On the Structure of Certain Galois Cohomology Groups

Ralph Greenberg[†]

To John Coates on the occasion of his 60th birthday

1 Introduction

Suppose that K is a finite extension of $\mathbb{Q}$ and that Σ is a finite set of primes of K. Let K_Σ denote the maximal extension of K unramified outside of Σ. We assume that Σ contains all archimedean primes and all primes lying over some fixed rational prime p. The Galois cohomology groups that we consider in this article are associated to a continuous representation

$$\rho : \mathrm{Gal}(K_\Sigma/K) \longrightarrow GL_n(R)$$

where R is a complete local ring. We assume that R is Noetherian and commutative. Let $\mathfrak{m}$ denote the maximal ideal of R. We also assume that the residue field $R/\mathfrak{m}$ is finite and has characteristic p. Thus, R is compact in its $\mathfrak{m}$-adic topology, as will be any finitely generated R-module. Let $\mathcal{T}$ be the underlying free R-module on which $\mathrm{Gal}(K_\Sigma/K)$ acts via ρ. We define $\mathcal{D} = \mathcal{T} \otimes_R \widehat{R}$, where $\widehat{R} = \mathrm{Hom}(R, \mathbb{Q}_p/\mathbb{Z}_p)$ is the Pontryagin dual of R with a trivial action of $\mathrm{Gal}(K_\Sigma/K)$. Thus, $\mathcal{D}$ is a discrete abelian group which is isomorphic to $\widehat{R}^n$ as an R-module and which has a continuous R-linear action of $\mathrm{Gal}(K_\Sigma/K)$ given by ρ.

The Galois cohomology groups $H^i(K_\Sigma/K, \mathcal{D})$, where $i \geq 0$, can be considered as discrete R-modules too. The action of $\mathrm{Gal}(K_\Sigma/K)$ on $\mathcal{D}$ is R-linear and so, for any $r \in R$, the map $\mathcal{D} \to \mathcal{D}$ induced by multiplication by r induces a corresponding map on $H^i(K_\Sigma/K, \mathcal{D})$. This defines the R-module structure. It is not hard to prove that these Galois cohomology groups are cofinitely generated over R. That is, their Pontryagin duals are finitely generated R-modules. We will also consider the subgroup defined by

$$Ш^i(K, \Sigma, \mathcal{D}) = \ker\big(H^i(K_\Sigma/K, \mathcal{D}) \to \prod_{v\in\Sigma} H^i(K_v, \mathcal{D})\big).$$

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Here K_v denotes the v-adic completion of K. Thus, $\text{Ш}^i(K,\Sigma,\mathcal{D})$ consists of cohomology classes which are locally trivial at all primes in Σ and is easily seen to be an R-submodule of $H^i(K_\Sigma/K,\mathcal{D})$. Of course, it is obvious that $\text{Ш}^0(K,\Sigma,\mathcal{D})=0$. It turns out that $\text{Ш}^i(K,\Sigma,\mathcal{D})=0$ for $i\geq 3$ too. However, the groups $\text{Ш}^1(K,\Sigma,\mathcal{D})$ and $\text{Ш}^2(K,\Sigma,\mathcal{D})$ can be nontrivial and are rather mysterious objects in general.

Suppose that one has a surjective, continuous ring homomorphism $\phi: R\to\mathcal{O}$, where $\mathcal{O}$ is a finite, integral extension of $\mathbb{Z}_p$. Such homomorphisms exist if R is a domain and has characteristic 0. Then $\mathcal{P}_\phi=\ker(\phi)$ is a prime ideal of R. One can reduce the above representation modulo $\mathcal{P}_\phi$ to obtain a representation $\rho_\phi:\mathrm{Gal}(K_\Sigma/K)\longrightarrow GL_n(\mathcal{O})$ which is simply the composition of ρ with the homomorphism $GL_n(R)\to GL_n(\mathcal{O})$ induced by ϕ. Thus, ρ is a deformation of ρ_ϕ and one can think of ρ as a family of such representations. The underlying Galois module for ρ_ϕ is $T_\phi=\mathcal{T}/\mathcal{P}_\phi\mathcal{T}$. This is a free $\mathcal{O}$-module of rank n. Let $D_\phi=T_\phi\otimes_{\mathcal{O}}\widehat{\mathcal{O}}$, where $\widehat{\mathcal{O}}$ is the Pontryagin dual of $\mathcal{O}$ with trivial Galois action. The Pontryagin dual of $R/\mathcal{P}_\phi$ is $\widehat{R}[\mathcal{P}_\phi]$, the submodule of $\widehat{R}$ annihilated by $\mathcal{P}_\phi$. Since $R/\mathcal{P}_\phi\cong\mathcal{O}$, we have $\widehat{R}[\mathcal{P}_\phi]\cong\widehat{\mathcal{O}}$. One can identify D_ϕ with $\mathcal{D}[\mathcal{P}_\phi]$. We can compare the cohomology of D_ϕ with $\mathcal{D}$ since one has a natural homomorphism

$$H^i(K_\Sigma/K,D_\phi)=H^i(K_\Sigma/K,\mathcal{D}[\mathcal{P}_\phi])\longrightarrow H^i(K_\Sigma/K,\mathcal{D})[\mathcal{P}_\phi].$$

However, unless one makes certain hypotheses, this homomorphism may fail to be injective and/or surjective. Note also that all of the representation ρ_ϕ have the same residual representation, namely $\overline{\rho}$, the reduction of ρ modulo $\mathfrak{m}$. This gives the action of $\mathrm{Gal}(K_\Sigma/K)$ on $T_\phi/\mathfrak{m}T_\phi\cong\mathcal{T}/\mathfrak{m}\mathcal{T}$ or, alternatively, on the isomorphic Galois modules $D_\phi[\mathfrak{m}]\cong\mathcal{D}[\mathfrak{m}]$.

Assume that R is a domain. Let X denote the Pontryagin dual of $H^1(K_\Sigma/K,\mathcal{D})$. One can derive a certain lower bound for $\mathrm{rank}_R(X)$ by using Tate's theorems on global Galois cohomology groups. Let Y denote the torsion R-submodule of X. The main result of this paper is to show that if $\mathrm{rank}_R(X)$ is equal to the lower bound and if R and ρ satisfy certain additional assumptions, then the associated prime ideals for Y are all of height 1. Thus, under certain hypotheses, we will show that X has no nonzero pseudo-null R-submodules. By definition, a finitely generated, torsion R-module Z is said to be *"pseudo-null"* if the localization $Z_{\mathcal{P}}$ is trivial for every prime ideal $\mathcal{P}$ of R of height 1, or, equivalently, if the associated prime ideals for Z have height at least 2.

If the Krull dimension of R is $d=m+1$, where $m\geq 0$, then it is known that R contains a subring Λ such that *(i)* Λ is isomorphic to either $\mathbb{Z}_p[[T_1,...,T_m]]$ or $\mathbb{F}_p[[T_1,...,T_{m+1}]]$, depending on whether R has characteristic 0 or p, and *(ii)* R is finitely generated as a Λ-module. (See theorem 6.3 in [D].) One important assumption that we will often make is that R is reflexive as a Λ-module. We then say that R is a reflexive domain. It turns out that this does not depend on the choice of the subring Λ. An equivalent, intrinsic way of stating this assumption is the following: $R = \bigcap_{\mathcal{P}} R_{\mathcal{P}}$, *where* $\mathcal{P}$ *varies over all prime ideals of* R *of height 1.* Here $R_{\mathcal{P}}$ denotes the localization of R at $\mathcal{P}$, viewed as a subring of the fraction field $\mathcal{K}$ of R. Such rings form a large class. For example, if R is integrally closed, then R is reflexive. Or, if R is Cohen-Macaulay, then R will actually be a free Λ-module and so will also be reflexive. We will also say that a finitely generated, torsion-free R-module X is reflexive if $X = \bigcap_{\mathcal{P}} X_{\mathcal{P}}$, where $\mathcal{P}$ again varies over all the prime ideals of R of height 1 and $X_{\mathcal{P}} = X \otimes_R R_{\mathcal{P}}$ considered as an R- submodule of the $\mathcal{K}$-vector space $X \otimes_R \mathcal{K}$.

We will use the following standard terminology throughout this paper. If A is a discrete R-module, let $X = \widehat{A}$ denote its Pontryagin dual. We say that A is a cofinitely generated R-module if X is finitely generated as an R-module, A is a cotorsion R-module if X is a torsion R-module, and A is a cofree R-module if X is a free R-module. We define $\mathrm{corank}_R(A)$ to be $\mathrm{rank}_R(X)$. Similar terminology will be used for Λ-modules. Although it is not so standard, we will say that A is coreflexive if X is reflexive, either as an R-module or as a Λ-module, and that A is co-pseudo-null if X is pseudo-null. For most of these terms, it doesn't matter whether the ring is Λ or a finite, integral extension R of Λ. For example, as we will show in section 2, A is a coreflexive R-module if and only if it is a coreflexive Λ-module. A similar statement is true for co-pseudo-null modules. However, the module $\mathcal{D}$ defined above for a representation ρ is a cofree R-module and a coreflexive, but not necessarily cofree, Λ-module, assuming that R is a reflexive domain.

Assume that X is a torsion-free R-module. Then, if r is any nonzero element of R, multiplication by r defines an injective map $X \to X$. The corresponding map on the Pontryagin dual is then surjective. Thus, $A = \widehat{X}$ will be a divisible R-module. Conversely, if A is a divisible R-module, then X is torsion-free. If R is a finite, integral extension of Λ, then A is divisible as an R-module if and only if A is divisible as a Λ-module. The kernel of multiplication by an element $r \in R$ will be denoted by $A[r]$. More generally, if I is any ideal of R or Λ, we let $A[I] = \{a \in A \mid ia = 0 \text{ for all } i \in I\}$.

Suppose v is a prime of K. Let $\overline{K}, \overline{K}_v$ denote algebraic closures of the indicated fields and let $\mathrm{G}_K = \mathrm{Gal}(\overline{K}/K)$, $\mathrm{G}_{K_v} = \mathrm{Gal}(\overline{K}_v/K_v)$. We can fix an embedding $\overline{K} \to \overline{K}_v$ and this induces continuous homomorphisms $\mathrm{G}_{K_v} \to \mathrm{G}_K \to \mathrm{Gal}(K_\Sigma/K)$. Thus, we get a continuous R-linear action of G_{K_v} on $\mathcal{T}$ and on $\mathcal{D}$. Define $\mathcal{T}^* = \mathrm{Hom}(\mathcal{D}, \mu_{p^\infty})$, where μ_{p^∞} denotes the group of p-power roots of unity. Note that $\mathcal{T}^*$ is a free R-module of rank n. Choosing a basis, the natural action of $\mathrm{Gal}(K_\Sigma/K)$ on $\mathcal{T}^*$ is given by a continuous homomorphism $\rho^* : \mathrm{Gal}(K_\Sigma/K) \longrightarrow GL_n(R)$. Consider the action of G_{K_v} on $\mathcal{T}^*$. The set of G_{K_v}-invariant elements $(\mathcal{T}^*)^{\mathrm{G}_{K_v}} = \mathrm{Hom}_{\mathrm{G}_{K_v}}(\mathcal{D}, \mu_{p^\infty})$ is an R-submodule. The following theorem is the main result of this paper.

Theorem 1. *Suppose that* R *is a reflexive domain. Suppose also that* $\mathcal{T}^*$ *satisfies the following two local assumptions:*

(a) For every prime $v \in \Sigma$, the R-module $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is reflexive.

(b) There is at least one non-archimedean prime $v_o \in \Sigma$ such that $(\mathcal{T}^*)^{G_{K_{v_o}}} = 0$.

Then $Ш^2(K, \Sigma, \mathcal{D})$ is a coreflexive R-module. If $Ш^2(K, \Sigma, \mathcal{D}) = 0$, then the Pontryagin dual of $H^1(K_\Sigma/K, \mathcal{D})$ has no nonzero, pseudo-null R-submodules.

The proof of this theorem will be given in section 6, but some comments about the role of various assumptions may be helpful here. The assumption that R is a domain is not essential. It suffices to just assume that R contains a formal power series ring Λ over either $\mathbb{Z}_p$ or $\mathbb{F}_p$ and that R is a finitely generated, reflexive module over Λ. Then $\mathcal{D}$ will be a coreflexive Λ-module. In fact, it is precisely that assumption which is needed in the argument. In particular, it implies that if π is an irreducible element of Λ, then $\mathcal{D}[\pi]$ is a divisible module over the ring $\Lambda/(\pi)$. Coreflexive Λ-modules are characterized by that property. (See corollary 2.6.1.) The assertion that $Ш^2(K, \Sigma, \mathcal{D})$ is also a coreflexive Λ-module implies that it is Λ-divisible, but is actually a much stronger statement. Reflexive Λ-modules are a rather small subclass of the class of torsion-free Λ-modules.

The conclusion in theorem 1 concerning $H^1(K_\Sigma/K, \mathcal{D})$ can be expressed in another way which seems quite natural. It suffices to consider it just as a Λ-module. The ring Λ is a UFD and so we can say that two nonzero elements of Λ are relatively prime if they have no irreducible factor in common. We make the following definition.

Definition. *Assume that A is a discrete Λ-module. We say that A is an "almost divisible" Λ-module if there exists a nonzero element $\theta \in \Lambda$ with the following property: If $\lambda \in \Lambda$ is a nonzero element relatively prime to θ, then $\lambda A = A$.*

If A is a cofinitely generated Λ-module, then it is not hard to see that A is an almost divisible Λ-module if and only if the Pontryagin dual of A has no nonzero pseudo-null Λ-submodules. (See proposition 2.4.) Under the latter condition, one could take θ to be any nonzero annihilator of the torsion Λ-submodule Y of $X = \widehat{A}$, e.g., a generator of the characteristic ideal of Y. Thus, theorem 1 asserts that, under certain assumptions, the Λ-module $H^1(K_\Sigma/K, \mathcal{D})$ is almost divisible.

The main local ingredient in the proof is to show that $H^1(K_v, \mathcal{D})$ is an almost divisible Λ-module for all $v \in \Sigma$. It is local assumption *(a)* that guarantees this. In fact, it is sufficient to assume that $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is reflexive as a Λ-module for all $v \in \Sigma$. This implies that the map $H^2(K_v, \mathcal{D}[P]) \longrightarrow H^2(K_v, \mathcal{D})$ is injective for all but a finite number of prime ideals P in Λ of height 1; the almost divisibility of $H^1(K_v, \mathcal{D})$ follows from that. The hypothesis that $Ш^2(K, \Sigma, \mathcal{D}) = 0$ then allows us to deduce that the map $H^2(K_\Sigma/K, \mathcal{D}[P]) \longrightarrow H^2(K_\Sigma/K, \mathcal{D})$ is injective for all but finitely many such P's, which implies the almost divisibility of $H^1(K_\Sigma/K, \mathcal{D})$.

Both local assumptions *(a)* and *(b)* are used in the proof that $\text{Ш}^2(K, \Sigma, \mathcal{D})$ is a coreflexive Λ-module. Assumption *(b)* obviously implies that $(\mathcal{T}^*)^{\mathrm{Gal}(K_\Sigma/K)} = 0$. That fact, in turn, implies that the global-to-local map defining $\text{Ш}^2(K, \Sigma, \mathcal{D})$ is surjective. Such a surjectivity statement plays an important role in our proof of theorem 1. We will discuss the validity of the local assumptions at the end of section 5. Local assumption *(a)* is easily verified for archimedean primes if p is odd, but is actually not needed in that case. It is needed when $p = 2$ and, unfortunately, could then fail to be satisfied. For non-archimedean primes, the local assumptions are often satisfied simply because $(\mathcal{T}^*)^{G_{K_v}} = 0$ for *all* such $v \in \Sigma$. However, there are interesting examples where this fails to be true for at least some v's in Σ and so it is too restrictive to make that assumption.

