Title: Inverse Electromagnetic Problems

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Introduction

In this chapter we consider inverse boundary problems for electromagnetic waves. The goal is to determine the electromagnetic parameters of a medium by making measurements at the boundary of the medium. We concentrate on fixed energy problems. We first discuss the case of electrostatics, which is called Electrical Impedance Tomography (EIT). This is also called Calderón problem since the mathematical formulation of the problem and the first results in the multi-dimensional case were due to A.P. Calderón [11]. In this case the electromagnetic parameter is the conductivity of the medium and the equation modelling the problem is the conductivity equation. Then we discuss the more general case of recovering all the electromagnetic parameters of the medium, the electric permittivity, magnetic permeability and electrical conductivity of the medium by making boundary measurements and the equation modelling the problem is the full system of Maxwell's equations. Finally we consider the problem of determining electromagnetic inclusions and obstacles from electromagnetic boundary measurements. A common feature of the problems we study is that they are fixed energy problems. The type of electromagnetic waves that we use to probe the medium are complex geometrical optics solutions to Maxwell's equations.

Electrical Impedance Tomography

The problem that Calderón proposed was whether one can determine the electrical conductivity of a medium by making voltage and current measurements at the boundary of the medium. Calderón was motivated by oil prospection. In the 40's he worked as an engineer for Yacimientos Petrolíferos Fiscales (YPF), the state oil company of Argentina, and he thought about this problem then although he did not publish his results until many years later. For applications of electrical methods in geophysics see [48]. EIT also arises in medical imaging given that human organs and tissues have quite different conductivities. One potential application is the early diagnosis of breast cancer [47]. The conductivity of a malignant breast tumor is typically 0.2 mho which is significantly higher than normal tissue which has been typically measured at 0.03 mho. For other medical imaging applications see [18].

We now describe more precisely the mathematical problem. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary (many of the results we will describe are valid for domains with Lipschitz boundaries). The isotropic electrical conductivity of Ω is represented by a bounded and positive function $\gamma(x)$. In the absence of sinks or sources of current and given a voltage potential on the boundary $f \in H^{\frac{1}{2}}(\partial\Omega)$ the induced potential $u \in H^1(\Omega)$ solves the Dirichlet problem

$$\nabla \cdot (\gamma \nabla u) = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f. \tag{1}$$

The Dirichlet to Neumann map, or voltage to current map, is given by

$$\Lambda_\gamma(f) = \left(\gamma \frac{\partial u}{\partial \nu}\right)\Big|_{\partial\Omega} \tag{2}$$

where ν denotes the unit outer normal to $\partial\Omega$.

The inverse problem of EIT is to determine γ knowing Λ_γ. It is difficult to find a systematic way of prescribing voltage measurements at the boundary to be able to find the conductivity. Calderón took instead a different route. Using the divergence theorem we have

$$Q_\gamma(f) := \int_\Omega \gamma |\nabla u|^2 dx = \int_{\partial\Omega} \Lambda_\gamma(f) f \, dS \tag{3}$$

where dS denotes surface measure and u is the solution of (1). In other words $Q_\gamma(f)$ is the quadratic form associated to the linear map $\Lambda_\gamma(f)$, and to know $\Lambda_\gamma(f)$ or $Q_\gamma(f)$ for all $f \in H^{\frac{1}{2}}(\partial\Omega)$ is equivalent. $Q_\gamma(f)$ measures the energy needed to maintain the potential f at the boundary. Calderón's point of view is that if one looks at $Q_\gamma(f)$ the problem is changed to finding enough solutions $u \in H^1(\Omega)$ of the conductivity equation in order to find γ in the interior. He carried out this approach for the linearized EIT problem at constant conductivity. He used the harmonic functions $e^{x\cdot\rho}$ with $\rho \in \mathbb{C}^n, \rho \cdot \rho = 0$.

Complex geometrical optics solutions with a linear phase

Sylvester and Uhlmann [41; 42] constructed in dimension $n \geq 2$ complex geometrical optics (CGO) solutions of the conductivity equation for C^2 conductivities that behave like Calderón exponential solutions for large frequencies. This can be reduced to constructing solutions in the whole space (by extending $\gamma = 1$ outside a large ball containing Ω) for the Schrödinger equation with potential.

