On the Structure of Selmer Groups

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1 Introduction

Our objective in this paper is to prove a rather broad generalization of some classical theorems in Iwasawa theory. We begin by recalling two of those old results. The first is a theorem of Iwasawa, which we state in terms of Galois cohomology. Suppose that K is a totally real number field, that F is a totally complex quadratic extension of K, and that ψ is the nontrivial character of $\mathrm{Gal}(F/K)$, viewed as a character of the absolute Galois group G_K. Let p be an odd prime and let D be the Galois module which is isomorphic to $\mathbf{Q}_p/\mathbf{Z}_p$ as a group and on which G_K acts by ψ. Let K_∞ denote the cyclotomic $\mathbf{Z}_p$-extension of K. Thus $\Gamma = \mathrm{Gal}(K_\infty/K)$ is isomorphic to $\mathbf{Z}_p$. Define $S(K_\infty, D)$ to be the subgroup of $H^1(K_\infty, D)$ consisting of everywhere unramified cocycle classes. As is usual in Iwasawa theory, we can view $S(K_\infty, D)$ as a discrete Λ-module, where $\Lambda = \mathbf{Z}_p[[\Gamma]]$ is the completed group algebra for Γ over $\mathbf{Z}_p$. Iwasawa's theorem asserts that the Pontryagin dual of $S(K_\infty, D)$ has no nonzero, finite Λ-submodule.

The Selmer group for the above Galois module D, as defined in [Gr1], is precisely $S(K_\infty, D)$. One can identify it with $\mathrm{Hom}(X^{(\psi)}, D)$, where X is a certain Galois group on which $\mathrm{Gal}(F/K)$ acts. To be precise, one takes $X = \mathrm{Gal}(L_\infty/F_\infty)$, where $F_\infty = FK_\infty$ and L_∞ denotes the maximal, abelian pro-p extension of F_∞ which is unramified at all primes of F_∞. There is a canonical action of $G = \mathrm{Gal}(F_\infty/K)$ on X (defined by conjugation). Furthermore, one has $G \cong \Delta \times \Gamma$, where $\Delta = \mathrm{Gal}(F/K)$ (identified with $\mathrm{Gal}(F_\infty/K_\infty)$ and Γ is as above. We define $X^{(\psi)}$ to be $e_\psi X$, where $e_\psi \in \mathbf{Z}_p[\Delta]$ is the idempotent for ψ. Iwasawa proved that $X^{(\psi)}$ has no nonzero, finite Λ-submodules. The theorem stated above is equivalent to that result. We should also mention the well-known fact that X is a finitely-generated, torsion Λ-module, an earlier theorem of Iwasawa. Thus, the same statements are true for $X^{(\psi)}$ and so we say that $S(K_\infty, D)$ is *cofinitely generated* and *cotorsion* as a Λ-module.

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To state the second classical result, suppose that K is any number field and that E is an elliptic curve defined over K with good, ordinary reduction at the primes of K lying above p. The p-primary subgroup $\mathrm{Sel}_E(K_\infty)_p$ of the Selmer group for E over K_∞ is again a discrete Λ-module. Its Pontryagin dual $X_E(K_\infty)$ is a finitely-generated Λ-module. Mazur conjectured that $X_E(K_\infty)$ is a torsion Λ-module. If this is so, and if one makes the additional assumption that $E(K)$ has no element of order p, then one can show that $X_E(K_\infty)$ has no nonzero, finite Λ-submodule.

The above results take the following form: $\mathcal{S}$ is a certain discrete Λ-module. The Pontryagin dual $\mathcal{X} = \mathrm{Hom}(\mathcal{S}, \mathbf{Q}_p/\mathbf{Z}_p)$ is finitely generated as a Λ-module. The results assert that $\mathcal{X}$ has no nonzero finite Λ-submodule. An equivalent statement about $\mathcal{S}$ is the following: *There exists a nonzero element $\theta \in \Lambda$ such that $\pi\mathcal{S} = \mathcal{S}$ for all irreducible elements $\pi \in \Lambda$ not dividing θ.* We then say that $\mathcal{S}$ is an *"almost divisible"* Λ-module. Note that Λ is isomorphic to $\mathbf{Z}_p[[T]]$, a formal power series ring over $\mathbf{Z}_p$ in one variable, and hence is a unique factorization domain. Thus, one can equivalently say that $\lambda\mathcal{S} = \mathcal{S}$ for all $\lambda \in \Lambda$ which are relatively prime to θ. This definition makes sense in a much more general setting, as we now describe.

Suppose that Λ is isomorphic to a formal power series ring over $\mathbf{Z}_p$, or over $\mathbf{F}_p$, in a finite number of variables. Suppose that $\mathcal{S}$ is a discrete Λ-module and that its Pontryagin dual $\mathcal{X}$ is finitely generated. We then say that $\mathcal{S}$ is cofinitely generated. We say that $\mathcal{S}$ is *almost divisible* if any one of the five equivalent statements given below is satisfied. In the statements, the set of prime ideals of Λ of height 1 is denoted by $\mathrm{Spec}_{ht=1}(\Lambda)$. Note that since Λ is a UFD, all such prime ideals Π are principal. Also, if we say *almost all*, we mean *all but finitely many*. The notation $\mathcal{X}[\Pi]$ denotes the Λ-submodule of $\mathcal{X}$ consisting of elements annihilated by Π. This is also denoted by $\mathcal{X}[\pi]$, where π is a generator of Π.

- *We have $\Pi\mathcal{S} = \mathcal{S}$ for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$.*

- *There exists a nonzero element θ in Λ such that $\pi\mathcal{S} = \mathcal{S}$ for all irreducible elements π of Λ not dividing θ.*

- *We have $\mathcal{X}[\Pi] = 0$ for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$.*

- *The set $\mathrm{Ass}_\Lambda(\mathcal{Y})$ of associated prime ideals for the torsion Λ-submodule $\mathcal{Y}$ of $\mathcal{X}$ contains only prime ideals of height 1.*

- *The Λ-module $\mathcal{X}$ has no nonzero, pseudo-null submodules.*

If it is not sufficiently clear from the context, we may say that $\mathcal{S}$ *is almost divisible as a Λ-module*, or just *almost Λ-divisible*. We refer the reader to [Gr4] for further discussion, including an explanation of the equivalence of all of the above statements.

We will consider "Selmer groups" that arise in the following very general context. Suppose that K is a finite extension of $\mathbf{Q}$ and that Σ is a finite set of primes of K. Let K_Σ denote the maximal extension of K unramified outside of Σ. We assume that Σ contains all archimedean primes and all primes lying over some fixed rational prime p. The Selmer groups that we consider in this article are associated to a continuous representation

$$\rho : \mathrm{Gal}(K_\Sigma/K) \longrightarrow GL_n(R)$$

where R is a complete Noetherian local ring. Let $\mathfrak{M}$ denote the maximal ideal of R. We assume that the residue field $R/\mathfrak{M}$ is finite and has characteristic p. Hence R is compact in its $\mathfrak{M}$-adic topology. Let $\mathcal{T}$ be the underlying free R-module on which $\mathrm{Gal}(K_\Sigma/K)$ acts via ρ. We define $\mathcal{D} = \mathcal{T} \otimes_R \widehat{R}$, where $\widehat{R} = \mathrm{Hom}(R, \mathbf{Q}_p/\mathbf{Z}_p)$ is the Pontryagin dual of R with a trivial action of $\mathrm{Gal}(K_\Sigma/K)$. That Galois group acts on $\mathcal{D}$ through its action on the first factor $\mathcal{T}$. Thus, $\mathcal{D}$ is a discrete abelian group which is isomorphic to $\widehat{R}^n$ as an R-module and which has a continuous R-linear action of $\mathrm{Gal}(K_\Sigma/K)$.

The Galois cohomology group $H^1(K_\Sigma/K, \mathcal{D})$ can be considered as a discrete R-module too. It is a cofinitely generated R-module in the sense that its Pontryagin dual is finitely generated as an R-module. (See Prop. 3.2 in [Gr4].) Suppose that one specifies an R-submodule $L(K_v, \mathcal{D})$ of $H^1(K_v, \mathcal{D})$ for each $v \in \Sigma$. We will denote such a specification by $\mathcal{L}$ for brevity. Let

$$P(K, \mathcal{D}) \;=\; \prod_{v \in \Sigma} H^1(K_v, \mathcal{D}) \qquad \textit{and} \qquad L(K, \mathcal{D}) \;=\; \prod_{v \in \Sigma} L(K_v, \mathcal{D}) \quad .$$

Thus, $L(K, \mathcal{D})$ is an R-submodule of $P(K, \mathcal{D})$. Let $Q_{\mathcal{L}}(K, \mathcal{D}) = P(K, \mathcal{D})\big/L(K, \mathcal{D})$. Thus,

$$Q_{\mathcal{L}}(K, \mathcal{D}) \;=\; \prod_{v \in \Sigma} Q_{\mathcal{L}}(K_v, \mathcal{D}), \qquad \textit{where} \quad Q_{\mathcal{L}}(K_v, \mathcal{D}) \;=\; H^1(K_v, \mathcal{D})\big/L(K_v, \mathcal{D}) \quad .$$

The natural global-to-local restriction maps for $H^1(\;\cdot\;, \mathcal{D})$ induce a map

$$\phi_{\mathcal{L}} :\; H^1(K_\Sigma/K, \mathcal{D}) \longrightarrow Q_{\mathcal{L}}(K, \mathcal{D}) \quad . \tag{1}$$

The kernel of $\phi_{\mathcal{L}}$ will be denoted by $S_{\mathcal{L}}(K, \mathcal{D})$. It is the *"Selmer group"* for $\mathcal{D}$ over K corresponding to the specification $\mathcal{L}$.

It is clear that $S_{\mathcal{L}}(K, \mathcal{D})$ is an R-submodule of $H^1(K_\Sigma/K, \mathcal{D})$ and so is also a discrete, cofinitely generated R-module. For a fixed set Σ, the smallest possible Selmer group occurs when we take $L(K_v, \mathcal{D}) = 0$ for all $v \in \Sigma$. The Selmer group corresponding to that choice will be denoted by $\text{Ш}^1(K, \Sigma, \mathcal{D})$. That is,

$$\text{Ш}^1(K, \Sigma, \mathcal{D}) \;=\; \ker\Big(H^1(K_\Sigma/K, \mathcal{D}) \longrightarrow \prod_{v \in \Sigma} H^1(K_v, \mathcal{D})\Big)$$

Obviously, we have $Ш^1(K, \Sigma, \mathcal{D}) \subseteq S_{\mathcal{L}}(K, \mathcal{D})$ for any choice of the specification $\mathcal{L}$.