The hypothesis that $\text{Ш}^2(K, \Sigma, \mathcal{D}) = 0$ is quite interesting in itself. Under the assumptions in theorem 1, $\text{Ш}^2(K, \Sigma, \mathcal{D})$ will be coreflexive, and hence divisible, as an R-module. Therefore, the statement that $\text{Ш}^2(K, \Sigma, \mathcal{D}) = 0$ would then be equivalent to the seemingly weaker statement that $\mathrm{corank}_R\big(\text{Ш}^2(K, \Sigma, \mathcal{D})\big) = 0$. Just for convenience, we will give a name to that statement.

Hypothesis L: $\text{Ш}^2(K, \Sigma, \mathcal{D})$ *is a cotorsion R-module.*

Of course, it is only under certain assumptions that this statement implies that $\text{Ш}^2(K, \Sigma, \mathcal{D})$ actually vanishes. We will now describe two equivalent formulations of hypothesis L which are more easily verified in practice. To state the first one, let $\mathcal{D}^* = \mathcal{T}^* \otimes_R \widehat{R}$. Then we will show that

$$\mathrm{corank}_R\big(\text{Ш}^2(K, \Sigma, \mathcal{D})\big) = \mathrm{corank}_R\big(\text{Ш}^1(K, \Sigma, \mathcal{D}^*)\big) \tag{1}$$

This will be proposition 4.4. Thus, one reformulation of hypothesis L is the assertion that $\text{Ш}^1(K, \Sigma, \mathcal{D}^*)$ is a cotorsion R-module. This formulation has the advantage that it is easier to study H^1 and hence Ш^1. We should mention that even under strong hypotheses like those in theorem 1, it is quite possible for $\text{Ш}^1(K, \Sigma, \mathcal{D}^*)$ to be a nonzero, cotorsion R-module.

A second equivalent formulation can be given in terms of the R-corank of $H^1(K_\Sigma/K, \mathcal{D})$. As we mentioned before, we will derive a lower bound on this corank by using theorems of Tate. Those theorems concern finite Galois modules, but can be extended to Galois modules such as $\mathcal{D}$ in a straightforward way. The precise statement is given in proposition 4.3. It is derived partly from a formula for the Euler-Poincaré characteristic. For $i \geq 0$, we let $h_i = \mathrm{corank}_R\big(H^i(K_\Sigma/K, \mathcal{D})\big)$. Let r_2 denote the number of complex primes of K. For each real prime v of K, let $n_v^- = \mathrm{corank}_R\big(\mathcal{D}/\mathcal{D}^{G_{K_v}}\big)$. Then

$$h_1 = h_0 + h_2 + \delta.$$

where $\delta = r_2 n + \sum_{v \text{ real}} n_v^-$. The Euler-Poincaré characteristic is $h_0 - h_1 + h_2 = -\delta$. Thus, h_1 is essentially determined by h_0 and h_2 since the quantity δ is usually easy to evaluate. On the other hand, one gets a lower bound on h_2 by studying the global-to-local map

$$\gamma : H^2(K_\Sigma/K, \mathcal{D}) \longrightarrow P^2(K, \Sigma, \mathcal{D}),$$

where $P^2(K, \Sigma, \mathcal{D}) = \prod_{v \in \Sigma} H^2(K_v, \mathcal{D})$. The cokernel of γ is determined by Tate's theorems: $\text{coker}(\gamma) \cong H^0(K_\Sigma/K, \mathcal{T}^*)^\wedge$. Thus, one can obtain a certain lower bound for h_2 and hence for h_1. In proposition 4.3, we give this lower bound in terms of the ranks or coranks of various H^0's. The assertion that h_1 is equal to this lower bound is equivalent to the assertion that $\ker(\gamma)$ has R-corank 0, which is indeed equivalent to hypothesis L.

The local duality theorem of Poitou and Tate asserts that the Pontryagin dual of $H^2(K_v, \mathcal{D})$ is isomorphic to $H^0(K_v, \mathcal{T}^*) = (\mathcal{T}^*)^{G_{K_v}}$. Thus, if we assume that $(\mathcal{T}^*)^{G_{K_v}} = 0$ for all non-archimedean $v \in \Sigma$, then $H^2(K_v, \mathcal{D}) = 0$ for all such v. If we also assume that p is odd, then obviously $H^2(K_v, \mathcal{D}) = 0$ for all archimedean v. Under these assumptions, $P^2(K, \Sigma, \mathcal{D}) = 0$ and Hypothesis L would then be equivalent to the assertion that $H^2(K_\Sigma/K, \mathcal{D}) = 0$.

The validity of Hypothesis L seems to be a very subtle question. We will discuss this at the end of section 6. It can fail to be satisfied if R has Krull dimension 1. If R has characteristic 0, then, apart from simple counterexamples constructed by extension of scalars, it is not at all clear what one should expect when the Krull dimension is greater than 1. However, one can construct nontrivial counterexamples where R has arbitrarily large Krull dimension and R has characteristic p.

Theorem 1 has a number of interesting consequences in classical Iwasawa theory. These will be the subject of a subsequent paper. We will just give an outline of some of them here. In fact, our original motivation for this work was to improve certain results in our earlier paper [Gr89]. There we considered the cyclotomic $\mathbb{Z}_p$-extension K_∞ of a number field K and a discrete $\text{Gal}(K_\Sigma/K)$-module D isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^n$ as a $\mathbb{Z}_p$-module. We obtain such a Galois module from a vector space V of dimension n over $\mathbb{Q}_p$ which has a continuous $\mathbb{Q}_p$-linear action of $\text{Gal}(K_\Sigma/K)$. Let T be a Galois-invariant $\mathbb{Z}_p$-lattice in V and let $D = V/T$. The Galois action defines a representation $\rho_o : \text{Gal}(K_\Sigma/K) \to \text{Aut}_{\mathbb{Z}_p}(T) \cong GL_n(\mathbb{Z}_p)$. Since only primes of K lying above p can ramify in K_∞/K, we have $K_\infty \subset K_\Sigma$. One therefore has a natural action of $\Gamma = \text{Gal}(K_\infty/K)$ on the Galois cohomology groups $H^i(K_\Sigma/K_\infty, D)$ for any $i \geq 0$. Now $H^i(K_\Sigma/K_\infty, D)$ is also a $\mathbb{Z}_p$-module. One can then regard $H^i(K_\Sigma/K_\infty, D)$ as a discrete Λ-module, where $\Lambda = \mathbb{Z}_p[[\Gamma]]$, the completed $\mathbb{Z}_p$-group algebra for Γ. The ring Λ is isomorphic to the formal power series ring $\mathbb{Z}_p[[T]]$ in one variable and is a complete Noetherian local domain of Krull dimension 2. The modules $H^i(K_\Sigma/K_\infty, D)$ are cofinitely generated over Λ.

Propositions 4 and 5 in [Gr89] assert that if p is an odd prime, then $H^2(K_\Sigma/K_\infty, D)$ is a cofree Λ-module, and if $H^2(K_\Sigma/K_\infty, D) = 0$, then the Pontryagin dual of $H^1(K_\Sigma/K_\infty, D)$ contains no nonzero, finite Λ-modules. One consequence of theorem 1 is the following significantly more general result. We allow p to be *any* prime and K_∞/K to be *any* Galois extension such that $\Gamma = \mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_p^m$ for some $m \geq 1$. For any $i \geq 0$, we define $Ш^i(K_\infty, \Sigma, D)$ to be the subgroup of $H^i(K_\Sigma/K_\infty, D)$ consisting of cocycle classes which are locally trivial at all primes of K_∞ lying above the primes in Σ. Again, Γ acts continuously on those Galois cohomology groups and so we can regard them as modules over the ring $\Lambda = \mathbb{Z}_p[[\Gamma]]$. This ring is now isomorphic to the formal power series ring $\mathbb{Z}_p[[T_1, ..., T_m]]$ in m variables and has Krull dimension $d = m+1$. The group $Ш^i(K_\infty, \Sigma, D)$ is a Λ-submodule of $H^i(K_\Sigma/K_\infty, D)$. All of these Λ-modules are cofinitely generated.

Theorem 2. *Let p be a prime. Suppose that K_∞/K is any $\mathbb{Z}_p^m$-extension, where $m \geq 1$. Then $Ш^2(K_\infty, \Sigma, D)$ is a coreflexive Λ-module. If $Ш^2(K_\infty, \Sigma, D) = 0$, then the Pontryagin dual of $H^1(K_\Sigma/K_\infty, D)$ has no nonzero, pseudo-null Λ-submodules.*

The results proved in [Gr89] which were mentioned above concern the case where K_∞ is the cyclotomic $\mathbb{Z}_p$-extension of K. For odd p, one then has $Ш^2(K_\infty, \Sigma, D) = H^2(K_\Sigma/K_\infty, D)$. The assertion about cofreeness follows since $m = 1$ and so a cofinitely generated Λ-module A is coreflexive if and only if it is cofree. (See remark 2.6.2.) Also, A is co-pseudo-null if and only if it is finite. In that special case, theorem 2 is more general only because it includes $p = 2$.

The relationship to theorem 1 is based on a version of Shapiro's lemma which relates the above cohomology groups to those associated with a suitably defined $\mathrm{Gal}(K_\Sigma/K)$-module $\mathcal{D}$. We can regard Γ as a subgroup of the multiplicative group $\Lambda^\times$ of Λ. This gives a homomorphism $\Gamma \to GL_1(\Lambda)$. and hence a representation over Λ of $\mathrm{Gal}(K_\Sigma/K)$ of rank 1 factoring through Γ. We will denote this representation by κ. Define $\mathcal{T} = T \otimes_{\mathbb{Z}_p} \Lambda$. Thus, $\mathcal{T}$ is a free Λ-module of rank n. We let $\mathrm{Gal}(K_\Sigma/K)$ act on $\mathcal{T}$ by the representation $\rho = \rho_o \otimes \kappa^{-1}$. We then define, as before, $\mathcal{D} = \mathcal{T} \otimes_\Lambda \widehat{\Lambda}$, which is a cofree Λ-module with a Λ-linear action of $\mathrm{Gal}(K_\Sigma/K)$. The Galois action is through the first factor $\mathcal{T}$. We will say that $\mathcal{D}$ is induced from D via the $\mathbb{Z}_p^m$-extension K_∞/K. Sometimes we will use the notation: $\mathcal{D} = \mathrm{Ind}_{K_\infty/K}(D)$. Of course, the ring R is now Λ which is certainly a reflexive domain. We have the following comparison theorem.

Theorem 3. *For $i \geq 0$, $H^i(K_\Sigma/K, \mathcal{D}) \cong H^i(K_\Sigma/K_\infty, D)$ as Λ-modules.*

There is a similar comparison theorem for the local Galois cohomology groups which is compatible with the isomorphism in theorem 3 and so, for any $i \geq 0$, one obtains an isomorphism

$$Ш^i(K, \Sigma, \mathcal{D}) \cong Ш^i(K_\infty, \Sigma, D) \tag{2}$$

as Λ-modules. In particular, one can deduce from (1) and (2) that $Ш^2(K_\infty, \Sigma, D)$ has the same Λ-corank as $Ш^1(K_\infty, \Sigma, D^*)$, where $D^* = \mathrm{Hom}(T, \mu_{p^\infty})$.

Both of the local assumptions in theorem 1 turn out to be automatically satisfied for $\mathcal{D}$ and so theorem 2 is indeed a consequence of theorem 1. The verification of those assumptions is rather straightforward. The most subtle point is the consideration of primes that split completely in K_∞/K, including the archimedean primes of K if $p = 2$. For any v which does not split completely, one sees easily that $(\mathcal{T}^*)^{G_{K_v}} = 0$. Thus, hypothesis *(b)* is satisfied since at least one of the primes of K lying over p must be ramified in K_∞/K; one could take v_o to be one of those primes. If v does split completely, then one shows that $(\mathcal{T}^*)^{G_{K_v}}$ is a direct summand in the free Λ-module $\mathcal{T}^*$. This implies that the corresponding quotient, the complementary direct summand, is also a free Λ-module and hence reflexive.

As a consequence, we can say that $Ш^2(K_\infty, \Sigma, D)$ is a coreflexive Λ-module. We believe that it is reasonable to make the following conjecture.

Conjecture L. *Suppose that K_∞ is an arbitrary $\mathbb{Z}_p^m$-extension of a number field K, Σ is any finite set of primes of K containing the primes lying above p and ∞, and D is a $\mathrm{Gal}(K_\Sigma/K)$-module which is isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^n$ as a group for some $n \geq 1$. Then $Ш^2(K_\infty, \Sigma, D) = 0$.*

That is, hypothesis L should hold for $\mathcal{D} = \mathrm{Ind}_{K_\infty/K}(D)$. Equivalently, $Ш^1(K_\infty, \Sigma, D^*)$ should be a cotorsion Λ-module. Furthermore, it turns out that the global-to-local map γ is now actually surjective. The Λ-module $P^2(K, \Sigma, \mathcal{D})$ can, in general, be nonzero and even have positive Λ-corank. To be precise, only primes v of K which split completely in K_∞/K can make a nonzero contribution to $P^2(K, \Sigma, \mathcal{D})$. The contribution to the Λ-corank can only come from the non-archimedean primes. If v is a non-archimedean prime which splits completely in K_∞/K, then we have $\mathrm{corank}_\Lambda\big(H^2(K_v, \mathcal{D})\big) = \mathrm{corank}_{\mathbb{Z}_p}\big(H^2(K_v, D)\big)$ and this can be positive.

As an illustration, consider the special case where $D = \mu_{p^\infty}$. In this case, $D^* = \mathbb{Q}_p/\mathbb{Z}_p$ (with trivial Galois action). One then has the following concrete description of $Ш^1(K_\infty, \Sigma, D^*)$. Let L_∞ denote the maximal, abelian, pro-p-extension of K_∞ which is unramified at all primes. Let L'_∞ be the subfield in which all primes of K_∞ split completely. Then we have

$$Ш^1(K_\infty, \Sigma, \mathbb{Q}_p/\mathbb{Z}_p) = \mathrm{Hom}\big(\mathrm{Gal}(L'_\infty/K_\infty), \mathbb{Q}_p/\mathbb{Z}_p)\big)$$

It is known that $\mathrm{Gal}(L_\infty/K_\infty)$ is a finitely generated, torsion Λ-module. (This is a theorem of Iwasawa if $m = 1$ and is proved in [Gr73] for arbitrary m.) Hence the same thing is true for the quotient Λ-module $\mathrm{Gal}(L'_\infty/K_\infty)$. Therefore, $Ш^1(K_\infty, \Sigma, \mathbb{Q}_p/\mathbb{Z}_p)$ is indeed Λ-cotorsion. Thus, conjecture L is valid for $D = \mu_{p^\infty}$ for an arbitrary $\mathbb{Z}_p^m$-extension K_∞/K. Note also that $\mathrm{corank}_{\mathbb{Z}_p}\big(H^2(K_v, \mu_{p^\infty})\big) = 1$ for any non-archimedean prime v. Hence, if Σ

contains non-archimedean primes which split completely in K_∞/K, then $H^2(K_\Sigma/K_\infty, \mu_{p^\infty})$ will have a positive Λ-corank. Since $\text{Ш}^2(K_\infty, \Sigma, \mu_{p^\infty}) = 0$, as just explained, it follows that $\text{corank}_\Lambda\big(H^2(K_\Sigma/K_\infty, \mu_{p^\infty})\big)$ is precisely the number of such primes, i.e., the cardinality of Υ. Therefore, the Λ-corank of $H^1(K_\Sigma/K_\infty, \mu_{p^\infty})$ will be equal to $r_1 + r_2 + |\Upsilon|$. Non-archimedean primes that split completely in a $\mathbb{Z}_p^m$-extension can exist. For example, let K be an imaginary quadratic field and let K_∞ denote the so-called "anti-cyclotomic" $\mathbb{Z}_p$-extension of K. Thus, K_∞ is a Galois extension of $\mathbb{Q}$ and $\text{Gal}(K_\infty/\mathbb{Q})$ is a dihedral group. One sees easily that if v is any prime of K not lying over p which is inert in $K/\mathbb{Q}$, then v splits completely in K_∞/K.

