing material with nonsingular index of refraction $n(x)$, from all measurements made at the boundary $\partial\Omega$, but also an active device, interpreted as a collection of sources and sinks within D.

Among the phenomena described here is that finite energy solutions to the single coating construction must satisfy certain "hidden" boundary conditions at ∂D. For the Helmholtz equation, this is the Neumann boundary condition at ∂D^-, and it follows that waves which propagate inside D and are incident to ∂D^- behave as if the boundary were perfectly reflecting. Thus, the cloaking structure on the exterior of D produces a "virtual surface" at ∂D^-. However, for Maxwell's equations with electric permittivity $\varepsilon(x)$ and magnetic permeability $\mu(x)$, the situation is more complicated. The hidden boundary condition forces the tangential components of both the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ to vanish on ∂D^-. For cloaking passive objects, for which the internal current $J = 0$, this condition can be satisfied, but for generic J, finite energy time-harmonic solutions fail to exist, and thus the single coating construction is insufficient for invisibility. In practice, even for cloaking passive objects, this may degrade the effective invisibility.

We find two ways of dealing with this difficulty. One is to simply augment the single coating construction around a ball by adding a perfect electrical conductor (PEC) lining at ∂D, in order to make the object inside the coating material to appear like a passive object. (Such a lining was apparently incorporated, although claimed to be unnecessary, into the code used in [CPSSP] in an effort to stabilize the numerics.) However, for the sake of brevity, the necessary weak formulation of the boundary value problem for this setup will not be considered in this paper.

Alternatively, one can introduce a more elaborate construction, which we refer to as the *double coating*. Mathematically, this corresponds to a singular Riemannian metric which degenerates in the same way as one approaches ∂D from both sides; physically it would correspond to surrounding *both* the inner and outer surfaces of D with appropriately matched metamaterials. We show that, for the double coating, no lining is necessary and full invisibility holds for arbitrary active devices, at all nonzero frequencies, for both Helmholtz and Maxwell. It is even possible for the field to be identically zero outside of D while nonzero within D, and vice versa.

Finally, we also analyze cloaking within an infinitely long cylinder, $D \subset \mathbb{R}^3$. In the main result of §7 and §8, we show that the cylinder D becomes invisible at all frequencies if we use a double coating together with the so-called *soft and hard* (SHS) boundary condition on ∂D. For the origin and properties of the SHS condition and a description of how the SHS condition can be physically implemented, see [HLS, Ki1, Ki2, Li].

We point out that there is some confusion in the physical literature concerning the theoretical possibility of invisibility. By this we mean uniqueness theorems for the inverse problem of recovering the electromagnetic parameters from boundary information (near field) or scattering (far field) at a single frequency, or for all frequencies. There is a vast literature on this subject. We only mention here mathematical results directly related to the one mentioned in [Le, SMJCPSS]. The Helmholtz operator at non-zero energy for isotropic media is given by $\Delta + k^2 n(x)$, where $n(x)$ is the index of refraction and $k \neq 0$. Unique determination of $n(x)$ from boundary data for a single frequency k, under suitable regularity assumptions on $n(x)$ and in dimension $n \geq 3$, was proved in [SyU], with a similar result for the acoustic wave equation in [N]. (See [U] for a survey of related results). The article [N] was referred to in [Le, SMJCPSS] as showing that perfect invisibility was not possible. However, the results of [SyU, N] for the Helmholtz equation are valid only under the assumption that the medium is isotropic and that the index of refraction is bounded. This does not contradict the possibility of invisibility for an anisotropic index of refraction, nor for an unbounded isotropic index of refraction. The constructions of [GLU3, Le, PSS1] and the present paper violate both of these conditions. We also point out that the counterexamples given in [GLU1, Sec. 4] yield invisibility for the Helmholtz equation, in dimension $n \geq 3$, for certain isotropic *negative* indices of refraction which are highly singular (and negative) on $\Omega \setminus D$.

We also note here that, at fixed energy, the Cauchy data is equivalent to the inverse scattering data. The connection between the fixed energy inverse scattering data, the Dirichlet-to-Neumann map and the Cauchy data is discussed, for instance, in [N] for the Schrödinger equation and in [U] for the Helmholtz equation in anisotropic media. The scattering operator is well defined for the degenerate metrics defined here; see, e.g., [M].

There is a large literature (see [U]) on uniqueness in the Calderón problem for isotropic conductivities under the assumption of positive upper and lower bounds for σ. It was noted by Luc Tartar (see [KV] for an account) that uniqueness fails badly if anisotropic tensors are allowed, since if $F : \overline{\Omega} \longrightarrow \overline{\Omega}$ is a smooth diffeomorphism with $F|_{\partial\Omega} = id$, then $F_*\sigma$ and σ have the same Dirichlet-to-Neumann map (and Cauchy data.) Note that since ε and μ transform in the same way, this already constitutes a form of invisibility, i.e., from the Cauchy data one cannot distinguish between the EM material parameter pairs ε, μ and $\widetilde{\varepsilon} = F_*\varepsilon$, $\widetilde{\mu} = F_*\mu$.

Thus, uniqueness for anisotropic media, in the mathematical literature, has come to mean uniqueness up to a pushforward by a (sufficiently regular) map F. Such uniqueness in the Calderón problem is known under various regularity assumptions on the anisotropic conductivity in two dimensions [S, N1, SuU, ALP] and in three dimensions or higher [LaU, LeU, LTU], but for all of these results it is assumed that the eigenvalues of $\sigma(x)$ are bounded below and above by positive constants. Related to the Calderón problem is the Gel'fand problem, which uses Cauchy data at all frequencies, rather than at a fixed one; for this problem, uniqueness results are also available, e.g., [BeK, KK], with a detailed exposition in [KKL]. For example, in the anisotropic inverse conductivity problem as above, Cauchy data at *all* frequencies determines the tensor up to a diffeomorphism $F : \overline{\Omega} \longrightarrow \overline{\Omega}$.

Thus, a key point in the current works on invisibility that allows one to avoid the known uniqueness theorems for the Calderón problem is the lack of positive lower and upper bounds on the eigenvalues of these symmetric tensor fields. In this paper, as in [GLU3, Le, PSS1], the lower bound condition is violated near ∂D, and there fails to be a global diffeomorphism F relating the pairs of material parameters having the same Cauchy data.

For Maxwell's equations, all of our constructions are made within the context of the permittivity and permeability tensors ε and μ being conformal to each other, i.e., multiples of each other by a positive scalar function; this condition has been studied in detail in [KLS]. For Maxwell's equations in the time domain, this condition corresponds to polarization-independent wave velocity. In particular, all isotropic media are included in this category. This seemingly special condition arises naturally from our construction, since the pushforward $(\widetilde{\varepsilon}, \widetilde{\mu})$ of an isotropic pair (ε, μ) by a diffeomorphism need not be isotropic but does satisfy this conformality. For both mathematical and practical reasons, it would be very interesting to understand cloaking for general anisotropic materials in the absence of this assumption.

We believe that our results suggest improvements which can be made in physical implementations of cloaking. In the very recent experiment [SMJCPSS], the configuration corresponds to a thin slice of of an infinite cylinder, inside of which a homogeneous, highly conducting disk was placed in order to be cloaked. This corresponds to the single coating with the metric g_2 (see §2) on D being a constant multiple of the Euclidian metric. The analysis here suggests that lining the inside surface ∂D^- of the coating with a material implementing the SHS boundary condition [HLS, Ki1, Ki2, Li] should result in less observable scattering than occurs without the SHS lining, improving the partial invisibility already observed.

The paper is organized as follows. In §2 we describe the single and double coating constructions. We then establish cloaking for the Helmholtz equation at all frequencies in §3. The notion of a finite energy solution for the single coating is defined in §§3.2 and then the key step for showing invisibility is Proposition 3.5. We discuss the Helmholtz equation for the double coating In §§3.3; there we define the notion of a weak solution and the Neumann boundary condition at the inner surface of the cloaked region. The key step for invisibility for Helmholtz at all frequencies in the presence of the double coating is Proposition 3.11.

In §4 we study invisibility at all frequencies for Maxwell's equations. We define the notion of finite energy solutions for the single and double coatings. In §5 we demonstrate invisibility for Maxwell's at all frequencies for the double coating; see Proposition 5.1. In §6 we show that, for the single coating construction, the Cauchy data for Maxwell's equations must vanish on the surface of the cloaked region, showing that generically finite energy solutions for Maxwell's equations in the cloaked region do not exist. In §7 we consider an infinite cylindrical domain and show invisibility at all frequencies for Maxwell's equations for the double coating; the key result is Proposition 7.1. In §8, we consider how to cloak the cylinder, treating its surface as an obstacle with the SHS boundary condition. Finally, in §9, we briefly indicate how general the constructions can be made. In particular, we note that a modification the double coating allows one to change the topology of the domain and yet maintain invisibility.

We would like to thank Bob Kohn for bringing the papers [Le, PSS1] to our attention, and Ismo Lindell for discussions concerning the SHS boundary condition.

2. Geometry and basic constructions

The material parameters of electromagnetism, namely the conductivity, $\sigma(x)$; electrical permittivity, $\varepsilon(x)$; and magnetic permeability, $\mu(x)$, all transform as a product of a contravariant symmetric 2-tensor and a $(+1)-$density. That is, if $F : \Omega_1 \longrightarrow \Omega_2, \quad y = F(x)$, is a diffeomorphism between domains in $\mathbb{R}^n$, then $\sigma(x) = (\sigma^{jk}(x))$ on Ω_1 pushes forward to $(F_*\sigma)(y)$ on Ω_2, given by

$$(F_*\sigma)^{jk}(y) = \frac{1}{\det\left[\frac{\partial F^j}{\partial x^k}(x)\right]} \sum_{p,q=1}^{n} \frac{\partial F^j}{\partial x^p}(x) \, \frac{\partial F^k}{\partial x^q}(x)\sigma^{pq}(x) \Bigg|_{x=F^{-1}(y)}, \tag{1}$$

with the same transformation rule for the other material parameters. It was observed by Luc Tartar (se [KV]) that it follows that if F is a diffeomorphism of a domain Ω fixing $\partial\Omega$, then σ and $\widetilde{\sigma} := F_*\sigma$ have the same Dirichlet-to-Neumann map, producing infinite-dimensional families of indistinguishable conductivities. On the other hand, a Riemannian metric $g = (g_{jk}(x))$ is a covariant symmetric two-tensor. Remarkably, in dimension three or higher, a material parameter tensor and a Riemannian metric can be associated with each other by

$$\sigma^{jk} = |g|^{1/2} g^{jk}, \quad \text{or} \quad g^{jk} = |\sigma|^{2/(n-2)}\sigma^{jk}, \tag{2}$$

where $(g^{jk}) = (g_{jk})^{-1}$ and $|g| = \det(g)$. Using this correspondence, examples of singular anisotropic conductivities in $\mathbb{R}^n, n \geq 3$, that are indistinguishable from a constant isotropic conductivity, in that they have the same Dirichlet-to-Neumann map, were given in [GLU3]. The two constructions there are based on two different types degenerations of Riemannian metrics, whose singular limits can be considered as coming from singular changes of variables. The singular conductivities arising from these metrics via the above correspondence are then indistinguishable from a constant isotropic σ. In the current paper, we will continue to examine one of these constructions, corresponding to pinching off a neck of a Riemannian manifold; we refer to it as the single coating. We also introduce another construction, referred to as the double coating. We start by giving basic examples of each of these.

For both examples, let $\Omega = B(0,2) \subset \mathbb{R}^3$, the ball of radius 2 and center 0, be the domain at the boundary of which we make our observations; $D = B(0,1) \subset \Omega$ the region to be cloaked; and $\Sigma = \partial D = \mathbb{S}^2$ the boundary of the cloaked region.

Single coating construction: We begin by recalling an example from [GLU3, PSS1]; the two dimensional examples in [Le, V] are either essentially the same or closely related in structure.

For the single coating, we blow up 0 using the map

$$F_1 : \overline{B}(0,2) \setminus \{0\} \to \overline{\Omega} \setminus \overline{D}, \quad F_1(x) = (\frac{r}{2}+1)\frac{x}{r}, \quad r = |x|, \quad 0 < r \le 2. \tag{3}$$

On $\overline{B}(0,2)$, let $(g_e)_{ij} = \delta_{ij}$ be the Euclidian metric, corresponding to constant isotropic material parameters; via the map F_1, g_e pushes forward, i.e., pulls back by F^{-1}, to a metric on $\overline{\Omega} \setminus \overline{D}$,

$$\widetilde{g}_1 = (F_1)_* g_e := (F_1^{-1})^*(g_e) \quad .$$

Introducing the boundary normal coordinates (ω, τ) in the annulus $\Omega \setminus \overline{D}$, where $\omega = (\omega^1, \omega^2)$ are local coordinates on $\Sigma = \mathbb{S}^2$ and $\tau > 0$ is the distance in metric $\widetilde{g}_1$ to Σ, we have, from (3),

$$\widetilde{g}_1 = \tau^2\, d\omega^2 + d\tau^2, \quad \tau = 2(r-1). \tag{4}$$

Here $d\omega^2 = h_{\alpha\beta}(\omega) d\omega^\alpha d\omega^\beta$ is the standard metric on $\mathbb{S}^2$, induced by the Euclidian metric on $\mathbb{R}^3$. Note that $\widetilde{g}_1$ has the following properties:

Consider a local g_e-orthonormal frame (∂_r, v, w) on $\overline{\Omega} \setminus \overline{D}$ consisting of the radial vector

$$\partial_r = \frac{\partial}{\partial r} = \frac{x^j}{r}\frac{\partial}{\partial x^j}$$

and two vector fields v, w. Then,

$$\begin{aligned} &\widetilde{g}_1(\partial_r, \partial_r) = 4, \quad \widetilde{g}_1(\partial_r, v) = \widetilde{g}_1(\partial_r, w) = 0, \quad \widetilde{g}_1(w, v) = 0, \\ &\frac{\widetilde{g}_1(v,v)}{(r-1)^2} \in [c_1, c_2], \quad \frac{\widetilde{g}_1(w,w)}{(r-1)^2} \in [c_1, c_2], \end{aligned} \tag{5}$$

where $c_1, c_2 > 0$. Thus, $\widetilde{g}_1$ has one eigenvalue bounded from below (with eigenvector corresponding to the radial direction) and two eigenvalues that are of order $(r-1)^2$ (with eigenspace $span\{v, w\}$). In Euclidean coordinates, we have that, for $|\widetilde{g}_1| = \det(\widetilde{g}_1)$,

$$\begin{aligned} &|\widetilde{g}_1(x)|^{1/2} \sim C_1 (r-1)^2, \\ &|\widetilde{g}_1^{ij} \nu_i| \le C_2, \quad \nu_i = \frac{2x}{r} = 2(\partial_r)_i. \end{aligned} \tag{6}$$

Here and below we use Einstein's summation convention, summing over indices appearing both as sub- and super-indices in formulae, and $\nu = (\nu_1, \nu_2, \nu_3)$ denotes the unit co-normal vectors of the surfaces $\{x \in \Omega \setminus \overline{D} : \ |x| = s\}$, $1 < s < 2$, with respect to the metric $\widetilde{g}$.

On D, we simply let $\widetilde{g}_2$ be the Euclidian metric. Together, the pair $(\widetilde{g}_1, \widetilde{g}_2)$ define a singular Riemannian metric on $\overline{\Omega}$,

$$\widetilde{g} = \begin{cases} \widetilde{g}_1, \ x \in \Omega \setminus \overline{D}, \\ \widetilde{g}_2, \ x \in D, \end{cases}$$

which is singular on Σ^+, i.e., as one approaches Σ from $\Omega \setminus \overline{D}$; in the sequel, we will identify the metric $\widetilde{g}$ and the corresponding pair $(\widetilde{g}_1, \widetilde{g}_2)$.

To unify notation for the two basic constructions, we will denote in the single coating case $M_1 = \Omega$, $M_2 = D$ and let M be the disjoint union $M = M_1 \cup M_2$. Also, for notational unity with the double coating, we let $\gamma_1 = \{0\} \subset M_1$, $\gamma_2 = \emptyset \subset M_2$, and $\gamma = \gamma_1 \cup \gamma_2$. Moreover, we denote $N_1 = \Omega \setminus \overline{D}$, $N_2 = D$, $\Sigma = \partial D$, and $N = N_1 \cup \Sigma \cup N_2 := \Omega \subset \mathbb{R}^3$.

Double coating construction: The double coating refers to a metric on Ω that is degenerate on both sides of Σ and has the same limit as one approaches Σ from both sides.

We now introduce notation, shared with the single coating, that will be used throughout for the double coating. Let $M_1 = \Omega = B(0, 2)$, which is compact with boundary, and $M_2 := \mathbb{S}^3_{1/\pi}$, the 3-sphere of radius $1/\pi$, which is compact without boundary, and again let $M = M_1 \cup M_2$ be their disjoint union. For the double coating, let $\gamma_1 = \{0\} \subset M_1$, $\gamma_2 = \{NP\} \subset M_2$, where NP is a chosen point, e.g., the North Pole of $\mathbb{S}^3_{1/\pi}$, and $\gamma = \gamma_1 \cup \gamma_2$. As in the single coating example, we let $N_1 = \Omega \setminus \overline{D} = B(0, 2) \setminus \overline{B}(0, 1)$, $N_2 = D = B(0, 1)$, $\Sigma = \partial D$, and $N = N_1 \cup \Sigma \cup N_2 \subset \mathbb{R}^3$. We take the diffeomorphism $F_1 : M_1 \setminus \gamma_1 \longrightarrow N_1$ to be the blow-up of γ_1 as in the single coating, while we blow-up γ_2 by defining $F_2 : M_2 \setminus \gamma_2 \longrightarrow N_2$ as follows. Denote by SP the point on Ω_2 antipodal to NP. Then the Riemannian normal coordinates centered at SP are defined on $B(0, 1) \subset T_{SP}\mathbb{S}^3 \simeq \mathbb{R}^3$,

$$\exp_{SP} : \ B(0, 1) \to M_2 \setminus \{NP\}.$$

Denote by F_2 the diffeomorphism

$$F_2 = (\exp_{SP})^{-1} : \ M_2 \setminus \{NP\} \to B(0, 1).$$

Introduce (local) spherical coordinates (ω, r) on $N_2 = B(0, 1)$, considered as a subset of $T_{SP}(\mathbb{S}^3)$, with $\omega = (\omega_1, \omega_2)$, $\omega \in \Sigma = \partial B(0, 1)$ and $0 \leq r \leq 1$. The standard metric g on $\mathbb{S}^3_{1/\pi}$ in these coordinates takes the form

$$g_2 = \frac{\sin^2(\pi r)}{\pi^2} \, d\omega^2 + dr^2, \tag{7}$$

where $d\omega^2$ is again the standard metric on $\mathbb{S}^2$.