Let $\gamma \in C^2(\mathbb{R}^n)$, γ strictly positive in $\mathbb{R}^n$ and $\gamma = 1$ for $|x| \geq R$, $R > 0$. Let $L_\gamma u = \nabla \cdot \gamma \nabla u$. Then we have

$$\gamma^{-\frac{1}{2}} L_\gamma(\gamma^{-\frac{1}{2}}) = \Delta - q, \quad q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}. \tag{4}$$

Therefore, to construct solutions of $L_\gamma u = 0$ in $\mathbb{R}^n$ it is enough to construct solutions of the Schrödinger equation $(\Delta - q)u = 0$ with q of the form (4). The next result states the existence of complex geometrical optics solutions for the Schrödinger equation associated to any bounded and compactly supported potential.

Theorem 1. *([[41; 42]]) Let* $q \in L^\infty(\mathbb{R}^n)$*,* $n \geq 2$*, with* $q(x) = 0$ *for* $|x| \geq R > 0$*. Let* $-1 < \delta < 0$*. There exists* $\epsilon(\delta)$ *and such that for every* $\rho \in \mathbb{C}^n$ *satisfying* $\rho \cdot \rho = 0$ *and* $\frac{\|(1+|x|^2)^{1/2}q\|_{L^\infty(\mathbb{R}^n)}+1}{|\rho|} \leq \epsilon$ *there exists a unique solution to*

$$(\Delta - q)u = 0$$

of the form

$$u = e^{x\cdot\rho}(1 + \psi_q(x, \rho)) \tag{5}$$

with $\psi_q(\cdot, \rho) \in L^2_\delta(\mathbb{R}^n)$*. Moreover* $\psi_q(\cdot, \rho) \in H^2_\delta(\mathbb{R}^n)$ *and for* $0 \leq s \leq 2$ *there exists* $C = C(n, s, \delta) > 0$ *such that* $\|\psi_q(\cdot, \rho)\|_{H^s_\delta} \leq \frac{C}{|\rho|^{1-s}}$*.*

Here $L^2_\delta(\mathbb{R}^n) = \{f; \int(1 + |x|^2)^\delta |f(x)|^2 dx < \infty\}$ with the norm given by $\|f\|^2_{L^2_\delta} = \int(1+|x|^2)^\delta |f(x)|^2 dx$ and $H^m_\delta(\mathbb{R}^n)$ denotes the corresponding Sobolev space. Note that for large $|\rho|$ these solutions behave like Calderón's exponential solutions. If 0 is not a Dirichlet eigenvalue for the Schrödinger equation we can also define the DN map

$$\Lambda_q(f) = \frac{\partial u}{\partial \nu}|_{\partial\Omega}$$

where u solves

$$(\Delta - q)u = 0; \quad u|_{\partial\Omega} = f.$$

More generally we can define the set of Cauchy data for the Schrödinger equation as the set

$$\mathbb{C}_q = \left\{ \left(u\Big|_{\partial\Omega}, \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} \right) \right\}, \tag{6}$$

where $u \in H^1(\Omega)$ is a solution of

$$(\Delta - q)u = 0 \text{ in } \Omega. \tag{7}$$

We have $\mathbb{C}_q \subseteq H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$. If 0 is not a Dirichlet eigenvalue of $\Delta - q$, then $\mathbb{C}_q$ is the grapht of the DN map.

The Calderón problem in dimension $n \geq 3$

The identifiability question in EIT was resolved for smooth enough isotropic conductivities. The result is

Theorem 2. *([41]) Let $\gamma_i \in C^2(\overline{\Omega})$, γ_i strictly positive, $i = 1, 2$. If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ then $\gamma_1 = \gamma_2$ in $\overline{\Omega}$.*

In dimension $n \geq 3$ this result is a consequence of a more general result. Let $q \in L^\infty(\Omega)$.

Theorem 3. *([41]) Let $q_i \in L^\infty(\Omega)$, $i = 1, 2$. Assume $\mathbb{C}_{q_1} = \mathbb{C}_{q_2}$, then $q_1 = q_2$.*

Theorem 2 has been extended to conductivities having 3/2 derivatives in some sense in [37], [7]. Uniqueness for conormal conductivies in $C^{1+\epsilon}$ was shown in [16]. It is an open problem whether uniqueness holds in dimension $n \geq 3$ for Lipschitz or less regular conductivities. For conormal potentials with singularities including almost a delta function of an hypersurface, uniqueness was shown in [16]. Stability for EIT using CGO solutions was shown by Alessandrini [1], and a reconstruction method was proposed by Nachman [**?**].