As a second illustration, consider the Galois module $D = \mathbb{Q}_p/\mathbb{Z}_p$ with a trivial action of $\text{Gal}(K_\Sigma/K)$. For an arbitrary $\mathbb{Z}_p^m$-extension K_∞/K, it is not hard to see that $\text{Ш}^2(K_\infty, \Sigma, D) = H^2(K_\Sigma/K_\infty, D)$. This is so because $H^2(K_v, \mathbb{Q}_p/\mathbb{Z}_p) = 0$ for all primes v of K. Let M_∞^Σ denote the maximal abelian pro-p-extension of K_∞ contained in K_Σ. Then $H^1(K_\Sigma/K_\infty, D) = \text{Hom}\big(\text{Gal}(M_\infty^\Sigma/K_\infty), \mathbb{Q}_p/\mathbb{Z}_p\big)$, which is just the Pontryagin dual of $\text{Gal}(M_\infty^\Sigma/K_\infty)$. In this case, $n = 1$ and $n_v^- = 0$ for all real primes. Conjecture L is therefore equivalent to the statement that the Λ-module $\text{Gal}(M_\infty^\Sigma/K_\infty)$ has rank r_2. Theorem 3 together with other remarks we have made has the following consequence.

Theorem 4. *Let p be a prime. Suppose that K_∞/K is any $\mathbb{Z}_p^m$-extension, where $m \geq 1$. Then $\text{Gal}(M_\infty^\Sigma/K_\infty)$ is a finitely generated Λ-module and $\text{rank}_\Lambda\big(\text{Gal}(M_\infty^\Sigma/K_\infty)\big) \geq r_2$. If $\text{rank}_\Lambda\big(\text{Gal}(M_\infty^\Sigma/K_\infty)\big) = r_2$, then $\text{Gal}(M_\infty^\Sigma/K_\infty)$ has no nonzero pseudo-null Λ-submodules.*

Let M_∞ denote the maximal abelian pro-p-extension of K_∞ which is unramified at all primes of K_∞ not lying above p or ∞. One can show that $\text{Gal}(M_\infty^\Sigma/M_\infty)$ is a torsion Λ-module and so the equality in the above theorem is equivalent to the assertion that the Λ-rank of $\text{Gal}(M_\infty/K_\infty)$ is equal to r_2. Note that $M_\infty = M_\infty^\Sigma$ if one takes $\Sigma = \{v \mid v|p \text{ or } v|\infty\}$. In that case, the above theorem is proved in [NQD]. A somewhat different, but closely related, result is proved in [Gr78]. Theorem 1 can be viewed as a rather broad generalization of these results in classical Iwasawa theory.

The statement that $\text{corank}_\Lambda\big(\text{Gal}(M_\infty/K_\infty)\big) = r_2$ is known as the *Weak Leopoldt Conjecture for K_∞/K*. That name arises from the fact that if one considers a $\mathbb{Z}_p$-extension K_∞/K and the Galois module $D = \mathbb{Q}_p/\mathbb{Z}_p$, the conjecture is equivalent to the following assertion:

Let K_n denote the n-th layer in the $\mathbb{Z}_p$-extension K_∞/K. Let M_n be the compositum of all $\mathbb{Z}_p$-extensions of K_n. Let $\delta_n = \text{rank}_{\mathbb{Z}_p}\big(\text{Gal}(M_n/K_n)\big) - r_2p^n$. Then δ_n is bounded as $n \to \infty$.

The well-known conjecture of Leopoldt would assert that $\delta_n = 1$ for all n.

If a $\mathbb{Z}_p^m$-extension K_∞ of K contains μ_{p^∞}, then the Galois modules μ_{p^∞} and $\mathbb{Q}_p/\mathbb{Z}_p$ are isomorphic over K_∞. Since conjecture L is valid for $D = \mu_{p^\infty}$, it is then also valid for $D = \mathbb{Q}_p/\mathbb{Z}_p$. One deduces easily that conjecture L is valid for $D = \mathbb{Q}_p/\mathbb{Z}_p$ if one just assumes that K_∞ contains the cyclotomic $\mathbb{Z}_p$-extension of K. Thus, under that assumption, it follows unconditionally that $\mathrm{Gal}(M_\infty^\Sigma/K_\infty)$ has no nonzero pseudo-null Λ-submodules. If K_∞ is the cyclotomic $\mathbb{Z}_p$-extension of K, then this result was originally proved by Iwasawa. It is theorem 18 in [Iw73]. He showed that that Galois group indeed has Λ-rank r_2 and deduced the non-existence of finite Λ-submodules from that.

There is a long history behind the topics discussed in this article. We have already mentioned Iwasawa's theorem in [Iw73]. A similar, but less general, result is proved in his much earlier paper [Iw59]. There he assumes a special case of Leopoldt's conjecture. Those theorems of Iwasawa were generalized in [Gr78], [NQD], and [Pe84] for similarly-defined Galois groups, but over $\mathbb{Z}_p^m$-extensions of a number field. The generalization to Galois cohomology groups for arbitrary Galois modules of the form $D = V/T$ has also been considered by several authors, e.g., see [Sch], [Gr89], and [J]. The conjecture concerning the possible vanishing of $H^2(K_\Sigma/K_\infty, D)$, and its relevance to the question of finite submodules, can be found in those references. Perrin-Riou has a substantial discussion of these issues in [Pe95], Appendice B, referring to that conjecture as the *Conjecture de Leopoldt faible* because it generalizes the assertion of the same name mentioned before. We also want to mention that the idea of proving the non-existence of nonzero pseudo-null submodules under an assumption like hypothesis L was inspired by the thesis of McConnell [McC].

Considerable progress has been made in one important special case, namely $D = E[p^\infty]$, where E is an elliptic curve defined over $\mathbb{Q}$. If one takes K_∞ to be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, where p is an odd prime, then conjecture L was verified in [C-M] under certain hypotheses. This case is now settled completely; a theorem of Kato asserts that $H^2(K_\Sigma/K_\infty, E[p^\infty]) = 0$ if K_∞ is the cyclotomic $\mathbb{Z}_p$-extension of K, $K/\mathbb{Q}$ is assumed to be abelian and p is assumed to be odd. Kato's theorem applies more generally when $D = V/T$ and V is the p-adic representation associated to a cuspform.

More recently, similar types of questions have been studied when K_∞/K is a p-adic Lie extension. The ring Λ is then non-commutative. Nevertheless, Venjakob has defined the notion of pseudo-nullity and proved the non-existence of nonzero pseudo-null submodules in certain Galois groups. We refer the readers to [Ve] for a discussion of this situation. In [C-S], Coates and Sujatha study the group $\text{Ш}^1(K_\infty, \Sigma, E[p^\infty])$, where E is an elliptic curve defined over K. Those authors refer to this group as the *"fine Selmer group"* for E over K_∞ and conjecture that it is actually a co-pseudo-null Λ-module under certain assumptions.

Another topic which we intend to study in a future paper concerns the structure of a Selmer group $\mathrm{Sel}_{\mathcal{D}}(K)$ which can be attached to the representation ρ under certain assumptions. This Selmer group will be an R-submodule of $H^1(K_\Sigma/K, \mathcal{D})$ defined by imposing certain local conditions on the cocycles. Theorem 1 can then be effectively used to prove that the Pontryagin dual of $\mathrm{Sel}_{\mathcal{D}}(K)$ has no nonzero pseudo-null R-submodules under various sets of assumptions. One crucial assumption will be that $\mathrm{Sel}_{\mathcal{D}}(K)$ is a cotorsion R-module. Such a theorem is useful in that one can then study how the Selmer group behaves under specialization, i.e., reducing the representation ρ modulo a prime ideal $\mathcal{P}$ of R.

The study of Iwasawa theory in the context of a representation ρ was initiated in [Gr94]. More recently, Nekovar has taken a rather innovative point of view towards studying *large* representations and the associated cohomology and Selmer groups, introducing his idea of Selmer complexes [Nek]. It may be possible to give nice proofs of some of the theorems in this paper from such a point of view. In section 9.3 of his article, Nekovar does give such proofs in the context of classical Iwasawa theory. (See his proposition 9.3.1 and corollary 9.3.2.)

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2 Some Module Theory.

Theorem 1 and some of the other theorems mentioned in the introduction concern modules over a complete Noetherian local domain R. This section will include a variety of module-theoretic results that will be useful in the proofs. In particular, we will point out that several properties, such as pseudo-nullity or reflexivity, can be studied by simply considering the modules as Λ-modules. The main advantage of doing so is that Λ is a regular local ring and so has the following helpful property: *Every prime ideal of Λ of height 1 is principal.* This is useful in proofs by induction on the Krull dimension. Such arguments would work for any regular, Noetherian local ring. It seems worthwhile to state and prove various results in greater generality than we really need. However, in some cases, we haven't determined how general the theorems can be.

We will use the notation $\mathrm{Spec}_{ht=1}(R)$ to denote the set of prime ideals of height 1 in a ring R. The terminology *"almost all"* means *all but finitely many.* If I is any ideal of R, we will let $V(I)$ denote the set of prime ideals of R containing I.

A. Behavior of ranks and coranks under specialization. Consider a finitely generated module X over an integral domain R. If $\mathcal{K}$ is the fraction field of R, then $\mathrm{rank}_R(X) = \dim_{\mathcal{K}}(X \otimes_R \mathcal{K})$. The following result holds:

Proposition 2.1. *Let* $r = \mathrm{rank}_R(X)$. *Then* $\mathrm{rank}_{R/\mathcal{P}}(X/\mathcal{P}X) \geq r$ *for every prime ideal*

$\mathcal{P}$ *of* R. *There exists a nonzero ideal* I *of* R *such that* $\operatorname{rank}_{R/\mathcal{P}}(X/\mathcal{P}X) > r$ *if and only if* $\mathcal{P} \in V(I)$. *In particular,* $\operatorname{rank}_{R/\mathcal{P}}(X/\mathcal{P}X) = r$ *for all but finitely many prime ideals* $\mathcal{P} \in \operatorname{Spec}_{ht=1}(R)$.

Proof. We prove a somewhat more general result by a linear algebra argument. Suppose that $s \geq r$. We will show that there is an ideal I_s with the property:

$$\operatorname{rank}_{R/\mathcal{P}}(X/\mathcal{P}X) > s \iff \mathcal{P} \in V(I_s)$$

The ideal I_s will be a Fitting ideal. Suppose that X has g generators as an R-module. Thus X is a quotient of the free R-module $F = R^g$. Therefore, one has an exact sequence of R-modules

$$R^h \xrightarrow{\phi} R^g \xrightarrow{\psi} X \longrightarrow 0$$

The map ϕ is multiplication by a certain $g \times h$ matrix α. Let f denote the rank of the matrix α. The R-rank of the image of ϕ is equal to f and so we have $r = g - f$. By matrix theory, there is at least one $f \times f$-submatrix (obtained by omitting a certain number of rows and/or columns) of the matrix α whose determinant is nonzero, but there is no larger square submatrix with nonzero determinant.

For every prime ideal $\mathcal{P}$ of R, the above exact sequence induces a free presentation of $X/\mathcal{P}X$.

$$(R/\mathcal{P})^h \xrightarrow{\phi_{\mathcal{P}}} (R/\mathcal{P})^g \xrightarrow{\psi_{\mathcal{P}}} X/\mathcal{P}X \longrightarrow 0$$

The second term is $F/\mathcal{P}F$ and exactness at that term follows from the fact that the image of $\mathcal{P}F$ under ψ is $\mathcal{P}X$. The homomorphism $\phi_{\mathcal{P}}$ is multiplication by the matrix $\alpha_{\mathcal{P}}$, the reduction of α modulo $\mathcal{P}$. We have $\operatorname{rank}_{R/\mathcal{P}}(X/\mathcal{P}X) = g - \operatorname{rank}(\alpha_{\mathcal{P}})$. The description of the rank in terms of the determinants of submatrices shows that $\operatorname{rank}(\alpha_{\mathcal{P}}) \leq \operatorname{rank}(\alpha)$ for every prime ideal $\mathcal{P}$ of R. If $g \geq s \geq r$, let $e = g - s$ so that $0 \leq e \leq f$. Let I_s denote the ideal in R generated by the determinants of all $e \times e$ submatrices of the matrix α. If $e = 0$, then take $I_s = R$. Since $e \leq f$, it is clear that I_s is a nonzero ideal. Then $\alpha_{\mathcal{P}}$ has rank $< e$ if and only if $I_s \subseteq \mathcal{P}$. This implies that $\operatorname{rank}_{R/\mathcal{P}}(X/\mathcal{P}X) > g - e = s$ if and only if $I_s \subseteq \mathcal{P}$ as stated. Finally, we recall the simple fact that if I is any nonzero ideal in a Noetherian domain R, then there can exist only finitely many prime ideals of R of height 1 which contain I. ■

Corollary 2.1.1. *Let* X_1 *and* X_2 *be finitely generated* R*-modules. Suppose that* $\phi : X_1 \to X_2$ *is an* R*-module homomorphism. Let* $r_1 = \operatorname{rank}_R\big(\ker(\phi)\big)$ *and* $r_2 = \operatorname{rank}_R\big(\operatorname{coker}(\phi)\big)$. *For every prime ideal* $\mathcal{P}$ *of* R, *let* $\phi_{\mathcal{P}} : X_1/\mathcal{P}X_1 \to X_2/\mathcal{P}X_2$ *be the induced map. There exists a nonzero ideal* I *of* R *such that*

$$\operatorname{rank}_{R/\mathcal{P}}\big(\ker(\phi_{\mathcal{P}})\big) = r_1, \quad \operatorname{rank}_{R/\mathcal{P}}\big(\operatorname{coker}(\phi_{\mathcal{P}})\big) = r_2$$

if $\mathcal{P} \notin V(I)$. These equalities hold for almost all $\mathcal{P} \in \mathrm{Spec}_{ht=1}(R)$

Proof. Let $X = \mathrm{coker}(\phi) = X_2/\phi(X_1)$. The cokernel of $\phi_{\mathcal{P}}$ is isomorphic $X/\mathcal{P}X$ and so the statement about the cokernels follows from proposition 2.1. Now the $(R/\mathcal{P})$-rank of the kernel of $\phi_{\mathcal{P}}$ is determined by the $(R/\mathcal{P})$-ranks of $X_1/\mathcal{P}X_1$, $X_2/\mathcal{P}X_2$, and $\mathrm{coker}(\phi_{\mathcal{P}})$. We can apply proposition 2.1 to X, X_1 and X_2, which gives certain nonzero ideals of R in each case. Take I to be the intersection of those ideals. ■

Remark 2.1.2. Consider the special case where X_1 and X_2 are free R-modules. Then the map ϕ is given by a matrix and the behavior of the ranks of the kernels and cokernels in the above corollary is determined by the rank of the matrix and its reduction modulo $\mathcal{P}$ as in the proof of proposition 2.1. The following consequence will be useful later.