Observe that $\widetilde{g}_2 = (F_2)_*(g_2)$, as one approaches Σ^- on $B(0, 1)$, has very similar properties to $\widetilde{g}_1$ on $B(0, 2) \setminus B(0, 1)$ as one approaches Σ^+. Indeed, again consider the radial vector $\partial_r = \frac{\partial}{\partial r} = \frac{x^j}{r}\frac{\partial}{\partial x^j}$ at $x \in N_2$ and two vectors v, w such that in Euclidean metric (∂_r, v, w) is a local orthonormal frame. Then, as follows from (7), at $x \in N_2$ with, say, $1/2 < r < 1$,

$$\begin{aligned} &\widetilde{g}_2(\partial_r, \partial_r) = 1, \quad \widetilde{g}_2(\partial_r, v) = \widetilde{g}_2(\partial_r, w) = 0, \\ &\widetilde{g}_2(w, v) = 0, \quad \frac{\widetilde{g}_2(v, v)}{(1-r)^2}, \ \frac{\widetilde{g}_2(w, w)}{(1-r)^2} \in [c_1, c_2], \end{aligned}$$

where $c_1, c_2 > 0$. Thus, $\widetilde{g}_2$ has one eigenvalue bounded from below (with eigenvector corresponding to the radial direction) and two eigenvalues that are of order $(1-r)^2$, and with respect to the Euclidean coordinates on N_2,

$$|\widetilde{g}_2(x)|^{1/2} \le C_1(1-r)^2, \quad |\widetilde{g}_2^{ij}\nu_i| \le C_2, \quad \nu_i = -\frac{x_i}{r} = -(\partial_r)_i, \ \frac{1}{2} < r < 1. \quad (8)$$

Set $\widetilde{g}_1 = (F_1)_* g_e$ on N_1, where F_1 is defined as for the single coating example. Together, these define a singular metric $\widetilde{g} = (\widetilde{g}_1, \widetilde{g}_2)$ on the entire ball $N = N_1 \cup N_2 \cup \Sigma = B(0, 2)$. Comparing (4) and (7), we see that, in the Fermi coordinates [1] associated to Σ, $|\widetilde{g}|^{1/2}\widetilde{g}^{ij}$ is Lipschitz continuous on N; note also that $|\widetilde{g}|^{1/2}\widetilde{g}_{ij}$ is not invertible at $\partial B(0, 1)$.

Although they are distinct, each of these constructions may be summarized as follows. The domain Ω, which we will refer to as N, decomposes as $N = N_1 \cup \Sigma \cup N_2$, where $N_1 = \Omega \setminus \overline{D}$, $N_2 = D$ and $\Sigma = \partial D$. N_1 and N_2 are manifolds with boundary, with $\partial N_1 = \partial\Omega \cup \partial D^+ = \partial N \cup \Sigma^+$ and $\partial N_2 = \Sigma^-$, where the superscripts $\pm$ are used when considering limits from the exterior or interior of the cloaked region. The singular electromagnetic material parameters on N will correspond to a singular Riemannian metric $\widetilde{g} = (\widetilde{g}_1, \widetilde{g}_2)$, arising as the pushforward of a (nonsingular) Riemannian metric $g = (g_1, g_2)$ on a manifold with two components, $M = M_1 \cup M_2$, via a map $F : M \setminus \gamma \longrightarrow N$,

$$F(x) = \begin{cases} F_1(x), \, x \in M_1 \setminus \gamma, \\ F_2(x), \, x \in M_2 \setminus \gamma. \end{cases}$$

Here, M_1 and M_2 are disjoint, with $\overline{M_1}$ diffeomorphic to $\overline{N}$; $\gamma_1 = \gamma \cap M_1$ is either a point (the point being blown up) for the single and double coatings, or a line (for the cloaking of an infinite cylinder in §7,8); and $\gamma_2 = \gamma \cap M_2$ is either empty (for the single coating) or a point (for the double coating) or a line (for the cylinder.) In §9, we will show that such constructions exist in great generality, and for this reason the proofs will be expressed in terms of analysis on M and N.

In this generality, we say that $(M, N, F, \gamma, \Sigma, g)$ is a *coating construction* if (M, g) is a (nonsingular) Riemannian manifold, $\gamma \subset M$ and $\Sigma \subset N$ are as above, and $F : M \setminus \gamma \to N \setminus \Sigma$ is diffeomorphism of either type. This then defines a singular Riemannian metric $\widetilde{g}$ everywhere on $N \setminus \Sigma = N_1 \cup N_2$, by

$$\widetilde{g} = \begin{cases} \widetilde{g}_1 := F_{1*} g_1, \, x \in N_1, \\ \widetilde{g}_2 := F_{2*} g_2, \, x \in N_2. \end{cases}$$

[1] Recall that the Fermi coordinates associated to Σ are (ω, τ), where $\omega = (\omega^1, \omega^2)$ are local coordinates on Σ and $\tau = \tau(x)$ is the distance from x to Σ with respect to the metric $\widetilde{g}$, multiplied by $+1$ in N_1 and -1 in N_2.

If we introduce Fermi coordinates (ω, τ) near Σ as above, the $\widetilde{g}$ satisfies (5),(6) or (8), with $r-1$ replaced by τ, for the single and double coatings, resp. From these, one sees that $|\widetilde{g}|^{1/2}g^{jk}$ has a jump discontinuity across Σ for the single coating and is Lipschitz for the double coating. Note that in both examples, $N = \Omega = B(0,2)$, so that N and M_1 have the same topology. However, in a direct extension of the double coating construction, described in §9, the domain N containing the cloaked region N_2 need not even be diffeomorphic to M_1.

3. The Helmholtz equation

We are interested in invisibility of a cloaked region with respect to the Cauchy data of solutions of the Helmholtz equation,

$$(\Delta_g + k^2)u = f \quad \text{in } \Omega, \tag{9}$$

where f represents a collection of sources and sinks. The *Cauchy data* $\mathcal{C}^k_{g,f}$ consists of the set of pairs of boundary measurements $(u|_{\partial\Omega}, \partial_\nu u|_{\partial\Omega})$ where u ranges over solutions to (9) in some function or distribution space (discussed below). Let $(M, N, F, \gamma, \Sigma, g)$ be a single coating construction as in §2. For the moment, as in the Introduction, we continue to refer to N as Ω, N_2 as D and Σ^+ as ∂D^+; we may assume that $M_1 = N$, $M_2 = D$ and $F_2 = id$, so that $\widetilde{g}_2 = g_2$ is a (nonsingular) Riemannian metric on D. Thus, $\widetilde{g}$ is a Riemannian metric on Ω, singular on $\Omega \setminus D$, resulting from blowing up the metric g_1 on Ω with respect to a point O and inserting the (D, g_2) into the resulting "hole".

We wish to show that $\mathcal{C}^k_{\widetilde{g},\widetilde{f}} = \mathcal{C}^k_{g,0}$ for all frequencies $0 < k < \infty$, if $\text{supp}(\widetilde{f}) \subset D$ and k is not a Neumann eigenvalue of (D, g_2). Due to the singularity of $\widetilde{g}$, it is necessary to consider nonclassical solutions to (9), and we will see that the exact notion of weak solution is crucial. Furthermore, a hidden Neumann boundary condition on ∂D^- is required for the existence of finite energy solutions. Physically, this means that the coating on $\Omega \setminus \overline{D}$ makes the inner boundary ∂D^- appear to be a perfectly reflecting "sound-hard surface" for waves propagating in D, while, from the exterior, the cloaked device is invisible; that is, measurements of solutions of the Helmholtz equation at $\partial\Omega$ cannot distinguish between $(\Omega, \widetilde{g})$ and (Ω, g).

3.1. $k = 0$ and weak solutions. First consider the case when $k = 0$ and $f = 0$. As described in the Introduction, this situation was treated in [GLU3] in the context of electrical impedance tomography. There, it sufficed to consider as weak solutions those L^∞ functions satisfying (9) (for the metric $\widetilde{g}$) in the sense of distributions. It was shown that, for given Dirichlet data h on $\partial\Omega$, (9) has a unique such solution, $\widetilde{u}$, which must, by removable singularity considerations, be constant on D. These same conclusions would have held if we had considered the larger class of spatial H^1 weak solutions (defined below). However, for $k > 0$ or $f \neq 0$, we will see that this notion of weak solution is inappropriate.

3.1.1. $k > 0$ and spatial H^1 solutions.

Definition 1. *$\widetilde{u}$ is a* spatial H^1 *solution to the Dirichlet problem for the Helmholtz equation,*

$$(\Delta_{\widetilde{g}} + k^2)\widetilde{u} = \widetilde{f} \quad \text{on } \Omega,\ \widetilde{u}|_{\partial\Omega} = h \tag{10}$$

if

$$\widetilde{u} \in H^1(\Omega, dx) \tag{11}$$

and

$$\partial_i(|\widetilde{g}|^{1/2}\widetilde{g}^{ij}\partial_j\widetilde{u}) + k^2|\widetilde{g}|^{1/2}\widetilde{u} = |\widetilde{g}|^{1/2}\widetilde{f} \quad \text{in } H^{-1}(\Omega, dx). \tag{12}$$

Here, for $s \in \mathbb{R}$, $H^s(\Omega, dx)$ refers to the standard Sobolev space of distributions with s derivatives in $L^2(\Omega, dx)$. Note that (11), together with the properties of the metric tensor given in §2, implies that $|\widetilde{g}|^{1/2}\widetilde{g}^{ij}\partial_i\widetilde{u} \in L^2(\Omega, dx)$.

Later in our analysis (see (36)), we will see that (12) implies that the normal derivative of $\widetilde{u}$ on ∂D^- vanishes,

$$\partial_r\widetilde{u}|_{\partial D^-} = 0.$$

On the other hand, the fact that $\widetilde{u} \in H^1(\Omega, dx)$ implies that

$$\widetilde{u}|_{\partial D^-} = \widetilde{u}|_{\partial D^+} = \text{constant} := u(O),$$

with u the solution to $(\Delta_g + k^2)u = 0$ in $\partial\Omega$, $u|_{\partial\Omega} = h$, where the first equality follows from the trace theorem for H^1 functions and the second from considerations similar to those in [GLU3, Prop. 1]. Note that, for generic k and h, $u(O) \neq 0$. Thus, $\widetilde{u}_2 := \widetilde{u}|_D$ needs to be a solution of the overdetermined elliptic boundary value problem on $(D, \widetilde{g}_2)$,

$$(\Delta + k^2)\widetilde{u}_2 = 0, \quad \partial_\nu\widetilde{u}_2|_{\partial D} = 0,\ \widetilde{u}_2|_{\partial D} = \text{constant} \neq 0. \tag{13}$$

Clearly, for generic $k > 0$ there exists no solution to (13) and therefore there is no weak solution to (10) in the sense of Definition 1. Rather, one needs to use an H^1 norm adapted to the singular Riemannian metric $\widetilde{g}$; this is in fact physically natural, being essentially the energy of the wave. We formulate the correct notion in the next section.

3.2. Finite energy solutions for the single coating. We now give a more satisfactory definition of weak solution, restricting the notion to those solutions that are physically meaningful in that they have finite energy.

We now revert to the notation of $M, N, \ldots$ when discussing the single coating construction, i.e., let $(M, N, F, \gamma, \Sigma, g)$ denote a single coating as in §2. Our first task is to understand in what sense the expression $|\widetilde{g}|^{1/2}\widetilde{g}^{ij}\partial_i\widetilde{u}$ is rigorously defined.

To this end, define for $\widetilde{\phi} \in C^\infty(\overline{N})$

$$\|\widetilde{\phi}\|_X^2 := \int_N (|\widetilde{g}|^{1/2}\widetilde{g}^{ij}\partial_i\widetilde{\phi}\partial_j\widetilde{\phi} + |\widetilde{g}|^{1/2}|\widetilde{\phi}|^2)\,dx.$$

Let

$$H^1(N, |\widetilde{g}|^{1/2}dx) = X := \mathrm{cl}_X(C^\infty(\overline{N}))$$

be the completion of $C^\infty(\overline{N})$ with respect to the norm $\|\cdot\|_X$. We note that $H^1(N, |\widetilde{g}|^{1/2}dx) \subset L^2(N, |\widetilde{g}|^{1/2}dx)$, so we can consider its elements as measurable functions on N.

Lemma 1. *The map*

$$\phi \longrightarrow D_{\widetilde{g}}\widetilde{\phi} = (D_{\widetilde{g}}^j\widetilde{\phi})_{j=1}^3 = (|\widetilde{g}|^{1/2}\widetilde{g}^{ij}\partial_i\widetilde{\phi})_{j=1}^3, \quad \phi \in C^\infty(\overline{N}),$$

has a bounded extension

$$D_{\widetilde{g}} : H^1(N, |\widetilde{g}|^{1/2}dx) \to \mathcal{M}(N; \mathbb{R}^3),$$

where $\mathcal{M}(N; \mathbb{R}^3)$ denotes the space of $\mathbb{R}^3$-valued signed Borel measures on N. Moreover, for $\widetilde{u} \in X$, we have, in the sense of Borel measures

$$(D_{\widetilde{g}}\widetilde{u})(\Sigma) = 0. \tag{14}$$

Proof. Let $\widetilde{\phi} \in C^\infty(\overline{N})$ and $\widetilde{\eta} \in C(\overline{N})$. Then $D_{\widetilde{g}}^j\widetilde{\phi} \in L^\infty(N)$. Let $\phi = \widetilde{\phi} \circ F$, $\eta = \widetilde{\eta} \circ F \in L^\infty(\Omega)$. Then,

$$\begin{aligned}\int_N (D_{\widetilde{g}}^j\widetilde{\phi})\,\widetilde{\eta}\,dx &= \int_{N\setminus\Sigma} (D_{\widetilde{g}}^j\widetilde{\phi})\,\widetilde{\eta}\,dx \\ &= \int_{M_1\setminus\gamma_1} |g|^{1/2}g^{kl}\frac{\partial y^j}{\partial x^l}\partial_k\phi\,\eta\,dx + \int_{M_2} |g|^{1/2}g^{kl}\frac{\partial y^j}{\partial x^l}\partial_k\phi\,\eta\,dx.\end{aligned}$$

As the metric g is bounded from above and below, and $\frac{\partial y^j}{\partial x^l} = O(r^{-1})$ on M_1 and $= \delta_l^j$ on M_2, we have

$$\begin{aligned}\left|\int_N (D_{\widetilde{g}}^j\widetilde{\phi})\,\widetilde{\eta}\,dx\right| &\le C_0(\|\phi\|_{H^1(M_1,dx)}\|\eta/r\|_{L^2(M_1,dx)} + \|\phi\|_{H^1(M_2,dx)}\|\eta\|_{L^2(M_2,dx)}) \\ &\le C_1||\widetilde{\phi}||_X||\widetilde{\eta}||_{C(N)}d_{\widetilde{g}}^{1/2}(\mathrm{supp}(\widetilde{\eta}), \Sigma),\end{aligned}$$

where $d_{\widetilde{g}}$ is the distance on N with respect to the metric $\widetilde{g}$. This shows the existence of the bounded extension $D_{\widetilde{g}} : H^1(N, |\widetilde{g}|^{1/2}dx) \to \mathcal{M}(N; \mathbb{R}^3)$. Also, if we consider functions $\widetilde{\eta}$ supported in small neighborhoods of Σ, we see that (14) follows. □

We also need the following auxiliary result

Lemma 2. *Assume that $\widetilde u$ is a measurable function on N such that*

$$\widetilde u \in L^2(N,\, |\widetilde g|^{1/2}dx), \tag{15}$$

$$\widetilde u|_{N\setminus\Sigma} \in H^1_{loc}(N\setminus\Sigma, dx), \tag{16}$$

$$\int_{N\setminus\Sigma} |\widetilde g|^{1/2}\widetilde g^{ij}\partial_i\widetilde u\partial_j\widetilde u\, dx < \infty. \tag{17}$$

Then $\widetilde u \in H^1(N, |\widetilde g|^{1/2}dx)$.

Note that, due to the fact that $\widetilde g$ is bounded and positive definite on any compact subset of $N\setminus\Sigma$, condition (16) in fact follows from conditions (15), (17) and is included for the convenience of future references.

Proof. Consider first the case when $\widetilde u = 0$ in N_1.

First, the condition (17) implies that $\widetilde v = \widetilde u|_{N_2} \in H^1(N_2, dx)$. Let $f = v|_\Sigma \in H^{1/2}(\Sigma)$ and $E^f \in H^1(N_1, dx)$ be an extension of f. Let $\chi \in C_0^\infty(\mathbb{R})$ be a cut-off function with $\chi(t) = 1$ for $|t| < \frac{1}{2}$ and $\chi(t) = 0$ for $|t| > 1$. We introduce Fermi coordinates near Σ as in §2, (τ,ω), $\tau \in (0,2)$, $\omega = (\omega_1,\omega_2) \in \Sigma$.