Other applications

We give a short list of other applications to inverse problems using the CGO solutions described above for the Schrödinger equation.

Quantum Scattering

In dimension $n \geq 3$ and in the case of a compactly supported electric potential, uniqueness for the fixed energy scattering problem was proven in [32], [34], [38]. For compactly supported potentials knowledge of the scattering amplitude at fixed energy is equivalent to knowing the Dirichlet-to-Neumann map for the Schrödinger equation measured on the boundary of a large ball containing the support of the potential (see [43] for an account). Then Theorem 3 implies the result. Melrose [30] suggested a related proof that uses the density of products of scattering solutions. Applications of CGO solutions to the 3-body problem were given in [44].

Optics

The DN map associated to the Helmholtz equation $-\Delta + k^2 n(x)$ with an isotropic index of refraction n determines uniquely a bounded index of refraction in dimension $n \geq 3$.

Optical tomography in the diffusion approximation

In this case we have $\nabla \cdot \mathrm{d}(x)\nabla u - \sigma_a(x)u - i\omega u = 0$ in Ω where u represents the density of photons, d the diffusion coefficient, and σ_a the optical absorption. Using Theorem 2 one can show in dimension three or higher that if $\omega \neq 0$ one can recover both d and σ_a from the corresponding DN map. If $\omega = 0$ then one can recover one of the two parameters.

Photoacoustic Tomography

Applications of CGO solutions to quantitative photoacoustic tomography were given in [4], [5].

The partial data problem in dimension $n \geq 3$

In several applications in EIT one can only measure currents and voltages on part of the boundary. Substantial progress has been made recently on the problem of whether one can determine the conductivity in the interior by measuring the DN map on part of the boundary.

The paper [10] used the method of Carleman estimates with a linear weight to prove that, roughly speaking, knowledge of the DN map in “half” of the boundary is enough to determine uniquely a C^2 conductivity. The regularity assumption on the conductivity was relaxed to $C^{1+\epsilon}, \epsilon > 0$ in [26]. Stability estimates for the uniqueness result of [10] were given in [17].

The result [10] was substantially improved in [25]. The latter paper contains a global identifiability result where it is assumed that the DN map is measured on any open subset of the boundary of a strictly convex domain for all functions supported, roughly, on the complement. The key new ingredient is the construction of a larger class of CGO solutions than the ones considered in the previous sections. These have the form

$$u = e^{\tau(\phi + i\psi)}(a + r), \tag{8}$$

where $\nabla\phi \cdot \nabla\psi = 0, |\nabla\phi|^2 = |\nabla\psi|^2$ and ϕ is a limiting Carleman weights (LCW). Moreover a is smooth and non-vanishing and $\|r\|_{L^2(\Omega)} = O(\frac{1}{\tau})$, $\|r\|_{H^1(\Omega)} = O(1)$. Examples of LCW are the linear phase $\phi(x) = x \cdot \omega, \omega \in S^{n-1}$, used previously, and the non-linear phase $\phi(x) = \ln|x - x_0|$, where $x_0 \in \mathbf{R}^n \setminus \overline{\mathrm{ch}\,(\Omega)}$ which was used in [25]. Here $\mathrm{ch}\,(\Omega)$ denotes the convex hull of Ω. All the LCW in $\mathbb{R}^n$ were characterized in [15]. In two dimensions any harmonic function is a LCW.

The CGO solutions used in [25] are of the form

$$u(x, \tau) = e^{\log|x - x_0| + id(\frac{x - x_0}{|x - x_0|}, \omega)}(a + r) \tag{9}$$

where x_0 is a point outside the convex hull of Ω, ω is a unit vector and $d(\frac{x-x_0}{|x-x_0|}, \omega)$ denote distance. We take directions ω so that the distance function is smooth for $x \in \overline{\Omega}$. These are called *complex spherical waves* since the level sets of the real part of the pase are spheres centered at x_0. Further applications of these type of waves are given below. A reconstruction method based on the uniqueness proof of [25] was proposed in [33].

The Two Dimensionsal Case

In EIT Astala and Päivärinta [2], in a seminal contribution, have extended significantly the uniqueness result of [31] for conductivities having two derivatives in an appropriate sense and the result of [8] for conductivities having one derivative in appropriate sense, by proving that any L^∞ conductivity in two dimensions can be determined uniquely from the DN map. The proof of [2] relies also on construction of CGO solutions for the conductivity equation with L^∞ coefficients and the $\overline{\partial}$ method. This is done by transforming the conductivity equation to a quasi-regular map.