Suppose that X_1 and X_2 are free R-modules. Then for every prime ideal $\mathcal{P}$ of R, we have $\mathrm{rank}_{R/\mathcal{P}}\big(\ker(\phi_{\mathcal{P}})\big) \geq \mathrm{rank}_R\big(\ker(\phi)\big)$.

This can also be easily deduced from the corollary. A similar inequality holds for the cokernels of ϕ and $\phi_{\mathcal{P}}$.

Suppose that R is a complete Noetherian local domain with finite residue field. Then X is compact and its Pontryagin dual $A = \widehat{X}$ is a cofinitely generated, discrete R-module. The Pontryagin dual of $X/\mathcal{P}X$ is $A[\mathcal{P}]$, the set of elements of A annihilated by $\mathcal{P}$. Thus, one has $\mathrm{corank}_{R/\mathcal{P}}(A[\mathcal{P}]) = \mathrm{rank}_{R/\mathcal{P}}(X/\mathcal{P}X)$. If A_1 and A_2 are two cofinitely generated R-modules and $\psi : A_1 \to A_2$ is a R-module homomorphism, then there is an R-module homomorphism $\phi : X_2 \to X_1$, where $X_1 = \widehat{A_1}$, $X_2 = \widehat{A_2}$ and ϕ is the adjoint of ψ. The kernel and cokernel of ψ are dual, respectively, to the cokernel and kernel of ϕ. We will say that A_1 and A_2 are R-isogenous if there exists an R-module homomorphism ψ such that $\ker(\psi)$ and $\mathrm{coker}(\psi)$ are both R-cotorsion. We then refer to ψ as an R-isogeny. It is easy to see that R-isogeny is an equivalence relation on cofinitely generated R-modules.

Remark 2.1.3. The above proposition and corollary can be easily translated into their "dual" versions for discrete, cofinitely generated R-modules. For example,

1. *If $r = \mathrm{corank}_R(A)$, then $\mathrm{corank}_{R/\mathcal{P}}(A[\mathcal{P}]) \geq r$ for every prime ideal $\mathcal{P}$ of R. There exists a nonzero ideal I of R with the following property: $\mathrm{corank}_{R/\mathcal{P}}(A[\mathcal{P}]) = r$ if and only if $I \not\subseteq \mathcal{P}$. The equality $\mathrm{corank}_{R/\mathcal{P}}(A[\mathcal{P}]) = r$ holds for almost all $\mathcal{P} \in \mathrm{Spec}_{ht=1}(R)$.*

2. *Suppose that A_1 and A_2 are cofinitely generated, discrete R-modules and that $\psi : A_1 \to A_2$ is an R-module homomorphism. Let $c_1 = \mathrm{corank}_R\big(\ker(\psi)\big)$ and $c_2 = \mathrm{corank}_R\big(\mathrm{coker}(\psi)\big)$. For every prime ideal $\mathcal{P}$ of R, let $\psi_{\mathcal{P}} : A_1[\mathcal{P}] \to A_2[\mathcal{P}]$ be the induced map. There exists a nonzero ideal I of R such that*

$$\mathrm{corank}_{R/\mathcal{P}}\big(\ker(\psi_{\mathcal{P}})\big) = c_1, \quad \mathrm{corank}_{R/\mathcal{P}}\big(\mathrm{coker}(\psi_{\mathcal{P}})\big) = c_2$$

if $\mathcal{P} \notin V(I)$. In particular, if ψ is an R-isogeny, then $\psi_{\mathcal{P}}$ is an $(R/\mathcal{P})$-isogeny if $\mathcal{P} \notin V(I)$.

Remark 2.1.2 can also be translated to the discrete version and asserts that if the above A_1 and A_2 are cofree R-modules, then

$$\mathrm{corank}_{R/\mathcal{P}}\big(\ker(\psi_{\mathcal{P}})\big) \geq c_1, \quad \mathrm{corank}_{R/\mathcal{P}}\big(\mathrm{coker}(\psi_{\mathcal{P}})\big) \geq c_2$$

for every prime ideal $\mathcal{P}$ of R.

Remark 2.1.4. As mentioned before, if I is a nonzero ideal in a Noetherian domain R, then there exist only finitely many prime ideals $\mathcal{P} \in \mathrm{Spec}_{ht=1}(R)$ which contain I. This is only important if R has infinitely many prime ideals of height 1. Suppose that R is a finite, integral extension of a formal power series ring Λ, as we usually consider in this article. Then if the Krull dimension of R is at least 2, the set of prime ideals of R of height 1 is indeed infinite. This follows from the corresponding fact for the ring Λ which will have the same Krull dimension. In fact, if $\mathcal{Q}$ is any prime ideal of R of height at least 2, then $\mathcal{Q}$ contains infinitely many prime ideals of R of height 1. Corollary 2.5.1 provides a useful strengthening of this fact when $R = \Lambda$. It will also be useful to point out that in the ring Λ, assuming its Krull dimension d is at least 2, there exist infinitely many prime ideals P of height 1 with the property that Λ/P is also a formal power series ring. The Krull dimension of Λ/P will be $d-1$.

The ideal I occurring in proposition 2.1 is not unique. In the special case where X is a torsion R-module, so that $r = 0$, one can take $I = \mathrm{Ann}_R(X)$. That is:

Proposition 2.2. *Suppose that X is a finitely generated, torsion R-module and that $\mathcal{P}$ is a prime ideal of R. Then* $\mathrm{rank}_{R/\mathcal{P}}(X/\mathcal{P}X) > 0$ *if and only if* $\mathrm{Ann}_R(X) \subseteq \mathcal{P}$.

Proof. This follows by a simple localization argument. Let $R_{\mathcal{P}}$ be the localization of R at $\mathcal{P}$. Then $\mathcal{M} = \mathcal{P}R_{\mathcal{P}}$ is the maximal ideal of $R_{\mathcal{P}}$. Let $X_{\mathcal{P}} = X \otimes_R R_{\mathcal{P}}$, the localization of X at $\mathcal{P}$. Then $\mathrm{rank}_{R/\mathcal{P}}(X/\mathcal{P}X) = \dim_k(X_{\mathcal{P}}/\mathcal{M}X_{\mathcal{P}})$, where k denotes the residue field $R_{\mathcal{P}}/\mathcal{M}$. We have $\mathrm{rank}_{R/\mathcal{P}}(X/\mathcal{P}X) = 0 \iff X_{\mathcal{P}} = \mathcal{M}X_{\mathcal{P}} \iff X_{\mathcal{P}} = 0$, the last equivalence following from Nakayama's Lemma. Finally, $X_{\mathcal{P}} = 0$ if and only if $\mathrm{Ann}_R(X) \not\subseteq \mathcal{P}$. ■

Remark 2.2.1. Proposition 2.2 can be easily restated in terms of the discrete, cofinitely generated, cotorsion R-module $A = \widehat{X}$. Note that $\mathrm{Ann}_R(A) = \mathrm{Ann}_R(X)$. As we will discuss below, the height of the prime ideals $\mathcal{P}$ for which $A[\mathcal{P}]$ fails to be $(R/\mathcal{P})$-cotorsion is of some significance, and specifically whether such prime ideals can have height 1.

A contrasting situation occurs when X is a torsion-free R-module. We then have the following simple result.

Proposition 2.3. *Assume that X is a finitely generated, torsion-free R-module and that $\mathcal{P}$ is a prime ideal of R of height 1 which is also a principal ideal. Then* $\mathrm{rank}_{R/\mathcal{P}}(X/\mathcal{P}X) = \mathrm{rank}_R(X)$. *In particular, if R is a regular local ring, then* $\mathrm{rank}_{R/\mathcal{P}}(X/\mathcal{P}X) = \mathrm{rank}_R(X)$ *for all* $\mathcal{P} \in \mathrm{Spec}_{ht=1}(R)$.

Proof. The assumption about $\mathcal{P}$ implies that the localization $R_{\mathcal{P}}$ is a discrete valuation ring and hence a principal ideal domain. Therefore $X_{\mathcal{P}}$ is a free $R_{\mathcal{P}}$-module of finite rank. Letting $k = R_{\mathcal{P}}/\mathcal{M}$ again, it is then clear that $\dim_k(X_{\mathcal{P}}/\mathcal{M}X_{\mathcal{P}}) = \mathrm{rank}_{R_{\mathcal{P}}}(X_{\mathcal{P}})$. The above equality follows from this. ■

If X is a free R-module, then the situation is better. One then has the obvious equality $\mathrm{rank}_{R/\mathcal{P}}(X/\mathcal{P}X) = \mathrm{rank}_R(X)$ for all prime ideals $\mathcal{P}$ of R.

B. Associated prime ideals and pseudo-nullity. Assume that X is a finitely generated, torsion R-module. A prime ideal $\mathcal{P}$ of R is called an associated prime ideal for X if $\mathcal{P} = \mathrm{Ann}_R(x)$ for some nonzero element $x \in X$. Assuming that R is Noetherian, there are only finitely many associated prime ideals for X. We say that X is a *pseudo-null* R-module if no prime ideal of R associated with X has height 1. If R has Krull dimension 1, then every nonzero prime ideal has height 1 and so a pseudo-null R-module must be trivial. If R is a local ring of Krull dimension 2 and has finite residue field, then X is a pseudo-null R-module if and only if X is finite.

If R is a finite, integral extension of a Noetherian domain Λ, then an R-module X can be viewed as a Λ-module. We say that a prime ideal $\mathcal{P}$ of R lies over a prime ideal P of Λ if $P = \mathcal{P} \cap \Lambda$. The height of $\mathcal{P}$ in R will then be the same as the height of P in Λ. For a given prime ideal P of Λ, there exist only finitely many prime ideals $\mathcal{P}$ lying over P. It is clear that if $\mathcal{P}$ is an associated prime ideal for the R-module X and if $\mathcal{P}$ lies over P, then P is an associated prime ideal for the Λ-module X. Conversely, if P is an associated prime ideal for the Λ-module X, then there exists at least one prime ideal $\mathcal{P}$ of R lying over P which is an associated prime ideal for the R-module X. To see this, consider the R-submodule $Y = X[P]$ which is nonzero. Suppose that the associated prime ideals of R for Y are $\mathcal{P}_1, ..., \mathcal{P}_t$. Let $P_i = \mathcal{P}_i \cap \Lambda$ for $1 \le i \le t$. Thus, each P_i is an associated prime ideal for the Λ-module Y and so $P \subseteq P_i$ for each i. There is some product of the $\mathcal{P}_i$'s which is contained in $\mathrm{Ann}_R(Y)$ and the corresponding product of the P_i's is contained in $\mathrm{Ann}_\Lambda(Y) = P$. Thus, $P_i \subseteq P$ for at least one i. This implies that $P_i = P$ and so, indeed, at least one of the prime ideals $\mathcal{P}_i$ lies over P. These observations justify the following statement:

1. *X is pseudo-null as an R-module if and only if X is pseudo-null as a Λ-module.*

The ring Λ is a UFD. Every prime ideal of height 1 is generated by an irreducible element of Λ. One can give the following alternative definition of pseudo-nullity:

2. *A finitely generated Λ-module X is pseudo-null if and only if* $\mathrm{Ann}(X)$ *contains two relatively prime elements.*

Another equivalent criterion for pseudo-nullity comes from the following observations. If Q is an associated prime ideal of X, then $X[Q] \neq 0$ and so $X[P] \neq 0$ for every ideal $P \subseteq Q$. If Q has height ≥ 2, then Q contains infinitely many prime ideals P of height 1. On the other hand, if the associated prime ideals for X all have height 1, then $X[P] = 0$ for all the non-associated prime ideals P of height 1. To summarize:

3. *A finitely generated Λ-module X has a nonzero pseudo-null Λ-submodule if and only if there exist infinitely many prime ideals $P \in \mathrm{Spec}_{ht=1}(\Lambda)$ such that $X[P] \neq 0$.*

If $A = \widehat{X}$, then $X[P] \neq 0$ if and only if $PA \neq A$. Hence, the above remarks imply the following result.

Proposition 2.4. *Suppose that A is a cofinitely generated, discrete Λ-module. The following three statements are equivalent:*

(a) $PA = A$ for almost all $P \in \mathrm{Spec}_{ht=1}(\Lambda)$.

(b) The Pontryagin dual of A has no nonzero pseudo-null Λ-submodules.

(c) A is an almost divisible Λ-module.

As mentioned before, if P has height 1, then $P = (\pi)$ where π is an irreducible element of Λ. The statement that $PA = A$ means that $\pi A = A$, i.e., A is *divisible by π*. Let Y denote the torsion Λ-submodule of $X = \widehat{A}$. Then, assuming statement *(b)*, one has $PA = A$ if and only if $P \notin \mathrm{Supp}(Y)$. In the definition of "almost divisible," one can take θ to be any nonzero element of Λ divisible by all irreducible elements π which generate prime ideals in $\mathrm{Supp}(Y)$, e.g., θ could be a generator of the characteristic ideal of the Λ-module Y.

One can ask about the behavior of pseudo-null modules under specialization. Here is one useful result.

Proposition 2.5. *Suppose that the Krull dimension of Λ is at least 3 and that X is a finitely generated, pseudo-null Λ-module. Then there exists infinitely many prime ideals $P \in \mathrm{Spec}_{ht=1}(\Lambda)$ such that X/PX is a pseudo-null (Λ/P)-module.*

Proof. One can consider Λ as a formal power series ring $\Lambda_o[[T]]$ in one variable, where the subring Λ_o is a formal power series ring (over either $\mathbb{Z}_p$ or $\mathbb{F}_p$) in one less variable. One can choose Λ_o so that X is a finitely generated, torsion module over Λ_o. (See Lemma 2 in [Gr78] if Λ has characteristic 0. The proof there works if Λ has characteristic p.) Since the Krull dimension of Λ_o is at least 2, there exist infinitely many prime ideals P_o of Λ_o of height 1.

The module X/P_oX will be a finitely generated, torsion (Λ_o/P_o)-module for all but finitely many such P_o's. Now $P_o = (\pi_o)$, where π_o is an irreducible element of Λ_o. Clearly, π_o is also irreducible in Λ. The ideal $P = \pi_o\Lambda$ is a prime ideal of height 1 in Λ. Since X/PX is finitely generated and torsion over Λ_o/P_o, and $\Lambda/P \cong (\Lambda_o/P_o)[[T]]$, it follows that X/PX is a pseudo-null (Λ/P)-module. ■

One surprising consequence concerns the existence of infinitely many height 1 prime ideals of a different sort.

Corollary 2.5.1. *Suppose that Λ has Krull dimension at least 2 and that X is a finitely generated, pseudo-null Λ-module. Then there exist infinitely many prime ideals $P \in \mathrm{Spec}_{ht=1}(\Lambda)$ such that $P \subset \mathrm{Ann}_\Lambda(X)$.*

Proof. We will argue by induction. If Λ has Krull dimension 2, the the result is rather easy to prove. In that case, one has $\mathfrak{m}_\Lambda^n \subset \mathrm{Ann}_\Lambda(X)$ for some $n > 0$. It suffices to prove that $\mathfrak{m}_\Lambda^n$ contains infinitely many irreducible elements which generate distinct ideals. First consider $\Lambda = \mathbb{Z}_p[[T]]$. There exist field extensions of $\mathbb{Q}_p$ of degree $\geq n$. For any such extension F, choose a generator over $\mathbb{Q}_p$ which is in a large power of the maximal ideal of F. Then its minimal polynomial over $\mathbb{Q}_p$ will be in $\mathfrak{m}_\Lambda^n$ and will be an irreducible elements of Λ. By varying the extension F or the generator, one obtains the desired irredicible elements of Λ. The same argument works for $\mathbb{F}_p[[S,T]]$ since the fraction field of $\mathbb{F}_p[[S]]$ also has finite, separable extensions of arbitrarily high degree.