In addition to the above assumptions about R, suppose that R is a domain. Let $d = m+1$ denote the Krull dimension of R, where $m \geq 0$. (We will assume that R is not a field. Our results are all trivial in that case.) A theorem of Cohen [Coh] implies that R is a finite, integral extension of a subring Λ which is isomorphic to one of the formal power series rings $\mathbf{Z}_p[[T_1, ..., T_m]]$ or $\mathbf{F}_p[[T_1, ..., T_{m+1}]]$, depending on whether R has characteristic 0 or p. Although such a subring is far from unique, it will be convenient to just fix a choice. A cofinitely generated R-module $\mathcal{S}$ will also be a cofinitely generated Λ-module. All the results that we will prove in this paper could be viewed as statements about the structure of the Selmer groups as R-modules. But they are equivalent to the corresponding statements about their structure as Λ-modules and that is how we will formulate and prove them. Those equivalences are discussed in some detail in [Gr4], section 2. In particular, if $\mathcal{S}$ is a discrete, cofinitely generated R-module, then we say that $\mathcal{S}$ is divisible (resp., almost divisible) as an R-module if $P\mathcal{S} = \mathcal{S}$ for all (resp., almost all) $P \in \mathrm{Spec}_{ht=1}(R)$. One result is that $\mathcal{S}$ is almost divisible as an R-module if and only if $\mathcal{S}$ is almost divisible as a Λ-module. (See statement **1** on page 372 of [Gr4].) A similar equivalence is true for divisibility, but quite easy to prove.

One basic assumption that we will make about R is that it contain a subring Λ of the form described in the previous paragraph, that R is finitely-generated as a Λ-module, and that R is also reflexive as a Λ-module. If these assumptions are satisfied, we say that R is a *"reflexive ring"*. In the case where R is also assumed to be a domain, one can equivalently require that R is the intersection of all its localizations at prime ideals of height 1. See part D, section 2 in [Gr4] for the explanation of the equivalence. In the literature, one sometimes finds the term *"weakly Krull domain"* for such a domain. The class of reflexive domains is rather large. For example, if R is integrally closed or Cohen-Macaulay, then it turns out that R is reflexive. There are important examples (from Hida theory), where R is not necessarily a domain, but is still a free (and hence reflexive) module over a suitable subring Λ.

The main results of this paper assert that if we make certain hypotheses about $\mathcal{D}$ and $\mathcal{L}$, then $S_{\mathcal{L}}(K, \mathcal{D})$ will be almost divisible. Some of the hypotheses are those needed for theorem 1 in [Gr4] which gives sufficient conditions for $H^1(K_\Sigma/K, \mathcal{D})$ itself to be almost divisible. That theorem will be stated later (as proposition 2.6.1.) and is our starting point. The basic approach for deducing the almost divisibility of a Λ-submodule of $H^1(K_\Sigma/K, \mathcal{D})$, defined by imposing local conditions corresponding to a specification $\mathcal{L}$, will be described in section 3. Some of the needed hypotheses will be discussed in section 2. We also state there some results from from [Gr5] concerning the surjectivity of the map $\phi_{\mathcal{L}}$. We will apply those results not just to $\mathcal{D}$, but also to the corresponding map for $\mathcal{D}[\Pi]$, where $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. Our main results concerning the almost divisibility of $S_{\mathcal{L}}(K, \mathcal{D})$ will be proved in section 4.1.

We show in section 4.2 how to prove the two classical theorems mentioned above from the point of view of this paper.

This paper is part of a series of papers concerning foundational questions in Iwasawa theory. The results discussed above depend on the results proved in [Gr4] and [Gr5], the first of this series. A subsequent paper will use the results we prove here to study the behavior of Selmer groups under specialization. In particular, one would like to understand how the *"characteristic ideal"* or *"characteristic divisor"* for a Selmer group associated to the representation $\rho_P : \mathrm{Gal}(K_\Sigma/K) \longrightarrow GL_n(R/P)$, the reduction of ρ modulo a prime ideal P of R, is related to the characteristic ideal or divisor associated to a Selmer group for ρ itself. Such a question has arisen many times in the past. Consequently, for the purpose of studying exactly that question, one can find numerous special cases of the results of this paper in the literature in Iwasawa theory.

2 Various Hypotheses.

The Galois module $\mathcal{T}$ is a free R-module and so we say that $\mathcal{D}$ is a cofree R-module. We also define $\mathcal{T}^* = \mathrm{Hom}(\mathcal{D}, \mu_{p^\infty})$. We can consider $\mathcal{T}^*$ as a module over the ring R^{op}, which is just R since that ring is commutative. It is clear that $\mathcal{T}^*$ is also a free R-module and that $\mathcal{D}^*$ is cofree. It will be simpler and more useful to formulate the hypotheses in terms of their structure as Λ-modules rather than R-modules.

2.1. Hypotheses involving reflexivity. Recall that Λ is isomorphic to a formal power series ring in a finite number of variables over either $\mathbf{Z}_p$ or $\mathbf{F}_p$. Reflexive Λ-modules play an important role here. A detailed discussion of the definition can be found in section 2, part C, of [Gr4].

RFX($\mathcal{D}$): *The Λ-module $\mathcal{T}$ is reflexive.*

Equivalently, since $\mathcal{T}$ is free as an R-module, RFX($\mathcal{D}$) means that the ring R is reflexive as a Λ-module. That is, R is a reflexive ring in the sense defined in the introduction. We are still assuming that R is a complete Noetherian local ring with finite residue field of characteristic p.

We will say that $\mathcal{D}$ is a coreflexive Λ-module if RFX($\mathcal{D}$) holds. This terminology is appropiate because the Pontryagin dual of $\mathcal{D}$ is the Λ-module $\mathcal{T}$. One important role of RFX($\mathcal{D}$) is to guarantee that $\mathcal{D}[\pi]$ is a divisible (Λ/Π)-module for all prime ideals $\Pi = (\pi)$ in Λ of height 1. That property is equivalent to requiring that $\mathcal{D}$ be coreflexive as a Λ-module. See corollary 2.6.1 in [Gr4] for the proof.

The next two hypotheses involve $\mathcal{T}^*$ and are of a local nature. They could be formulated just in terms of $\mathcal{D}$, but the statements would become more complicated. Note that if $\mathrm{RFX}(\mathcal{D})$ holds, then $\mathcal{T}^*$ is a reflexive Λ-module. We suppose that v is a prime of K and that K_v is the completion of K at v. We usually consider just the primes $v \in \Sigma$.

$\mathrm{LOC}_v^{(1)}(\mathcal{D})$: $\quad (\mathcal{T}^*)^{G_{K_v}} = 0$.

$\mathrm{LOC}_v^{(2)}(\mathcal{D})$: $\quad$ *The* Λ*-module* $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ *is reflexive.*

Assumptions $\mathrm{LOC}_v^{(1)}(\mathcal{D})$ and $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ play a crucial role in proving theorem 1 in [Gr4]. Just as in that result, we will usually assume $\mathrm{LOC}_v^{(1)}(\mathcal{D})$ for at least one non-archimedean prime $v \in \Sigma$ and $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ for all $v \in \Sigma$. One can find a general discussion of when those hypotheses are satisfied in part F, section 5 of *loc cit.* One obvious remark is that since $\mathcal{T}^*$ is a torsion-free Λ-module, $\mathrm{LOC}_v^{(1)}$ is satisfied if and only if $\mathrm{rank}_\Lambda\big((\mathcal{T}^*)^{G_{K_v}}\big) = 0$. It is also obvious that $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is at least torsion-free as a Λ-module. Furthermore, note that if $\mathrm{RFX}(\mathcal{D})$ is true, then $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ follows from $\mathrm{LOC}_v^{(1)}(\mathcal{D})$. Notice also that if $\mathrm{LOC}_v^{(1)}(\mathcal{D})$ and $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ are both true for some prime v, then $\mathrm{RFX}(\mathcal{D})$ is also true. Nevertheless, our propositions will often include $\mathrm{RFX}(\mathcal{D})$ as a hypothesis even though it may actually be implied by other hypotheses.

2.2. Locally trivial cocycle classes. The following much more subtle hypothesis is also needed in the proof of theorem 1 in [Gr4], where it is referred to as Hypothesis L. As we explain there, it can be viewed as a generalization of Leopoldt's Conjecture for number fields. That special case occurs when $\Lambda = \mathbf{Z}_p$, $\mathcal{D} = \mathbf{Q}_p/\mathbf{Z}_p$, and G_K acts trivially on $\mathcal{D}$. For the formulation, we define

$$Ш^2(K, \Sigma, \mathcal{D}) \;=\; \ker\Big(H^2(K_\Sigma/K, \mathcal{D}) \longrightarrow \prod_{v\in\Sigma} H^2(K_v, \mathcal{D})\Big),$$

which is a discrete, cofinitely-generated Λ-module.

$\mathrm{LEO}(\mathcal{D})$: $\quad$ *The* Λ*-module* $Ш^2(K, \Sigma, \mathcal{D})$ *is cotorsion.*

A long discussion about the validity of the above hypothesis can be found in the last few pages of [Gr4]. There are situations where it fails to be satisfied. Also, section 4 of that paper derives a natural lower bound on the Λ-corank of $H^1(K_\Sigma/K, \mathcal{D})$ from the duality theorems of Poitou and Tate. Hypothesis $\mathrm{LEO}(\mathcal{D})$ is equivalent to the statement that $\mathrm{corank}_\Lambda\big(H^1(K_\Sigma/K, \mathcal{D})\big)$ is equal to that lower bound. That equivalence is the content of propositions 4.3 and 4.4 in [Gr4].

2.3. Hypotheses involving $\mathcal{L}$. None of the hypotheses stated above involves the specification $\mathcal{L}$. We now mention two hypotheses which do involve $\mathcal{L}$, one of which implies the other. They are statements about the cokernel of the map $\phi_{\mathcal{L}}$ defined in the introduction. The first plays an important role in studying Selmer groups. The second appears weaker, but often is sufficient to imply the first.