For the partial data problem it is shown in [23] that for a two dimensional bounded domain the Cauchy data for the Schrödinger equation measured on an arbitrary open subset of the boundary determines uniquely the potential. This implies, for the conductivity equation, that if one measures the current fluxes at the boundary on an arbitrary open subset of the boundary produced by voltage potentials supported in the same subset, one can determine uniquely the conductivity. The paper [23] uses Carleman estimates with weights which are harmonic functions with non-degenerate critical points to construct appropriate complex geometrical optics solutions to prove the result.

For the Schrödinger equation Bukhgeim in a recent breakthrough [9] proved that a potential in $L^p(\Omega), p > 2$ can be uniquely determined from the set of Cauchy data as defined in (6). Assume now that $0 \in \Omega$. Bukhgeim constructs CGO solutions of the form

$$u_1(z,k) = e^{z^2 k}(1 + \psi_1(z,k)), \quad u_2(z,k) = e^{-\overline{z}^2 k}(1 + \psi_2(z,k)) \tag{10}$$

where $z, k \in \mathbb{C}$ and we have used the complex notation $z = x_1 + ix_2$. Moreover ψ_1 and ψ_2 decay uniformly in Ω, in an appropriate sense, for $|k|$ large. Note that the weight

z^2k in the exponential is a limiting Carleman weight since it is a harmonic function but it has a non-degenerate critical point at 0.

Anisotropic Conductivities

Anisotropic conductivities depend on direction. Muscle tissue in the human body is an important example of an anisotropic conductor. For instance cardiac muscle has a conductivity of 2.3 mho in the transverse direction and 6.3 in the longitudinal direction. The conductivity in this case is represented by a positive definite, smooth, symmetric matrix $\gamma = (\gamma^{ij}(x))$ on Ω.

Under the assumption of no sources or sinks of current in Ω, the potential u in Ω, given a voltage potential f on $\partial\Omega$, solves the Dirichlet problem

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\gamma^{ij}\frac{\partial u}{\partial x_j}\right) = 0 \text{ on } \Omega, \quad u|_{\partial\Omega} = f. \tag{11}$$

The DN map is defined by

$$\Lambda_\gamma(f) = \sum_{i,j=1}^{n} \nu^i \gamma^{ij} \frac{\partial u}{\partial x_j}\Big|_{\partial\Omega} \tag{12}$$

where $\nu = (\nu^1, \ldots, \nu^n)$ denotes the unit outer normal to $\partial\Omega$ and u is the solution of (11). The inverse problem is whether one can determine the matrix γ by knowing Λ_γ. Unfortunately, Λ_γ doesn't determine γ uniquely. Let $\psi : \overline{\Omega} \to \overline{\Omega}$ be a C^∞ diffeomorphism with $\psi|_{\partial\Omega} = Id$ where Id denotes the identity map. We have

$$\Lambda_{\widetilde{\gamma}} = \Lambda_\gamma \tag{13}$$

where

$$\widetilde{\gamma} = \left(\frac{(D\psi)^T \circ \gamma \circ (D\psi)}{|\det D\psi|}\right) \circ \psi^{-1}. \tag{14}$$

Here $D\psi$ denotes the (matrix) differential of ψ, $(D\psi)^T$ its transpose and the composition in (14) is to be interpreted as multiplication of matrices.

We have then a large number of conductivities with the same DN map: any change of variables of Ω that leaves the boundary fixed gives rise to a new conductivity with the same electrostatic boundary measurements. The question is then whether this is the only obstruction to unique identifiability of the conductivity.

In two dimensions this has been shown for $L^\infty(\Omega)$ conductivities in [3]. This is done by reducing the anisotropic problem to the isotropic one by using isothermal coordinates and using Astala and Päivärinta's result in the isotropic case [2]. Earlier results were for C^3 conductivities using the result of Nachman [31], for Lipschitz conductivities in [39] using the techniques of [8] and [40] for anisotropic conductivities close to constant.

In three or more dimensions this has been shown for real-analytic conductivities ion domains with real-analytic boundary. In fact this problem admits a geometric formulation on manifolds [29] and it has been proven for real-analytic manifolds with boundary [27]. New CGO solutions were constructed in [15] for anisotropic conductivities or metrics for which roughly speaking the metric or conductivity is Euclidean in one direction.