In the proof of proposition 2.5, it is clear that we can choose the P_o's so that Λ_o/P_o is also a formal power series ring. The same will then be true for Λ/P. Now assume that the Krull dimension of Λ is at least 3. Choose two elements $\theta_1, \theta_2 \in \mathrm{Ann}(X)$ such that θ_1 and θ_2 are relatively prime. The Λ-module $Y = \Lambda/(\theta_1, \theta_2)$ is then pseudo-null. Choose P so that $\overline{\Lambda} = \Lambda/P$ is a formal power series ring and so that $\overline{Y} = Y/PY$ is a pseudo-null $\overline{\Lambda}$-module. Let $\overline{\theta_1}$ and $\overline{\theta_2}$ denote the images of θ_1 and θ_2 in $\overline{\Lambda}$. Then $\overline{Y} = \overline{\Lambda}/(\overline{\theta_1}, \overline{\theta_2})$ and the fact that this is pseudo-null means that $\overline{\theta_1}$ and $\overline{\theta_2}$ are relatively prime in that ring. Clearly, the ideal $\mathrm{Ann}_{\overline{\Lambda}}(\overline{Y})$ in $\overline{\Lambda}$ is generated by $\overline{\theta_1}$ and $\overline{\theta_2}$. We assume that this ideal contains infinitely many prime ideals of $\overline{\Lambda}$ of height 1. Any such prime ideal has a generator of the form $\overline{\alpha_1}\overline{\theta_1} + \overline{\alpha_2}\overline{\theta_2}$, where $\overline{\alpha_1}, \overline{\alpha_2} \in \overline{\Lambda}$ are the images of $\alpha_1, \alpha_2 \in \Lambda$, say. Let $\eta = \alpha_1\theta_1 + \alpha_2\theta_2$. Then $\eta \in \mathrm{Ann}_\Lambda(X)$ and is easily seen to be an irreducible element of Λ. We can find infinitely many distinct prime ideals $(\eta) \subset \mathrm{Ann}_\Lambda(X)$ in this way. ■

C. Reflexive and coreflexive modules. Let $m \geq 0$. Suppose that the ring Λ is either $\mathbb{Z}_p[[T_1, ..., T_m]]$ (which we take to be $\mathbb{Z}_p$ if $m = 0$) or $\mathbb{F}_p[[T_1, ..., T_{m+1}]]$, so that the Krull-dimension of Λ is $m+1$. Suppose that X is a finitely generated, torsion-free Λ-module. Let $\mathcal{L}$ denote the fraction field of Λ. Let Λ_P be the localization of Λ at P. We can view the localization $X_P = X \otimes_\Lambda \Lambda_P$ as a subset of $\mathcal{V} = X \otimes_\Lambda \mathcal{L}$ which is a vector space over $\mathcal{L}$ of dimension $\mathrm{rank}_\Lambda(X)$. The *reflexive hull* of X is defined to be the Λ-submodule of $\mathcal{V}$ defined by $\widetilde{X} = \bigcap_P X_P$, where this intersection is over all prime ideals $P \in \mathrm{Spec}_{ht=1}(\Lambda)$ and Λ_P is the localization of Λ at P. Then $\widetilde{X}$ is also a finitely generated, torsion-free Λ-module, $X \subseteq \widetilde{X}$, and the quotient $\widetilde{X}/X$ is a pseudo-null Λ-module. Furthermore, suppose that X' is any finitely generated, torsion-free Λ-module such that $X \subseteq X'$ and X'/X is pseudo-null. Since X'/X is Λ-torsion, one can identify X' with a Λ-submodule of $\mathcal{V}$ containing X. Then $X' \subseteq \widetilde{X}$. We say that X is a *reflexive* Λ*-module* if $\widetilde{X} = X$. This is equivalent to the more usual definition that X is isomorphic to its Λ-bidual under the natural map. We will make several useful observations.

Suppose that R is a finite, integral extension of Λ. Let $\mathcal{K}$ denote the fraction field of R. We can define the notion of a reflexive R-module in the same way as above. If X is any finitely generated, torsion-free R-module, the *R-reflexive hull of* X is the R-submodule of the $\mathcal{K}$-vector space $X \otimes_R \mathcal{K}$ defined by $\widetilde{X} = \bigcap_{\mathcal{P}} X_{\mathcal{P}}$, where $\mathcal{P}$ runs over all the prime ideals of R of height 1. This is easily seen to coincide with the Λ-reflexive hull of X as defined above. One uses the fact that, with either definition, $\widetilde{X}$ is torsion-free as both an R-module and a Λ-module, $\widetilde{X}/X$ is pseudo-null as both an R-module and a Λ-module, and $\widetilde{X}$ is maximal with respect to those properties. We can define X to be a reflexive R-module if $\widetilde{X} = X$. But our remarks justify the following equivalence:

1. *An R-module X is reflexive as an R-module if and only if it is reflexive as a Λ-module.*

Thus, it suffices to consider Λ-modules. Suppose that X is a reflexive Λ-module and that Y is an arbitrary Λ-submodule of X. Both are torsion-free Λ-modules, but, of course, the quotient R-module X/Y may fail to be torsion-free. However, one can make the following important observation:

2. *The Λ-module Y is reflexive if and only if X/Y contains no nonzero pseudo-null Λ-submodules.*

This is rather obvious from the properties of the reflexive hull. Since X is assumed to be reflexive, we have $\widetilde{Y} \subseteq X$. Hence $\widetilde{Y}/Y$ is the maximal pseudo-null Λ-submodule of X/Y. Every pseudo-null Λ-submodule of X/Y is contained in $\widetilde{Y}/Y$. The observation follows from this.

The above observation provides a rather general construction of reflexive Λ-modules. To start, suppose that X is any reflexive Λ-module and that $\mathrm{rank}(X) = r$, e.g., $X = \Lambda^r$. If Y is a Λ-submodule of X such that X/Y is torsion-free, then X/Y certainly cannot contain a nonzero pseudo-null Λ-submodule. Thus Y must be reflexive. Consider the $\mathcal{L}$-vector space $\mathcal{V}$ defined before. It has dimension r over $\mathcal{L}$. Let $\mathcal{W}$ be any $\mathcal{L}$-subspace of $\mathcal{V}$. Let $Y = X \cap \mathcal{W}$. Then $\mathrm{rank}_\Lambda(Y) = \dim_{\mathcal{L}}(\mathcal{W})$. It is clear that X/Y is a torsion-free Λ-module and so the Λ-module Y will be reflexive. To see this, first note that X/Y is a torsion-free Λ-module.

Here is one important type of example.

3. *Suppose that a group G acts Λ-linearly on a reflexive Λ-module X. Then $Y = X^G$ must also be reflexive as a Λ-module.*

This is clear since G will act $\mathcal{L}$-linearly on $\mathcal{V}$ and, if we let $\mathcal{W}$ denote the subspace $\mathcal{V}^G$, then $Y = X \cap \mathcal{W}$.

Suppose that $m = 0$. Then Λ is either $\mathbb{Z}_p$ or $\mathbb{F}_p[[T]]$. Both are discrete valuation rings and have just one nonzero prime ideal, its maximal ideal, which has height 1. The module theory is quite simple, and every finitely generated, torsion-free Λ-module is free and hence reflexive. However, suppose that $m \geq 1$. Then Λ has infinitely many prime ideals of height 1. They are all principal since Λ is a UFD. We then have the following useful result. We always take the term reflexive to include the assumption that the module is finitely generated and torsion-free.

Proposition 2.6. *Assume that $m \geq 1$ and that X is a finitely generated Λ-module.*

- *(a) If X is a reflexive Λ-module and if $P \in \mathrm{Spec}_{ht=1}(\Lambda)$, then X/PX is a torsion-free (Λ/P)-module.*
- *(b) Conversely, if X/PX is a torsion-free (Λ/P)-module for almost all $P \in \mathrm{Spec}_{ht=1}(\Lambda)$, then X is a reflexive Λ-module.*

Proof. Suppose that X is reflexive and that P is any prime ideal of height 1 in Λ. Then $P = (\pi)$, where π is any irreducible element of Λ. Then $PX = \pi X$ is also a reflexive Λ-module. As observed above, it follows that X/PX contains no nonzero pseudo-null Λ-submodules. But any finitely generated, torsion (Λ/P)-module will be pseudo-null when considered as a Λ-module. This is clear because the annihilator of such a (Λ/P)-module will contain π as well as some nonzero element of Λ which is not divisible by π. Therefore, X/PX must indeed be a torsion-free as a (Λ/P)-module, proving part *(a)*.

Now, under the assumptions of *(b)*, we first show that X must be a torsion-free Λ-module. For if Y is the Λ-torsion submodule of X and if $P = (\pi)$ is any height 1 prime ideal, then the snake lemma implies that there is an injective map $Y/PY \to X/PX$. But, if Y is nonzero, so is Y/PY. Also, if $\lambda \in \Lambda$ is a nonzero annihilator of Y, then Y/PY is a torsion (Λ/P)-module for all but the finitely many prime ideals P of height 1 which contain λ. It follows that $Y = 0$. There are infinitely many such P's.

Let $Z = \widetilde{X}/X$. Then Z is a pseudo-null Λ-module. Assume Z is nonzero. Then there exist infinitely many prime ideals $P = (\pi)$ of Λ of height 1 such that $Z[\pi]$ is nonzero too. Clearly, $Z[\pi]$ is a torsion (Λ/P)-module. Consider the exact sequence

$$0 \to X \to \widetilde{X} \to Z \to 0$$

By the snake lemma, together with the fact that $\widetilde{X}$ is a torsion-free Λ-module, one obtains an injective map $Z[\pi] \to X/PX$. Therefore, for infinitely many P's, X/PX fails to be torsion-free as a (Λ/P)-module, contradicting the hypothesis. Hence $Z = 0$ and X is indeed reflexive. ■

The first part of proposition 2.6 is quite trivial for free modules. In fact, if R is any ring and X is a free R-module, then $X/\mathcal{P}X$ is a free $(R/\mathcal{P})$-module and will certainly be torsion-free if $\mathcal{P}$ is any prime ideal of R.

We often will use proposition 2.6 in its discrete form.

Corollary 2.6.1. *Suppose that $m \geq 1$ and that A is a cofinitely generated Λ-module.*

(a) If A is a coreflexive Λ-module, then $A[P]$ is a divisible (Λ/P)-module for every prime ideal P of Λ of height 1.

(b) Conversely, if $A[P]$ is a divisible (Λ/P)-module for almost all $P \in \mathrm{Spec}_{ht=1}(\Lambda)$, then A must be coreflexive as a Λ-module.

Remark 2.6.2. One simple consequence concerns the case where the Krull-dimension is 2, i.e. Λ is either $\mathbb{Z}_p[[T]]$ or $\mathbb{F}_p[[S,T]]$. Suppose that X is a reflexive Λ-module. The ring $\Lambda/(T)$ is isomorphic to either $\mathbb{Z}_p$ or $\mathbb{F}_p[[S]]$, both principal ideal domains. Since X/TX is a finitely-generated, torsion-free module over $\Lambda/(T)$, it is therefore a free module. Let $r = \mathrm{rank}_\Lambda(X)$. Proposition 2.3 implies that the rank of X/TX over $\Lambda/(T)$ is also equal to r. Hence X/TX can be generated as a $\Lambda/(T)$-module by exactly r elements. By Nakayama's lemma, X can be generated by r elements as a Λ-module and so it is a quotient of Λ^r. It follows that $X \cong \Lambda^r$. Therefore, we have the following well-known result:

If Λ has Krull dimension 2, then every reflexive Λ-module is free.

It follows that every coreflexive Λ-module is cofree when Λ has Krull dimension 2.

Remark 2.6.3. One can use proposition 2.6 to give examples of reflexive Λ-modules which are not free if the Krull dimension of Λ is at least 3. A torsion-free Λ-module of rank 1 will be isomorphic to an ideal in Λ and it is known that a reflexive ideal must be principal and hence free. Thus, our examples will have rank at least 2. We take $X = \Lambda^r$. Let Y be a Λ-submodule of X with the property that $Z = X/Y$ is a torsion-free Λ-module. Thus, as observed before, Y will be Λ-reflexive. Suppose that $P = (\pi)$ is any prime ideal of Λ of height 1. Then we have an exact sequence

$$0 \longrightarrow Y/PY \longrightarrow X/PX \longrightarrow Z/PZ \longrightarrow 0$$

of (Λ/P)-modules. We can choose P so that Λ/P is also a formal power series ring. Assume that Y is actually a free Λ-module. Then both Y/PY and X/PX would be free (Λ/P)-modules and hence reflexive. Thus, the quotient (Λ/P)-module Z/PZ would contain no nonzero pseudo-null (Λ/P)-submodules. However, it is easy to give examples of torsion-free Λ-modules Z which fail to have that property. As one simple example, suppose that Z is the maximal ideal $\mathfrak{m}_\Lambda$ of Λ. Then Λ/Z is annihilated by π and so we have

$$\pi Z \subsetneq \pi\Lambda \subset Z$$

Thus, $\pi\Lambda/\pi Z$ is a (Λ/P)-submodule of Z/PZ, has order p, and will be a pseudo-null (Λ/P)-module since that ring has Krull dimension at least 2. Take r to be the number of generators of Z as a Λ-module and take X as above. Then one has a surjective Λ-module homomorphism $X \to Z$. If we let Y denote the kernel of this homomorphism, then Y is a reflexive Λ-module, but cannot be free.

One can view this remark from the point of view of homological algebra. Nakayama's Lemma implies easily that projective Λ-modules are free. Let d denote the Krull dimension of Λ. Thus, as we just explained, the Λ-module $\mathfrak{m}_\Lambda$ cannot have projective dimension 1 if $d \geq 3$. In fact, one can show that $\mathfrak{m}_\Lambda$ has projective dimension $d-1$.

D. Reflexive domains. In general, if R is any commutative integral domain, we will say that R is a reflexive domain if

$$R = \bigcap_{\mathcal{P}} R_{\mathcal{P}},$$

where $\mathcal{P}$ varies over all prime ideal of R of height 1 and $R_{\mathcal{P}}$ denotes the localization of R at $\mathcal{P}$. If R contains Λ as a subring and is finitely generated as a Λ-module, then R is reflexive in the above sense precisely when R is reflexive as a Λ-module. This is implied by the following result. Note that $\mathcal{K} = R \otimes_\Lambda \mathcal{L}$ is the fraction field of R. We define $\widetilde{R} = \bigcap_{\mathcal{P}} R_{\mathcal{P}}$, where $\mathcal{P}$ varies over all prime ideals of R of height 1. Thus, $\widetilde{R}$ is a subring of $\mathcal{K}$ containing R and R is a reflexive domain if and only if $R = \widetilde{R}$.