Define, for $\varepsilon > 0$,

$$w_\varepsilon(x) = \begin{cases} v(x), & x \in N_2, \\ \chi(\frac{\tau}{\varepsilon})E^f(x), & x \in N_1. \end{cases}$$

Then $w_\varepsilon \in H^1(N, dx)$ and, using (3), (5), we see that

$$\lim_{\varepsilon\to 0}\int_{N\setminus\Sigma} |\widetilde g|^{1/2}[\widetilde g^{ij}\partial_i(w_\varepsilon - \widetilde u)\partial_j(w_\varepsilon - \widetilde u) + (w_\varepsilon - \widetilde u)^2]\, dx \tag{18}$$

$$= \lim_{\varepsilon\to 0}\int_{N_1} |\widetilde g|^{1/2}[\widetilde g^{ij}\partial_i w_\varepsilon\partial_j w_\varepsilon + |w_\varepsilon|^2]\, dx = 0,$$

Observe that the integrand vanishes outside the a neighborhood of Σ^+ of volume less than $C\varepsilon$. Next, divide the integral involving derivatives in the right-hand side of (18) into the terms involving components tangential and normal to the boundary, using the fact that $\tau = 2(r-1)$:

$$\int_{N_1\setminus\Sigma} |\widetilde g|^{1/2}\chi^2(\frac{\tau}{\varepsilon})\widetilde g^{\alpha\beta}\partial_{\omega_\alpha}E^f\, \partial_{\omega_\beta}E^f\, d\tau d\omega_1 d\omega_2,$$

and where α,β run over $\{1,2\}$,

$$\int_{N_1\setminus\Sigma} |\widetilde g|^{1/2}\left|\partial_\tau[\chi(\frac{\tau}{\varepsilon})E^f]\right|^2\, d\tau d\omega_1 d\omega_2.$$

As, by (5), $|\widetilde g|^{1/2}\widetilde g^{\alpha\beta}$ is bounded, the integral involving tangential derivatives tends to 0 due to the volume of the domain of integration. Again, by (5) we have $|\widetilde g|^{1/2} \le C\tau^2$; this, together with the volume estimate and the fact that $|\partial_\tau\chi(\frac{\tau}{\varepsilon})| \le C\tau^{-1}$, implies that the integral involving normal derivatives tends to 0 when $\varepsilon \to 0$. Similarly, we see that

$$\int_{N_1\setminus\Sigma} |\widetilde g|^{1/2}|\chi(\frac{\tau}{\varepsilon})E^f|^2 dx \to 0 \quad \text{for } \varepsilon \to 0.$$

The function $w_\varepsilon \in H^1(N,dx)$ can be approximated with an arbitrarily small error in $H^1(N,dx)$ by a $C^\infty(\overline{N})$ function, and we see that the same holds in the X-norm. Thus $w_\varepsilon \in H^1(N,|\widetilde g|^{1/2}dx)$, and the above limit shows that $\widetilde u \in H^1(N,|\widetilde g|^{1/2}dx)$.

Now let $\widetilde u$ be a measurable function in N satisfying (15), (16), and (17). Let χ_{N_2} be the characteristic function of N_2. As $\chi_{N_2}\widetilde u \in H^1(N,|\widetilde g|^{1/2}dx)$, it is enough to show that $\widetilde u - \chi_{N_2}\widetilde u \in H^1(N,|\widetilde g|^{1/2}dx)$. This means that it is enough to consider the case when $\widetilde u = 0$ in N_2. Clearly, we can restrict our attention to the case when $\widetilde u$ vanishes also near ∂N.

Now let $u_1 = \widetilde u \circ F$ in $M_1 \setminus \gamma_1$. Then we see that

$$\int_{M_1\setminus\gamma_1} |g|^{1/2} g^{ij}\partial_i(u_1)\partial_j(u_1)\,dx < \infty.$$

Let $w = \nabla u|_{M_1\setminus\gamma_1}$. Using a change of coordinates in integration and (15), we see that $u \in L^2(M_1\setminus\gamma_1,dx)$. Extending u_1 and w to functions u_1^e and w^e on γ_1, we obtain functions $u_1^e \in L^2(M_1,dx)$ and $\mathbb{R}^3$–valued function $w^e \in L^2(M_1,dx)$. Now $\nabla u_1^e - w^e \in H^{-1}(M_1,dx)$ is supported on γ_1. Since there are no non-zero $H^{-1}(M_1,dx)$ distributions supported on γ_1, we see that $\nabla u_1^e = w^e \in L^2(M_1,dx)$. Thus we see that $u_1^e \in H^1(M_1,dx)$. In the following we identify u_1 and u_1^e. As u_1 vanishes near ∂M_1, and γ_1 consists of a single point and thus is a $(2,1)$-polar set [Ma, pp.393–7], there are $\phi_j \in C_0^\infty(M_1\setminus\gamma_1)$ such that $\phi_j \to u_1$ in $H^1(M_1,dx)$ as $j\to\infty$, that is,

$$\lim_{j\to\infty}\int_{M_1} |g|^{1/2}[g^{ik}\partial_i(\phi_j-u)\partial_k(\phi_j-u)+(\phi_j-u)^2]\,dx = 0.$$

Now let $\widetilde\phi_j \in C_0^\infty(N)$, with $\mathrm{supp}(\widetilde\phi_j)\subset N_1$ and

$$\widetilde\phi_j = \begin{cases} \phi_j\circ F_1^{-1} & \text{in } N_1,\\ 0 & \text{in } N_2.\end{cases}$$

Then the previous equation implies that

$$\lim_{j\to\infty}\int_{N\setminus\Sigma} |\widetilde g|^{1/2}[\widetilde g^{ik}\partial_i(\widetilde\phi_j-\widetilde u)\partial_k(\widetilde\phi_j-\widetilde u)+(\widetilde\phi_j-\widetilde u)^2]\,dx =$$
$$\lim_{j\to\infty}\int_{N_1} |\widetilde g|^{1/2}[\widetilde g^{ik}\partial_i(\widetilde\phi_j-\widetilde u)\partial_k(\widetilde\phi_j-\widetilde u)+(\widetilde\phi_j-\widetilde u)^2]\,dx = 0,$$

where we use that $\widetilde u = 0$ in N_2.

This shows that $\widetilde\phi_j$ is a sequence converging in the X-norm and that the limit is $\widetilde u$. Thus $\widetilde u \in H^1(N,|\widetilde g|^{1/2}dx)$, proving the claim. □.

Although in this section (M,N,F,γ,Σ,g) continues to denote a single coating, we will see later that the following definition is also appropriate for the double coating construction.

Let $\widetilde f \in L^2(N,dx)$ be a function such that $\mathrm{supp}\,(\widetilde f)\cap\Sigma = \emptyset$.

Definition 2. *Let* $(M, N, F, \gamma, \Sigma, g)$ *be a coating construction. A measurable function* $\widetilde{u}$ *on* N *is a* finite energy *solution of the Dirichlet problem for the Helmholtz equation on* N*,*

$$(\Delta_{\widetilde{g}} + k^2)\widetilde{u} = \widetilde{f} \quad on\ N, \qquad (19)$$
$$\widetilde{u}|_{\partial N} = \widetilde{h},$$

if

$$\widetilde{u} \in L^2(N, |\widetilde{g}|^{1/2}dx); \qquad (20)$$
$$\widetilde{u}|_{N\setminus\Sigma} \in H^1_{loc}(N \setminus \Sigma, dx); \qquad (21)$$
$$\int_{N\setminus\Sigma} |\widetilde{g}|^{1/2}\widetilde{g}^{ij}\partial_i\widetilde{u}\partial_j\widetilde{u}\,dx < \infty, \qquad (22)$$
$$\widetilde{u}|_{\partial N} = \widetilde{h},;$$

and, for all $\widetilde{\psi} \in C^\infty(N)$ *with* $\widetilde{\psi}|_{\partial N} = 0$*,*

$$\int_N [-(D_{\widetilde{g}}\widetilde{u})\partial_j\widetilde{\psi} + k^2\widetilde{u}\widetilde{\psi}|\widetilde{g}|^{1/2}]dx = \int_N \widetilde{f}(x)\widetilde{\psi}(x)|\widetilde{g}|^{1/2}dx \qquad (23)$$

where the integral on the left hand side of (23) is defined by distribution-test function duality.

Note as before that condition (21) follows from (20), (22).

Cloaking by the single coating of arbitrary active devices, with respect to solutions of the Helmholtz equation at all frequencies, then follows from the following.

Theorem 1. *Let* $u = (u_1, u_2) : M \setminus \gamma \to \mathbb{R}$ *and* $\widetilde{u} : N \setminus \Sigma \to \mathbb{R}$ *be measurable functions such that* $u = \widetilde{u} \circ F$*. Let* $f = (f_1, f_2) : M \setminus \gamma \to \mathbb{R}$ *and* $\widetilde{f} : N \setminus \Sigma \to \mathbb{R}$ *be* L^2 *functions supported away from* γ *and* Σ *such that* $f = \widetilde{f} \circ F$*, and* $\widetilde{h} : \partial N \to \mathbb{R}$*,* $h : \partial M_1 \to \mathbb{R}$ *be such that* $h = \widetilde{h} \circ F_1$*.*

Then the following are equivalent:

1. *The function* $\widetilde{u}$*, considered as a measurable function on* N*, is a finite energy solution to the Helmholtz equation (19) with inhomogeneity* $\widetilde{f}$ *and Dirichlet data* $\widetilde{h}$ *in the sense of Definition 2.*
2. *The function* u *satisfies*

$$(\Delta_g + k^2)u_1 = f_1 \quad on\ M_1, \quad u_1|_{\partial M_1} = h, \qquad (24)$$

and

$$(\Delta_g + k^2)u_2 = f_2 \quad on\ M_2, \quad g^{jk}\nu_j\partial_k u_2|_{\partial M_2} = b, \qquad (25)$$

with $b = 0$*. Here* u_1 *denotes the continuous extension of* u_1 *from* $M_1 \setminus \gamma$ *to* M_1

Moreover, if u solves (24) and (25) with $b \neq 0$, then the function $\widetilde{u} = u \circ F^{-1} : N \setminus \Sigma \to \mathbb{R}$, considered as a measurable function on N, is not a finite energy solution to the Helmholtz equation.

Remarks. (i) It follows that the construction of [GLU1,PSS1] cloaks active devices from detection by unpolarized EM waves at all frequencies.

(ii) Observe that in (24) the right hand side f_1 is zero near γ_1. Thus u_1, considered as a distribution in a neighborhood of γ_1, has an extension on γ_1 that is C^∞ smooth function in a neighborhood of γ_1.

(iii) As noted previously, for the single coating case one may assume that $N_2 = M_2$ and $F|_{M_2}$ is the identity. Thus $\widetilde{u}|_{N_2} = u|_{M_2}$; hence, if $\widetilde{u}$ is a finite energy solution of the Helmholtz equation on N, we see that $u|_{M_2}$ satisfies the Neumann boundary condition on ∂M_2 and thus also $\widetilde{u}|_{N_2}$ automatically has to satisfy the Neumann condition on Σ^-. The Neumann boundary condition that appears on ∂N_2 means that, observed from the inside of the cloaked region N_2, the single coating construction has the effect of creating a virtual sound hard, i.e., perfectly reflecting, surface at Σ. Similarly, we will see later that there are hidden boundary conditions for Maxwell's equations in the presence of the single coating, but they are overdetermined and generally preclude such solutions existing.

Proof. First we proof that Helmholtz on M implies Helmholtz on N.

Let $f \in L^2(M, dx)$ be a function such that $\operatorname{supp}(f) \cap (\gamma \cup \partial M_1 \cup \partial M_2) = \emptyset$. Assume that a function u on M is a classical solution of (24) and (25). Notice that we have required here that u_2 on ∂M_2 satisfies the Neumann boundary condition at ∂M_2.

Again, define $\widetilde{u} = F_* u$ and $\widetilde{f} = f \circ F^{-1}$ on $N \setminus \Sigma$ and extend it, e.g., by setting it equal to zero on Σ. Note that then $\widetilde{f} \in L^2(N, dx)$ is supported away from Σ, and $\widetilde{u} \in L^2(N, |\widetilde{g}|^{1/2} dx)$ satisfies

$$(\Delta_{\widetilde{g}} + k^2)\widetilde{u}_1 = \widetilde{f}_1 = \widetilde{f}|_{N_1} \quad \text{in } N_1, \quad \widetilde{u}|_{\partial N} = \widetilde{h}, \tag{26}$$

and

$$(\Delta_{\widetilde{g}} + k^2)\widetilde{u}_2 = \widetilde{f}_2 = \widetilde{f}|_{N_2} \quad \text{in } N_2. \tag{27}$$

Let $\Sigma(\varepsilon)$ be the ε-neighborhood of Σ with respect to the metric $\widetilde{g}$. Let $\gamma(\varepsilon)$ be the ε-neighborhood of $\gamma \subset M_1$ with respect to the metric g. Let g_{bnd} and $\widetilde{g}_{bnd}$ be the induced metrics on $\partial\gamma(\varepsilon)$ and $\partial\Sigma(\varepsilon)$, correspondingly.

Clearly, the function $\widetilde{u}$ satisfies conditions (20), (21), and (22). By Lemma 2, we have that $\widetilde{u} \in H^1(N, |\widetilde{g}|^{1/2} dx)$, and $D_{\widetilde{g}}\widetilde{u}$ is thus well defined.

Using relations (5) for the normal component and (26), (27), and property (14) of $D_{\widetilde g}u$, we see that, for $\widetilde\psi \in C_0^\infty(N)$,

$$\begin{aligned}
&\int_N [-D_{\widetilde g}(\widetilde u)\partial_j\widetilde\psi + k^2\widetilde u\widetilde\psi|\widetilde g|^{1/2} - \widetilde f\widetilde\psi|\widetilde g|^{1/2}]dx \qquad (28)\\
&= \lim_{\varepsilon\to 0}\int_{N\setminus\Sigma(\varepsilon)} (-\widetilde g^{ij}\,\partial_i\widetilde u\,\partial_j\widetilde\psi + (k^2\widetilde u - \widetilde f)\widetilde\psi)|\widetilde g|^{1/2}dx\\
&= \lim_{\varepsilon\to 0}\Big(\int_{\partial\Sigma(\varepsilon)\cap N_2} + \int_{\partial\Sigma(\varepsilon)\cap N_1}\Big)(-\widetilde g^{ij}\,\nu_j\,\partial_i\widetilde u\,\widetilde\psi)|\widetilde g_{bnd}|^{1/2}dS\\
&= \lim_{\varepsilon\to 0}\int_{\Sigma(\varepsilon)\cap N_2} (-\widetilde g^{ij}\,\nu_j\partial_i\widetilde u_2\widetilde\psi)|\widetilde g_{bnd}|^{1/2}dS + \qquad (29)\\
&\quad + \lim_{\varepsilon\to 0}\int_{\partial\gamma(\varepsilon)} (-g^{ij}\,\partial_i u_1\,\nu_j(\widetilde\psi\circ F))|g_{bnd}|^{1/2}dS \qquad (30)\\
&= 0.
\end{aligned}$$

Indeed, the integral (29) in the right-hand side of this equation tends to 0 due to the boundary condition on Σ^- (25), and boundedness of $\widetilde\psi\circ F$. To analyze the integral (30) observe that, as $\mathrm{supp} f_1\cap\gamma_1 = \emptyset$, u_1 is infinitely smooth near γ_1. Thus all $\partial_i u_1$ and $\widetilde\psi\circ F$ are bounded near γ_1, while the area of $\partial\gamma(\varepsilon)$ is bounded by $C\varepsilon^2$. Hence we see that (23) is valid and thus

$$(\Delta_{\widetilde g} + k^2)\widetilde u = \widetilde f \quad \text{in } N$$

in the sense of the Definition 2.

Summarizing, so far we have proven that a (classical) solution to the Helmholtz equation on M yields, via the pushforward, a finite energy solution to the equation on N.

Next we consider the other direction and prove that the Helmholtz equation on N implies Helmholtz equation on M.

Assume that $\widetilde u$ satisfies Helmholtz equation (19) on $(N,\widetilde g)$ in the sense of Definition 2, with $\widetilde f\in L^2(N)$ supported away from Σ. In particular, $\widetilde u$ is a measurable function in N satisfying (15), (16), and (17).