Full Maxwell's Equations

Inverse Boundary Value Problems

In the present section, we consider the inverse boundary value problems for the full time-harmonic Maxwell's equations in a bounded domain, that is, to reconstruct three key electromagnetic parameters: electric permittivity $\varepsilon(x)$, conductivity $\sigma(x)$ and magnetic permeability $\mu(x)$, as functions of the spatial variables, from a specified set of electromagnetic field measurements taken on the boundary. To be more specific, let $E(x)$ and $H(x)$ denote the time-harmonic electric and magnetic fields inside the domain $\Omega \subset \mathbb{R}^3$. At the frequency $\omega > 0$, E and H satisfy the Maxwell's equations

$$\nabla \times E = i\omega\mu H, \quad \nabla \times H = -i\omega\gamma E \tag{15}$$

where $\gamma(x) = \varepsilon(x) + i\sigma(x)$. Assume that the parameters are L^∞ functions in Ω and, for some positive constants ε_m, ε_M, μ_m, μ_M and σ_M,

$$\varepsilon_m \le \varepsilon(x) \le \varepsilon_M, \quad \mu_m \le \mu(x) \le \mu_M, \quad 0 \le \sigma(x) \le \sigma_M \quad \text{for } \; x \in \overline{\Omega}. \tag{16}$$

To introduce the solution space, we define

$$H^1_{\text{Div}}(\Omega) := \left\{ u \in \left(H^1(\Omega)\right)^3 \;\middle|\; \text{Div}(\nu \times u|_{\partial\Omega}) \in H^{1/2}(\partial\Omega) \right\}$$

where on the boundary $\partial\Omega$, ν is the outer normal unit vector and Div denotes the surface divergence. Let $TH^{1/2}_{\text{Div}}(\partial\Omega)$ denote the Sobolev space obtained by taking natural tangential traces of functions in $H^1_{\text{Div}}(\Omega)$ on the boundary. It is well-known that (15) admits a unique solution $(E, H) \in H^1_{\text{Div}}(\Omega) \times H^1_{\text{Div}}(\Omega)$ with imposed boundary electric (or magnetic) condition $\nu \times E = f \in TH^{1/2}_{\text{Div}}(\partial\Omega)$ (or $\nu \times H = g \in TH^{1/2}_{\text{Div}}(\partial\Omega)$), except for a discrete set of resonant frequencies $\{\omega_n\}$ in the dissipative case, namely, $\sigma = 0$.

Then the inverse boundary value problem is to recover ε, σ and μ from the boundary measurements encoded as the well-defined impedance map

$$\Lambda^\omega : TH^{1/2}_{\text{Div}}(\partial\Omega) \to TH^{1/2}_{\text{Div}}(\partial\Omega)$$

$$f = \nu \times E|_{\partial\Omega} \mapsto \nu \times H|_{\partial\Omega}.$$

We remark that the impedance map Λ^ω is a natural analog of the Dirichlet-Neumann map for EIT, since it carries enough information of the electromagnetic energy in the system.

The underlying problem was first formulated in [13] and a local uniqueness result was obtained based on Calderón's linearization idea, that is, the parameters that are slightly perturbed from constants can be uniquely determined by the impedance map. For the global uniqueness and reconstruction of the parameters, the following result was proved in [35] and the proof was simplified later in [36] by introducing the so-called generalized Sommerfeld potentials.

Theorem 4 ([35], [36]). *Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with a $C^{1,1}$-boundary and a connected complement $\mathbb{R}^3 \backslash \overline{\Omega}$. Assume that ε, σ and μ are in $C^3(\mathbb{R}^3)$ satisfying the condition* (16) *in Ω and $\varepsilon(x) = \varepsilon_0$, $\mu(x) = \mu_0$ and $\sigma(x) = 0$ when $x \in \mathbb{R}^3 \backslash \overline{\Omega}$ for some constants ε_0 and μ_0. Assume that $\omega > 0$ is not a resonant frequency. Then the knowledge of Λ^ω determines the functions ε, σ and μ uniquely.*

A closely related problem to the one considered here is the inverse scattering problem of electromagnetism, that is, to reconstruct the unknown parameters from the far-field pattern of the scattered electromagnetic fields. It is shown in [14] that the refractive index $n(x)$ (corresponding to e.g., known constant μ but unknown $\varepsilon(x)$ and $\sigma(x)$) can be uniquely determined by the far-field patters of scattered electric fields satisfying

$$\nabla \times \nabla \times E - k^2 n(x) E = 0.$$

The approach is based on the ideas in [41] of constructing CGO type of solutions of the form $E = e^{ix\cdot\zeta}(\eta + R_\zeta)$ where $\zeta, \eta \in \mathbb{C}^3$, $\zeta \cdot \zeta = k^2$ and $\zeta \cdot \eta = 0$.