Proposition 2.7. *$\widetilde{R}$ is the reflexive hull of R as a Λ-module.*

Proof. Let P be a prime ideal of Λ of height 1. Let $\mathcal{P}_1, ..., \mathcal{P}_g$ be the prime ideals $\mathcal{P}$ of R such that $\mathcal{P} \cap \Lambda = P$. We let $R_P = R \otimes_\Lambda \Lambda_P$, which is the ring of fractions of R corresponding to the multiplicative set $\Lambda - P$. Then R_P is a subring of $\mathcal{K}$. The maximal ideals of R_P are $\mathcal{P}_i R_P, 1 \leq i \leq g$. The localization of R_P at $\mathcal{P}_i R_P$ is clearly $R_{\mathcal{P}_i}$ and so we have

$$R_P = \bigcap_{1 \leq i \leq g} R_{\mathcal{P}_i}$$

If $\mathcal{P}$ is any height 1 prime ideal of R, then $P = \mathcal{P} \bigcap \Lambda$ is a height 1 prime ideal of Λ. The proposition follows immediately. ■

Since $\widetilde{R}$ is also a finitely generated Λ-module, and hence an integral extension of Λ, we get the following corollary (which is actually a standard theorem; see corollary 11.4 in [E]).

Corollary 2.7.1. *If R is integrally closed, then R is reflexive.*

Suppose that R is a finite integral extension of Λ. Then it is known that R is a free Λ-module if and only if R is Cohen-Macaulay. (See proposition 2.2.11 in [B-H].) Any free Λ-module is reflexive, Thus, if R is Cohen-Macaulay, then R is reflexive. One simple type of example is $R = \Lambda[\theta]$, where θ is integral over Λ. Also, if R is regular or Gorenstein, then R is Cohen-Macaulay.

The first part of proposition 2.6 is valid for R-modules if R is assumed to be a reflexive domain. That is, if X is a finitely generated, reflexive R-module and $\mathcal{P} \in \mathrm{Spec}_{ht=1}(R)$, then $X/\mathcal{P}X$ is a torsion-free $(R/\mathcal{P})$-module. The same proof works once one notes that any prime ideal $\mathcal{P}$ of height 1 in a reflexive domain R must be reflexive as an R-module. This is easily verified.

Suppose that R is a complete Noetherian local ring, but is not necessarily a domain. We will say that R is a reflexive ring if it has the following properties: *(i)* R contains a subring Λ which is isomorphic to a formal power series ring over either $\mathbb{Z}_p$ or $\mathbb{F}_p$ and *(ii)* R is a finitely generated, reflexive module over Λ. One important example arises from Hida theory. The universal ordinary Hecke algebra $\mathfrak{h}$ for a given level contains a natural subring Λ isomorphic to the formal power series ring $\mathbb{Z}_p[[T]]$ in one variable and is actually a free Λ-module of finite rank. Thus this ring $\mathfrak{h}$ is reflexive, but is not necessarily a domain. In general, suppose that R satisfies *(i)* and R is a torsion-free Λ-module. Then R is a subring of the $\mathcal{L}$-algebra $R \otimes_\Lambda \mathcal{L}$ and the reflexive hull $\widetilde{R}$ of R as a Λ-module will be a reflexive ring.

E. Different choices of $\mathcal{D}$. In the introduction we considered a free R-module $\mathcal{T}$ and defined $\mathcal{D} = \mathcal{T} \otimes_R \widehat{R}$, a cofree R-module, which we will now denote by $\mathcal{D}_R$. This construction behaves well under specialization at any ideal I of R in the following sense. Consider the free (R/I)-module $\mathcal{T}/I\mathcal{T}$. Applying the construction, we get

$$(\mathcal{T}/I\mathcal{T}) \otimes_{R/I} \widehat{(R/I)} \cong \mathcal{T} \otimes_R (\widehat{R}[I]) \cong \mathcal{D}_R[I].$$

Another construction which will be useful later is to define $\mathcal{D}_\Lambda = \mathcal{T} \otimes_\Lambda \widehat{\Lambda}$. Both constructions can be applied to an arbitrary R-module $\mathcal{T}$. To see the relationship, note that we have $\mathcal{D}_\Lambda \cong \mathcal{T} \otimes_R \widehat{\Lambda}_R$ where $\widehat{\Lambda}_R = R \otimes_\Lambda \widehat{\Lambda}$, the R-module obtained from $\widehat{\Lambda}$ by extending scalars from Λ to R. We have $\widehat{\Lambda}_R \cong \widehat{R}$ if R is free as a Λ-module. In that case, it follows that $\mathcal{D}_R$ and $\mathcal{D}_\Lambda$ are isomorphic as R-modules. In general, one can only say that $\mathcal{D}_R$ and $\mathcal{D}_\Lambda$ are R-isogenous. Their R-coranks are equal to $\mathrm{rank}_R(\mathcal{T})$.

The Λ-module $\mathcal{D}_\Lambda$ is always coreflexive. To see this, let $P = (\pi)$ be any prime ideal of height 1 in Λ. Consider the exact sequence induced by multiplication by π.

$$0 \longrightarrow \widehat{\Lambda}[P] \longrightarrow \widehat{\Lambda} \xrightarrow{\pi} \widehat{\Lambda} \longrightarrow 0$$

Tensoring over Λ by $\mathcal{T}$, one gets a surjective homomorphism

$$(\mathcal{T}/P\mathcal{T}) \otimes_{\Lambda/P} (\widehat{\Lambda}[P]) \longrightarrow \mathcal{D}_\Lambda[P] \tag{3}$$

Note that $\widehat{\Lambda}[P] \cong \widehat{(\Lambda/P)}$. Since $\widehat{\Lambda}[P]$ is (Λ/P)-divisible, so is $(\mathcal{T}/P\mathcal{T}) \otimes_{\Lambda/P} (\widehat{\Lambda}[P])$ and that implies that $\mathcal{D}_\Lambda[P]$ is a divisible (Λ/P)-module. Corollary 2.6.1 then implies that $\mathcal{D}_\Lambda$ is coreflexive. We also remark that if $\mathcal{T}$ is assumed to be a torsion-free Λ-module, then proposition 2.3 implies that $\mathrm{rank}_{\Lambda/P}(\mathcal{T}/P\mathcal{T})$ and $\mathrm{corank}_{\Lambda/P}(\mathcal{D}_\Lambda[P])$ are both equal to $\mathrm{rank}_\Lambda(\mathcal{T})$ and so the map in (3) must be a (Λ/P)-isogeny.

Suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ are finitely generated R-modules. Let $\mathcal{D}_1 = \mathcal{T}_1 \otimes_R \widehat{R}$ and $\mathcal{D}_2 = \mathcal{T}_2 \otimes_R \widehat{R}$. We then have the following result.

Proposition 2.8. *Suppose that $\phi : \mathcal{T}_1 \to \mathcal{T}_2$ is an R-module homomorphism and that $\psi : \mathcal{D}_1 \to \mathcal{D}_2$ is the R-module homomorphism determined by $\psi(x \otimes y) = \phi(x) \otimes y$ for $x \in \mathcal{T}_1,\ y \in \widehat{R}$. Then* $\mathrm{corank}_R\big(\ker(\psi)\big) = \mathrm{rank}_R\big(\ker(\phi)\big)$. *A similar equality holds for the cokernels of ψ and ϕ.*

Proof. Let $\mathcal{T}_3$ denote the cokernel of ϕ. Let $D_3 = \mathcal{T}_3 \otimes_R \widehat{R}$. We then have exact sequences:

$$\mathcal{T}_1 \xrightarrow{\phi} \mathcal{T}_2 \longrightarrow \mathcal{T}_3 \longrightarrow 0, \qquad \mathcal{D}_1 \xrightarrow{\psi} \mathcal{D}_2 \longrightarrow \mathcal{D}_3 \longrightarrow 0$$

The second exact sequence follows from the first by tensoring each term with $\widehat{R}$. Since $\mathrm{corank}_R(\mathcal{D}_i) = \mathrm{rank}_R(\mathcal{T}_i)$ for each i, the stated equalities follow immediately. ■

The proposition is also valid if $\mathcal{D}_i$ is defined to be $\mathcal{T}_i \otimes_\Lambda \widehat{\Lambda}$ instead.

3 Cohomology Groups.

We consider a rather general situation. Suppose that R is a complete Noetherian local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of characteristic p. Suppose that $\mathcal{D}$

is a cofinitely generated R-module and that G is a profinite group which acts continuously and R-linearly on $\mathcal{D}$. Then the cohomology groups $H^i(G,\mathcal{D})$ are also R-modules. Now $\mathcal{D}[\mathfrak{m}]$ is a finite dimensional representation space for G over k and hence over $\mathbb{F}_p$. Denote the distinct, $\mathbb{F}_p$-irreducible subquotients by $\alpha_1, ..., \alpha_t$. We will assume throughout that the cohomology groups $H^i(G,\alpha_k)$ are finite for all $i \geq 0$ and for all k, $1 \leq k \leq t$. This is so if *(i)* $G = G_{K_v}$, where K_v is the v-adic completion of a number field K at any prime v, or if *(ii)* $G = \mathrm{Gal}(K_\Sigma/K)$, where Σ is any finite set of primes of K.

A. Properties inherited from $\mathcal{D}$. First we prove the following result which will be useful in subsequent arguments.

Proposition 3.1. *Let $\mathcal{C} = \mathcal{D}_1/\mathcal{D}_2$, where $\mathcal{D}_1$ and $\mathcal{D}_2$ are G-invariant R-submodules of $\mathcal{D}$. Then every $\mathbb{F}_p$-irreducible subquotient of $\mathcal{C}[\mathfrak{m}]$ is isomorphic to one of the α_k's.*

Proof. First note that $\mathcal{C}$ is a cofinitely generated R-module, and so $\mathcal{C}[\mathfrak{m}]$ is finite. Also, $\mathcal{D} = \bigcup_{n \geq 0} \mathcal{D}[\mathfrak{m}^n]$. It follows that $\mathcal{C}[\mathfrak{m}]$ is a subquotient of $\mathcal{D}[\mathfrak{m}^n]$ for some n. Hence it is enough to prove that the composition factors for the G-module $\mathcal{D}[\mathfrak{m}^n]$ are isomorphic to the α_k's. It suffices to verify this for $\mathcal{D}[\mathfrak{m}^{j+1}]/\mathcal{D}[\mathfrak{m}^j]$ for all $j \geq 0$. Let $\lambda_1, ..., \lambda_g$ be a set of generators for the ideal $\mathfrak{m}^j$. Then one can define an injective G-homomorphism $\mathcal{D}[\mathfrak{m}^{j+1}]/\mathcal{D}[\mathfrak{m}^j] \to \mathcal{D}[\mathfrak{m}]^g$ by mapping the coset of $x \in \mathcal{D}[\mathfrak{m}^{j+1}]$ to $(\lambda_1 x, ..., \lambda_g x)$. The assertion about the composition factors follows from this. ■

Corollary 3.1.1. *Let $i \geq 0$. If $H^i(G,\alpha_k) = 0$ for all k, $1 \leq k \leq t$, then $H^i(G,\mathcal{C}) = 0$ for every subquotient $\mathcal{C}$ of $\mathcal{D}$ as an $R[G]$-module.*

Proof. The hypothesis implies that $H^i(G,\mathcal{C}[\mathfrak{m}^n]) = 0$ for all $n \geq 0$. Since $\mathcal{C} = \varinjlim_n \mathcal{C}[\mathfrak{m}^n]$, it follows that $H^i(G,\mathcal{C}) = 0$ as stated. ■

Note that $H^0(G,\mathcal{D}) = \mathcal{D}^G$ is just an R-submodule of $\mathcal{D}$, and so is also a cofinitely generated R-module. More generally, we have

Proposition 3.2. *For any $i \geq 0$, $H^i(G,\mathcal{D})$ is a cofinitely generated R-module.*

Proof. We prove this by induction on the minimal number of generators of the maximal ideal $\mathfrak{m}$. Let λ be one element of such a generating set for $\mathfrak{m}$. Consider the two exact sequences

$$0 \to \mathcal{D}[\lambda] \to \mathcal{D} \to \lambda\mathcal{D} \to 0, \qquad 0 \to \lambda\mathcal{D} \to \mathcal{D} \to \mathcal{D}/\lambda\mathcal{D} \to 0$$

The first is induced by multiplication by λ; the second is obvious. If $\mathfrak{m}$ is principal, then $\mathcal{D}[\lambda] = \mathcal{D}[\mathfrak{m}]$ and $\mathcal{D}/\lambda\mathcal{D} = \mathcal{D}/\mathfrak{m}\mathcal{D}$ are both finite, and the hypothesis that the $H^i(G,\alpha_k)$'s are finite implies that $H^i(G,\mathcal{D}[\lambda])$ and $H^{i-1}(G,\mathcal{D}/\lambda\mathcal{D})$ are both finite. Thus the kernels of the two maps

$$H^i(G,\mathcal{D}) \to H^i(G,\lambda\mathcal{D}), \qquad H^i(G,\lambda\mathcal{D}) \to H^i(G,\mathcal{D})$$

are both finite. But the composite map $\mathcal{D} \to \lambda\mathcal{D} \to \mathcal{D}$ is multiplication by λ, and so the kernel of the composite map $H^i(G, \mathcal{D}) \to H^i(G, \lambda\mathcal{D}) \to H^i(G, \mathcal{D})$ is just $H^i(G, \mathcal{D})[\lambda]$, which is therefore finite. Thus, $H^i(G, \mathcal{D})[\mathfrak{m}]$ is finite, and hence, by Nakayama's lemma (the version for compact R-modules), $H^i(G, \mathcal{D})$ is cofinitely generated as a R-module.

If a minimal generating set for $\mathfrak{m}$ requires g generators, where $g > 1$, then the maximal ideal of $R/(\lambda)$ requires $g-1$ generators. The $R/(\lambda)$-modules $\mathcal{D}[\lambda]$ and $\mathcal{D}/\lambda\mathcal{D}$ are both cofinitely generated. And so, by induction, we can assume that the $R/(\lambda)$-modules $H^i(G, \mathcal{D}[\lambda])$ and $H^{i-1}(G, \mathcal{D}/\lambda\mathcal{D})$ are also cofinitely generated. The above argument then shows that the $R/(\lambda)$-module $H^i(G, \mathcal{D})[\lambda]$ is cofinitely generated, and hence so is $H^i(G, \mathcal{D})[\mathfrak{m}]$. Nakayama's lemma then implies that the R-module $H^i(G, \mathcal{D})$ is cofinitely generated. ∎

Various other properties of $\mathcal{D}$ are inherited by the Galois cohomology groups under certain hypotheses. Some are quite obvious. We assume in the rest of this section that R is a domain.

If $\mathcal{D}$ is R-cotorsion, then so is $H^i(G, \mathcal{D})$.

If $\mathcal{D}$ is a co-pseudo-null R-module, then so is $H^i(G, \mathcal{D})$.

As for the properties of divisibility or coreflexivity, these are also inherited under certain rather stringent hypotheses. We have the following result.

Proposition 3.3. *Suppose that $i \geq 0$. Suppose that $H^{i+1}(G, \alpha_k) = 0$ for $1 \leq k \leq t$.*

(a) *If $\mathcal{D}$ is a divisible R-module, then so is $H^i(G, \mathcal{D})$.*

(b) *If $\mathcal{D}$ is a coreflexive R-module, then so is $H^i(G, \mathcal{D})$.*

Note that the hypothesis that the $H^{i+1}(G, \alpha_k)$'s vanish is true if G has p-cohomological dimension equal to i. In particular, this hypothesis is true when $i = 2$ for $G = G_{K_v}$, where v is any non-archimedean prime of K, and for $G = \mathrm{Gal}(K_\Sigma/K)$ when p is an odd prime.