Let $u = \widetilde u\circ F$ and $f = \widetilde f\circ F$ on $M\setminus\gamma$. Then we have

$$(\Delta_g + k^2)u_1 = f_1 = f|_{M_1\setminus\gamma_1} \quad \text{in } M_1\setminus\gamma_1, \quad u_1|_{\partial M_1} = h \qquad (31)$$

and

$$(\Delta_g + k^2)u_2 = f_2 = f|_{M_2} \quad \text{in } M_2. \qquad (32)$$

By conditions (15), (16), and (17), we have that

$$\begin{aligned}
&|u|^2 \in L^1(M_1\setminus\gamma_1, |g|^{1/2}dx),\\
&g^{jk}(\partial_j u)(\partial_k u) \in L^1(M_1\setminus\gamma_1, |g|^{1/2}dx).
\end{aligned}$$

and thus $u_1 \in H^1(M_1 \setminus \gamma_1, dx)$. As before, we see that

$$(\Delta_g + k^2)u_1 = f_1 \quad \text{in } M_1, \quad u_1|_{\partial M_1} = h, \tag{33}$$

where f_1 is extends to have the value 0 at γ_1 and u_1 is smooth near γ_1.
Let us now consider the boundary condition on M_2. Since $\widetilde{u}$ satisfies (23), we see that for $\widetilde{\psi} \in C_0^\infty(N)$,

$$0 = \int_N [-D_{\widetilde{g}}\widetilde{u}\partial_j\widetilde{\psi} + k^2\widetilde{u}\widetilde{\psi}|\widetilde{g}|^{1/2} - \widetilde{f}\widetilde{\psi}|\widetilde{g})|^{1/2}]dx \tag{34}$$

$$= \lim_{\varepsilon\to 0}\int_{N\setminus\Sigma(\varepsilon)} (-\widetilde{g}^{ij}\,\partial_i\widetilde{u}\,\partial_j\widetilde{\psi} + (k^2\widetilde{u} - \widetilde{f})\widetilde{\psi})\ |\widetilde{g}|^{1/2}dx$$

$$= \lim_{\varepsilon\to 0}(\int_{\partial\Sigma(\varepsilon)\cap N_2} + \int_{\partial\Sigma(\varepsilon)\cap N_1})(-\widetilde{g}^{ij}\,\nu_j\,\partial_i\widetilde{u}\,\widetilde{\psi})|\widetilde{g}_{bnd}|^{1/2}dS$$

$$= \int_{\partial M_2} (-g^{ij}\,\nu_j\partial_i u_2|_{\partial M_2}\psi)|g_{bnd}|^{1/2}dS \tag{35}$$

$$+ \lim_{\varepsilon\to 0}\int_{\partial\gamma(\varepsilon)} (-g^{ij}\,\partial_i u_1\,\nu_j\psi)|g_{bnd}|^{1/2}dS$$

$$= \int_{\partial M_2} (-g^{ij}\,\nu_j\partial_i u|_{\partial M_2}\psi)|g_{bnd}|^{1/2}dS,$$

where $\psi = \widetilde{\psi} \circ F$. Here we use the fact that u_1 is a smooth function, implying that $\partial_i u_1$ is bounded and that $\psi = \widetilde{\psi} \circ F$ is bounded. As $\widetilde{\psi}|_{\partial M_2} \in C^\infty(\partial M_2)$ is arbitrary, this shows that

$$\widetilde{g}^{ij}\,\nu_j\partial_i\widetilde{u}_2|_{\partial M_2} = 0. \tag{36}$$

Thus, we have shown that the function u is a classical solution on M of

$$(\Delta_g + k^2)u_1 = f_1 \quad \text{in } M_1, \quad u_1|_{\partial M_1} = h \tag{37}$$

and

$$(\Delta_g + k^2)u_2 = f_2 \quad \text{in } M_2, \quad g^{jk}\nu_j\partial_k u_2|_{\partial M_2} = 0. \tag{38}$$

This proves the claim, and finishes the proof of Theorem 1. □

3.2.1. Operator theoretic definition of the Helmholtz equation. It is standard in quantum physics that a self-adjoint operator can be defined via the quadratic form corresponding to energy. In the case considered here, the energy associated with the wave operator is defined by the quadratic (Dirichlet) form A,

$$A[\widetilde{u}, \widetilde{u}] := \int_{N\setminus\Sigma} |\widetilde{g}|^{1/2}\widetilde{g}^{ij}\partial_i\widetilde{u}\partial_j\widetilde{u}\,dx, \quad \widetilde{u} \in \mathcal{D}(A) \tag{39}$$

As we deal with the sound-soft boundary ∂N or, more generally, with the source on ∂N of the form $\widetilde u|_{\partial N} = \widetilde h$, the domain $\mathcal{D}(A)$ of the form A should be taken as

$$\mathcal{D}(A) = H^1_0(N, |\widetilde g|^{1/2}dx) \subset X.$$

Thus, by standard techniques of operator theory, e.g., [Ka], the form A defines a positive selfadjoint operator, denoted $A_0 = -\Delta^D_{\widetilde g}$, on $L^2(N, |\widetilde g|^{1/2}dx)$. Next we recall this construction. We say that $\widetilde u \in H^1_0(N, |\widetilde g|^{1/2}dx)$ is in the domain of A_0, $\widetilde u \in \mathcal{D}(A_0)$ if there is an $\widetilde f \in L^2(N, |\widetilde g|^{1/2}dx)$ such that for all $\widetilde v \in H^1_0(N, |\widetilde g|^{1/2}dx)$,

$$A[\widetilde u, \widetilde v] = \int_N \widetilde f \widetilde v\, |\widetilde g|^{1/2}dx. \tag{40}$$

In this case, we define

$$A_0 \widetilde u = \widetilde f.$$

Proposition 1. *Assume that $-k^2$ is not in the spectrum of $\Delta^D_{\widetilde g}$. Then $\widetilde u$ is a finite energy solution to*

$$(\Delta_{\widetilde g} + k^2)\widetilde u = \widetilde f, \quad \widetilde u|_{\partial N} = \widetilde h \in H^{1/2}(\partial N)$$

if and only if

$$\widetilde u = E\widetilde h + (\Delta^D_{\widetilde g} + k^2)^{-1}(\widetilde f - (\Delta_{\widetilde g} + k^2)E\widetilde h), \tag{41}$$

where $E\widetilde h$ is an $H^1(N, dx)$-extension of $\widetilde h$ to N satisfying supp$\,(E\widetilde h) \subset \partial N \cup N_1$.

Proof. First we show that if $\widetilde u$ satisfies the conditions of Definition 2 then it satisfies (41). As $\widetilde \psi \in C^\infty(N)$, $\widetilde\psi|_{\partial N} = 0$, imply that $\widetilde\psi \in H^1_0(N, |g|^{1/2}dx)$, we see by (23) that $\widetilde u - E\widetilde h$ satisfies

$$\int_N (-D^j_{\widetilde g}(\widetilde u - E\widetilde h)\,\partial_j \widetilde v + k^2(\widetilde u - E\widetilde h)\widetilde v)\,dx = \int_{N\setminus\Sigma} |\widetilde g|^{1/2}(\widetilde f - (\Delta_{\widetilde g} + k^2)E\widetilde h)\widetilde v\,dx,$$

for any $\widetilde v \in C^\infty_0(N)$. By (14) and (22), this implies

$$\begin{aligned}&\int_{N\setminus\Sigma} |\widetilde g|^{1/2}\left(-\widetilde g^{ij}\partial_i(\widetilde u - E\widetilde h)\partial_j\widetilde v + k^2(\widetilde u - E\widetilde h)\widetilde v\right)\,dx \\ &= \int_{N\setminus\Sigma} |\widetilde g|^{1/2}(\widetilde f - (\Delta_{\widetilde g} + k^2)E\widetilde h)\widetilde v\,dx,\end{aligned} \tag{42}$$

for any $\widetilde v \in C^\infty_0(N)$. We need to show that (42) is valid for all $\widetilde v \in H^1_0(N, |g|^{1/2}dx)$. Observe that

$$\int_{N\setminus\Sigma} |\widetilde g|^{1/2}\left(-\widetilde g^{ij}\partial_i(E\widetilde h)\partial_j\widetilde v + k^2(E\widetilde h)\widetilde v\right)\,dx = \int_{N\setminus\Sigma} |\widetilde g|^{1/2}((\Delta_{\widetilde g} + k^2)E\widetilde h)\widetilde v\,dx,$$

where we use that $\mathrm{supp}(E\widetilde{h}) \subset \partial N \cup N_1$ and $\widetilde{v}|_{\partial N} = 0$. Thus, it remains to show that

$$\int_{N\setminus\Sigma} |\widetilde{g}|^{1/2} \left(-\widetilde{g}^{ij}\partial_i\widetilde{u}\partial_j\widetilde{v} + k^2\widetilde{u}\widetilde{v}\right) dx = \int_{N\setminus\Sigma} |\widetilde{g}|^{1/2}\widetilde{f}\widetilde{v}\, dx \tag{43}$$

for $\widetilde{v} \in H_0^1(N, |g|^{1/2}dx)$. Clearly, to show this it is enough to show that

$$\lim_{\varepsilon\to 0}\int_{N\setminus\Sigma_1(\varepsilon)} |\widetilde{g}|^{1/2} \left(-\widetilde{g}^{ij}\partial_i\widetilde{u}\partial_j\widetilde{v} + k^2\widetilde{u}\widetilde{v} - \widetilde{f}\widetilde{v}\right) dx = 0. \tag{44}$$

where $\Sigma_1(\varepsilon) = N_1 \cap \Sigma(\varepsilon)$.

Next we argue analogously to the reasoning that led to equation (28). Let $v = \widetilde{v}\circ F$, $f = \widetilde{f}\circ F$, and $u = \widetilde{u}\circ F$ in $M\setminus\gamma$. To clarify notations, denote $u_1 = u|_{M_1}$, $u_2 = u|_{M_2}$, $v_1 = v|_{M_1}$, $v_2 = v|_{M_2}$, and $f_1 = f|_{M_1}$, $f_2 = f|_{M_2}$. Then, by Proposition 1,

$$(\Delta_g + k^2)u_1 = f_1, \quad \text{in } M_1, \tag{45}$$
$$(\Delta_g + k^2)u_2 = f_2, \quad \text{in } M_2, \tag{46}$$
$$\partial_\nu u_2|_{\partial M_2} = 0, \tag{47}$$

and we see that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{N\setminus\Sigma_1(\varepsilon)} |\widetilde{g}|^{1/2} \left(-\widetilde{g}^{ij}\partial_i\widetilde{u}\partial_j\widetilde{v} + k^2\widetilde{u}\widetilde{v} - \widetilde{f}\widetilde{v}\right) dx \\
&= \lim_{\varepsilon\to 0}\int_{(M_1\setminus\gamma(\varepsilon))\cup M_2} |g|^{1/2} \left(-g^{ij}\partial_i u\partial_j v + k^2 uv - fv\right) dx \\
&= \int_{\partial\gamma(\varepsilon)} (-g^{ij}\nu_j\,\partial_i(u)\,v)|g_{bnd}|^{1/2}dS + \int_{\partial M_2} (-g^{ij}\nu_j\,\partial_i(u)\,v)|g_{bnd}|^{1/2}dS.
\end{aligned}$$

By (47), we have that

$$\int_{\partial M_2} (-g^{ij}\nu_j\,\partial_i(u)\,v)|\widetilde{g}_{bnd}|^{1/2}dS = 0. \tag{48}$$

Next we consider

$$I_1(\varepsilon) = \int_{\partial\gamma(\varepsilon)\cap M_1} (-g^{ij}\nu_j\,\partial_i(u)\,v)|\widetilde{g}_{bnd}|^{1/2}dS.$$

Note that $\lim_{\varepsilon\to 0} I_1(\varepsilon)$ exists as the limits (44) and (48) exists.

As $\mathrm{supp}\,(f)\cap\gamma = \emptyset$, we see that u_1 is smooth function near γ. Moreover, as $\widetilde{v} \in X$, we observe that $v_1 \in H^1(M_1\setminus\gamma, dx)$, and so v_1 can be extended to $v_1 \in H^1(M_1, dx)$. Hence, by the Sobolev embedding theorem, $v_1 \in L^6(M_1, dx)$. This allows us to deduce that

$$\liminf_{\varepsilon\to 0}\varepsilon^{-3/2}\int_{\partial\gamma(\varepsilon)} |v_1| \quad dS = 0. \tag{49}$$

Indeed,

$$\int_0^\varepsilon \left(\int_{\partial\gamma(r)} |v_1| dS(x)\right) dr = \int_{\gamma(\varepsilon)} |v_1| dx$$
$$\leq \left(\int_{\gamma(\varepsilon)} |v_1|^6 dx\right)^{1/6} \left(\int_{\gamma(\varepsilon)} dx\right)^{5/6} = o(\varepsilon^{5/2}).$$

Clearly, this inequality implies (49). Thus using boundedness of $g^{ij}\nu_j\partial_i u$ we see that

$$\liminf_{\varepsilon\to 0} \int_{\partial\gamma(\varepsilon)} (-g^{ij}\nu_j\,\partial_i(u)\,v|g_{bnd}|^{1/2})dS = 0.$$

As $\lim_{\varepsilon\to 0} I_1(\varepsilon)$ exists, this implies $\lim_{\varepsilon\to 0} I_1(\varepsilon) = 0$. As $\widetilde{u}|_{\partial N} = \widetilde{h}$ by Definition 2 we have shown that Definition 2 implies (41).

Next, consider the case when $\widetilde{u}$ satisfies (41). Since $\widetilde{u} \in X$, we see by (14) that

$$\int_N D^j_{\widetilde{g}}(\widetilde{u})\partial_j\widetilde{v}\,dx = \int_{N\setminus\Sigma} |\widetilde{g}|^{1/2}\widetilde{g}^{ij}\partial_i\widetilde{u}\partial_j\widetilde{v}\,dx \tag{50}$$

for all $\widetilde{v} \in C_0^\infty(N)$. Thus, by (41) we see that (43) is valid for $\widetilde{v} \in C_0^\infty(N)$, which implies condition (22). The other conditions in Definition 2 follow easily from (41). □

3.3. Helmholtz for the double coating. We now examine solutions to the Helmholtz equation in the presence of the double coating; we will establish full-wave invisibility at all nonzero frequencies. Unlike for the single coating, for the double coating no extra boundary conditions appear at Σ. Otherwise, the reasoning here parallels that in §3.2.

Throughout this section, $(M, N, F, \gamma, \Sigma, g)$ is a double coating construction.

3.3.1. Weak solutions for the double coating.. Suppose that $k \geq 0$ and $\widetilde{f} \in L^2(N, |\widetilde{g}|^{1/2}dx)$. We use the same notion of weak solution as for the single coating, saying that $\widetilde{u}$ is a *finite energy* solution of

$$(\Delta_{\widetilde{g}} + k^2)\widetilde{u} = \widetilde{f} \quad \text{in } N, \quad \widetilde{u}|_{\partial N} = \widetilde{h} \tag{51}$$

if $\widetilde{u}$ is a solution of the Dirichlet problem in the sense of Definition 2.

We start with analogues of the space $H^1(N, |\widetilde{g}|^{1/2}dx)$, and Lemmas 1 and 2. To this end define, for $\widetilde{\phi} \in C^\infty(\overline{N})$,

$$\|\widetilde{\phi}\|_Y^2 := \int_N (|\widetilde{g}|^{1/2}\widetilde{g}^{ij}\partial_i\widetilde{\phi}\partial_j\widetilde{\phi} + |\widetilde{g}|^{1/2}|\widetilde{\phi}|^2)\,dx.$$

Let

$$H^1(N, |\widetilde{g}|^{1/2}dx) = Y := \mathrm{cl}_Y(C^\infty(\overline{N}))$$

be the completion of $C^\infty(\overline{N})$ with respect to the norm $\|\cdot\|_Y$. Note that $H^1(N, |\widetilde{g}|^{1/2}dx) \subset L^2(N, |\widetilde{g}|^{1/2}dx)$, so we can consider its elements as measurable functions in N.

Lemma 3. *The map*

$$\phi \longrightarrow D_{\widetilde{g}}\widetilde{\phi} = (D^j_{\widetilde{g}}\widetilde{\phi})_{j=1}^3 = (|\widetilde{g}|^{1/2}\widetilde{g}^{ij}\partial_i\widetilde{\phi})_{j=1}^3, \quad \phi \in C^\infty(\overline{N}),$$

has a bounded extension

$$D_{\widetilde{g}} : H^1(N, |\widetilde{g}|^{1/2}dx) \to \mathcal{M}(N; \mathbb{R}^3),$$

where $\mathcal{M}(N;\mathbb{R}^3)$ denotes the space of $\mathbb{R}^3$-valued signed Borel measures on N. Moreover, for $\widetilde{u} \in Y$, we have

$$(D_{\widetilde{g}}\widetilde{u})(\Sigma) = 0. \tag{52}$$

If $\widetilde{u}$ is a measurable function on N such that

$$\widetilde{u} \in L^2(N, |\widetilde{g}|^{1/2}dx), \tag{53}$$

$$\widetilde{u}|_{N\setminus\Sigma} \in H^1_{loc}(N \setminus \Sigma, dx), \text{ and} \tag{54}$$

$$\int_{N\setminus\Sigma} |\widetilde{g}|^{1/2}\widetilde{g}^{ij}\partial_i\widetilde{u}\partial_j\widetilde{u}\, dx < \infty, \tag{55}$$

then $\widetilde{u} \in H^1(N, |\widetilde{g}|^{1/2}dx)$.

Proof. The proof here is essentially the same as of Lemmas 1 and 2. The only difference is that, as described in §2, the map

$$F : M \setminus \gamma \to N \setminus \Sigma$$

now consists of two maps,

$$F_i : M_i \setminus \gamma \to N_i, \quad i = 1, 2,$$

having similar structure to each other, namely that of the map F_1 in the single coating construction. (Recall that for the double coating construction, $\gamma_1 := \gamma \cap M_1$ is a point $O \in M_1$ and $\gamma_2 := \gamma \cap M_2$ a point $NP \in M_2$.)

Therefore, when proving that $\widetilde{u}$ satisfying (53)–(55) is in $H^1(N, |\widetilde{g}|^{1/2}dx)$, we can use the fact that, in this case, both $(1 - \chi_{N_1})\widetilde{u}$ and $(1 - \chi_{N_2})\widetilde{u}$ satisfy (53)–(55) and carry out the proof for each of them as for the $(1 - \chi_{N_2})\widetilde{u}$ term in the proof of Lemma 2.

Invisibility of active devices in the presence of the double coating with respect to the Helmholtz equation at all frequencies then follows from

Theorem 2. *Let $u = (u_1, u_2) : M \setminus \gamma \to \mathbb{R}$ and $\widetilde{u} : N \setminus \Sigma \to \mathbb{R}$ be measurable functions such that $u = \widetilde{u} \circ F$. Let $f = (f_1, f_2) : M \setminus \gamma \to \mathbb{R}$ and $\widetilde{f} : N \setminus \Sigma \to \mathbb{R}$ be L^2 functions supported away from γ and Σ such that $f = \widetilde{f} \circ F$. Then the following are equivalent:*

1. *The function $\widetilde{u}$, considered as a measurable function on N, is a finite energy solution to the Helmholtz equation (51) with inhomogeneity $\widetilde{f}$ and Dirichlet data $\widetilde{h}$ in the sense of Definition 2.*

2. We have

$$(\Delta_g + k^2)u_1 = f_1 \quad \text{on } M_1, \quad u|_{\partial M} = h := \widetilde{h} \circ F \tag{56}$$

and

$$(\Delta_g + k^2)u_2 = f_2 \quad \text{on } M_2. \tag{57}$$

Proof As in the proof of Theorem 1, we first prove that Helmholtz on M implies Helmholtz on N.