For Maxwell's equations (15), more generalized solutions of such type were constructed in [35] as follows.

Proposition 1 ([35]). *Suppose the parameters ε, σ and μ satisfy the condition in Theorem 4. Let $\eta, \theta, \zeta \in \mathbb{C}^3$ satisfy $\zeta \cdot \zeta = \omega^2$, $\zeta \times \eta = \omega\mu_0\theta$ and $\zeta \times \theta = -\omega\mu_0\eta$. Then for $|\zeta|$ large enough, the Maxwell's equation* (15) *admits a unique global solution (E, H) of the form*

$$E = e^{ix\cdot\zeta}(\eta + R_\zeta) \quad H = e^{ix\cdot\zeta}(\theta +_\zeta) \tag{17}$$

where $R_\zeta(x)$ and $Q_\zeta(x)$ belong to $(L^2_{-\delta}(\mathbb{R}^3))^3$ for $\delta \in [\frac{1}{2}, 1]$.

However, such vector CGO type solutions for both [14] and [35] do not have the property that R_ζ decays like $O(|\zeta|^{-1})$, which was a key ingredient in the proof of the uniqueness in the scalar case. The nature of this difficulty is that the vector-valued analogue of Faddeev's fundamental solution (for the scalar Schrödinger equation), used in the construction of (17), does not share the decaying property of it. In [14], this is tackled by constructing R_ζ that decays to zero in certain distinguished directions as $|\zeta|$ tends to infinity. By rotations, such special set of solutions are enough to determine the refractive index.

In [35], the approach to the final proof of uniqueness starts with the following identity obtained integrating by parts

$$\int_{\partial\Omega} \nu \times E \cdot \overline{H_0} + \Lambda^\omega(\nu \times E|_{\partial\Omega}) \cdot \overline{E_0} dS = i\omega \int_\Omega (\mu - \mu_0) H \cdot \overline{H_0} - (\gamma - \varepsilon_0) E \cdot \overline{E_0} dx \quad (18)$$

where (E, H) is an arbitrary solution of (15) while (E_0, H_0) is a solution in the free space where $\varepsilon = \varepsilon_0$, $\sigma = 0$ and $\mu = \mu_0$. It is shown that if one let ζ tend to infinity along a certain manifold (similar to the choices of directions and by rotations in [14]), the right-hand side of (18) has the asymptotic to be a nonlinear functional of unknown parameters ε, σ and μ. It results in a semilinear elliptic equation of the parameters and their uniqueness is a direct corollary of the unique continuation principle.

On the other hand, the article [36] reduces significantly the asymptotic estimates used in [35] by an augmenting technique, in which the Maxwell's equations are transformed into a matrix Schrödinger equation. To be more specific, denoting scalar functions $\Phi = \frac{i}{\omega}\nabla \cdot \gamma E$ and $\Psi = \frac{i}{\omega}\nabla \cdot \mu H$, we consider the following rescalization

$$X := \left(\frac{1}{\omega\gamma\mu^{1/2}}\Phi,\ \gamma^{1/2}E,\ \mu^{1/2}H,\ \frac{1}{\omega\mu\gamma^{1/2}}\Psi \right)^T \in (\mathcal{D}')^8. \quad (19)$$