Proof. The ring R is a finitely generated module over a formal power series ring Λ. A finitely generated R-module X is torsion-free as an R-module if and only if it is torsion-free as a Λ-module. Also, X is reflexive as an R-module if and only if it is reflexive as a Λ-module. Thus, we may prove the proposition by using only the Λ-module structure. Prime ideals of Λ of height 1 are principal. First we consider divisibility. Let $\lambda \in \Lambda$ be nonzero. Then we have the exact sequence

$$0 \to \mathcal{D}[\lambda] \to \mathcal{D} \to \mathcal{D} \to 0$$

induced by multiplication by λ. Hence we get an exact sequence

$$H^i(G, \mathcal{D}) \to H^i(G, \mathcal{D}) \to H^{i+1}(G, \mathcal{D}[\lambda])$$

The hypothesis in corollary 3.1.1 is satisfied for the index $i+1$ for the module $\mathcal{C} = \mathcal{D}[\lambda]$, and so we have $H^{i+1}(G, \mathcal{D}[\lambda]) = 0$. Thus multiplication by λ is surjective on $H^i(G, \mathcal{D})$, proving part *(a)* of the proposition.

Now we consider coreflexivity. Let $P = (\pi)$ be any prime ideal of height 1 in Λ. It suffices to show that $H^i(G, \mathcal{D})[P]$ is a divisible (Λ/P)-module for all such P. Then one can apply corollary 2.6.1 to get the conclusion. Now since $\mathcal{D}$ is Λ-divisible, we get an exact sequence

$$0 \to \mathcal{D}[P] \to \mathcal{D} \to \mathcal{D} \to 0$$

induced by multiplication by π. The corresponding cohomology sequence then gives a surjective map $H^i(G, \mathcal{D}[P]) \to H^i(G, \mathcal{D})[P]$ of (Λ/P)-modules. Corollary 2.6.1 implies that $\mathcal{D}[P]$ is (Λ/P)-divisible, and hence, by part *(a)*, so is $H^i(G, \mathcal{D}[P])$. It follows that $H^i(G, \mathcal{D})[P]$ is indeed divisible as a (Λ/P)-module, proving part *(b)*. ■

B. Behavior under specialization. If I is any ideal of R, then one has an obvious (R/I)-module homomorphism

$$H^i(G, \mathcal{D}[I]) \longrightarrow H^i(G, \mathcal{D})[I] \tag{4}$$

We will discuss the kernel and cokernel. Since $\mathcal{D}[I]^G = \mathcal{D}^G[I]$, this homomorphism is an isomorphism when $i = 0$. If $i \geq 1$, the simplest case to study is when I is a principal ideal and $\mathcal{D}$ is a divisible R-module. If $I = (\xi)$, then we consider the exact sequence induced by multiplication by ξ.

$$0 \longrightarrow \mathcal{D}[I] \longrightarrow \mathcal{D} \xrightarrow{\xi} \mathcal{D} \longrightarrow 0$$

The corresponding map on the cohomology groups is also induced by multiplication by ξ. This gives the exact sequence

$$0 \to H^{i-1}(G, \mathcal{D})/\xi H^{i-1}(G, \mathcal{D}) \to H^i(G, \mathcal{D}[I]) \to H^i(G, \mathcal{D})[I] \to 0 \tag{5}$$

Thus, when I is principal and $\mathcal{D}$ is divisible, the map (5) will at least be surjective. It suffices just to assume that $\mathcal{D}$ is divisible by the element ξ generating I. Here is one rather general and useful result for arbitrary ideals, valid even when $\mathcal{D}$ is not assumed to be divisible.

Proposition 3.4. *Suppose that $\mathcal{D}$ is a cofinitely generated R-module. Let $i \geq 0$. If $i > 0$, assume that $H^{i-1}(G, \alpha_k) = 0$ for $1 \leq k \leq t$. Suppose that I is any ideal of R. Then the map*

$$H^i(G, \mathcal{D}[I]) \longrightarrow H^i(G, \mathcal{D})[I]$$

is an isomorphism.

Proof. We've already remarked that the map is an isomorphism when $i = 0$. If $i > 0$, the assumption implies that $H^{i-1}(G, \mathcal{C}) = 0$ for every subquotient $\mathcal{C}$ of $\mathcal{D}$ as an $R[G]$-module.

Thus, if we have an injective map $\mathcal{C}_1 \to \mathcal{C}_2$ of such $R[G]$-modules, then $H^{i-1}(G, \mathcal{C}_2/\mathcal{C}_1) = 0$ and so the induced map $H^i(G, \mathcal{C}_1) \to H^i(G, \mathcal{C}_2)$ will be injective.

Suppose that $I = (\lambda)$ is a principal ideal. Multiplication by λ gives an exact sequence

$$0 \longrightarrow \mathcal{D}[\lambda] \xrightarrow{a} \mathcal{D} \xrightarrow{b} \lambda\mathcal{D} \longrightarrow 0$$

Let $\alpha : H^i(G, \mathcal{D}[\lambda]) \to H^i(G, \mathcal{D})$ and $\beta : H^i(G, \mathcal{D}) \to H^i(G, \lambda\mathcal{D})$ be the maps induced from a and b. The map α is injective and its image is the kernel of the map β. But the map $\gamma : H^i(G, \lambda\mathcal{D}) \to H^i(G, \mathcal{D})$ is also injective and so the maps β and $\gamma \circ \beta$ have the same kernel. The map $\gamma \circ \beta : H^i(G, \mathcal{D}) \to H^i(G, \mathcal{D})$ is just multiplication by λ. Therefore, the image of α is indeed $H^i(G, \mathcal{D})[\lambda]$, which proves the proposition if I is principal.

We will argue by induction on the minimum number of generators of I. Suppose that $\lambda_1, ..., \lambda_g$ is a minimal generating set for I, where $g > 1$. Let $I_1 = (\lambda_1, ..., \lambda_{g-1})$. Assume that the map $H^i(G, \mathcal{D}[I_1]) \to H^i(G, \mathcal{D})[I_1]$ is an isomorphism. Then so is the map $H^i(G, \mathcal{D}[I_1])[\lambda_g] \to (H^i(G, \mathcal{D})[I_1])[\lambda_g] = H^i(G, \mathcal{D})[I]$. Now $\mathcal{D}[I_1][\lambda_g] = \mathcal{D}[I]$ and so, as shown above, the map $H^i(G, \mathcal{D}[I]) \to H^i(G, \mathcal{D}[I_1])[\lambda_g]$ is an isomorphism. Composing these isomorphisms, we get the isomorphism stated in the proposition for I. ■

Remark 3.4.1. For the case $i = 1$, the assumption in proposition 3.4 is that the trivial $\mathbb{F}_p$-representation of G is not a composition factor in the $\mathbb{F}_p[G]$-module $\mathcal{D}[\mathfrak{m}]$. Assuming this is satisfied, we have $H^1(G, \mathcal{D}[\mathcal{P}]) \cong H^1(G, \mathcal{D})[\mathcal{P}]$ for every prime ideal $\mathcal{P}$ of R. Let $r = \mathrm{corank}_R\big(H^1(G, \mathcal{D})\big)$. Applying remark 2.1.3 to $A = H^1(G, \mathcal{D})$, we see that $\mathrm{corank}_{R/\mathcal{P}}\big(H^1(G, \mathcal{D}[P])\big) \geq r$ for all $\mathcal{P}$ and that equality holds for all $\mathcal{P} \notin V(I)$, where I is some nonzero ideal of R. A similar statement is true for any i under the assumptions of proposition 3.4.

Remark 3.4.2. Suppose now that $R = \Lambda$ and that $\mathcal{D}$ is a cofree Λ-module. Assume that P is a regular prime ideal of Λ, i.e., that the local ring Λ/P is regular. The ideal P can be generated by a regular sequence $\lambda_1, ..., \lambda_g$ of elements of Λ. (See proposition 2.2.4 in [B-H].) Define $P_0 = (0)$ and $P_j = (\lambda_1, ..., \lambda_j)$ for $1 \leq j \leq g$. Then P_j is a prime ideal for $j \geq 0$ and $\mathcal{D}[P_j]$ is a cofree, and hence divisible, (Λ/P_j)-module. Note that if $j \geq 1$, then$\mathcal{D}[P_j] = (\mathcal{D}[P_{j-1}])[\lambda_j]$ and multiplication by λ_j defines a surjective map on $\mathcal{D}[P_{j-1}]$. This induces a surjective map

$$H^i(G, \mathcal{D}[P_j]) \longrightarrow H^i(G, \mathcal{D}[P_{j-1}])[P_j]$$

Thus, $\mathrm{corank}_{\Lambda/P_j}\big(H^i(G, \mathcal{D}[P_j])\big) \geq \mathrm{corank}_{\Lambda/P_j}\big(H^i(G, \mathcal{D}[P_{j-1}])[P_j]\big)$. Remark 2.1.3 implies that $\mathrm{corank}_{\Lambda/P_j}\big(H^i(G, \mathcal{D}[P_{j-1}])[P_j]\big) \geq \mathrm{corank}_{\Lambda/P_{j-1}}\big(H^i(G, \mathcal{D}[P_{j-1}]\big)$. Since $\mathcal{D}[P_0] = \mathcal{D}$, we have proved that

$$\mathrm{corank}_{\Lambda/P}\big(H^i(G, \mathcal{D}[P])\big) \geq \mathrm{corank}_{\Lambda}\big(H^i(G, \mathcal{D})\big)$$

for all regular prime ideals of Λ. In particular, suppose that $\Lambda/P \cong \mathbb{Z}_p$. Then

$$\mathrm{corank}_\Lambda\big(H^i(G,\mathcal{D})\big) \leq \mathrm{corank}_{\mathbb{Z}_p}\big(H^i(G,\mathcal{D}[P])\big) \leq \dim_{\mathbb{F}_p}\big(H^i(G,\mathcal{D}[\mathfrak{m}_\Lambda])\big).$$

In the following proposition, we consider $\mathcal{D}$ just as a Λ-module and take $I = P$ to be a prime ideal of height 1. However, the result can be extended to a more general class of rings R as explained in remark 3.5.2 below.

Proposition 3.5. *Suppose that $\mathcal{D}$ is a cofinitely generated Λ-module. Let $i \geq 0$. Then, for almost all $P \in \mathrm{Spec}_{ht=1}(\Lambda)$, the kernel and cokernel of the map*

$$H^i(G,\mathcal{D}[P]) \longrightarrow H^i(G,\mathcal{D})[P]$$

are cotorsion (Λ/P)-modules and hence $H^i(G,\mathcal{D}[P])$ and $H^i(G,\mathcal{D})[P]$ will have equal (Λ/P)-coranks.

Proof. As already mentioned, the result is obvious for $i = 0$. We assume first that $\mathcal{D}$ is Λ-divisible. Suppose that $i \geq 1$. The map in question is surjective. Let π be a generator of P, which is a principal ideal. Since we are assuming that $\mathcal{D}$ is Λ-divisible, we can use (5) for $I = P$. As a Λ-module, $H^{i-1}(G,\mathcal{D})/\pi H^{i-1}(G,\mathcal{D})$ is a quotient of the cofinitely generated, cotorsion Λ-module $A = H^{i-1}(G,\mathcal{D})/H^{i-1}(G,\mathcal{D})_{\Lambda-\mathrm{div}}$. Let $J = \mathrm{Ann}_\Lambda(A)$. Then it is clear that if P does not contain J, then $H^{i-1}(G,\mathcal{D})/\pi H^{i-1}(G,\mathcal{D})$ is a cotorsion (Λ/P)-module. If $\mathcal{D}$ is not Λ-divisible, then one notes that $\mathcal{D}$ is Λ-isogenous to $\mathcal{D}_{\Lambda-\mathrm{div}}$ and so one can easily reduce to the Λ-divisible case. ■

Remark 3.5.1. A similar result holds for the cohomology groups associated to a finitely generated Λ-module $\mathcal{T}$. We assume that G acts continuously and Λ-linearly on $\mathcal{T}$ and that the cohomology groups $H^i(G,\alpha)$ are finite for every simple subquotient α of the G-module $\mathcal{T}/\mathfrak{m}_\Lambda\mathcal{T}$. The G-module $\mathcal{T}$ is now compact and so we consider the continuous cohomology groups. A discussion of their properties can be found in [NSW], chapter II, §3. Since $\mathcal{T} = \varprojlim_n \mathcal{T}/\mathfrak{m}^n\mathcal{T}$, an inverse limit of finite Galois modules, we have

$$H^i_{cts}(G,\mathcal{T}) = \varprojlim_n H^i(G,\mathcal{T}/\mathfrak{m}^n\mathcal{T})$$

This follows from corollary 2.3.5 in [NSW]. Note that our assumption that the $H^i(G,\alpha)$'s are finite is needed for this. It is not hard to show that $H^i_{cts}(G,\mathcal{T})$ is a finitely generated Λ-module. If P is a prime ideal of Λ, one has a natural map $H^i_{cts}(G,\mathcal{T}) \to H^i_{cts}(G,\mathcal{T}/P\mathcal{T})$.

Suppose that P is a prime ideal of height 1. Then we have the following compact version of proposition 3.5.

The kernel and cokernel of the map $H^i_{cts}(G,\mathcal{T})/PH^i_{cts}(G,\mathcal{T}) \longrightarrow H^i_{cts}(G,\mathcal{T}/P\mathcal{T})$ *are torsion* (Λ/P)*-modules for almost all* $P \in \mathrm{Spec}_{ht=1}(\Lambda)$.

The argument is analogous to that given above. Suppose that $P=(\pi)$. Assuming first that $\mathcal{T}$ is a torsion-free Λ-module, one considers the exact sequence

$$0 \longrightarrow \mathcal{T} \xrightarrow{\pi} \mathcal{T} \longrightarrow \mathcal{T}/P\mathcal{T} \longrightarrow 0$$

induced by multiplication by π. The map in question is induced by this exact sequence. It is injective and its cokernel is isomorphic to $H^{i+1}_{cts}(G,\mathcal{T})[\pi]$, which is a Λ-submodule of $H^{i+1}_{cts}(G,\mathcal{T})_{\Lambda-tors}$, the torsion Λ-submodule of $H^{i+1}_{cts}(G,\mathcal{T})$. Thus, this cokernel is indeed (Λ/P)-torsion for all but the finitely many $P \in \mathrm{Spec}_{ht=1}(\Lambda)$ containing the annihilator of $H^{i+1}_{cts}(G,\mathcal{T})_{\Lambda-tors}$. As before, one easily reduces the general case to the case where $\mathcal{T}$ is torsion-free.

Remark 3.5.2. Suppose that $\mathcal{D}$ is a cofinitely generated R-module, where R is a finite, integral extension of Λ. Let $\mathcal{K}$ be the field of fractions for R, a finite extension of the field of fractions $\mathcal{L}$ of Λ. We will assume that $\mathcal{K}/\mathcal{L}$ is a separable extension. One can prove that the kernel and cokernel of the map

$$H^i(G,\mathcal{D}[\mathcal{P}]) \longrightarrow H^i(G,\mathcal{D})[\mathcal{P}]$$

will be cotorsion $(R/\mathcal{P})$-modules for almost all $\mathcal{P} \in \mathrm{Spec}_{ht=1}(R)$ as follows. Assume that $P \in \mathrm{Spec}_{ht=1}(\Lambda)$ satisfies the conclusion of proposition 3.5 and is also unramified for the extension $\mathcal{K}/\mathcal{L}$ in the following sense: For all $\mathcal{P}$ lying over P, the maximal ideal in the localization $R_{\mathcal{P}}$ is generated by P. Fix one such $\mathcal{P}$. Consider the following commutative diagram

$$\begin{array}{ccc}
H^i(G,\mathcal{D}[\mathcal{P}]) & \xrightarrow{\alpha} & H^i(G,\mathcal{D})[\mathcal{P}] \\
\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta'} \\
H^i(G,\mathcal{D}[P]) & \xrightarrow{\alpha'} & H^i(G,\mathcal{D})[P]
\end{array}$$

The horizontal maps α and α' are defined in the obvious way. Both $\ker(\alpha')$ and $\mathrm{coker}(\alpha')$ are (Λ/P)-cotorsion by assumption. Thus, they are annihilated by some element $\lambda \in \Lambda - P$. The map β is induced from the inclusion $\mathcal{D}[\mathcal{P}] \to \mathcal{D}[P]$. Since P is assumed to be unramified, $\mathcal{P} \subset PR_{\mathcal{P}}$ and hence there exists an element $\gamma \in R-\mathcal{P}$ such that $\gamma\mathcal{P} \subseteq PR$. This implies that $\gamma\mathcal{D}[P] \subseteq \mathcal{D}[\mathcal{P}]$ and so γ annihilates $\mathcal{D}[P]/\mathcal{D}[\mathcal{P}]$. It follows that $\ker(\beta)$ and $\mathrm{coker}(\beta)$ are annihilated by γ. It is also clear that β' is injective and $\mathrm{coker}(\beta')$ is annihilated by

γ. A diagram chase then implies that $\ker(\alpha)$ and $\operatorname{coker}(\alpha)$ are annihilated by $\lambda\gamma$. Since this element of R is not in $\mathcal{P}$, it follows that the kernel and cokernel of α are cotorsion $(R/\mathcal{P})$-modules. This is true for all $\mathcal{P}$ lying over P.