SUR$(\mathcal{D},\mathcal{L})$: *The map $\phi_{\mathcal{L}}$ defining $S_{\mathcal{L}}(K,\mathcal{D})$ is surjective.*

An obvious necessary condition for this to be satisfied is the following equality for the coranks:

$$\text{CRK}(\mathcal{D},\mathcal{L}):\quad \text{corank}_\Lambda\big(H^1(K_\Sigma/K,\mathcal{D})\big) \;=\; \text{corank}_\Lambda\big(S_{\mathcal{L}}(K,\mathcal{D})\big) \;+\; \text{corank}_\Lambda\big(Q_{\mathcal{L}}(K,\mathcal{D})\big)\ .$$

This just means that $\text{coker}\big(\phi_{\mathcal{L}}\big)$ is a cotorsion Λ-module. Proposition 3.2.1 in [Gr5] shows that CRK$(\mathcal{D},\mathcal{L})$, together with various additional assumptions, actually implies SUR$(\mathcal{D},\mathcal{L})$. One has the following obvious inequality:

$$(2)\qquad \text{corank}_\Lambda\big(S_{\mathcal{L}}(K,\mathcal{D})\big) \;\geq\; \text{corank}_\Lambda\big(H^1(K_\Sigma/K,\mathcal{D})\big) \;-\; \text{corank}_\Lambda\big(Q_{\mathcal{L}}(K,\mathcal{D})\big)$$

Thus, CRK$(\mathcal{D},\mathcal{L})$ is equivalent to having equality here. Of course, CRK$(\mathcal{D},\mathcal{L})$, and hence SUR$(\mathcal{D},\mathcal{L})$, can fail simply because the quantity on the right side is negative. Verifying CRK$(\mathcal{D},\mathcal{L})$ is quite a difficult problem in many interesting cases.

It is worth recalling what the formulas for global and local Euler-Poincaré characteristics tell us about the coranks on the right side of (2). One can find proofs in section 4 of [Gr4]. For any prime v of K, we use the notation $\mathcal{D}(K_v)$ as an abbreviation for $H^0(K_v,\mathcal{D})$, a Λ-submodule of $\mathcal{D}$. Similarly, $\mathcal{D}(K)$ will denote $H^0(K,\mathcal{D})$. Let $r_1(K)$ and $r_2(K)$ denote the number of real primes and complex primes of K, respectively. We give formulas for the Λ-coranks of the global and local H^1's. For the global H^1, we have

$$\text{corank}_\Lambda\big(H^1(K_\Sigma/K,\mathcal{D})\big) \;=\; \text{corank}_\Lambda\big(\mathcal{D}(K)\big) \;+\; \text{corank}_\Lambda\big(H^2(K_\Sigma/K,\mathcal{D})\big) \;+\; \delta_\Lambda(K,\mathcal{D})\ \ ,$$

where $\delta_\Lambda(K,\mathcal{D}) = \big(r_1(K)+r_2(K)\big)\text{corank}_\Lambda\big(\mathcal{D}\big) - \sum_{v\ real}\text{corank}_\Lambda\big(\mathcal{D}(K_v)\big)$. Now assume that v is a non-archimedean prime. We will use the notation $\mathcal{D}^*$ to denote $\mathcal{T}^*\otimes_R \widehat{R}$. If v does not lie over p, then the local Euler-Poincaré characteristic is 0 and we have

$$\text{corank}_\Lambda\big(H^1(K_v,\mathcal{D})\big) \;=\; \text{corank}_\Lambda\big(\mathcal{D}(K_v)\big) \;+\; \text{corank}_\Lambda\big(\mathcal{D}^*(K_v)\big)\quad .$$

To justify replacing the Λ-corank of $H^2(K_v,\mathcal{D})$ by that of $\mathcal{D}^*(K_v)$ in the above formula as well as the formula below, one uses the fact that the Pontryagin dual of $H^2(K_v,\mathcal{D})$ is isomorphic to $H^0(K_v, \mathcal{T}^*)$. Proposition 3.10 in [Gr4] implies that the Λ-rank of $H^0(K_v, \mathcal{T}^*)$ is equal to the Λ-corank of $H^0(K_v, \mathcal{D}^*)$. If v lies over p, then we have

$$\mathrm{corank}_\Lambda\big(H^1(K_v, \mathcal{D})\big) \;=\; \mathrm{corank}_\Lambda\big(\mathcal{D}(K_v)\big) \;+\; \mathrm{corank}_\Lambda\big(\mathcal{D}^*(K_v)\big) \;+\; [K_v : \mathbf{Q}_p]\mathrm{corank}_\Lambda\big(\mathcal{D}\big) \quad .$$

For a non-archimedean prime v, one then needs to know the Λ-corank of $L(K_v, \mathcal{D})$ to determine the Λ-corank of $Q_{\mathcal{L}}(K_v, \mathcal{D})$.

2.4. Behavior under specialization. In some proofs, Selmer groups for $\mathcal{D}[\Pi]$, as well as for $\mathcal{D}$, will occur. Here Π is a prime ideal of Λ and $\mathcal{D}[\Pi]$ is a discrete, cofinitely-generated module over the ring Λ/Π. Various other modules over Λ/Π will arise. Now Λ/Π is a complete, Noetherian, local ring, and therefore (just as for R in the introduction), it is a finite, integral extension of a subring Λ' which is isomorphic to a formal power series ring over $\mathbf{Z}_p$ or $\mathbf{F}_p$. We fix such a choice for each Π and denote Λ' by Λ_Π. If Λ has Krull dimension d, then Λ_Π has Krull dimension $d-1$. Of course, some results could be easily stated or proved just in terms of Λ/Π itself.

Many of the above hypotheses are not preserved when the Λ-module $\mathcal{D}$ is replaced by the Λ_Π-module $\mathcal{D}[\Pi]$. For example, even if RFX($\mathcal{D}$) is satisfied, $\mathcal{D}[\Pi]$ may fail to be reflexive as a Λ_Π-module and so RFX($\mathcal{D}[\Pi]$) may fail to be satisfied. In general, all one can say is that RFX($\mathcal{D}$) implies that $\mathcal{D}[\Pi]$ is a divisible Λ_Π-module for all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. The situation is better for $\mathrm{LOC}_v^{(1)}(\mathcal{D})$ and LEO($\mathcal{D}$). We have the following equivalences.

- *Assume that* RFX($\mathcal{D}$) *is satisfied. Then* $\mathrm{LOC}_v^{(1)}(\mathcal{D})$ *is true if and only if* $\mathrm{LOC}_v^{(1)}(\mathcal{D}[\Pi])$ *is true for almost all* $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$.

- LEO($\mathcal{D}$) *is true if and only if* LEO($\mathcal{D}[\Pi]$) *is true for almost all* $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$.

These assertions follow easily from results in [Gr4]. For the first statement, one should see remarks 3.5.1 or 3.10.1 there. Note that $\mathrm{LOC}_v^{(1)}(\mathcal{D}[\Pi])$ and $\mathrm{LOC}_v^{(2)}(\mathcal{D}[\Pi])$ are statements about the (Λ/Π)-module $\mathrm{Hom}(\mathcal{D}[\Pi], \mu_{p^\infty}) \cong \mathcal{T}^*/\Pi\mathcal{T}^*$. The second of the above equivalences follows from lemma 4.4.1 and remark 2.1.3 in [Gr4]. We will prove a similar equivalence for CRK($\mathcal{D}, \mathcal{L}$) in section 3.4.

2.5. A result about almost divisibility. In addition to SUR($\mathcal{D}, \mathcal{L}$) and CRK($\mathcal{D}, \mathcal{L}$), there will be various other hypotheses concerning the specification $\mathcal{L}$. If $L(K_v, \mathcal{D})$ is Λ-divisible (resp., almost Λ-divisible) for all $v \in \Sigma$, then we will say that $\mathcal{L}$ is Λ-divisible (resp, almost Λ-divisible). Consider another specification $\mathcal{L}'$ for $\mathcal{D}$ and let $L'(K_v, \mathcal{D})$ denote the corresponding subgroup of $H^1(K_v, \mathcal{D})$ for each $v \in \Sigma$. We write $\mathcal{L}' \subseteq \mathcal{L}$ if $L'(K_v, \mathcal{D}) \subseteq L(K_v, \mathcal{D})$ for all $v \in \Sigma$. In particular, if $L'(K_v, \mathcal{D}) = L(K_v, \mathcal{D})_{\Lambda\text{-}div}$ for each $v \in \Sigma$, then we will refer to the specification $\mathcal{L}'$ as the maximal Λ-divisible subspecification of $\mathcal{L}$, which we denote simply by $\mathcal{L}_{div}$. One assumption that we will usually make is that $\mathcal{L}$ is almost divisible. Its importance is clear from the following proposition.

Proposition 2.5.1. *Assume that $\mathcal{L}'$ and $\mathcal{L}$ are specifications for $\mathcal{D}$ and that $\mathcal{L}' \subseteq \mathcal{L}$. Assume also that $\mathrm{SUR}(\mathcal{D}, \mathcal{L}')$ is true. Then $\mathrm{SUR}(\mathcal{D}, \mathcal{L})$ is also true, $S_{\mathcal{L}'}(K, \mathcal{D}) \subseteq S_{\mathcal{L}}(K, \mathcal{D})$, and*

$$S_{\mathcal{L}}(K, \mathcal{D}) \big/ S_{\mathcal{L}'}(K, \mathcal{D}) \;\cong\; \prod_{v \in \Sigma} L(K_v, \mathcal{D}) \big/ L'(K_v, \mathcal{D})$$

as Λ-modules. In particular, if $\mathrm{SUR}(\mathcal{D}, \mathcal{L}_{div})$ is true and $S_{\mathcal{L}_{div}}(K, \mathcal{D})$ is almost Λ-divisible, then $S_{\mathcal{L}}(K, \mathcal{D})$ is almost Λ-divisible if and only if $\mathcal{L}$ is almost Λ-divisible. If $\mathrm{SUR}(\mathcal{D}, \mathcal{L}_{div})$ is true and $S_{\mathcal{L}}(K, \mathcal{D})$ is almost Λ-divisible, then $\mathcal{L}$ must be almost Λ-divisible.