Let $f \in L^2(M, dx)$ be a function such that $\operatorname{supp}(f) \cap (\gamma \cup \partial M_1 \cup \partial M_2) = \emptyset$. Assume that a function $u = (u_1, u_2)$ on M is a classical solution of (56) and (57). Define $\widetilde{u} = F_* u$ and $\widetilde{f} = f \circ F^{-1}$ on $N \setminus \Sigma$ and extend it, e.g., by setting it equal to zero on Σ. Note that then $\widetilde{f} \in L^2(N, dx)$ is supported away of Σ. Then $\widetilde{u} \in L^2(N, |\widetilde{g}|^{1/2} dx)$ satisfies

$$(\Delta_{\widetilde{g}} + k^2)\widetilde{u}_1 = \widetilde{f}_1 = \widetilde{f}|_{N_1} \quad \text{in } N_1, \quad \widetilde{u}|_{\partial N} = \widetilde{h}, \tag{58}$$

and

$$(\Delta_{\widetilde{g}} + k^2)\widetilde{u}_2 = \widetilde{f}_2 = \widetilde{f}|_{N_2} \quad \text{in } N_2. \tag{59}$$

Let $\Sigma(\varepsilon)$ be the ε-neighborhood of Σ with respect to the metric $\widetilde{g}$. Let $\gamma_1(\varepsilon)$ be the ε-neighborhood of $\gamma_1 = \{0\} \subset M_1$ with respect to the metric g. Let $\gamma_2(\varepsilon)$ be the ε-neighborhood of $\gamma_2 = \{NP\} \subset M_2$ with respect to the metric g. Let g_{bnd} and $\widetilde{g}_{bnd}$ be the induced metrics on $\partial\gamma(\varepsilon)$ and $\partial\Sigma(\varepsilon)$, correspondingly.

Clearly, the function $\widetilde{u}$ satisfies conditions (20), (21), and (22). By Lemma 3, we have that $\widetilde{u} \in H^1(N, |\widetilde{g}|^{1/2} dx)$, and $D_{\widetilde{g}}\widetilde{u}$ is thus well defined.

Using relations (5), (6) in M_1 and (8) in M_2, it follows from (58), (59) that for $\widetilde{\psi} \in C_0^\infty(N)$,

$$\begin{aligned}
&\int_N [-(D_{\widetilde{g}})\widetilde{u}\partial_j\widetilde{\psi} + k^2\widetilde{u}\widetilde{\psi}|\widetilde{g}|^{1/2} - \widetilde{f}\widetilde{\psi}|\widetilde{g}|^{1/2}]dx \qquad (60)\\
&= \lim_{\varepsilon\to 0} \int_{N\setminus\Sigma(\varepsilon)} (-\widetilde{g}^{ij}\,\partial_i\widetilde{u}\,\partial_j\widetilde{\psi} + (k^2\widetilde{u} + \widetilde{f})\widetilde{\psi})|\widetilde{g}|^{1/2}dx\\
&= \lim_{\varepsilon\to 0}\Big(\int_{\partial\Sigma(\varepsilon)\cap N_2} + \int_{\partial\Sigma(\varepsilon)\cap N_1}\Big)(-\widetilde{g}^{ij}\,\nu_j\,\partial_i\widetilde{u}\,\widetilde{\psi})|\widetilde{g}_{bnd}|^{1/2}dS\\
&= \lim_{\varepsilon\to 0}\int_{\partial\gamma_1(\varepsilon)} (-g^{ij}\,\partial_i u_1\,\nu_j(\widetilde{\psi}\circ F))|g_{bnd}|^{1/2}dS\\
&\quad + \lim_{\varepsilon\to 0}\int_{\partial\gamma_2(\varepsilon)} (-g^{ij}\,\partial_i u_2\,\nu_j(\widetilde{\psi}\circ F))|g_{bnd}|^{1/2}dS\\
&= 0.
\end{aligned}$$

Indeed, both terms in the right-hand side of (60) tend to 0 by the same arguments as the term $\int_{\partial\gamma(\varepsilon)}(-g^{ij}\,\nu_j\,\partial_i u_1\,(\widetilde{\psi}\circ F))|g_{bnd}|^{1/2}dS$ in (28). Hence we see that (23) is valid and thus

$$(\Delta_{\widetilde{g}} + k^2)\widetilde{u} = \widetilde{f} \quad \text{in } N$$

in the sense of the Definition 2.

So far, we have proven that a (classical) solution to the Helmholtz equation on M yields a finite energy solution to the equation on N. Next, we prove the converse, i.e., that the Helmholtz equation on N implies Helmholtz equation on M.

Assume that $\widetilde{u}$ satisfies Helmholtz equation (19) on $(N, \widetilde{g})$ in the sense of Definition 2, with $\widetilde{f} \in L^2(N)$ supported away from Σ. In particular, $\widetilde{u}$ is a measurable function in N satisfying (15), (16), and (17).

Let $u = \widetilde{u} \circ F$ and $f = \widetilde{f} \circ F$ on $M \setminus \gamma$. Then we have

$$(\Delta_g + k^2)u_1 = f_1 = f|_{M_1 \setminus \gamma_1} \quad \text{in } M_1 \setminus \gamma_1, \quad u_1|_{\partial M_1} = h \tag{61}$$

and

$$(\Delta_g + k^2)u_2 = f_2 = f|_{M_2 \setminus \gamma_2} \quad \text{in } M_2 \setminus \gamma_2. \tag{62}$$

By conditions (15), (16), and (17), we have that

$$|u_i|^2 \in L^1(M_i \setminus \gamma_i, |g|^{1/2} dx),$$
$$g_i^{jk} (\partial_j u_i)(\partial_k u_i) \in L^1(M_i \setminus \gamma_i, |g|^{1/2} dx), i = 1, 2.$$

Thus $u_i \in H^1(M_i \setminus \gamma_i, dx)$. As before, we see that then

$$\begin{aligned} &(\Delta_g + k^2)u_1 = f_1 \quad \text{in } M_1, \quad u_1|_{\partial M_1} = h, \\ &(\Delta_g + k^2)u_2 = f_2 \quad \text{in } M_2, \end{aligned} \tag{63}$$

where f_i is extended to have the value 0 at γ_i and u_i are smooth near γ_i.

Since $\widetilde{u}$ satisfies (23), we see that for $\widetilde{\psi} \in C_0^\infty(N)$,

$$\begin{aligned}
0 &= \int_N [-D_{\widetilde{g}} \widetilde{u} \partial_j \widetilde{\psi} + k^2 \widetilde{u} \widetilde{\psi} |\widetilde{g}|^{1/2} - \widetilde{f} \widetilde{\psi} |\widetilde{g}|)|^{1/2}] dx \\
&= \lim_{\varepsilon \to 0} \int_{N \setminus \Sigma(\varepsilon)} (-\widetilde{g}^{ij} \, \partial_i \widetilde{u} \, \partial_j \widetilde{\psi} + (k^2 \widetilde{u} + \widetilde{f}) \widetilde{\psi}) \; |\widetilde{g}|^{1/2} dx \\
&= \lim_{\varepsilon \to 0} \left(\int_{\partial \Sigma(\varepsilon) \cap N_2} + \int_{\partial \Sigma(\varepsilon) \cap N_1} \right) (-\widetilde{g}^{ij} \, \partial_i \widetilde{u}|_{\partial \Sigma(\varepsilon)} \, \nu_j \widetilde{\psi}) |\widetilde{g}_{bnd}|^{1/2} dS(x) \\
&= \lim_{\varepsilon \to 0} \int_{\partial \gamma_1(\varepsilon)} (-g^{ij} \, \partial_i u_1 \, \nu_j \psi) |g_{bnd}|^{1/2} dS(x) \\
&\quad + \lim_{\varepsilon \to 0} \int_{\partial \gamma_2(\varepsilon)} (-g_s^{ij} \, \partial_i u_2 \, \nu_j \psi) |g_{bnd}|^{1/2} dS(x) \\
&= 0,
\end{aligned}$$

where $\psi = \widetilde{\psi} \circ F$. Here as in the proof of Proposition 1, we use the fact that u_1 is smooth function implying that $\partial_i u_1$ is bounded.

Thus, we have shown that the function u is a classical solution on M of

$$(\Delta_g + k^2)u_1 = f_1 \quad \text{in } M_1, \quad u_1|_{\partial M_1} = h \tag{64}$$

and

$$(\Delta_g + k^2)u_2 = f_2 \quad \text{in } M_2. \tag{65}$$

This proves the claim. □

Next we prove a result that is not necessary for the proof but gives, in the case of the double coating, an alternative treatment of the distribution $D_{\widetilde{g}}\widetilde{u}$, simpler than before.

Lemma 4. *In the double coating construction, the term*

$$|\widetilde{g}|^{1/2}\widetilde{g}^{ij}\partial_i\widetilde{u} \in \mathcal{D}'(N, dx), \tag{66}$$

appearing in Definition 2 as $D_{\widetilde{g}}\widetilde{u}$ is well-defined as a sum of products of Sobolev distributions and Lipschitz functions.

Proof. The problem we need to consider is here is that $L^2(N, |\widetilde{g}|^{1/2}dx)$ contains functions that are not locally integrable with respect to measure dx and thus we do not immediately see that distribution derivatives $\partial_j\widetilde{u}$ in N are well defined. We deal with this by applying condition (22). To do this, let $u = \widetilde{u}\circ F : M\setminus\gamma \to \mathbb{R}$. Using (20), (21), (22) and changing variables in the integration, one sees that

$$\int_{M\setminus\gamma} |g|^{1/2}g^{ij}(\partial_i u)\partial_j u)\, dx < \infty.$$

As g is bounded from above and below, this implies that $u \in H^1(M\setminus\gamma, dx) \subset L^6(M\setminus\gamma), dx)$. Furthermore, changing variables again implies that

$$\int_{N\setminus\Sigma} |\widetilde{g}|^{1/2}|\widetilde{u}|^6\, dx < \infty,$$

so that $\widetilde{u} \in L^6(N, \det(\widetilde{g})^{1/2}dx)$. Now in the boundary normal coordinates (ω, τ) near Σ, $\tau(x) = \mathrm{dist}_{\mathbb{R}^3}(x, \Sigma)$, we have

$$\tau^{-2}|\widetilde{g}|^{1/2} \in [c_1, c_2], \quad c_1, c_2 > 0,$$

and thus

$$\begin{aligned}\int_N |\widetilde{u}|\, dx &= \int_N |\widetilde{u}|\tau(x)^{1/3}\tau(x)^{-1/3}\, dx \\ &\le \|\widetilde{u}\,\tau^{1/3}\|_{L^6(N,dx)}\|\tau(x)^{-1/3}\|_{L^{6/5}(N,dx)} \\ &\le \|\widetilde{u}\|_{L^6(N,\tau^2dx)}\|\tau(x)^{-2/5}\|_{L^1(N,dx)} \\ &\le C\|\widetilde{u}\|_{L^6(N,|\widetilde{g}|^{1/2}dx)} < \infty,\end{aligned}$$

cf. the discussion at the end of §§3.2.3. A similar computation shows that $\widetilde{u} \in L^p(N, dx)$ for some $p > 1$, and thus $\partial_j\widetilde{u} \in W^{-1,p}(N, dx)$. As is shown at the end of §2 that

$$|\widetilde{g}|^{1/2}\widetilde{g}^{jk} \in C^{0,1}(N), \tag{67}$$

multiplication by $|\widetilde{g}|^{1/2}\widetilde{g}^{jk}$ maps $W^{1,p'} \longrightarrow W^{1,p'}$ and thus, by duality,

$$|\widetilde{g}|^{1/2}\widetilde{g}^{jk}\partial_j\widetilde{u} \in W^{-1,p}(N, dx),$$

i.e., the distribution (66) is defined as a sum of products of Lipschitz functions and $W^{-1,p}$-distributions □

3.4. Coating with a lining: a physical surface . In the previous sections we have considered the Helmholtz equation in a domain $N \subset \mathbb{R}^3$, equipped with a metric $\widetilde{g}$ that is singular at a surface Σ. Later, for Maxwell's equations, we will need to consider Σ as a "physical" surface, i.e., an obstacle on which we have to impose a boundary condition. To motivate these constructions, we consider next, for the Helmholtz equation, what happens when we have such a physical surface at Σ. More precisely, we consider the Helmholtz equation in the domain $N \setminus \Sigma = N_1 \cup N_2$ where, on the both sides of the boundary of Σ, that is, on $\Sigma_+ = \partial N_1 \setminus \partial N$ and on $\Sigma_- = \partial N_2$, we impose a degenerate boundary condition of Neumann type. In physical terms, this corresponds to having a material, sound hard surface located at Σ, separating space into two open components, N_1 and N_2. Although we will not need this, it can in fact be shown that $\widetilde{u}$ is a solution in the sense of Def. 3 iff it is in the sense of Def. 2.

3.4.1. Weak solutions for the double coating with Neumann boundary conditions. In the following, we consider a double coating (M, N, F, Σ, g). Suppose that $k \geq 0$ and $\widetilde{f} \in L^2(N, |\widetilde{g}|^{1/2}dx)$.

Definition 3. *We say that $\widetilde{u}$ is a* finite energy *solution of the boundary value problem with degenerate Neumann boundary conditions at Σ,*

$$(\Delta_{\widetilde{g}} + k^2)\widetilde{u} = \widetilde{f} \quad \text{in } N \setminus \Sigma, \tag{68}$$

$$\widetilde{u}|_{\partial N} = \widetilde{h} \tag{69}$$

$$|\widetilde{g}|^{1/2}\partial_\nu \widetilde{u}|_{\Sigma_+} = 0, \quad |\widetilde{g}|^{1/2}\partial_\nu \widetilde{u}|_{\Sigma_-} = 0, \tag{70}$$

if $\widetilde{u}$ is a measurable function in $N \setminus \Sigma$ such that

$$\widetilde{u} \in L^2(N \setminus \Sigma, |\widetilde{g}|^{1/2}dx); \tag{71}$$

$$\partial_j \widetilde{u} \in H^1_{loc}(N \setminus \Sigma, dx); \tag{72}$$

$$\int_{N\setminus\Sigma} |\widetilde{g}|^{1/2}\widetilde{g}^{ij}\partial_i\widetilde{u}\partial_j\widetilde{u}\, dx < \infty; \tag{73}$$

$$(\Delta_{\widetilde{g}} + k^2)\widetilde{u} = \widetilde{f} \text{ in some neighborhood of } \partial N, \tag{74}$$

$$\widetilde{u}|_{\partial N} = \widetilde{h};$$

and finally,

$$\int_{N\setminus\Sigma} \left(-\widetilde{g}^{ij}\,\partial_i\widetilde{u}\,\partial_j\widetilde{\psi} + (k^2 - \widetilde{f})\widetilde{u}\widetilde{\psi}\right)|\widetilde{g}|^{1/2}dx = 0 \tag{75}$$

for all

$$\widetilde{\psi} = \begin{cases} \widetilde{\psi}_1(x),\, x \in N_1, \\ \widetilde{\psi}_2(x),\, x \in N_2, \end{cases}$$

with $\widetilde{\psi}_1 \in C^\infty(\overline{N}_1)$ vanishing near the exterior boundary $\partial N = \partial N_1 \setminus \Sigma$ and $\widetilde{\psi}_2 \in C^\infty(\overline{N}_2)$.

Invisibility for the double coating with a physical surface at Σ, with respect to the Helmholtz equation at all frequencies, is a consequence of the following analogue of Theorem 2 :

Theorem 3. *Let $u = (u_1, u_2) : M \setminus \gamma \to \mathbb{R}$ and $\widetilde{u} : N \setminus \Sigma \to \mathbb{R}$ be measurable functions such that $u = \widetilde{u} \circ F$. Let $f = (f_1, f_2) : M \setminus \gamma \to \mathbb{R}$ and $\widetilde{f} : N \setminus \Sigma \to \mathbb{R}$ be L^2 functions supported away from Σ and γ such that $f = \widetilde{f} \circ F$, and $\widetilde{h} : \partial N \to \mathbb{R}$, $h : \partial M_1 \to \mathbb{R}$ be such that $h = \widetilde{h} \circ F_1$..*

Then the following are equivalent:

1. *The function $\widetilde{u}$, considered as a measurable function on $N \setminus \Sigma$, is a finite energy solution of (68) with Neumann boundary conditions at Σ and inhomogeneity $\widetilde{f}$ in the sense of Definition 3.*
2. *The function u satisfies*

$$(\Delta_g + k^2)u_1 = f_1 \quad \text{on } M_1, \quad u|_{\partial M_1} = h := \widetilde{h} \circ F \tag{76}$$

and

$$(\Delta_g + k^2)u_2 = f_2 \quad \text{on } M_2. \tag{77}$$

Proof. The proof is identical to that of Theorem 2. □

Remark. Let $\widetilde{g}$ be a singular metric on N corresponding to a double coating. The implication of Theorems 2 and 3 is that the solutions $\widetilde{u}$ in $N \setminus \Sigma$ coincide in the following cases:

1. We have the metric $\widetilde{g}$ on N, singular at the virtual surface Σ.
2. We have the metric $\widetilde{g}$ on $N \setminus \Sigma$ and a sound hard physical surface at Σ, in the sense of Definition 3.

Similar results can be proven when the metric $\widetilde{g}$ in N corresponds to a single coating.

4. Maxwell's equations

4.1. Geometry and definitions. Let us start with a general Riemannian manifold (M, g), possibly with a non-empty boundary, and consider how to define Maxwell's equations on M. We follow the treatment in [KLS], using, however, slightly different notation.

Using the metric g, we define a permittivity and permeability by setting

$$\varepsilon^{jk} = \mu^{jk} = |g|^{1/2} g^{jk}, \quad \text{on } M.$$

Although defined with respect to local coordinates, ε and μ are in fact invariantly defined, and transform as a product of a $(+1)-$density and a contravariant symmetric two-tensor.