Such rescalization is particularly chosen so that one has, under conditions on Φ and Ψ, the equivalence between Maxwell's equations (15) and a Dirac system about X

$$(P(i\nabla) - k + V)\, X = 0, \qquad P(i\nabla) := i \begin{pmatrix} 0 & \nabla\cdot & 0 & 0 \\ \nabla & 0 & \nabla\times & 0 \\ 0 & -\nabla\times & 0 & \nabla \\ 0 & 0 & \nabla\cdot & 0 \end{pmatrix}. \tag{20}$$

where $k = \omega(\varepsilon_0\mu_0)^{1/2}$ and $V \in (C^\infty(\mathbb{R}^3))^8$ (Here we assume the unknown parameters are C^∞). For a more detailed argument on the rescalization, we refer the readers to [12; 24]. Moreover, the operator $(P(i\nabla) - k + V)$ is related to the matrix Schrödinger operator by

$$(P(i\nabla) - k + V)\left(P(i\nabla) + k - V^T\right) = -(\Delta + k^2)\mathbf{1}_8 + Q \tag{21}$$

where $\mathbf{1}_8$ is the identity matrix and the potential $Q \in (C^\infty(\mathbb{R}^3))^{8\times 8}$ is compactly supported. Therefore, the generalized Sommerfeld potential Y defined by $X = \left(P(i\nabla) + k - V^T\right) Y$, satisfies the Schrödinger equation

$$-(\Delta + k^2)Y + QY = 0, \tag{22}$$

for which we can construct the CGO solution for some constant vector $y_{0,\zeta}$

$$Y_\zeta = e^{ix\cdot\zeta}(y_{0,\zeta} + v_\zeta) \tag{23}$$

where v_ζ decays to zero as $O(|\zeta|^{-1})$. The rest of the proof is based on the identity

$$-i\int_{\partial\Omega} Y_0^* \cdot P(\nu)X dS = \int_\Omega Y_0^* \cdot QY dx \tag{24}$$

where Y_0^* annihilates $P(i\nabla)+k$ and $P(\nu)$ is the matrix with $i\nabla$ replaced by ν in $P(i\nabla)$. Then substitute the CGO solution Y_ζ into the identity and let Y_0^* depend on ζ in an appropriate way. Taking $|\zeta|$ to infinity, the left-hand side of (24) can be computed from the impedance map Λ^ω and the right-hand side converges to functionals of Q. Such functionals carry the information of the unknown parameters and the reconstruction of each of them is possible when proper directions, along which ζ diverges, are chosen.

For the partial data problem, namely, to determine the parameters from the impedance map only made on part of the boundary, there are not as many results as in the scalar case. It is shown in [12] that if the measurements $\Lambda^{\omega}(f)$ is taken only on a nonempty open subset Γ of $\partial\Omega$ for $f = \nu \times E|_{\partial\Omega}$ supported in γ, where the inaccessible part $\overline{\partial\Omega\backslash\Gamma}$ is part of a plane or a sphere, the electromagnetic parameters can still be uniquely determined. Combined with the augmenting argument in [36], the proof in [12] generalized the reflection technique used in [22], where the restriction on the shape of inaccessible part comes from. As for another well-known method in dealing with partial data problems based on the Carleman estimates [10; 25], there are however significant difficulties in generalizing the method to the full system of Maxwell's equations, e.g., the CGO solutions constructed using Carleman estimates.

In the anisotropic setting, where the electromagnetic parameters depend on direction and are regarded as matrix-valued functions, one of the uniqueness results was obtained in [24] for Maxwell's equations on certain admissible Riemannian manifolds. Such manifold has a product structure and includes compact manifolds in Euclidean space, hyperbolic space and $\mathbb{S}^3$ minus a point, and also sufficiently small sub-manifolds of conformally flat manifolds as examples. A construction of CGO solutions based on direct Fourier arguments was provided with a suitable uniqueness result.

Identifying Electromagnetic Obstacles by the Enclosure Method

As another application of the important CGO solutions for scalar conductivity equations and Helmholtz equations, in [20], the enclosure method was introduced to determine the shape of an obstacle or inclusion embedded in a bounded domain with known background parameters like conductivity or sound speed, from the boundary measurements of electric currents or sound waves. The fundamental idea of this method is to implement the low penetrating ability of CGO plane waves due to its rapidly decaying property away from the key planes. The energies associated with such waves show little evidence of the existence of the inclusion unless the key planes have intersection with it. These planes will enclose the inclusion from each direction and the convex hull can be reconstructed. The method was improved in [19] by the complex spherical waves constructed in [25] to enclose some non-convex part of the shape of electrostatic inclusions. For the application on more generalized systems of two variables, in which case more choices of CGO solutions are available, we refer the article [45]. Numerical simulations of the approach were done in [21; 19].