Thus, under certain assumptions, the structure of $S_{\mathcal{L}}(K, \mathcal{D})$ can be related to that of $S_{\mathcal{L}_{div}}(K, \mathcal{D})$ and the quotient Λ-modules $L(K_v, \mathcal{D}) \big/ L(K_v, \mathcal{D})_{div}$ for $v \in \Sigma$. Since all of those quotients are cofinitely-generated, cotorsion Λ-module for all $v \in \Sigma$, it follows that $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$ is true if and only if $\mathrm{CRK}(\mathcal{D}, \mathcal{L}_{div})$ is true.

Proof. Most of the statements are clear from the definitions. For the isomorphism, consider the following maps:

$$H^1(K_\Sigma/K, \mathcal{D}) \xrightarrow{\phi_{\mathcal{L}'}} Q_{\mathcal{L}'}(K, \mathcal{D}) \xrightarrow{\psi} Q_{\mathcal{L}}(K, \mathcal{D})$$

where ψ is the natural map, the canonical homomorphism whose kernel is the direct product in the proposition. The map ψ is surjective and the composition is $\phi_{\mathcal{L}}$. If $\phi_{\mathcal{L}'}$ is surjective, then it follows that $\phi_{\mathcal{L}}$ is also surjective and that $S_{\mathcal{L}}(K, \mathcal{D}) \big/ S_{\mathcal{L}'}(K, \mathcal{D})$ is isomorphic to $\ker(\psi)$. The stated isomorphism follows immediately. For the final statements, one takes $\mathcal{L}' = \mathcal{L}_{div}$. Note that if $S_{\mathcal{L}}(K, \mathcal{D})$ is almost divisible, and if one assumes $\mathrm{SUR}(\mathcal{D}, \mathcal{L}_{div})$, then there is a surjective homomorphism from $S_{\mathcal{L}}(K, \mathcal{D})$ to $L(K_v, \mathcal{D}) \big/ L(K_v, \mathcal{D})_{\Lambda\text{-}div}$, which must therefore be almost divisible too. This implies that $L(K_v, \mathcal{D})$ is then almost divisible. Thus, $\mathcal{L}$ is almost divisible. Moreover, if a discrete, cofinitely-generated Λ-module $\mathcal{S}$ contains an almost divisible Λ-submodule $\mathcal{S}'$, then it is clear that $\mathcal{S}$ is almost divisible if and only if $\mathcal{S}/\mathcal{S}'$ is almost divisible. ∎

2.6. The main results in [Gr4] and [Gr5]. The following result is proved in [Gr4] . It is part of the theorem 1 which we alluded to before. It plays a crucial role in this paper because we will study when $S_{\mathcal{L}}(K, \mathcal{D})$ is almost divisible as a Λ-module under the assumption that $H^1(K_\Sigma/K, \mathcal{D})$ is almost divisible, as outlined in the next section.

Proposition 2.6.1. *Suppose that* $\mathrm{RFX}(\mathcal{D})$ *and* $\mathrm{LEO}(\mathcal{D})$ *are satisfied, that* $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ *is satisfied for all* v *in* Σ*, and that there exists a non-archimedean prime* $\eta \in \Sigma$ *such that* $\mathrm{LOC}_\eta^{(1)}(\mathcal{D})$ *is satisfied. Then* $H^1(K_\Sigma/K, \mathcal{D})$ *is an almost divisible* Λ*-module.*

Another part of theorem 1 is the following.

Proposition 2.6.2. *Suppose that* $\mathrm{RFX}(\mathcal{D})$ *is satisfied, that* $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ *is satisfied for all* v *in* Σ*, and that there exists a non-archimedean prime* $\eta \in \Sigma$ *such that* $\mathrm{LOC}_\eta^{(1)}(\mathcal{D})$ *is satisfied. Then* $Ш^2(K, \Sigma, \mathcal{D})$ *is a coreflexive* Λ*-module.*

The conclusion in this result has the interesting consequence that the Pontryagin dual of $Ш^2(K, \Sigma, \mathcal{D})$ is torsion-free as a Λ-module. It follows that $Ш^2(K, \Sigma, \mathcal{D})$ is Λ-divisible. Hence either $Ш^2(K, \Sigma, \mathcal{D})$ has positive Λ-corank or $Ш^2(K, \Sigma, \mathcal{D}) = 0$ under the assumptions in proposition 2.6.2.

We now state the main result that we need from [Gr5]. It is proposition 3.2.1 there.

Proposition 2.6.3. *Suppose that* $\mathcal{D}$ *is divisible as a* Λ*-module. Assume that* $\mathrm{LEO}(\mathcal{D})$, $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$*, and also at least one of the following additional assumptions is satisfied.*

(a) $\mathcal{D}[\mathfrak{m}]$ *has no subquotient isomorphic to* μ_p *for the action of* G_K*,*

(b) $\mathcal{D}$ *is a cofree* Λ*-module and* $\mathcal{D}[\mathfrak{m}]$ *has no quotient isomorphic to* μ_p *for the action of* G_K*,*

(c) There is a prime $\eta \in \Sigma$ *satisfying the following properties: (i)* $H^0(K_\eta, \mathcal{T}^*) = 0$*, and (ii)* $Q_{\mathcal{L}}(K_\eta, \mathcal{D})$ *is divisible as a* Λ*-module.*

Then $\phi_{\mathcal{L}}$ *is surjective.*

As mentioned in the introduction, we will apply the above result not just to $\mathcal{D}$, but also to $\mathcal{D}[\Pi]$ for prime ideals Π of Λ of height 1. Fortunately, if $\mathcal{D}$ is itself coreflexive as a Λ-module, then $\mathcal{D}[\Pi]$ is divisible as a (Λ/Π)-module, and hence satisfies the first hypothesis in the above proposition.

3 An Outline.

3.1. An exact sequence. Assume that $\mathrm{SUR}(\mathcal{D}, \mathcal{L})$ is satisfied. We will denote $\phi_{\mathcal{L}}$ just by ϕ, although we will continue to indicate the $\mathcal{L}$ for other objects. We have an exact sequence

$$0 \longrightarrow S_{\mathcal{L}}(K, \mathcal{D}) \longrightarrow H^1(K_\Sigma/K, \mathcal{D}) \xrightarrow{\phi} Q_{\mathcal{L}}(K, \mathcal{D}) \longrightarrow 0 \tag{3}$$

of discrete Λ-modules. Suppose that $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$ and that π is a generator of Π. Applying the snake lemma to the exact sequence (3) and to the endomorphisms of each of the above modules induced by multiplication by π, we obtain the following important exact sequence

$$H^1(K_\Sigma/K, \mathcal{D})[\pi] \xrightarrow{\alpha_\Pi} Q_{\mathcal{L}}(K, \mathcal{D})[\pi] \longrightarrow S_{\mathcal{L}}(K, \mathcal{D})/\pi S_{\mathcal{L}}(K, \mathcal{D}) \longrightarrow H^1(K_\Sigma/K, \mathcal{D})/\pi H^1(K_\Sigma/K, \mathcal{D})$$

Now assume additionally that $H^1(K_\Sigma/K, \mathcal{D})$ is an almost divisible Λ-module. The last term in the above exact sequence is then trivial for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. Therefore, under these assumptions, the assertion that $S_{\mathcal{L}}(K, \mathcal{D})$ is almost divisible is equivalent to the assertion that α_Π is surjective for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. We study the surjectivity of α_Π by considering the (Λ/Π)-module $\mathcal{D}[\pi]$.

Remark 3.1.1. One can ask if $\mathfrak{m}S_{\mathcal{L}}(K, \mathcal{D}) = S_{\mathcal{L}}(K, \mathcal{D})$, where $\mathfrak{m}$ denotes the maximal ideal of Λ. This would mean that the Pontryagin dual $\mathcal{X}$ of $S_{\mathcal{L}}(K, \mathcal{D})$ has no nonzero, finite Λ-submodules. For if $\mathcal{Z}$ is the maximal finite Λ-submodule of $\mathcal{X}$, then $\mathcal{Z}[\mathfrak{m}] = \mathcal{X}[\mathfrak{m}]$ is the Pontryagin dual of $S_{\mathcal{L}}(K, \mathcal{D})\big/\mathfrak{m}S_{\mathcal{L}}(K, \mathcal{D})$. This is trivial if and only if $\mathcal{Z}$ itself is trivial.

One sees easily that if $\mathcal{Z} \neq 0$, then $\mathcal{Z}[\pi] \neq 0$ for all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. As a consequence, if one can show that α_Π is surjective for infinitely many Π's in $\mathrm{Spec}_{ht=1}(\Lambda)$, then it would follow that $\mathfrak{m}S_{\mathcal{L}}(K, \mathcal{D}) = S_{\mathcal{L}}(K, \mathcal{D})$. This observation is especially useful if Λ has Krull dimension 2. In that case, it follows that $S_{\mathcal{L}}(K, \mathcal{D})$ is almost divisible if and only if α_Π is surjective for infinitely many Π's in $\mathrm{Spec}_{ht=1}(\Lambda)$. $\Diamond$

Remark 3.1.2. Let $\mathcal{X}$ again be the Pontryagin dual of $S_{\mathcal{L}}(K, \mathcal{D})$. We assume that the Krull dimension of Λ is at least 2. Let $\mathcal{Y}$ denote the maximal pseudo-null Λ-submodule of $\mathcal{X}$. Corollary 2.5.1 in [Gr4] implies that there exist infinitely many prime ideals Π in $\mathrm{Spec}_{ht=1}(\Lambda)$ such that $\mathcal{Y} = \mathcal{X}[\Pi]$. Let us denote the orthogonal complement of $\mathcal{Y}$ by $S_{\mathcal{L}}(K, \mathcal{D})_{adiv}$. Since $\mathcal{X}/\mathcal{Y}$ has no nontrivial pseudo-null Λ-submodules, it follows that $S_{\mathcal{L}}(K, \mathcal{D})_{adiv}$ is the maximal, almost divisible Λ-submodule of $S_{\mathcal{L}}(K, \mathcal{D})$. Furthermore, if $\mathcal{Y} = \mathcal{X}[\Pi]$ and $\Pi = (\pi)$, then we have

$$\pi S_{\mathcal{L}}(K, \mathcal{D}) \ = \ S_{\mathcal{L}}(K, \mathcal{D})_{adiv} \ .$$