Remark. In $\mathbb{R}^3$ with the Euclidean metric $g_{jk} = \delta_{jk}$, we have $\varepsilon^{jk} = \mu^{jk} = \delta^{jk}$. If we would like to define a generalization of isotropic media on a general Riemannian manifold, it would be as

$$\varepsilon^{jk} = \alpha(x)^{-1}|g|^{1/2}g^{jk},$$
$$\mu^{jk} = \alpha(x)|g|^{1/2}g^{jk},$$

on M, where $\alpha(x)$ is a positive scalar function. However, in the following we assume for simplicity that $\alpha = 1$.

In the following we consider the electric and magnetic fields, E and H, as differential 1-forms, given in some local coordinates by

$$E = E_j dx^j, \quad H = H_j dx^j,$$

and J, the internal current, as a 2-form.

Now consider the time harmonic Maxwell's equations on (M, g) at frequency k. They can be written invariantly as

$$dE = ik *_g H, \quad dH = -ik *_g E + J \tag{78}$$

where $*_g : C^\infty(\Omega^j M) \longrightarrow C^\infty(\Omega^{3-j} M)$ denotes the Hodge-operator on j-forms, $0 \le j \le 3$, given on 1-forms by

$$\begin{aligned} *_g(E_j dx^j) &= \frac{1}{2}|g|^{1/2}g^{jl}E_j\, s_{lpq}dx^p \wedge dx^q \\ &= \frac{1}{2}\varepsilon^{jl}E_j\, s_{lpq}dx^p \wedge dx^q \end{aligned} \tag{79}$$

where s_{lpq} is the Levi-Civita permutation symbol, $s_{lpq} = 1$ if (l, p, q) even permutation of $(1, 2, 3)$, $s_{lpq} = -1$ if (l, p, q) odd permutation of $(1, 2, 3)$, and zero otherwise. Thus

$$*_g(E_j dx^j) = (\varepsilon^{j3}E_j)\, dx^1 \wedge dx^2 - (\varepsilon^{j2}E_j)\, dx^1 \wedge dx^3 + (\varepsilon^{j1}E_j)\, dx^2 \wedge dx^3.$$

Next, we want to write these equations in arbitrary coordinates so that they resemble the traditional Maxwell equations. The idea is that we want to have expressions which specialize, in the case of the Euclidean metric on $\mathbb{R}^3$, to expressions involving *curl* and the matrices ε^{jk} and μ^{jk}. To write equations in such a form, let us introduce, for $H = H_j dx^j$, the notation

$$(\mathrm{curl}\, H)^l = s^{lpq}\frac{\partial}{\partial x^p}H_q \quad .$$

The exterior derivative

$$d(H_j dx^j) = \frac{\partial H_j}{\partial x^k}\, dx^k \wedge dx^j$$

may then be written as

$$dH = \frac{1}{2}(\mathrm{curl}\, H)^l\, s_{lpq}dx^p \wedge dx^q. \tag{80}$$

Combining (79) and (80) we see that Maxwell equations (78) can be written as

$$(\mathrm{curl}\, E)^l = ik\, \mu^{jl} H_j,$$
$$(\mathrm{curl}\, H)^l = -ik\, \varepsilon^{jl} E_j + J^l.$$

Below, we denote also

$$(\nabla \times E)^j = (\mathrm{curl}\, E)^j,$$

and usually denote the standard volume element of $\mathbb{R}^3$ by $dV_0(x)$.

There are many boundary conditions that makes the boundary value problem for Maxwell's equations on a domain well posed. For example:

- Electric boundary condition:

$$\nu \times E|_{\partial M} = 0,$$

  where ν is the Euclidean normal vector of ∂M. Physically this corresponds to lining the boundary with a perfectly conducting material.
- Magnetic boundary condition:

$$\nu \times H|_{\partial M} = 0,$$

  where ν is the Euclidean normal vector of ∂M. In other words, the tangential components of the magnetic field vanish.
- Soft and hard surface (SHS) boundary condition [HLS, Ki1, Ki2, Li]:

$$\zeta \cdot E|_{\partial M} = 0 \quad \text{and} \quad \zeta \cdot H|_{\partial M} = 0$$

  where $\zeta = \zeta(x)$ is a tangential vector field on ∂M, that is, $\zeta \times \nu = 0$. In other words, the part of the tangential component of the electric field E that is parallel to ζ vanishes, and the same is true for the magnetic field H. This can be physically realized by having a surface with thin parallel gratings [HLS, Ki1, Ki2, Li].

4.2. Definition of solutions of Maxwell equations. Assume that $k \in \mathbb{R} \setminus \{0\}$. We will define finite energy solutions for Maxwell's equations in the same way for both the single and double coatings.

Let $(M, N, F, \gamma, \Sigma, g)$ be either a single or double coating construction, as in §2, denoting as usual $\widetilde{g} = F_* g$ on $N \setminus \Sigma$. On M and $N \setminus \Sigma$, we then define permittivity and permeability tensors by setting

$$\varepsilon^{jk} = \mu^{jk} = |g|^{1/2} g^{jk}, \quad \text{on } M,$$
$$\widetilde{\varepsilon}^{jk} = \widetilde{\mu}^{jk} = |\widetilde{g}|^{1/2} \widetilde{g}^{jk}, \quad \text{on } N \setminus \Sigma.$$

Let J be a smooth internal current 2-form on M that is supported away from ∂M.

4.3. Finite energy solutions for single and double coatings. The definition of finite energy solution is the same for both coatings. On M, the parameters ε and μ are bounded from below and above, so Maxwell's equations,

$$\nabla\times E = ik\mu(x)H,\quad \nabla\times H = -ik\varepsilon(x)E + J \quad \text{in } M, \qquad (81)$$
$$R(\nu, E, H)|_{\partial M} = b$$

are defined in the sense of distributions in the usual way. Here, ν denotes the Euclidian unit normal vector of ∂M and $R(\cdot,\cdot,\cdot)$ is a boundary value operator corresponding to the boundary conditions of interest, e.g., $R(\nu, E, H) = \nu\times E$ for the electric boundary condition.

If J is smooth, Maxwell's equations imply that $E, H \in C^\infty(M)$.

Next, we consider Maxwell's equations on N. Let $\widetilde J$ be a smooth 2-form on N that is supported away form $\partial N\cup\Sigma$.

Definition 4. *Let $(M, N, F, \gamma, \Sigma, g)$ be either a single or double coating. We say that $(\widetilde E, \widetilde H)$ is a finite energy solution to Maxwell's equations on N,*

$$\nabla\times\widetilde E = ik\widetilde\mu(x)\widetilde H,\quad \nabla\times\widetilde H = -ik\widetilde\varepsilon(x)\widetilde E + \widetilde J \quad \textit{on } N, \qquad (82)$$

if $\widetilde E$, $\widetilde H$, $\widetilde D := \widetilde\varepsilon\,\widetilde E$ and $\widetilde B := \widetilde\mu\,\widetilde H$ are forms in N with $L^1(N, dx)$-coefficients satisfying

$$\|\widetilde E\|^2_{L^2(N,|\widetilde g|^{1/2}dV_0(x))} = \int_N \widetilde\varepsilon^{jk}\,\widetilde E_j\,\overline{\widetilde E_k}\,dV_0(x) < \infty, \qquad (83)$$

$$\|\widetilde H\|^2_{L^2(N,|\widetilde g|^{1/2}dV_0(x))} = \int_N \widetilde\mu^{jk}\,\widetilde H_j\,\overline{\widetilde H_k}\,dV_0(x) < \infty; \qquad (84)$$

$(\widetilde E, \widetilde H)$ is a classical solution of Maxwell's equations on a neighborhood $U\subset\overline{N}$ of ∂N:

$$\nabla\times\widetilde E = ik\widetilde\mu(x)\widetilde H,\quad \nabla\times\widetilde H = -ik\varepsilon(x)\widetilde E + \widetilde J \quad \textit{in } U,$$
$$R(\nu, \widetilde E, \widetilde H)|_{\partial N} = \widetilde b;$$

and finally,

$$\int_N ((\nabla\times\widetilde h)\cdot\widetilde E - ik\widetilde h\cdot\widetilde\mu(x)\widetilde H)\,dV_0(x) = 0,$$
$$\int_N ((\nabla\times\widetilde e)\cdot\widetilde H + \widetilde e\cdot(ik\widetilde\varepsilon(x)\widetilde E - \widetilde J))\,dV_0(x) = 0$$

for all $\widetilde e, \widetilde h \in C_0^\infty(\Omega^1 N)$.

Here, $C_0^\infty(\Omega^1 N)$ denotes smooth 1-forms on N whose supports do not intersect ∂N, and the inner product "$\cdot$" denotes the Euclidean inner product.

Remark. The fact that $\widetilde{E}, \widetilde{H}$ are solutions of (82) in the sense of Def. 4 implies that they are distributional solutions in the usual sense. Thus they also satisfy the divergence equations,

$$\nabla\cdot \widetilde{\varepsilon}\widetilde{E} = \frac{1}{ik}\nabla\cdot \widetilde{J}, \quad \nabla\cdot \widetilde{\mu}\widetilde{H} = 0, \tag{85}$$

in the sense of distributions.

5. Full wave invisibility for the double coating

In this section, $(M, N, F, \gamma, \Sigma, g)$ denotes a double coating construction. Invisibility for active devices enclosed in the double coating, with respect to Maxwell's equations at all frequencies, is a consequence of:

Theorem 4. *Let E and H be 1-forms with measurable coefficients on $M \setminus \gamma$ and $\widetilde{E}$ and $\widetilde{H}$ be 1-forms with measurable coefficients on $N \setminus \Sigma$ such that $E = F^*\widetilde{E}$, $H = F^*\widetilde{H}$. Let J and $\widetilde{J}$ be 2-forms with smooth coefficients on $M \setminus \gamma$ and $N \setminus \Sigma$ that are supported away from γ and Σ.*

Then the following are equivalent:

1. *The 1-forms $\widetilde{E}$ and $\widetilde{H}$ on N form a finite energy solution of Maxwell's equations*

$$\nabla \times \widetilde{E} = ik\widetilde{\mu}(x)\widetilde{H}, \quad \nabla \times \widetilde{H} = -ik\widetilde{\varepsilon}(x)\widetilde{E} + \widetilde{J} \quad \textit{on } N, \tag{86}$$
$$R(\nu, \widetilde{E}, \widetilde{H})|_{\partial N} = b.$$

2. *The 1-forms E and H on M satisfy Maxwell's equations*

$$\nabla \times E = ik\mu(x)H, \quad \nabla \times H = -ik\varepsilon(x)E + J \quad \textit{on } M_1,$$
$$R(\nu, E, H)|_{\partial N} = b$$

and

$$\nabla \times E = ik\mu(x)H, \quad \nabla \times H = -ik\varepsilon(x)E + J \quad \textit{on } M_2.$$

Proof. First we prove that Maxwell's equations on M imply Maxwell equations on N

Assume now that the 1-forms E and H are classical solutions of Maxwell's equations on $M = M_1 \cup M_2$,

$$\nabla \times E = ik\mu(x)H, \quad \nabla \times H = -ik\varepsilon(x)E + J \quad \text{on } M = M_1 \cup M_2,$$
$$R(\nu, E, H)|_{\partial N} = b. \tag{87}$$

Since J vanishes near γ, ellipticity implies that E and H are smooth near γ.

Define on $N \setminus \Sigma$ the forms $\widetilde{E} = (F^{-1})^*E$, $\widetilde{H} = (F^{-1})^*H$, and $\widetilde{J} = (F^{-1})^*J$.

Then $\widetilde{E}$ satisfies the Maxwell's equations on $N \setminus \Sigma$,

$$\nabla \times \widetilde{E} = ik\widetilde{\mu}(x)\widetilde{H}, \quad \nabla \times \widetilde{H} = -ik\widetilde{\varepsilon}(x)\widetilde{E} + \widetilde{J} \quad \text{on } N \setminus \Sigma, \tag{88}$$

Again, let $\Sigma(t)$ be the t-neighborhood of Σ with respect to the metric $\widetilde{g}$ and $\gamma(t)$ the t-neighborhood of γ with respect to g. Let $I_t : \partial\gamma(t) \to M$ be the identity embedding. Denote by ν be the unit normal vector of $\partial\Sigma(t)$ and $\partial\gamma(t)$ in Euclidean metric.

Now, writing $E = E_j(x)dx^j$ on M, we see using the transformation rule for differential 1-forms that the form $\widetilde{E} = (F^{-1})^*E$ is in local coordinates is

$$\widetilde{E} = \widetilde{E}_j(\widetilde{x})d\widetilde{x}^j = (DF^{-1})_j^k(\widetilde{x})\,E_k(F^{-1}(\widetilde{x}))d\widetilde{x}^j, \quad \widetilde{x} \in N \setminus \Sigma,$$

and, using $F_t = F \circ I_t : \partial\gamma(t) \to \partial\Sigma(t)$, we have

$$\widetilde{I}^*(\widetilde{E}_j(x)dx^j) = (DF_t^{-1})_j^k(\widetilde{x})\,E_k(F^{-1}(\widetilde{x}))\,d\widetilde{x}^j, \quad \widetilde{x} = F(x) \tag{89}$$

Let us now do computations in the Euclidean coordinates. In the Euclidean metric g_e, the matrix DF_t^{-1} satisfies

$$\|DF_t^{-1}\|_{(T\partial\Sigma(t),g_e)\to(T\partial\gamma(t),g_e)} \le Ct, \tag{90}$$

and since E is smooth near γ we see

$$|\nu \times \widetilde{E}(y)|_{\mathbb{R}^3} \le Ct, \quad y \in \partial\Sigma(t).$$

Thus using (88) we see that for $\widetilde{h} \in C_0^\infty(\Omega^1 N)$

$$\begin{aligned} &\int_N ((\nabla \times \widetilde{h}) \cdot \widetilde{E} - ik\widetilde{h} \cdot \widetilde{\mu}(x)\widetilde{H})\,dV_0(x) \\ &= \lim_{t\to 0} \int_{N\setminus\Sigma(t)} ((\nabla \times \widetilde{h}) \cdot \widetilde{E} - ik\widetilde{h} \cdot \widetilde{\mu}(x)\widetilde{H})\,dV_0(x) \\ &= -\lim_{t\to 0} \int_{\partial\Sigma(t)} (\nu \times \widetilde{E}) \cdot \widetilde{h}\,dS(x) = 0. \end{aligned} \tag{91}$$

Thus, we have shown that

$$\nabla \times \widetilde{E} = ik\widetilde{\mu}(x)\widetilde{H} \quad \text{in } N \tag{92}$$

in the sense of Definition 4. Similarly, we see that

$$\nabla \times \widetilde{H} = -ik\widetilde{\varepsilon}(x)\widetilde{E} + \widetilde{J} \quad \text{in } N \tag{93}$$

in the same finite energy sense.

Next we show that Maxwell's equations on N implies Maxwell's equations on M. Let $U \subset M$ be a bounded neighborhood of γ and $W = F(U \setminus \gamma) \cup \Sigma$ be a neighborhood of Σ such that $\mathrm{supp}\,(\widetilde{J}) \cap W = \emptyset$.

Assume that $\widetilde{E}$ and $\widetilde{H}$ form a finite energy solution of Maxwell's equations (86) on (N, g) in finite energy sense with a source $\widetilde{J}$ supported away from Σ, implying in particular that

$$\widetilde{\varepsilon}^{jk}\widetilde{E}_j\overline{\widetilde{E}_k} \in L^1(W, dx), \quad \widetilde{\mu}^{jk}\widetilde{H}_j\overline{\widetilde{H}_k} \in L^1(W, dx).$$

Define $E = F^*\widetilde{E}$, $H = F^*\widetilde{H}$ and $J = F^*\widetilde{J}$ on $M \setminus \gamma$. We have

$$\nabla \times E = ik\mu(x)H, \quad \nabla \times H = -ik\varepsilon(x)E + J \quad \text{in } M \setminus \gamma$$

and

$$\varepsilon^{jk} E_j \overline{E_k} \in L^1(U \setminus \gamma,\, dV_0(x)), \quad \mu^{jk} H_j \overline{H_k} \in L^1(U \setminus \gamma,\, dV_0(x)).$$

As ε and μ on M are bounded from above and below, these imply that

$$\begin{aligned} &\nabla \times E \in L^2(U \setminus \gamma,\, dV_0(x)), \quad \nabla \times H \in L^2(U \setminus \gamma,\, dV_0(x)),\\ &\nabla\cdot (\varepsilon E) = 0, \quad \nabla\cdot (\mu H) = 0 \quad \text{in } U \setminus \gamma. \end{aligned}$$

Let $E^e, H^e \in L^2(U,\, dV_0(x))$ be measurable extensions of E and H to γ. Then

$$\begin{aligned} &\nabla \times E^e - ik\mu(x)H^e = 0 \quad \text{in } U \setminus \gamma,\\ &\nabla \times E^e - ik\mu(x)H^e \in H^{-1}(U,\, dV_0(x)),\\ &\nabla \times H^e + ik\varepsilon(x)E^e = 0 \quad \text{in } U \setminus \gamma,\\ &\nabla \times H^e + ik\varepsilon(x)E^e \in H^{-1}(U,\, dV_0(x)). \end{aligned}$$

Since γ is a subset with (Hausdorff) dimension 1 of the 3-dimensional domain U, it has zero capacitance. Thus, the Lipschitz functions on U that vanish on γ are dense in $H^1(U)$, see [KKM, Thm 4.8 and remark 4.2(4)], or [AF, Thm. 3.28]. Thus there are no non-zero distributions in $H^{-1}(U)$ supported on γ. Hence we see that

$$\nabla \times E^e - ik\mu(x)H^e = 0, \quad \nabla \times H^e + ik\varepsilon(x)E^e = 0 \quad \text{in } U.$$

This also implies that

$$\nabla\cdot (\varepsilon E^e) = 0, \quad \nabla\cdot (\mu H^e) = 0 \quad \text{in } U.$$

These imply that E^e and H^e are in C^∞ smooth in U.

Summarizing, E and H have unique continuous extensions to γ, and the extensions are classical solutions to Maxwell's equations.