For the full time-harmonic system of Maxwell's equations, the enclosure method is generalized in [49] to identify the electromagnetic obstacles embedded in lossless background media. Suppose the obstacle D satisfies $\overline{D} \subset \Omega$ and $\Omega \backslash \overline{D}$ is connected. It is embedded in an lossless electromagnetic medium and therefore the EM fields in $\Omega \backslash \overline{D}$ satisfy

$$\nabla \times E = i\omega\mu H, \quad \nabla \times H = -i\omega\varepsilon E, \tag{25}$$

with perfect magnetic obstacle condition $\nu \times H|_{\partial D} = 0$. With well-defined boundary impedance map denoted by Λ_D^ω on $\partial\Omega$ for non-resonant frequency ω, the inverse problem aims to recover the convex hull of D. The candidates of the probing waves are among the CGO solutions for the background medium, of the form

$$E_0 = \varepsilon^{1/2} e^{\tau(x\cdot\rho - t) + i\sqrt{\tau^2+\omega^2}x\cdot\rho^\perp}(\eta + R_\tau), \quad H_0 = \mu^{1/2} e^{\tau(x\cdot\rho - t) + i\sqrt{\tau^2+\omega^2}x\cdot\rho^\perp}(\theta + Q_\tau) \tag{26}$$

where the planes used to enclose the obstacle are level sets $\{x \cdot \rho = t\}$. It is possible to compute, from the impedance map Λ_D^ω, an energy difference between two systems: the domain with obstacle and the background domain without an obstacle, for the same boundary CGO inputs. This is denoted as an indicator function given by

$$I_\rho(\tau,t) := i\omega\int_{\partial\Omega}(\nu\times E_0)\cdot\overline{(\Lambda_D^\omega-\Lambda_\emptyset^\omega)(\nu\times E_0)\times\nu}dS. \tag{27}$$

Since that as $\tau\to\infty$, the CGO EM fields (26) decay to zero exponentially on the half space $\{x\cdot\rho<t\}$ and grow exponentially on the other half, one would expect $\lim_{\tau\to\infty}I_\rho(\tau,t)=0$, i.e., no energy detection, as long as D stays in $\{x\cdot\rho<t\}$. On the other hand, if D has any intersection with the opposite closed half space $\{x\cdot\rho\geq 0\}$, the limit should not any longer be small. This provides a way by testing different $\rho\in\mathbb{S}^2$ and $t>0$ to detect where the boundary of D lies. However, for the full system of Maxwell's equation, a difficulty arises when showing the non-vanishing property of the indicator function in the latter case. This is again mainly because that the CGO solutions' remainder terms R_τ and Q_τ do not decay. To address this, one can choose the relatively free incoming constant fields $\eta=\eta_\tau$ and $\theta=\theta_\tau$ share different asymptotic speeds as τ tends to infinity. In this way, one can prove that the lower bound of the indicator function is dominated by the CGO magnetic energy in D, which is never vanishing. Hence the enclosure method is developed. We would like to point out that in [49], the construction of CGO solutions for the system is based on the augmenting technique in [36] and the choice of constant fields η_τ and θ_τ is similar to that in [14; 35; 36].

A natural improvement of the enclosure method as in the scalar case is to examine the reconstruction of non-convex part of the shape of D. The complex spherical waves constructed in [25] using Carleman estimates are CGO solutions with nonlinear phase $\ln|x-x_0|$ where $x_0\in\mathbb{R}^3\backslash\overline{\Omega}$, with spherical level sets. When replacing the linear-phase-CGO solutions in the enclosure method by complex spherical waves, the obstacle or the inclusion is enclosed by the exterior of spheres. However, for Maxwell's equations, the Carleman estimates argument hasn't been carried out yet. Instead, it is shown, in [49], that one can implement the Kelvin transformation

$$T:\ x\ \mapsto\ R^2\frac{x-x_0}{|x-x_0|^2}+x_0,\quad x_0\in\mathbb{R}^3\backslash\overline{\Omega},\ R>0,$$

which maps spheres passing x_0 to planes. The invariance of Maxwell's equations under T makes it possible to compute the impedance map associated to the image domain $T(\Omega)$ and apply the enclosure method there with linear-phase-CGO solutions. This is equivalent to enclosing in the original domain with spheres, which are pre-images of the planes. We notice that the pull back of the linear-phase-CGO fields in the image space are complex spherical fields in the original space with LCW

$$\varphi(x)=R^2\frac{(x-x_0)\cdot\rho}{|x-x_0|^2}+x_0\cdot\rho.$$

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