This equality will be true for infinitely many Π's in $\mathrm{Spec}_{ht=1}(\Lambda)$. $\Diamond$

3.2. The cokernel of α_Π. Now let us assume just that $\mathcal{D}$ is divisible and cofinitely generated as a Λ-module. We then have an exact sequence

$$0 \longrightarrow \mathcal{D}[\pi] \longrightarrow \mathcal{D} \longrightarrow \mathcal{D} \longrightarrow 0$$

induced by multiplication by π. If $\mathcal{D}$ arises from a representation ρ as described in the introduction, and if R is a domain, then $\mathcal{D}$ will certainly be Λ-divisible. As a consequence, the following global and local *"specialization"* maps are surjective:

$$h_\Pi : H^1(K_\Sigma/K, \mathcal{D}[\pi]) \longrightarrow H^1(K_\Sigma/K, \mathcal{D})[\pi], \qquad h_{\Pi,v} :\ H^1(K_v, \mathcal{D}[\pi]) \longrightarrow H^1(K_v, \mathcal{D})[\pi]$$

We can compare the exact sequence (3) with an analogous sequence for $\mathcal{D}[\pi]$, viewed as a (Λ/Π)-module. For this purpose, we define a specification $\mathcal{L}_\Pi$ for $\mathcal{D}[\pi]$ as follows: For each $v \in \Sigma$, let us take

$$L(K_v, \mathcal{D}[\pi]) \ = \ h_{\Pi,v}^{-1}\big(\ L(K_v, \mathcal{D})\ \big)$$

which is a (Λ/Π)-submodule of $H^1(K_v, \mathcal{D}[\pi])$. If we think of $\mathcal{L}$ as fixed, we will refer to the specification $\mathcal{L}_\Pi$ just defined as the *"$\mathcal{L}$-maximal specification for $\mathcal{D}[\pi]$"*. Using the analogous notation to that for $\mathcal{D}$, we have

$$P(K, \mathcal{D}[\pi]) = \prod_{v\in\Sigma} H^1(K_v, \mathcal{D}[\pi]), \qquad Q_{\mathcal{L}_\Pi}(K, \mathcal{D}[\pi]) = P(K, \mathcal{D}[\pi])\big/L(K, \mathcal{D}[\pi])$$

where $L(K, \mathcal{D}[\pi]) = \prod_{v\in\Sigma} L(K_v, \mathcal{D}[\pi])$. We can then define the corresponding global-to-local map

$$\phi_{\mathcal{L}_\Pi} :\ H^1(K_\Sigma/K, \mathcal{D}[\pi]) \ \longrightarrow\ Q_{\mathcal{L}_\Pi}(K, \mathcal{D}[\pi])$$

We will usually denote the map $\phi_{\mathcal{L}_\Pi}$ simply by ϕ_Π. The product of the $h_{\Pi,v}$'s for $v \in \Sigma$ defines a map $b_\Pi :\ P(K, \mathcal{D}[\pi]) \to P(K, \mathcal{D})[\pi]$. The image of $L(K, \mathcal{D}[\pi])$ under b_Π is contained in $L(K, \mathcal{D})$ and so we get a well-defined map

$$q_\Pi :\ Q_{\mathcal{L}_\Pi}(K, \mathcal{D}[\pi]) \ \longrightarrow\ Q_{\mathcal{L}}(K, \mathcal{D})[\pi] \ .$$

The definition of $\mathcal{L}_\Pi$ implies that q_Π is injective. Furthermore, it is easy to see that if $L(K, \mathcal{D})$ is divisible by π, then the map $c_\Pi :\ P(K, \mathcal{D})[\pi] \to Q_{\mathcal{L}}(K, \mathcal{D})[\pi]$ will be surjective. Since the map b_Π is also surjective, it would then follow that $c_\Pi \circ b_\Pi$ is also surjective. This would imply that q_Π is surjective. Thus, assuming that $\mathcal{D}$ is Λ-divisible and that $\mathcal{L}$ is almost Λ-divisible, we see that q_Π is then an isomorphism for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$.

We have the following commutative diagram whose rows are exact:

$$\begin{tikzcd}
0 \arrow[r] & S_{\mathcal{L}_\Pi}(K, \mathcal{D}[\pi]) \arrow[r] \arrow[d, "s_\Pi"] & H^1(K_\Sigma/K, \mathcal{D}[\pi]) \arrow[r, "\phi_\Pi"] \arrow[d, "h_\Pi"] & Q_{\mathcal{L}_\Pi}(K, \mathcal{D}[\pi]) \arrow[d, "q_\Pi"] \\
0 \arrow[r] & S_{\mathcal{L}}(K, \mathcal{D})[\pi] \arrow[r] & H^1(K_\Sigma/K, \mathcal{D})[\pi] \arrow[r, "\alpha_\Pi"] & Q_{\mathcal{L}}(K, \mathcal{D})[\pi]
\end{tikzcd}$$

The map s_Π is defined since, by definition, the image of $S_{\mathcal{L}_\Pi}(K, \mathcal{D}[\pi])$ under the map h_Π is contained in $S_{\mathcal{L}}(K, \mathcal{D})[\pi]$. Although it is not needed now, we remark in passing that the injectivity of the map q_Π and the surjectivity of the map h_Π imply that s_Π is also surjective. The important consequence for us is that q_Π maps $\mathrm{im}(\phi_\Pi)$ isomorphically to $\mathrm{im}(\alpha_\Pi)$ and induces an isomorphism

$$\mathrm{coker}(\alpha_\Pi) \;\cong\; \mathrm{coker}(\phi_\Pi) \tag{4}$$

for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. To be precise, this is true whenever $L(K, \mathcal{D})$ is divisible by a generator of Π. In particular, for such Π, the surjectivity of α_Π and ϕ_Π would be equivalent. These remarks are valid just under the assumptions that $\mathcal{D}$ is Λ-divisible and that $\mathcal{L}$ is almost divisible. Our approach is then to find suitable additional hypotheses which guarantee the surjectivity of ϕ_Π for almost all prime ideals Π of Λ of height 1.

To summarize, if we make the assumptions that $\mathcal{D}$ is Λ-divisible, that $H^1(K_\Sigma/K, \mathcal{D})$ is almost Λ-divisible, that the specification $\mathcal{L}$ is almost Λ-divisible, and that $\mathrm{SUR}(\mathcal{D}, \mathcal{L})$ holds, then $S_{\mathcal{L}}(K, \mathcal{D})$ is almost divisible if and only if ϕ_Π is surjective for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$.

3.3. The case where $\phi_{\mathcal{L}}$ is not surjective. We will prove the theorem of Iwasawa stated in the introduction in section 4.2. In most cases, it will turn out that the map $\phi_{\mathcal{L}}$ occurring in that situation is surjective. However, this is not so if $F = K(\mu_p)$. Nevertheless, we will prove Iwasawa's theorem by using the following modification of the observations in sections 3.1 and 3.2.

In the exact sequence (3), we can simply replace $Q_{\mathcal{L}}(K, \mathcal{D})$ by the image of $\phi = \phi_{\mathcal{L}}$, which we will denote by $Q'_{\mathcal{L}}(K, \mathcal{D})$. We then can consider the map

$$\alpha'_\Pi \;:\; H^1(K_\Sigma/K, \mathcal{D})[\pi] \;\longrightarrow\; Q'_{\mathcal{L}}(K, \mathcal{D})[\pi] \quad .$$

Applying the snake lemma as before, and assuming that $H^1(K_\Sigma/K, \mathcal{D})$ is almost Λ-divisible, we see that $S_{\mathcal{L}}(K, \mathcal{D})$ is almost Λ-divisible if and only if α'_Π is surjective for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$.

In addition to assuming that $H^1(K_\Sigma/K, \mathcal{D})$ is almost Λ-divisible, let us assume that $\mathcal{D}$ is Λ-divisible and that the specification $\mathcal{L}$ is almost Λ-divisible. If $H^1(K_\Sigma/K, \mathcal{D})$ is divisible by π, then so is $Q'_{\mathcal{L}}(K, \mathcal{D})$. Therefore, $Q'_{\mathcal{L}}(K, \mathcal{D})$ is almost Λ-divisible. We have an exact sequence

$$0 \longrightarrow Q'_{\mathcal{L}}(K, \mathcal{D}) \longrightarrow Q_{\mathcal{L}}(K, \mathcal{D}) \longrightarrow \mathrm{coker}(\phi) \longrightarrow 0$$

and hence, by the snake lemma, the map $\beta_\Pi : Q_{\mathcal{L}}(K, \mathcal{D})[\pi] \to \mathrm{coker}(\phi)[\pi]$ is surjective for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. The kernel of β_Π is $Q'_{\mathcal{L}}(K, \mathcal{D})[\pi]$.

The map q_Π sends $\mathrm{im}(\phi_\Pi)$ to $\mathrm{im}(\alpha_\Pi) = \mathrm{im}(\alpha'_\Pi)$. For almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$, we have isomorphisms

$$\mathrm{coker}(\alpha'_\Pi) \;\cong\; \ker\big(\beta_\Pi \circ q_\Pi\big)/\mathrm{im}(\phi_\Pi) \;\cong\; \ker\Big(\mathrm{coker}(\phi_\Pi) \to \mathrm{coker}(\phi)[\pi]\Big)$$

and we therefore conclude that $S_{\mathcal{L}}(K, \mathcal{D})$ is almost Λ-divisible if and only if the map

$$\mathrm{coker}(\phi_\Pi) \;\longrightarrow\; \mathrm{coker}(\phi)[\pi] \tag{5}$$

is injective for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. Note that (5) is surjective for almost all Π's because that is true for the maps β_Π and q_Π.

3.4. Behavior of the corank hypothesis under specialization. We complete the discussion in section 2.4. We want to justify the following equivalence.

- $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$ *is true if and only if* $\mathrm{CRK}(\mathcal{D}[\Pi], \mathcal{L}_\Pi)$ *is true for almost all* $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$.