6. Cauchy data for the single coating must vanish

In this section $(M, N, F, \gamma, \Sigma, g)$ denotes a single coating construction. The following gives the counterpart for Maxwell's equations of the hidden Neumann boundary condition on ∂M_2 that appeared for the Helmholtz equation.

Theorem 5. *Let E and H be 1-forms with measurable coefficients on $M \setminus \gamma$ and $\widetilde{E}$ and $\widetilde{H}$ be 1-forms with measurable coefficients on $N \setminus \Sigma$ such that $E = F^*\widetilde{E}$, $H = F^*\widetilde{H}$. Let J and $\widetilde{J}$ be 2-forms with smooth coefficients on $M \setminus \gamma$ and $N \setminus \Sigma$, that are supported away from γ and Σ.*

Then the following are equivalent:

1. The 1-forms $\widetilde{E}$ and $\widetilde{H}$ on N satisfy Maxwell's equations

$$\begin{aligned}&\nabla\times\widetilde{E}=ik\widetilde{\mu}(x)\widetilde{H},\quad \nabla\times\widetilde{H}=-ik\widetilde{\varepsilon}(x)\widetilde{E}+\widetilde{J}\quad \text{on } N, \\ &\nu\times\widetilde{E}|_{\partial N}=f\end{aligned} \tag{94}$$

in the sense of Definition 4.

2. The forms E and H satisfy Maxwell's equations on M,

$$\begin{aligned}&\nabla\times E=ik\mu(x)H,\quad \nabla\times H=-ik\varepsilon(x)E+J\quad \text{on } M_1, \\ &\nu\times E|_{\partial M_1}=f\end{aligned} \tag{95}$$

and

$$\nabla\times E=ik\mu(x)H,\quad \nabla\times H=-ik\varepsilon(x)E+J\quad \text{on } M_2 \tag{96}$$

with Cauchy data

$$\nu\times E|_{\partial M_2}=b^e,\quad \nu\times H|_{\partial M_2}=b^h \tag{97}$$

that satisfies $b^e=b^h=0$.

Moreover, if E and H solve (95), (96), and (97) with non-zero b^e or b^h, then the fields $\widetilde{E}$ and $\widetilde{H}$ are not solutions of Maxwell equations on N in the sense of Definition 4.

Proof. Assume first that the 1-forms E and H are classical solutions of Maxwell's equations in M. Moreover, assume that both E and H satisfy homogeneous boundary condition

$$\nu\times E|_{\partial M_2}=0,\quad \nu\times H|_{\partial M_2}=0, \tag{98}$$

that is, for the field in M_2 the Cauchy data on ∂M_2 vanishes. (Here, ν again denotes the Euclidean unit normal of these surfaces.)

Again, define on $N\setminus\Sigma$ forms $\widetilde{E}=(F^{-1})^*E$, $\widetilde{H}=(F^{-1})^*H$, and $\widetilde{J}=(F^{-1})^*J$. Then $\widetilde{E}$ satisfies Maxwell's equations on $N\setminus\Sigma$,

$$\nabla\times\widetilde{E}=ik\widetilde{\mu}(x)\widetilde{H},\quad \nabla\times\widetilde{H}=-ik\widetilde{\varepsilon}(x)\widetilde{E}+\widetilde{J}\quad \text{in } N\setminus\Sigma, \tag{99}$$

Again, let $\Sigma(t)$ be the t-neighborhood of Σ in the $\widetilde{g}$-metric and $\gamma(t)$ be the t-neighborhood of γ in the g-metric.

Arguing as in (90) and below, we see that

$$|\nu\times\widetilde{E}(y)|_{\mathbb{R}^3}\leq Ct,\quad y\in\partial\Sigma(t)\cap N_2. \tag{100}$$

Recall that $\Sigma_1(\varepsilon)=N_1\cap\Sigma(\varepsilon)$. Then, using (88) we see that for $\widetilde{h}\in C_0^\infty(\Omega^1 N)$,

$$\begin{aligned}&\int_N((\nabla\times\widetilde{h})\cdot\widetilde{E}-ik\widetilde{h}\cdot\widetilde{\mu}(x)\widetilde{H})\,dV_0(x) \\ &=\lim_{t\to 0}\int_{(N\setminus\Sigma_1(t)}((\nabla\times\widetilde{h})\cdot\widetilde{E}-ik\widetilde{h}\cdot\widetilde{\mu}(x)\widetilde{H})\,dV_0(x) \\ &=-\lim_{t\to 0}\int_{\partial\Sigma_1(t)}(\nu\times\widetilde{E})\cdot\widetilde{h}\,dS(x)-\int_{\partial M_2}(\nu\times\widetilde{E})\cdot\widetilde{h}\,dS(x)=0\end{aligned} \tag{101}$$

where we used (100) and (98).

Thus, we have shown that

$$\nabla \times \widetilde{E} = ik\widetilde{\mu}(x)\widetilde{H} \quad \text{on } N \tag{102}$$

in the sense of Definition 4. Similarly, we see that

$$\nabla \times \widetilde{H} = -ik\widetilde{\varepsilon}(x)\widetilde{E} + \widetilde{J} \quad \text{on } N, \tag{103}$$

also in the sense of Definition 4.

Next we show that Maxwell's equations on N imply Maxwell's equations on M. Assume that $\widetilde{E}$ and $\widetilde{H}$ form a finite energy solution of Maxwell's equations (94) on (N, g). Again, define on $M \setminus \gamma$ forms $E = F^*\widetilde{E}$, $H = F^*\widetilde{H}$, and $J = F^*\widetilde{J}$.

As before, we see that E and H satisfy Maxwell's equations on $M_1 \setminus \gamma_1$ and the E and H are in $L^2(M_1, dV_0(x))$. Using the removable of singularity arguments as in the case of double coating, we see that E and H have extensions E^e and H^e in M_1 that are classical solutions of

$$\nabla \times E^e - ik\mu(x)H^e = 0 \quad \text{on } M_1, \tag{104}$$

$$\nabla \times H^e + ik\varepsilon(x)E^e = J \quad \text{on } M_1. \tag{105}$$

Note that (104) implies that, for the original field $\widetilde{E}$,

$$\lim_{t\to 0}\int_{\partial\Sigma(t)\cap N_1} (\nu \times \widetilde{E})\cdot \widetilde{h}\, dS(x) = \lim_{t\to 0}\int_{\partial\gamma(t)\cap M_1} (\nu \times E)\cdot h\, dS(x) = 0 \tag{106}$$

where $h = F^*\widetilde{h}$.

Moreover, Maxwell's equations hold in the interior of M_2:

$$\nabla \times E - ik\mu(x)H = 0, \quad \nabla \times H + ik\varepsilon(x)E = J \quad \text{on } M_2.$$

Let us start to analyze what the validity of the equation $\nabla \times \widetilde{E} - ik\widetilde{\mu}(x)\widetilde{H} = 0$ on N in the sense of Definition 4 implies about the boundary values on ∂M_2. Using (106), we see that for $\widetilde{h} \in C_0^\infty(\Omega^1 N)$

$$0 = \int_N ((\nabla \times \widetilde{h})\cdot \widetilde{E} - ik\widetilde{h}\cdot \widetilde{\mu}(x)\widetilde{H})\, dV_0(x) \tag{107}$$

$$= \lim_{t\to 0}\int_{(N\setminus \Sigma_1(t)} ((\nabla \times \widetilde{h})\cdot \widetilde{E} - ik\widetilde{h}\cdot \widetilde{\mu}(x)\widetilde{H})\, dV_0(x)$$

$$= -\left[\lim_{t\to 0}\int_{\partial\Sigma_1(t)} (\nu \times \widetilde{E})\cdot \widetilde{h}\, dS(x) + \int_{\partial N_2} (\nu \times \widetilde{E})\cdot \widetilde{h}\, dS(x)\right]$$

$$= 0 - \int_{\partial N_2} (\nu \times E)\cdot \widetilde{h}\, dS(x). \tag{108}$$

This shows $\nu \times E|_{\partial M_2} = 0$. Similarly, the equation $\nabla \times \widetilde{H} + ik\widetilde{\varepsilon}(x)\widetilde{E} = \widetilde{J}$ holding on N in the finite energy sense implies that $\nu \times H|_{\partial M_2} = 0$. □

Assume that E and H satisfy the time-harmonic Maxwell's equations on $M_2 \subset \mathbb{R}^3$ such that the Cauchy data $(\nu \times E|_{\partial M_2}, \nu \times H|_{\partial M_2})$ vanishes. By continuing E and H by zero to $\mathbb{R}^3 \setminus M_2$ we obtain solutions of Maxwell's equation in $\mathbb{R}^3$. Thus J must be a current for which there exist solutions of Maxwell's equations in $\mathbb{R}^3$ both satisfying the Sommerfeld radiation condition and vanishing outside N_2. Such currents are nowhere dense in $L^2(N_2)$, as then the fields E and H corresponding to J satisfy the Sommerfeld radiation condition and, using Stokes' theorem, we see that the source J is orthogonal to all (vector-valued) Green's functions $G_e(\cdot\,, y, k; a)$ with $y \in \mathbb{R}^3 \setminus \overline{M}_2$ and $a \in \mathbb{R}^3$. Here, the Green's function $(G_e(\cdot\,, y, k; a), G_h(\cdot\,, y, k; a))$ satisfies Maxwell's equations in $\mathbb{R}^3$ with current $a\delta_y$ and the Sommerfeld radiation condition.

We thus conclude that finite energy solutions to Maxwell's equations on N with the single coating exist only if the Cauchy data $(\nu \times E|_{\partial M_2}, \nu \times H|_{\partial M_2})$ vanishes on the inner surface of the cloaked region. Thus, finite energy solutions do not exist for generic sources, i.e., internal currents J, in the cloaked region.

7. Cloaking an infinite cylindrical domain

We now consider an infinite cylindrical domain, $N = B_2(0, 2) \times \mathbb{R}$ for simplicity, with the double coating. Here, $B_2(0, r) \subset \mathbb{R}^2$ is Euclidian disc with center 0 and radius r. Numerics for cloaking an infinite cylinder have been presented in [CPSSP]. This may also provide a picture of the cloaking that was physically implemented with a "sliced cylinder" geometry in [SMJCPSS], although precise modelling has not been carried out. With this limitation in mind, the physical interpretation of Theorems 6 and ref{single coating with Maxwell obstacle below is that the cloaking would be more effective with the insertion of a liner to implement the SHS boundary conditions which are necessary for the existence of finite energy solutions.

Here, we modify the treatment from §2 to the noncompact setting, blowing up a line and trying to obtain an infinitely long, invisible cable.

Let

$$M_1 = B_2(0, 2) \times \mathbb{R}, \quad \gamma_1 = \{(0, 0)\} \times \mathbb{R} \subset M_1,$$
$$M_2 = S^2 \times \mathbb{R}, \quad \gamma_2 = \{NP\} \times \mathbb{R} \subset M_2$$

Let $M = M_1 \cup M_2$, $\gamma = \gamma_1 \cup \gamma_2$,

$$N_1 = B_2(0, 2) \times \mathbb{R} \setminus (\overline{B}_2(0, 1) \times \mathbb{R}),$$
$$N_2 = B_2(0, 1) \times \mathbb{R},$$
$$\Sigma = \partial B_2(0, 1) \times \mathbb{R},$$

and $N = B_2(0, 2) \times \mathbb{R} = N_1 \cup N_2 \cup \Sigma$. Let

$$F = (F_1, F_2) : M \setminus \gamma \to N \setminus \Sigma$$

be such that

$$F_1 : M_1 \setminus \gamma_1 \to N_1,$$
$$F_2 : M_2 \setminus \gamma_2 \to N_2.$$

are diffeomorphisms. Let $X : B_2(0,2) \times \mathbb{R} \setminus \{(0,0)\} \times \mathbb{R}) \to (r,\theta,z)$ be the standard cylindrical coordinates on M_1. We assume that F is stretching only in radial direction, that is,

$$X(F(X^{-1}(r,\theta,z))) = (F_1(r),\theta,z). \tag{109}$$

Similarly, on M_2 we have variables (r,θ,z), where $r = \mathrm{dist}(x, SP)$ and we assume that F has a form analogous to (109) in M_2. For simplicity, let g_1 be the Euclidean metric on M_1 and g_2 the product of standard metric on S^2 and standard metric of $\mathbb{R}$ on M_2. Let $\widetilde{g} = F_* g$ on $N \setminus \Sigma$, so that $(M, N, F, \gamma, \Sigma, g)$ is a double coating construction in this context.

On M and $N \setminus \Sigma$ we define permittivity and permeability by setting

$$\begin{aligned} \varepsilon^{jk} &= \mu^{jk} = |g|^{1/2} g^{jk}, \quad \text{on } M_1 \cup M_2, \\ \widetilde{\varepsilon}^{jk} &= \widetilde{\mu}^{jk} = |\widetilde{g}|^{1/2} \widetilde{g}^{jk}, \quad \text{on } N \setminus \Sigma. \end{aligned}$$

By finite energy solutions of Maxwell's equations on N we will mean one-forms $\widetilde{E}$ and $\widetilde{H}$ satisfying the conditions of Definition 4.1, where we emphasize the assumption that $\widetilde{D} := \widetilde{\varepsilon}\widetilde{E}$ and $\widetilde{B} := \widetilde{\mu}\widetilde{H}$ are in $L^1(N \setminus \Sigma, dx)$, making the integrals at the end of Definition 4.1 well defined.

To formulate the results, we need to define the restrictions of fields on the lines $\gamma_1 \subset M_1$ and $\gamma_2 \subset M_2$. First, assume that the 1-forms E and H on M are classical solutions to Maxwell's equations on M,

$$\begin{aligned} &\nabla \times E = ik\mu(x)H, \quad \text{in } M = M_1 \cup M_2, \\ &\nabla \times H = -ik\varepsilon(x)E + J, \quad \text{in } M = M_1 \cup M_2, \\ &\nu \times E|_{\partial M_1} = f, \end{aligned} \tag{110}$$

where J is supported away from $\gamma = \gamma_1 \cup \gamma_2$. Note that then E and H are C^∞ near γ, and thus we can define the restrictions of the vertical components of the fields on $\gamma_1 \subset M_1$,

$$\zeta \cdot E|_{\gamma_1} = b_1^e, \quad \zeta \cdot H|_{\gamma_1} = b_1^h, \tag{111}$$

where $\zeta := (0,0,1) = \frac{\partial}{\partial z}$, $z := x^3$.

Similarly, we can define b_2^e and b_2^h to be the restrictions on $\gamma_2 \subset M_2$,

$$\zeta \cdot E|_{\gamma_2} = b_2^e, \quad \zeta \cdot H|_{\gamma_2} = b_2^h. \tag{112}$$

Note that $b_j^e = b_j^e(z)$ and $b_j^h = b_j^h(z)$, $j = 1, 2$, depend only on z.

Theorem 6. *Let E and H be 1-forms with measurable coefficients on $M \setminus \gamma$ and $\widetilde{E}$ and $\widetilde{H}$ be 1-forms with measurable coefficients on $N \setminus \Sigma$ such that $E = F^*\widetilde{E}$, $H = F^*\widetilde{H}$. Let J and $\widetilde{J}$ be 2-forms with smooth coefficients on $M \setminus \gamma$ and $N \setminus \Sigma$, that are supported away from γ and Σ, respectively*

Then the following are equivalent:

1. *On N, the 1-forms $\widetilde{E}$ and $\widetilde{H}$ satisfy Maxwell's equations*

$$\nabla\times\widetilde{E}=ik\widetilde{\mu}(x)\widetilde{H},\quad \nabla\times\widetilde{H}=-ik\widetilde{\varepsilon}(x)\widetilde{E}+\widetilde{J}\quad \textit{in } N, \tag{113}$$
$$\nu\times\widetilde{E}|_{\partial N}=f$$

and $\widetilde{E}$ and $\widetilde{H}$ are finite energy solutions.

2. *On M, the forms E and H are classical solutions to Maxwell's equations (110) on M, with data*

$$b_1^e=\zeta\cdot E|_{\gamma_1},\quad b_2^e=\zeta\cdot E|_{\gamma_2},\quad b_1^h=\zeta\cdot H|_{\gamma_1},\quad b_2^h=\zeta\cdot H|_{\gamma_2}, \tag{114}$$

that satisfy

$$b_1^e(z)=b_2^e(z)\quad \textit{and}\quad b_1^h(z)=b_2^h(z),\quad z\in\mathbb{R}. \tag{115}$$

Moreover, if E and H solve (110) with restrictions (114) that do not satisfy (115), then then the fields $\widetilde{E}$ and $\widetilde{H}$ are not finite energy solutions of Maxwell equations on N.

Proof. First we show that the equations on M imply that the equations hold on N. Assume that the forms E and H satisfy Maxwell's equations (110) in M in the classical sense, with traces (114) that satisfy (115). Then E and H are C^∞ smooth near γ.

Define 1-forms $\widetilde{E},\widetilde{H}$ and 2-form $\widetilde{J}$ on $N\setminus\Sigma$ by $\widetilde{E}=(F^{-1})^*E$, $\widetilde{H}=(F^{-1})^*H$, and $\widetilde{J}=((F^{-1})^*J$. Then $\widetilde{E}$ satisfies Maxwell's equations on $N\setminus\Sigma$,

$$\nabla\times\widetilde{E}=ik\widetilde{\mu}(x)\widetilde{H},\quad \nabla\times\widetilde{H}=-ik\widetilde{\varepsilon}(x)\widetilde{E}+\widetilde{J}\quad \text{in } N\setminus\Sigma. \tag{116}$$

A simple computation shows that $\widetilde{E}$, $\widetilde{H}$, $\widetilde{D}=\widetilde{\varepsilon}\widetilde{E}$, and $\widetilde{B}=\widetilde{\mu}\widetilde{H}$ are forms on N with $L^1(N,dx)$ coefficients. Again, let $\Sigma(t)$ be the t-neighborhood of Σ in $\widetilde{g}$-metric and $\gamma(t)$ be the t-neighborhood of γ in g-metric. Let $I_t:\partial\gamma(t)\to M$ be the identity embedding. Denote by ν be the unit normal vector of $\partial\Sigma(t)$ and $\partial\gamma(t)$ in Euclidean metric.

Now, writing $E=E_j(x)dx^j$ on M, we see as above using $F_t=F\circ I_t:\partial\gamma(t)\to\partial\Sigma(t)$, we have in local coordinates formula (89). Let us next do computations in the Euclidean coordinates. Using (109), the angular direction $\eta:=\partial_\theta$, and vertical direction $\zeta=\partial_z$, we see that the matrix $DF_t^{-1}(x)$ satisfies

$$\begin{aligned}
&|\eta\cdot(DF_t^{-1}(x)\eta)|_{\mathbb{R}^3}\leq Ct,\quad x\in\partial\Sigma(t),\\
&|\zeta\cdot(DF_t^{-1}(x)\zeta)|_{\mathbb{R}^3}=1,\quad x\in\partial\Sigma(t),\\
&\zeta\cdot(DF_t^{-1}(x)\eta)=0,\quad x\in\partial\Sigma(t),\\
&\eta\cdot(DF_t^{-1}(x)\zeta)=0,\quad x\in\partial\Sigma(t).
\end{aligned}$$

This implies that only angular components of $\widetilde{E}$ vanish on Σ, and we have

$$\begin{aligned}
&|\eta\cdot\widetilde{E}|_{\mathbb{R}^3}\leq Ct,\quad x\in\partial\Sigma(t), \qquad (117)\\
&\lim_{t\to 0}\zeta\cdot\widetilde{E}|_{\partial\Sigma(t)\cap N_j}=\widetilde{b}_j^e,\quad j=1,2,\\
&\lim_{t\to 0}\zeta\cdot\widetilde{H}|_{\partial\Sigma(t)\cap N_j}=\widetilde{b}_j^h,\quad j=1,2
\end{aligned}$$

where, for $(x^1,x^2,x^3)\in\Sigma\subset N$, we denote

$$\widetilde{b}_j^e(x^1,x^2,x^3)=b_j^e(x^3),\quad \widetilde{b}_j^h(x^1,x^2,x^3)=b_j^h(x^3),\quad j=1,2.$$

Thus, using (116) we see that for $\widetilde{h}\in C_0^\infty(\Omega^1 N)$

$$\begin{aligned}
&\int_N ((\nabla\times\widetilde{h})\cdot\widetilde{E}-ik\widetilde{h}\cdot\widetilde{\mu}(x)\widetilde{H})\,dV_0(x) \qquad (118)\\
&=\lim_{t\to 0}\int_{N\setminus\Sigma(t)} ((\nabla\times\widetilde{h})\cdot\widetilde{E}-ik\widetilde{h}\cdot\widetilde{\mu}(x)\widetilde{H})\,dV_0(x)\\
&=-\lim_{t\to 0}\int_{\partial\Sigma(t)}(\nu\times\widetilde{E})\cdot\widetilde{h}\,dS(x),\\
&=-\int_\Sigma(\nu\times(\widetilde{b}_1^e-\widetilde{b}_2^e)\zeta)\cdot\widetilde{h}\,dS(x)\\
&=0
\end{aligned}$$

where ν is the Euclidian unit normal of $\partial N_2=\Sigma$. This shows that Maxwell's equations are satisfied on N. Observe that if $\widetilde{b}_1^e\neq\widetilde{b}_2^e$, there exists a test function $\widetilde{h}$ such that the last integral is nonzero, precluding the existence of a finite energy solution. Similar considerations are valid for the equation $\nabla\times\widetilde{H}=-ik\widetilde{\varepsilon}\widetilde{E}+\widetilde{J}$.

On the other hand, assume that 1-forms $\widetilde{E}$ and $\widetilde{H}$ satisfy on N Maxwell's equations (113) in the finite energy sense. Then, as E and H are forms with $L^2(M)$-valued coefficients that satisfy Maxwell's equations in $M_1\setminus\gamma_1$ and $M_2\setminus\gamma_2$, we see that they have to satisfy Maxwell's equations in M_1 and M_2, and thus they are C^∞-smooth forms near γ_1 and γ_2. As $\widetilde{E}$ and $\widetilde{H}$ are finite energy solutions on N, the above arguments show that $b_1^e=b_2^e$ and $b_1^h=b_2^h$. This finishes the proof of Theorem 6. □

Remark. If E, H, and J on M are solutions of Maxwell's equations as in Proposition 6 (1) such that conditions (114) are not satisfied, then the proof of Proposition 7.1 shows that the fields $\widetilde{E}=F_*E$, $\widetilde{H}=F_*H$, and $\widetilde{J}=F_*J$ extend to forms with $L^1(N,dx)$ coefficients that satisfy

$$\begin{aligned}
&\nabla\times\widetilde{E}=ik\widetilde{B}+\widetilde{K}_{new}\quad\text{on } N,\\
&\nabla\times\widetilde{H}=-ik\widetilde{D}+\widetilde{J}+\widetilde{J}_{new}\quad\text{on } N
\end{aligned}\qquad (119)$$

in the sense of distributions. Here, $\widetilde{B}=\widetilde{\mu}\widetilde{H}$ and $\widetilde{D}=\widetilde{\varepsilon}\widetilde{E}$ are 2-forms with measurable coefficients and $\widetilde{K}_{new}=s_e\delta_\Sigma$ and $\widetilde{J}_{new}=s_h\delta_\Sigma$ where δ_Σ is a measure supported on Σ and s_e and s_h are smooth 2-forms.

Similarly, if E, H, and J on M are solutions of Maxwell's equations as in Theorem 6 (2) with non-vanishing Cauchy data (97), we see that that $\widetilde{E}$, $\widetilde{H}$, and $\widetilde{J}$ on N satisfy equations (119) with distributional sources $\widetilde{K}_{new}$ and $\widetilde{J}_{new}$ defined as above.

8. Cloaking a cylinder with the SHS boundary condition

Next, we consider N_2 as an obstacle, while the domain N_1 is equipped with a metric corresponding to the single coating. Motivated by the conditions at Σ in the previous section, we impose the soft-and-hard boundary condition on the boundary of the obstacle. To this end, let us give still one more definition of weak solutions, appropriate for this construction. We consider only solutions on the set N_1; nevertheless, we continue to denote $\partial N = \partial N_1 \setminus \Sigma$.

Definition 5. *Let $(M_1, N_1, F, \gamma_1, \Sigma, g_1)$ be a single coating construction. We say that the 1-forms $\widetilde{E}$ and $\widetilde{H}$ are* finite energy solutions *of Maxwell's equations on N_1 with the soft-and-hard (SHS) boundary conditions on Σ,*

$$\nabla \times \widetilde{E} = ik\widetilde{\mu}(x)\widetilde{H}, \quad \nabla \times \widetilde{H} = -ik\widetilde{\varepsilon}(x)\widetilde{E} + \widetilde{J} \quad \text{on } N_1, \tag{120}$$

$$\eta \cdot \widetilde{E}|_\Sigma = 0, \quad \eta \cdot \widetilde{H}|_\Sigma = 0, \tag{121}$$

$$\nu \times \widetilde{E}|_{\partial N} = f,$$

if $\widetilde{E}$ and $\widetilde{H}$ are 1-forms on N_1 and $\widetilde{\varepsilon}\widetilde{E}$ and $\widetilde{\mu}\widetilde{H}$ are 2-forms with measurable coefficients satisfying

$$\|\widetilde{E}\|^2_{L^2(N_1,|\widetilde{g}|^{1/2}dV_0)} = \int_{N_1} \widetilde{\varepsilon}^{jk}\, \widetilde{E}_j\, \overline{\widetilde{E}_k}\, dV_0(x) < \infty, \tag{122}$$

$$\|\widetilde{H}\|^2_{L^2(N_1,|\widetilde{g}|^{1/2}dV_0)} = \int_{N_1} \widetilde{\mu}^{jk}\, \widetilde{H}_j\, \overline{\widetilde{H}_k}\, dV_0(x) < \infty; \tag{123}$$

Maxwell's equation are valid in the classical sense in a neighborhood U of ∂N:

$$\nabla \times \widetilde{E} = ik\widetilde{\mu}(x)\widetilde{H}, \quad \nabla \times \widetilde{H} = -ik\varepsilon(x)\widetilde{E} + \widetilde{J} \quad \text{in } U,$$
$$\nu \times \widetilde{E}|_{\partial N} = f;$$

and finally,

$$\int_{N_1} ((\nabla \times \widetilde{h}) \cdot \widetilde{E} - ik\widetilde{h} \cdot \widetilde{\mu}(x)\widetilde{H})\, dV_0(x) = 0,$$
$$\int_{N} ((\nabla \times \widetilde{e}) \cdot \widetilde{H} + \widetilde{e} \cdot (ik\widetilde{\varepsilon}(x)\widetilde{E} - \widetilde{J}))\, dV_0(x) = 0,$$

for all $\widetilde{e}, \widetilde{h} \in C_0^\infty(\Omega^1 N_1)$ satisfying

$$\eta \cdot \widetilde{e}|_\Sigma = 0, \quad \eta \cdot \widetilde{h}|_\Sigma = 0, \tag{124}$$

where $\eta = \partial_\theta$ is the angular vector field that is tangential to Σ.

We have the following invisibility result.

In this section $(M_1, N_1, F, \gamma_1, \Sigma)$ is a coating configuration corresponding to single coating of a cylindrical obstacle $B_2(0,1) \times \mathbb{R}$.

Theorem 7. *Let E and H be 1-forms with measurable coefficients on $M_1 \setminus \gamma_1$ and $\widetilde{E}$ and $\widetilde{H}$ be 1-forms with measurable coefficients on N_1 such that $E = F^*\widetilde{E}$, $H = F^*\widetilde{H}$. Let J and $\widetilde{J}$ be 2-forms with smooth coefficients on $M_1 \setminus \gamma_1$ and $N_1 \setminus \Sigma$, that are supported away from γ_1 and Σ.*

Then the following are equivalent:

1. *On N_1, the 1-forms $\widetilde{E}$ and $\widetilde{H}$ satisfy Maxwell's equations (120) with SHS boundary conditions (121) in the sense of Definition 5.*
2. *On M_1, the forms E and H are classical solutions of Maxwell's equations,*

$$\begin{aligned}
&\nabla \times E = ik\mu(x)H, \quad \text{in } M_1 \\
&\nabla \times H = -ik\varepsilon(x)E + J, \quad \text{in } M_1, \\
&\nu \times E|_{\partial M_1} = f.
\end{aligned} \tag{125}$$

Proof. First, assume that the forms E and H satisfy Maxwell's equations (125) in M_1. Then E satisfies identities (117). Considerations similar to those yielding formula (118) imply that

$$\begin{aligned}
&\int_{N_1} ((\nabla \times \widetilde{h}) \cdot \widetilde{E} - ik\widetilde{h} \cdot \widetilde{\mu}(x)\widetilde{H})\, dV_0(x) \\
&= \lim_{t\to 0} \int_{N_1 \setminus \Sigma(t)} ((\nabla \times \widetilde{h}) \cdot \widetilde{E} - ik\widetilde{h} \cdot \widetilde{\mu}(x)\widetilde{H})\, dV_0(x) \\
&= -\lim_{t\to 0} \int_{\partial\Sigma(t)\cap N_1} (\nu \times \widetilde{E}) \cdot \widetilde{h}\, dS(x), \\
&= -\lim_{t\to 0} \int_{\partial\Sigma(t)\cap N_1} (\nu \times ((\eta \cdot \widetilde{E})\eta + (\zeta \cdot \widetilde{E})\zeta) \cdot \widetilde{h}\, dS(x), \\
&= 0
\end{aligned} \tag{126}$$

for a test function $\widetilde{h}$ satisfying (124).

Similar analysis for $\widetilde{H}$ shows that 1-forms $\widetilde{E}$ and $\widetilde{H}$ satisfy Maxwell's equations with SHS boundary conditions in the sense of Definition 5.

Next, we show that equations on N_1 imply equations on M_1. Assume that 1-forms $\widetilde{E}$ and $\widetilde{H}$ satisfy Maxwell's equations with SHS boundary conditions, and internal current $\widetilde{J}$, in the sense of Definition 5. Then E and H are classical solutions of Maxwell's equation in $M_1 \setminus \gamma_1$. Let $U \subset M_1$ be a neighborhood of γ_1 and $W = F(U \setminus \gamma_1) \cup \Sigma$ be a neighborhood of Σ in N_1 such that $\operatorname{supp}(\widetilde{J}) \cap W = \emptyset$. Then we have

$$\widetilde{\varepsilon}^{jk}\widetilde{E}_j\overline{\widetilde{E}_k} \in L^1(W, dV_0(x)), \quad \widetilde{\mu}^{jk}\widetilde{H}_j\overline{\widetilde{H}_k} \in L^1(W, dV_0(x)).$$

Define $E = F^*\widetilde{E}$, $H = F^*\widetilde{H}$ and $J = F^*\widetilde{J}$ on $M_1 \setminus \gamma_1$. Again, we see that E, H, and J satisfy Maxwell's equations on $U \setminus \gamma$, and as above we see that E and H have measurable extensions on γ, $E^e, H^e \in L^2(U, dV_0(x))$, such that $\nabla \times E^e - ik\mu(x)H^e$ and $\nabla \times H^e + ik\varepsilon(x)E^e$ are distributions in $H^{-1}(U, dV_0)$ supported on γ_1. As before, we see obtain

$$\nabla \times E^e - ik\mu(x)H^e = 0, \quad \nabla \times H^e + ik\varepsilon(x)E^e = 0 \quad \text{in } U.$$

This shows that E and H are classical solutions of Maxwell's equations on M_1.
□

Similar analysis can be done in the case when we have a physical surface $\Sigma = S^1 \times \mathbb{R}$ dividing $\mathbb{R}^3$ into two regions, having the SHS boundary conditions on both sides, and we define the material parameters according to double coating construction, i.e., on both sides of the surface.

9. Appendix: Single and double coating for arbitrary domains and metrics

The constructions of §2 and the results that follow easily extend to general domains and metrics. Let us assume that $\Omega \subset \mathbb{R}^3$ now is an arbitrary domain with smooth boundary, equipped with an arbitrary smooth Riemannian metric, $g = g_{ij}(x)$. This defines the Laplace operator Δ_g with, say Dirichlet boundary condition, cf. Remark 3.6. Choose a point $O \in \Omega$ to be blown up, and assume that the injectivity radius

of (Ω, g) at O is larger than $3a$ for some $a > 0$. Let $B(O, r)$ denote a metric ball of (M, g) with center O and radius r. Introduce Riemannian normal coordinates in $B(O, 3a) \subset \Omega$:

$$x = (x^1, x^2, x^3) \to (\tau, \omega), \tau > 0, \omega \in \mathbb{S}^2 \subset T_O\Omega,$$

so that $x = \exp_O(\tau\omega)$. Let $f(\tau) : [0, 3a] \to [a, 3a]$ be a smooth strictly increasing function coinciding with $\tau/2 + a$ near $\tau = 0$ and with τ for $\tau > 2a$.

Define, in these coordinates,

$$F : B(O, 3a) \setminus \{O\} \to B(O, 3a) \setminus B(O, a), \quad (\tau, \omega) \to (f(\tau), \omega).$$

We extend F by the identity to $\Omega \setminus B(O, 3a)$ and obtain a diffeomorphism

$$F_1 : \Omega \setminus \{O\} \to N_1 = \Omega \setminus B(O, a).$$

Consider the metric $\widetilde{g} = F_{1*}g$ in N_1. Observe that surfaces lying at distance τ from $\partial B(O, a)$ with respect to the metric $\widetilde{g}$ coincide with surfaces lying at distance $f(\tau) - a$ from $\partial B(O, a)$ with respect to the metric g. Therefore, the directions normal to these surfaces are the same with respect to the metrics g and $\widetilde{g}$. In particular, the direction of these normals, in the metric $\widetilde{g}$, is transversal to $\partial B(O, a)$. Thus, equations (5) remain valid if we use $\tau - a$ instead of $r - 1$. Similarly, we again have the estimate $|\widetilde{g}|^{1/2} \leq C_1(\tau - a)^2$.

One may also extend the double coating construction as follows. Let (D, g_D) be a compact Riemannian manifold without boundary, and choose a point $NP \in D$. Using Riemannian normal coordinates centered at NP, introduce, similar to the above, a diffeomorphism

$$F_2 : D \setminus \{NP\} \to N_2 = D \setminus \overline{B}(NP, b),$$

where we assume that $3b$ is smaller than injectivity radius of D. Pulling back the metric g_D, we get a metric $\widetilde{g_D}$ on $D \setminus \overline{B}(NP, b)$ with the same properties near $\partial B(NP, b)$ as $\widetilde{g}$ has near $\partial B(O, a)$.

Observe that, as we are inside the injectivity radii, $\partial B(O,a)$ and $\partial B(NP,b)$ are both diffeomorphic to $\mathbb{S}^2$, with diffeomorphisms given by $exp_O(a\omega)$ and $\exp_{NP}(b\omega)$. Thus, $\partial B(O,a)$ and $\partial B(NP,b)$ are diffeomorphic to each other. Gluing these boundaries, we obtain a smooth manifold $N = N_1 \cup N_2 \cup \Sigma$ with a Riemannian metric singular on Σ which, as one approaches Σ, satisfies conditions (5). This makes it possible to carry out all of the preceding analysis for the double coating.

Note that if D is diffeomorphic to S^3 (as earlier), then N is diffeomorphic to $\Omega \simeq M_1$. If however D has a non-trivial topology, N may have topology different from that of Ω. However, due to the full-wave invisibility, one is unable to observe this change of topology from observations made at $\partial\Omega$. Note that this is in contrast to the uniqueness result that holds for C^ω Riemannian manifolds [LTU].

Similar generalizations of the single coating construction are possible when ∂D is diffeomorphic to $\mathbb{S}^2$.

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