According to remark 2.1.3 in [Gr4], $\mathrm{coker}(\phi)$ is Λ-cotorsion if and only if $\mathrm{coker}\big(\alpha_\Pi\big)$ is (Λ/Π)-cotorsion for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. If we assume that $\mathcal{L}$ is almost Λ-divisible, then we have the isomorphism (4) for almost all Π's. Since Λ/Π is a finitely-generated Λ_Π-module, it follows that $\mathrm{coker}(\phi)$ is Λ-cotorsion if and only if $\mathrm{coker}\big(\phi_\Pi\big)$ is Λ_Π-cotorsion for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$, which is the stated equivalence.

The assumption that $\mathcal{L}$ is almost Λ-divisible is not needed. Suppose that $\Pi = (\pi)$ is an arbitrary element of $\mathrm{Spec}_{ht=1}(\Lambda)$. Referring to the discussion in section 3.2, we have an injective map

$$\mathrm{coker}\big(\phi_\Pi\big) \;\longrightarrow\; \mathrm{coker}\big(\alpha_\Pi\big) \quad . \tag{6}$$

induced by q_Π. Furthermore, the cokernel of (6) is isomorphic to $\mathrm{coker}\big(q_\Pi\big)$. The stated equivalence will follow if we show that $\mathrm{coker}\big(q_\Pi\big)$ is (Λ/Π)-cotorsion for almost all Π's. Using the notation from section 3.2, we have $\mathrm{coker}\big(q_\Pi\big) = \mathrm{coker}\big(c_\Pi\big)$. We then obtain another injective map

$$\mathrm{coker}\big(q_\Pi\big) \;\longrightarrow\; L(K, \mathcal{D})/\pi L(K, \mathcal{D}) \quad .$$

Thus, it suffices to show that $L(K, \mathcal{D})/\pi L(K, \mathcal{D})$ is (Λ/Π)-cotorsion for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$.

In general, suppose that $\mathcal{A}$ is a discrete, cofinitely-generated Λ-module and that $\mathcal{X}$ is the Pontryagin dual of $\mathcal{A}$. Let $\mathcal{Z}$ denote the maximal pseudo-null Λ-submodule of $\mathcal{X}$ and let $\mathcal{B}$ denote the orthogonal complement of $\mathcal{Z}$. Thus, $\mathcal{B} \subseteq \mathcal{A}$. Let $\mathcal{C} = \mathcal{A}/\mathcal{B}$. The Pontryagin duals of $\mathcal{B}$ and $\mathcal{C}$ are $\mathcal{X}/\mathcal{Z}$ and $\mathcal{Z}$, respectively. It follows that $\mathcal{B}$ is the maximal almost Λ-divisible Λ-submodule of $\mathcal{A}$. Furthermore, by definition, $\mathcal{Z}$ is annihilated by a nonzero element of Λ relatively prime to π, and so $\mathcal{Z}[\pi]$ is a torsion (Λ/Π)-module. Thus, $\mathcal{C}/\pi\mathcal{C}$ is (Λ/Π)-cotorsion. If we choose Π so that $\pi\mathcal{B} = \mathcal{B}$, it follows that $\mathcal{A}/\pi\mathcal{A} \cong \mathcal{C}/\pi\mathcal{C}$. Applying these considerations to $\mathcal{A} = L(K, \mathcal{D})$, we see that $L(K, \mathcal{D})/\pi L(K, \mathcal{D})$ is indeed (Λ/Π)-cotorsion for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$.

4 Sufficient conditions for almost divisibility.

We will prove a rather general result in section 4.1. Section 4.2 discusses the verification of various hypotheses in that result. Section 4.3 will concern a special case (although still quite general) where several of the hypotheses are automatically satisfied.

4.1. The main theorem. We prove the following result.

Proposition 4.1.1. *Suppose that* $\mathrm{RFX}(\mathcal{D})$ *and* $\mathrm{LEO}(\mathcal{D})$ *are satisfied, that* $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ *is satisfied for all* v *in* Σ*, and that there exists a non-archimedean prime* $\eta \in \Sigma$ *such that* $\mathrm{LOC}_\eta^{(1)}(\mathcal{D})$ *is satisfied. Suppose also that* $\mathcal{L}$ *is almost divisible, that* $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$ *is satisfied, and also that at least one of the following additional assumptions is satisfied.*

- *(a)* $\mathcal{D}[\mathfrak{m}]$ *has no subquotient isomorphic to* μ_p *for the action of* G_K*,*
- *(b)* $\mathcal{D}$ *is a cofree* Λ*-module and* $\mathcal{D}[\mathfrak{m}]$ *has no quotient isomorphic to* μ_p *for the action of* G_K*,*
- *(c) There is a prime* $\eta \in \Sigma$ *which satisfies* $\mathrm{LOC}_\eta^{(1)}(\mathcal{D})$ *and such that* $Q_{\mathcal{L}}(K_\eta, \mathcal{D})$ *is coreflexive as a* Λ*-module.*

Then $S_{\mathcal{L}}(K, \mathcal{D})$ *is an almost divisible* Λ*-module.*

Proof. First of all, $\mathrm{RFX}(\mathcal{D})$, $\mathrm{LEO}(\mathcal{D})$, and the assumptions about $\mathrm{LOC}_v^{(1)}$ and $\mathrm{LOC}_v^{(2)}$ are sufficient to imply that $H^1(K_\Sigma/K, \mathcal{D})$ is an almost divisible Λ-module. This follows from proposition 2.6.1. Secondly, since $\mathrm{RFX}(\mathcal{D})$ holds, $\mathcal{D}$ is certainly Λ-divisible. We can apply proposition 2.6.3 to conclude that $\mathrm{SUR}(\mathcal{D}, \mathcal{L})$ is satisfied too.

Thus, as described in section 3.1, it suffices to show that the map

$$\alpha_\Pi : \ H^1(K_\Sigma/K, \mathcal{D})[\pi] \ \longrightarrow \ Q_{\mathcal{L}}(K, \mathcal{D})[\pi]$$

is surjective for almost all $\Pi = (\pi)$ in $\mathrm{Spec}_{ht=1}(\Lambda)$. In the rest of this proof, we will exclude finitely many Π's in $\mathrm{Spec}_{ht=1}(\Lambda)$ in each step, and altogether just finitely many. We will follow the approach outlined in section 3, reducing the question to studying $\mathrm{coker}(\phi_\Pi)$ and then applying proposition 2.6.3. We want to apply that proposition to $\mathcal{D}[\pi]$ and so must verify the appropiate hypotheses. At each step, we consider just the Π's which have not been already excluded. As described in section 2, we regard various (Λ/Π)-modules as modules over a certain subring Λ_Π.

Since $\mathrm{RFX}(\mathcal{D})$ holds for $\mathcal{D}$, it follows that $\mathcal{D}[\pi]$ is a divisible (Λ/Π)-module. Corollary 2.6.1 in [Gr4] justifies that assertion. Therefore, $\mathcal{D}[\pi]$ is also divisible as a Λ_Π-module. Furthermore, the assumption $\mathrm{LEO}(\mathcal{D})$ means that $\text{Ш}^2(K, \Sigma, \mathcal{D})$ is Λ-cotorsion. Consequently, $\text{Ш}^2(K, \Sigma, \mathcal{D})[\pi]$ is a cotorsion (Λ/Π)-module for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. This follows from remark 2.1.3 in [Gr4]. The same is true for $\text{Ш}^2(K, \Sigma, \mathcal{D}[\pi])$ according to lemma 4.1.1 in [Gr4]. Recall that Λ/Π is finitely-generated as a Λ_Π-module. It follows that $\mathrm{LEO}(\mathcal{D}[\pi])$ holds for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$.

The fact that $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$ is satisfied implies that $\mathrm{CRK}(\mathcal{D}[\Pi], \mathcal{L}_\Pi)$ is satisfied for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. This follows from section 3.4. Thus, we can assume from here on that $\mathrm{coker}(\phi_\Pi)$ is Λ_Π-cotorsion. Now we consider the additional assumptions. Each implies the corresponding assumption in proposition 2.6.3. Once we verify that, it will then follow that ϕ_Π is surjective for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. Hence the same thing will be true for α_Π. This will prove that $S_{\mathcal{L}}(K, \mathcal{D})$ is indeed almost divisible as a Λ-module.

First assume that *(a)* is satisfied. Let $\mathfrak{m}_\Pi$ denote the maximal ideal of Λ_Π. Using proposition 3.8 in [Gr4], it follows that $\mathcal{D}[\pi][\mathfrak{m}_\Pi]$ indeed has no subquotient isomorphic to μ_p. Now assume that *(b)* is satisfied. Then $\mathcal{D}[\pi]$ is cofree as a (Λ/Π)-module. Since Π is principal, Λ/Π is a complete intersection, and hence a Cohen-Macaulay domain. It follows that Λ/Π is a free Λ_Π-module. Hence $\mathcal{D}[\pi]$ is cofree as a Λ_Π-module. Furthermore, $\mathcal{D}[\mathfrak{m}] = \mathcal{D}[\pi][\mathfrak{m}]$ has no quotient isomorphic to μ_p for the action of G_K. Remark 3.2.2 in [Gr5] implies that the same thing is true for $\mathcal{D}[\pi][\mathfrak{m}_\Pi]$. Thus, the assumption *(b)* in proposition 2.6.3 for the Λ_Π-module $\mathcal{D}[\pi]$ is indeed satisfied.

Now assume that *(c)* is satisfied. As pointed out in section 2.4, $\mathrm{LOC}^{(1)}_\eta(\mathcal{D}[\pi])$ is satisfied for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. Since $\mathcal{D}$ is Λ-divisible and $L(K_\eta, \mathcal{D})$ is almost divisible, we have

$$Q_{\mathcal{L}_\Pi}(K_\eta, \mathcal{D}[\pi]) \;\cong Q_{\mathcal{L}}(K_\eta, \mathcal{D})[\pi]$$

for almost all Π's. It suffices to have $L(K_\eta, \mathcal{D})$ divisible by π. The assumption that $Q_{\mathcal{L}}(K_\eta, \mathcal{D})$ is a coreflexive Λ-module then implies that $Q_{\mathcal{L}_\Pi}(K_\eta, \mathcal{D}[\pi])$ is (Λ/Π)-divisible, and hence Λ_Π-divisible, which is the only assumption in proposition 2.6.3*(c)* left to verify. ■

Now we consider *"non-primitive"* Selmer groups. Suppose that Σ_0 is a subset of Σ consisting of non-archimedean primes. Consider the map

$$\phi_{\mathcal{L},\Sigma_0} \ : \ H^1(K_\Sigma/K,\mathcal{D}) \ \longrightarrow \ \prod_{v\in\Sigma-\Sigma_0} Q_{\mathcal{L}}(K_v,\mathcal{D}) \quad .$$

We denote the kernel of $\phi_{\mathcal{L},\Sigma_0}$ by $S_{\mathcal{L}}^{\Sigma_0}(K,\mathcal{D})$. We refer to this group as the non-primitive Selmer group corresponding to the specification $\mathcal{L}$ and the set Σ_0. It is defined just as $S_{\mathcal{L}}(K,\mathcal{D})$, but one omits the local conditions for the specification $\mathcal{L}$ corresponding to the primes $v\in\Sigma_0$. Of course, we have the obvious inclusion $S_{\mathcal{L}}(K,\mathcal{D})\subseteq S_{\mathcal{L}}^{\Sigma_0}(K,\mathcal{D})$ and the corresponding quotient $S_{\mathcal{L}}^{\Sigma_0}(K,\mathcal{D})/S_{\mathcal{L}}(K,\mathcal{D})$ is isomorphic to a Λ-submodule of $\prod_{v\in\Sigma_0} Q_{\mathcal{L}}(K_v,\mathcal{D})$. In effect, $S_{\mathcal{L}}^{\Sigma_0}(K,\mathcal{D})$ is the Selmer group corresponding to a new specification $\mathcal{L}'$, where we simply replace $L(K_v,\mathcal{D})$ by $L'(K_v,\mathcal{D}) = H^1(K_v,\mathcal{D})$ for all $v\in\Sigma_0$. Thus, we now have $Q_{\mathcal{L}'}(K_v,\mathcal{D})=0$ for $v\in\Sigma_0$. The following corollary then follows immediately from proposition 4.1.1*(c)*.

Corollary 4.1.2. *Suppose that* $\mathrm{RFX}(\mathcal{D})$ *and* $\mathrm{LEO}(\mathcal{D})$ *are satisfied, that* $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ *is satisfied for all v in Σ, and that there exists a non-archimedean prime $\eta\in\Sigma_0$ such that* $\mathrm{LOC}_\eta^{(1)}(\mathcal{D})$ *is satisfied. Suppose also that $\mathcal{L}$ is almost divisible and that* $\mathrm{CRK}(\mathcal{D},\mathcal{L})$ *is satisfied. Then $S_{\mathcal{L}}^{\Sigma_0}(K,\mathcal{D})$ is an almost divisible Λ-module.*

Remark 4.1.3. Suppose that η is a non-archimedean prime not dividing p. Regarding $\mathcal{D}[\mathfrak{m}]$ as an $\mathbf{F}_p$-representation space for G_{K_η}, suppose that it has no subquotients isomorphic to μ_p. According to proposition 3.1 in [Gr4], the G_{K_η}-module $\mathcal{D}[\mathfrak{m}^t]$ has the same property for all $t\geq 1$. The local duality theorems imply that $H^0(K_\eta,\mathcal{D}[\mathfrak{m}^t])$ and $H^2(K_\eta,\mathcal{D}[\mathfrak{m}^t])$ both vanish, and therefore that $H^1(K_\eta,\mathcal{D}[\mathfrak{m}^t])=0$. It follows that $H^1(K_\eta,\mathcal{D})=0$. If we let $\Sigma_0=\{\eta\}$, then we have $S_{\mathcal{L}}^{\Sigma_0}(K,\mathcal{D})=S_{\mathcal{L}}(K,\mathcal{D})$. The hypothesis $\mathrm{LOC}_\eta^{(1)}(\mathcal{D})$ is also satisfied. Consequently, if the other assumptions in corollary 4.1.2 are satisfied, it follows that $S_{\mathcal{L}}(K,\mathcal{D})$ is almost divisible as a Λ-module.

4.2. Verifying the hypotheses. We will discuss the various hypotheses in proposition 4.1.1. Some of them are already needed for propositions 2.6.1 and 2.6.3, and we may simply refer to discussions in [Gr4] and [Gr5]. We have nothing additional to say about $\mathrm{RFX}(\mathcal{D})$. If $\mathcal{D}$ is R-cofree, then that hypothesis is just that R is a reflexive ring.

The local hypotheses. There is a discussion of the verification of $\mathrm{LOC}_v^{(1)}(\mathcal{D})$ and $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ in section 5, part F of [Gr4]. Most commonly, $\mathrm{LOC}_v^{(1)}(\mathcal{D})$ is satisfied for all non-archimedean primes $v\in\Sigma$ simply because $H^0(K_v,\mathcal{T}^*)=0$ for those v's. That is a rather mild condition, although we mention one kind of example in section 4.3 where it may fail to be so. Such examples were one motivation for introducing $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ as a hypothesis in [Gr4]. Another motivation is that for archimedean primes, $H^0(K_v, \mathcal{T}^*)$ is often nontrivial, but $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ may still be satisfied. The archimedean primes are only an issue when $p = 2$.

The hypotheses $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$ *and* $\mathrm{LEO}(\mathcal{D})$. Of course, the validity of $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$ is related to the choice of the specification $\mathcal{L}$. We will discuss one rather natural way of choosing a specification below. In [Gr5], one defines $c_{\mathcal{L}}(K, \mathcal{D})$ to be the Λ-corank of the cokernel of $\phi_{\mathcal{L}}$. Thus, $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$ means that $c_{\mathcal{L}}(K, \mathcal{D}) = 0$. As discussed in the introduction to [Gr5], one has an equation

$$s_{\mathcal{L}}(K, \mathcal{D}) \;=\; b_1(K, \mathcal{D}) - q_{\mathcal{L}}(K, \mathcal{D}) + c_{\mathcal{L}}(K, \mathcal{D}) + \mathrm{corank}_{\Lambda}\big(Ш^2(K, \Sigma, \mathcal{D})\big) \;\;,$$

where $s_{\mathcal{L}}(K, \mathcal{D})$ and $q_{\mathcal{L}}(K, \mathcal{D})$ are the Λ-coranks of $S_{\mathcal{L}}(K, \mathcal{D})$ and $Q_{\mathcal{L}}(K, \mathcal{D})$, respectively. The integer $b_1(K, \mathcal{D})$ is defined just in terms of the Euler-Poincaré characteristic for $\mathcal{D}$ and the Λ-coranks of some local Galois cohomology groups, and does not depend on $\mathcal{L}$. It occurs in proposition 4.3 in [Gr4]. One then has a lower bound

$$s_{\mathcal{L}}(K, \mathcal{D}) \;\geq\; b_1(K, \mathcal{D}) - q_{\mathcal{L}}(K, \mathcal{D})$$

and equality means that both $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$ and $\mathrm{LEO}(\mathcal{D})$ are satisfied. We also remark that section 6, part D in [Gr4] is a discussion of $\mathrm{LEO}(\mathcal{D})$, which is called hypothesis L there.

The additional assumptions in proposition 4.1.1. Remark 3.2.2 in [Gr5] discusses the additional assumptions *(a)* and *(b)*. It includes some observations when $\mathcal{D}$ arises from an n-dimensional representation ρ of $\mathrm{Gal}(K_\Sigma/K)$ over a ring R, as in the introduction. One observation is that if $n \geq 2$ and if the residual representation $\widetilde{\rho}$ is irreducible over the finite field $R/\mathfrak{M}$, then hypothesis *(a)* is satisfied. The residual representation gives the action of $\mathrm{Gal}(K_\Sigma/K)$ on $\mathcal{D}[\mathfrak{M}]$. Another observation in that remark is that $\mathcal{D}[\mathfrak{m}]$ has a quotient isomorphic to μ_p if and only if $\mathcal{D}[\mathfrak{M}]$ has such a quotient.

As an illustration, if there is a prime ideal $\mathfrak{p}$ of R such that $\mathcal{T}/\mathfrak{p}\mathcal{T}$ is isomorphic to the p-adic Tate module $T_p(E)$ for an elliptic curve E defined over K, then $\mathcal{D}[\mathfrak{M}] \cong E[p]$ as Galois modules. It is not uncommon for $E[p]$ to be irreducible. This just means that E has no isogenies of degree p defined over K. Furthermore, the Weil pairing $E[p] \times E[p] \to \mu_p$ is Galois-equivariant. Hence $E[p]$ has a quotient isomorphic to μ_p if and only if $E(K)$ has a point of order p. Also, $E[p]$ has a subquotient isomorphic to μ_p if and only if E is isogenous over K to an elliptic curve E' such that $E'(K)$ has a point of order p.

We now discuss hypothesis *(c)*. This will be useful if $\mathcal{D}[\mathfrak{m}]$ has a quotient or subquotient isomorphic to μ_p for the action of G_K. We will assume that η is a non-archimedean prime in Σ and that $\mathrm{LOC}_\eta^{(1)}(\mathcal{D})$ is satisfied. The issue is the coreflexivity of $Q_{\mathcal{L}}(K_\eta, \mathcal{D})$ as a Λ-module.

Suppose that we assume that $H^1(K_\eta, \mathcal{D})$ is Λ-coreflexive and that $L(K_\eta, \mathcal{D})$ is almost Λ-divisible. The first assumption implies that $H^1(K_\eta, \mathcal{T}^*)$ is a reflexive Λ-module. Also,the

Λ-module $Q(K_\eta, \mathcal{T}^*)$ is the Pontryagin dual of $L(K_\eta, \mathcal{D})$, and so the second assumption implies that $Q(K_\eta, \mathcal{T}^*)$ contains no nonzero, pseudo-null Λ-submodules. Combining these observations, it follows that $L(K_\eta, \mathcal{T}^*)$ is a reflexive Λ-module. Therefore, it follows that its Pontryagin dual $Q_{\mathcal{L}}(K_\eta, \mathcal{D})$ is coreflexive as a Λ-module.

Section 5, part D of [Gr4] gives some sufficient conditions for $H^1(K_\eta, \mathcal{D})$ to be coreflexive. One condition requires the assumption that μ_p is not a quotient of $\mathcal{D}[\mathfrak{m}]$ as a G_{K_η}-module. However, that assumption clearly implies assumption *(a)* in proposition 4.1.1. Another more subtle sufficient condition is given in proposition 5.9 in *loc cit.* It involves $\mathcal{T}^* \otimes_\Lambda \widehat{\Lambda}$ which is denoted by $\mathcal{D}^*$ there. We are assuming that $H^0(K_\eta, \mathcal{T}^*) = 0$. Equivalently, that means that $H^0(K_\eta, \mathcal{D}^*)$ is Λ-cotorsion. We denote that module more compactly by $\mathcal{D}^*(K_\eta)$. Its Pontryagin dual $\widehat{\mathcal{D}^*(K_\eta)}$ is a torsion Λ-module. The result from [Gr4] is that if $\mathcal{D}$ is Λ-cofree and if every associated prime ideal for the torsion Λ-module $\widehat{\mathcal{D}^*(K_\eta)}$ has height at least 3, then $H^1(K_\eta, \mathcal{D})$ is coreflexive as a Λ-module. Section 4.3 will discuss some cases where this hypothesis is satisfied.

Even if $H^1(K_\eta, \mathcal{D})$ fails to be coreflexive, it is still possible for the quotient Λ-module $Q_{\mathcal{L}}(K_\eta, \mathcal{D})$ to be coreflexive. Consider the following natural way to specify a choice of $L(K_\eta, \mathcal{D})$. Suppose that $\mathcal{C}_\eta$ is a G_{K_η}-invariant Λ-submodule of $\mathcal{D}$ and that we have $H^2(K_\eta, \mathcal{C}_\eta) = 0$. Then we can define

$$L(K_\eta, \mathcal{D}) \;=\; \mathrm{im}\big(\; H^1(K_\eta, \mathcal{C}_\eta) \;\longrightarrow\; H^1(K_\eta, \mathcal{D}) \;\big) \quad .$$

Let $\mathcal{E}_\eta = \mathcal{D}/\mathcal{C}_\eta$. The the map $H^1(K_\eta, \mathcal{D}) \to H^1(K_\eta, \mathcal{E}_\eta)$ is surjective and its kernel is $L(K_\eta, \mathcal{D})$. If $\eta \nmid p$, then one can take $\mathcal{C}_\eta = 0$ and hence $L(K_\eta, \mathcal{D}) = 0$. This is often a useful choice. If $\eta | p$, then one often will make a nontrivial choice of $\mathcal{C}_\eta$. This kind of definition occurs in [Gr2] for primes above p when a Galois representation ρ satisfies something we called a "Panchiskin condition." (See section 4 in *loc cit.*) Under the stated assumptions, we have

$$Q_{\mathcal{L}}(K_\eta, \mathcal{D}) \;\cong\; H^1(K_\eta, \mathcal{E}_\eta)$$

as Λ-modules. Propositions 5.8 and 5.9 from [Gr4] then give the following result.

Proposition 4.2.1. *In addition to the assumption that $H^2(K_\eta, \mathcal{C}_\eta) = 0$, suppose that either one of the following assumptions is satisfied.*

(i) *$\mathcal{E}_\eta$ is Λ-coreflexive and $\mathcal{E}_\eta[\mathfrak{m}]$ has no subquotient isomorphic to μ_p as a G_{K_η}-module,*

(ii) *$\mathcal{E}_\eta$ is Λ-cofree and every associated prime ideal for the Λ-module $\widehat{\mathcal{E}^*_\eta(K_\eta)}$ has height at least 3.*

Then the Λ-module $Q_{\mathcal{L}}(K_\eta, \mathcal{D})$ is coreflexive.

Concerning *(i)*, note that it may be satisfied even if assumption *(a)* in proposition 4.1.1 fails to be satisfied. One such situation will be mentioned in section 4.3. We will also want $L(K_\eta, \mathcal{C}_\eta)$ to be almost Λ-divisible. The following result follows immediately from proposition 5.3 in [Gr4].

Proposition 4.2.2. *Assume that $\mathcal{C}_\eta$ is Λ-coreflexive and that $H^2(K_\eta, \mathcal{C}_\eta) = 0$. Then $H^1(K_\eta, \mathcal{C}_\eta)$ is almost Λ-divisible. Hence the image of $H^1(K_\eta, \mathcal{C}_\eta)$ in $H^1(K_\eta, \mathcal{D})$ is also almost Λ-divisible.*

4.3. Twist deformations. Suppose that T is a free $\mathbf{Z}_p$-module of rank n which has an action of $\mathrm{Gal}(K_\Sigma/K)$. Thus, we have a continuous homomorphism $\mathrm{Gal}(K_\Sigma/K) \to \mathrm{Aut}_{\mathbf{Z}_p}(T)$. Suppose also that K_∞/K is a Galois extension such that $\mathrm{Gal}(K_\infty/K) \cong \mathbf{Z}_p^m$ for some $m \geq 1$. We let $\Lambda = \mathbf{Z}_p[[\mathrm{Gal}(K_\infty/K]]$ denote the completed group ring for $\mathrm{Gal}(K_\infty/K)$ over $\mathbf{Z}_p$. Thus, Λ is isomorphic to a formal power series ring in m variables over $\mathbf{Z}_p$. In this situation, one can define a free Λ-module $\mathcal{T}$ of rank n together with a homomorphism $\rho : \mathrm{Gal}(K_\Sigma/K) \to \mathrm{Aut}_\Lambda(\mathcal{T})$. This is described in section 5 of [Gr5] in detail, where $\mathcal{T}$ is denoted by $T \otimes \kappa$. Here κ is the natural embedding of Γ into $\Lambda^\times$ and one thinks of $\mathcal{T}$ as the twist of T by the $\Lambda^\times$-valued character κ. Just as in the introduction, taking $R = \Lambda$, one can define $\mathcal{D} = \mathcal{T} \otimes_\Lambda \widehat{\Lambda}$. This discrete, Λ-cofree $\mathrm{Gal}(K_\Sigma/K)$-module $\mathcal{D}$ is denoted by $D \otimes \kappa$ in [Gr5], where $D = T \otimes_{\mathbf{Z}_p} (\mathbf{Q}_p/\mathbf{Z}_p)$.

Obviously, $\mathrm{RFX}(\mathcal{D})$ is satisfied. Furthermore, it is shown in part **F** of section 5 in [Gr4] that $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ is satisfied for all v in Σ. Lemma 5.2.2 in [Gr5] shows that $\mathrm{LOC}_\eta^{(1)}(\mathcal{D})$ is satisfied for at least one η in Σ. In fact, that hypothesis holds for any prime η which does not split completely in K_∞/K. In particular, $\mathrm{LOC}_\eta^{(1)}(\mathcal{D})$ is satisfied for at least one prime η lying over p.

It is reasonable to conjecture that $\mathrm{LEO}(\mathcal{D})$ is satisfied. This is stated as conjecture 5.2.1 in [Gr5] and is equivalent to conjecture L stated in the introduction to [Gr4]. Section 5.2 in [Gr4] discusses its validity. It is proved in certain special cases. In the rest of this discussion, we will assume that $\mathrm{LEO}(\mathcal{D})$ is satisfied, that one has chosen an almost Λ-divisible specification $\mathcal{L}$, and that $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$ is also satisfied. Thus, we can apply proposition 4.1.1 if we verify any of the additional assumptions *(a)*, *(b)*, or *(c)*. Since $\mathcal{D}$ is Λ-cofree, it suffices to verify either *(b)* or *(c)*.

As for *(b)*, we merely point out that $\mathcal{D}[\mathfrak{m}] \cong D[p] \cong T/pT$. One can illustrate this case when $T = T_p(E)$, the p-adic Tate module for an elliptic curve defined over K. We then have $T/pT \cong E[p]$. The properties of the Weil pairing $E[p] \times E[p] \to \mu_p$ show that assumption *(b)* is satisfied if and only if $E(K)$ has no element of order p. Thus, if we assume that $E(K)[p] = 0$, that LEO($\mathcal{D}$) is indeed satisfied, that $\mathcal{L}$ is almost Λ-divisible, and that CRK($\mathcal{D}, \mathcal{L}$) is satisfied, then $S_{\mathcal{L}}(K, \mathcal{D})$ is almost Λ-divisible.

As another illustration, suppose that K is totally real, that F is an imaginary quadratic extension of K, that $T = \mathbf{Z}_p$, and that G_K acts on T by that nontrivial character ψ factoring through $\mathrm{Gal}(F/K)$. According to Leopoldt's conjecture for K and p, we have $m = 1$ and K_∞ is just the cyclotomic $\mathbf{Z}_p$-extension of K. We take $\mathcal{L}$ to be the trivial specification: $L(K_v, \mathcal{D}) = 0$ for all $v \in \Sigma$. Thus, $S_{\mathcal{L}}(K, \mathcal{D}) = \text{Ш}^1(K, \Sigma, \mathcal{D})$. In this case, just as explained in illustration 5.2.6 in [Gr5], LEO($\mathcal{D}$) and CRK($\mathcal{D}, \mathcal{L}$) are both satisfied. Assumption *(b)* just means that $F \neq K(\mu_p)$. Thus, proposition 4.1.1 implies that $S_{\mathcal{L}}(K, \mathcal{D})$ is almost Λ-divisible, except in the case $F = \mathbf{Q}(\mu_p)$. Of course, that case only occurs when $[K(\mu_p) : K] = 2$.

Assumption *(c)* also fails in the case where $F = \mathbf{Q}(\mu_p)$. In fact, the map $\phi_{\mathcal{L}}$ is not surjective. However, one can use the result from section 3.3 to settle this case. One shows that the map (5) is injective for infinitely many $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$ by showing that both groups have the same order. The map is surjective, and will therefore be injective. Since Λ has Krull dimension 2, proving injectivity for infinitely many Π's is sufficient, as explained in remark 3.1.1.

The above illustrations are very closely related to the classical theorems mentioned in the beginning of the introduction. The Selmer groups $S_{\mathcal{L}}(K, \mathcal{D})$ which we just considered are isomorphic as Λ-modules to the ones arising in those theorems. The relationship is based on a version of Shapiro's lemma. One finds this discussed without proof in the introduction to [Gr4]. In particular, see the discussion surrounding theorem 2 in that paper. We intend to justify this relationship completely in [Gr6]

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