The conclusion of proposition 3.5 is true for almost all $P \in \mathrm{Spec}_{ht=1}(\Lambda)$. It remains to show that almost all $P \in \mathrm{Spec}_{ht=1}(\Lambda)$ are unramified in $\mathcal{K}/\mathcal{L}$. Let S denote the integral closure of R in $\mathcal{K}$. Then it is known that S is finitely generated as a Λ-module. (See theorem 6.4 in [D].) Let $\omega_1, ..., \omega_n$ be a fixed basis for $\mathcal{K}$ over $\mathcal{L}$ contained in R. Then for almost all $P \in \mathrm{Spec}_{ht=1}(\Lambda)$, the localizations R_P and S_P coincide and are free Λ_P-modules with basis $\omega_1, ..., \omega_n$. Assume that P has this property. Now Λ_P is a discrete valuation ring and $R_P = S_P$ is a Dedekind ring. Since $\mathcal{K}/\mathcal{L}$ is separable, the discriminant of this extension for the fixed basis is nonzero, and the prime ideal P is unramified if it doesn't contain this discriminant. It clearly follows that only finitely many $P \in \mathrm{Spec}_{ht=1}(\Lambda)$ can be ramified in $\mathcal{K}/\mathcal{L}$.

C. Almost divisibility. Suppose that $i \geq 1$ and that $P = (\pi)$ is a prime ideal of Λ of height 1. Then, according to (5), the map $H^i(G, \mathcal{D}[P]) \longrightarrow H^i(G, \mathcal{D})[P]$ will be injective if and only if $H^{i-1}(G, \mathcal{D})/\pi H^{i-1}(G, \mathcal{D}) = 0$, assuming that $\mathcal{D}$ is divisible by π. Thus, we have the following useful equivalence.

Proposition 3.6. *Suppose that $\mathcal{D}$ is an almost divisible, cofinitely generated Λ-module. Let $i \geq 1$. Then the Λ-module $H^{i-1}(G, \mathcal{D})$ is almost divisible if and only if the map*

$$H^i(G, \mathcal{D}[P]) \longrightarrow H^i(G, \mathcal{D})[P]$$

is injective for almost all $P \in \mathrm{Spec}_{ht=1}(\Lambda)$.

Here is one important special case.

Proposition 3.7. *Suppose that $\mathcal{D}$ is a coreflexive Λ-module on which G acts. Let $i \geq 0$. Assume that $H^{i+2}(G, \alpha_k) = 0$ for $1 \leq k \leq t$. If $H^{i+1}(G, \mathcal{D}) = 0$, then $H^i(G, \mathcal{D})$ is an almost divisible Λ-module.*

Proof. By proposition 3.6, it certainly suffices to show that $H^{i+1}(G, \mathcal{D}[P]) = 0$ for almost all $P \in \mathrm{Spec}_{ht=1}(\Lambda)$. This follows if we show that $H^{i+1}(G, \mathcal{D}[P])$ is both (Λ/P)-cotorsion and (Λ/P)-divisible. Since $H^{i+1}(G, \mathcal{D}) = 0$, proposition 3.5 implies the first statement for all but finitely many height 1 prime ideals P. By corollary 2.6.1, $\mathcal{D}[P]$ is a divisible (Λ/P)-module, and proposition 3.3 then implies the (Λ/P)-divisibility of $H^{i+1}(G, \mathcal{D}[P])$ for every height 1 prime ideal P of Λ. ■

D. Replacing R by its reflexive closure. Now suppose that $\mathcal{T}$ is a free R-module of rank n and that G is a group which acts continuously and R-linearly on $\mathcal{T}$. Then G acts continuously and $\widetilde{R}$-linearly on $\widetilde{\mathcal{T}} = \mathcal{T} \otimes_R \widetilde{R}$. If R is a finite extension of Λ, then the above proposition implies that $\widetilde{\mathcal{T}}$ is the reflexive hull of $\mathcal{T}$ as a Λ-module. Both R and $\widetilde{R}$ are complete Noetherian local rings. As in the introduction, we define discrete G-modules $\mathcal{D} = \mathcal{T} \otimes_R \widehat{R}$ and $\widetilde{\mathcal{D}} = \widetilde{\mathcal{T}} \otimes_{\widetilde{R}} \widehat{\widetilde{R}}$. Then $\mathcal{D}$ is an R-module, $\widetilde{\mathcal{D}}$ is an $\widetilde{R}$-module, both are cofinitely generated Λ-modules, $\mathcal{D}$ is a divisible Λ-module, $\widetilde{\mathcal{D}}$ is a coreflexive Λ-module, and there is a surjective G-equivariant Λ-module homomorphism $\widetilde{\mathcal{D}} \to \mathcal{D}$ whose kernel $\mathcal{C}$ is a co-pseudo-null Λ-module.

The hypothesis in proposition 3.3 for $\mathcal{D}$ and for $\widetilde{\mathcal{D}}$ are equivalent. To explain this, let $\mathfrak{m}_\Lambda$ denote the maximal ideal of Λ, $\widetilde{\mathfrak{m}}$ the maximal ideal of $\widetilde{R}$. Then we can regard $\mathcal{D}[\mathfrak{m}_\Lambda]$ as a finite-dimensional representation space for G over the residue field $\Lambda/\mathfrak{m}_\Lambda \cong \mathbb{F}_p$ and $\mathcal{D}[\widetilde{\mathfrak{m}}]$ as such a representation space over $\widetilde{R}/\widetilde{\mathfrak{m}}$ and hence over $\mathbb{F}_p$. We then have the following observation.

Proposition 3.8. *The $\mathbb{F}_p$-representations spaces $\mathcal{D}[\mathfrak{m}]$, $\widetilde{\mathcal{D}}[\widetilde{\mathfrak{m}}]$, $\mathcal{D}[\mathfrak{m}_\Lambda]$, and $\widetilde{\mathcal{D}}[\mathfrak{m}_\Lambda]$ for G have the same irreducible subquotients.*

Proof. First note that $\mathcal{D}$ is a quotient of $\widetilde{\mathcal{D}}$. Also, for any nonzero $\lambda \in \Lambda$, one has $\widetilde{\mathcal{D}}/\widetilde{\mathcal{D}}[\lambda] \cong \widetilde{\mathcal{D}}$. One can choose λ so that $\mathcal{C} \subseteq \widetilde{\mathcal{D}}[\lambda]$. Since $\widetilde{\mathcal{D}}/\mathcal{C} \cong \mathcal{D}$, it is clear that $\widetilde{\mathcal{D}}$ is isomorphic to a subquotient of $\mathcal{D}$. Hence proposition 3.1 implies that $\mathcal{D}[\mathfrak{m}]$ and $\widetilde{\mathcal{D}}[\mathfrak{m}]$ have the same irreducible subquotients.

Now $\mathfrak{m}_\Lambda \subseteq \mathfrak{m}$ and so $\mathcal{D}[\mathfrak{m}] \subseteq \mathcal{D}[\mathfrak{m}_\Lambda]$. Also, the fact that $R/\mathfrak{m}_\Lambda R$ is finite implies that $\mathfrak{m}^t \subseteq \mathfrak{m}_\Lambda R$ for some $t \geq 1$. Hence $\mathcal{D}[\mathfrak{m}_\Lambda] \subseteq \mathcal{D}[\mathfrak{m}^t]$. Proposition 3.1 again implies that $\mathcal{D}[\mathfrak{m}_\Lambda]$ and $\mathcal{D}[\mathfrak{m}]$ have the same irreducible subquotients. The same argument applies to $\widetilde{\mathcal{D}}[\mathfrak{m}_\Lambda]$ and $\widetilde{\mathcal{D}}[\widetilde{\mathfrak{m}}]$. The proposition follows from these observations. ∎

The surjective homomorphism $\widetilde{\mathcal{D}} \to \mathcal{D}$ induces a map $H^i(G, \widetilde{\mathcal{D}}) \to H^i(G, \mathcal{D})$ for any $i \geq 0$. Since $H^i(G, \mathcal{C})$ and $H^{i+1}(G, \mathcal{C})$ are co-pseudo-null, the same will be true for both the kernel and the cokernel of that induced map. Proposition 3.3 then has the following consequence.

Proposition 3.9. *Suppose that $i \geq 0$. Suppose that $H^{i+1}(G, \alpha_k) = 0$ for $1 \leq k \leq t$. Then the map $H^i(G, \widetilde{\mathcal{D}}) \to H^i(G, \mathcal{D})$ is surjective, $H^i(G, \mathcal{D})$ is Λ-divisible, $H^i(G, \widetilde{\mathcal{D}})$ is Λ-coreflexive, and the Pontryagin dual of $H^i(G, \widetilde{\mathcal{D}})$ is precisely the reflexive hull of the Pontryagin dual of $H^i(G, \mathcal{D})$.*

Proof. Note that $H^{i+1}(G, \mathcal{C}) = 0$ by proposition 3.8 and corollary 3.1.1. This implies the surjectivity. The divisibility of $H^i(G, \mathcal{D})$ and coreflexivity of $H^i(G, \widetilde{\mathcal{D}})$ follow from propositions 3.8 and 3.3. The Pontryagin dual of $H^i(G, \mathcal{D})$ is a torsion-free Λ-module which is mapped injectively into the Pontryagin dual of $H^i(G, \widetilde{\mathcal{D}})$. The corresponding quotient Λ-module is a submodule of the Pontryagin dual of $H^i(G, \mathcal{C})$, and so it is pseudo-null. The final statement follows from this. ■

Remark 3.9.1. If $\mathcal{D}$ is not coreflexive, then the same would often be true for $H^i(G, \mathcal{D})$. Suppose, for example, that $i = 1$ and that $H^0(G, \alpha_k) = H^2(G, \alpha_k) = 0$ for $1 \le k \le t$. Then, if $H^1(G, \mathcal{D}[\mathfrak{m}]) \neq 0$, it follows that $H^1(G, \mathcal{C}) \neq 0$ and that the map $H^i(G, \widetilde{\mathcal{D}}) \to H^i(G, \mathcal{D})$ will have a nonzero kernel. In that case, proposition 3.9 implies that $H^i(G, \mathcal{D})$ is non-reflexive.

E. Relationship between $H^i(G, \mathcal{D})$ and $H^i_{cts}(G, \mathcal{T})$. Consider an arbitrary finitely generated R-module $\mathcal{T}$ on which a group G acts continuously and R-linearly. We assume that $H^i(G, \alpha)$ is finite for all $i \ge 0$ and all simple subquotients α of the finite G-module $\mathcal{T}/\mathfrak{m}\mathcal{T}$.

Proposition 3.10. *Let $\mathcal{D} = \mathcal{T} \otimes_R \widehat{R}$. Then $\operatorname{rank}_R\big(H^i_{cts}(G, \mathcal{T})\big) = \operatorname{corank}_R\big(H^i(G, \mathcal{D})\big)$ for all $i \ge 0$.*

Proof. The statement concerns $\mathcal{D} = \mathcal{D}_R$. Note that the simple subquotients α of the G-module $\mathcal{D}[\mathfrak{m}]$ are among those for $\mathcal{T}/\mathfrak{m}\mathcal{T}$ and so the corresponding cohomology groups are finite. To prove the equality, it is enough to consider the rank and corank over the subring Λ of R. We replace $\mathcal{D}_R$ by $\mathcal{D}_\Lambda = \mathcal{T} \otimes_\Lambda \widehat{\Lambda}$. This module is R-isogenous to $\mathcal{D}_R$ and so the corresponding cohomology groups will have the same coranks.

If Λ has Krull dimension 1, then the argument is straightforward. The maximal ideal $\mathfrak{m}_\Lambda$ of Λ is then principal. Letting $A_n = \mathcal{T}/\mathfrak{m}_\Lambda^n\mathcal{T}$, we have $A_n \cong \mathcal{D}[\mathfrak{m}_\Lambda^n]$ for any $n \ge 0$. One can relate the rank or corank in question to the growth of the finite groups $H^i(G, A_n)$ as $n \to \infty$. If Λ has Krull dimension > 1, there are infinitely many prime ideals of Λ of height 1. We then use an induction argument on the Krull dimension. Let $r = \operatorname{corank}_\Lambda\big(H^i(G, \mathcal{D})\big)$ and $s = \operatorname{rank}_\Lambda\big(H^i(G, \mathcal{T})\big)$. According to proposition 3.5, the (Λ/P)-corank of $H^i(G, \mathcal{D}[P])$ will be equal to r for almost all $P \in \operatorname{Spec}_{ht=1}(\Lambda)$. As pointed out in part **E** of section 2, one has a surjective (Λ/P)-homomorphism

$$(\mathcal{T}/P\mathcal{T}) \otimes_{\Lambda/P} \widehat{(\Lambda/P)} \to \mathcal{D}[P]$$

For almost all P's, the (Λ/P)-coranks of these modules will be equal, the kernel will therefore be (Λ/P)-cotorsion, and hence $H^i\big(G, (\mathcal{T}/P\mathcal{T}) \otimes_{\Lambda/P} \widehat{(\Lambda/P)}\big)$ will also have (Λ/P)-corank equal to r. We can choose such a P so that Λ/P is also a formal power series ring. The Krull dimension will be reduced by 1 and so we assume, inductively, that the (Λ/P)-rank of $H^i(G, \mathcal{T}/P\mathcal{T})$ is equal to r too. This will be true for an infinite set of P's in $\operatorname{Spec}_{ht=1}(\Lambda)$. However, according to remark 3.5.1, $H^i(G, \mathcal{T}/P\mathcal{T})$ will have (Λ/P)-rank equal to s for all but finitely many such P's. Therefore, $r = s$. ■

Remark 3.10.1. We want to mention another argument for the case $i = 0$ based on proposition 2.8. Let $\mathcal{D} = \mathcal{D}_R$. We will assume that G is topologically finitely generated. Let

$g_1, ..., g_t \in G$ generate a dense subgroup of G. Consider the map $\phi : \mathcal{T} \longrightarrow \mathcal{T}^t$ defined by $\phi(x) = \big((g_1 - 1)x, ..., (g_t - 1)x\big)$ for all $x \in \mathcal{T}$. The induced map $\psi : \mathcal{D} \longrightarrow \mathcal{D}^t$, as defined in proposition 2.8, is given by the same formula, but for $x \in \mathcal{D}$ instead. This definition implies that $\ker(\phi) = H^0(G, \mathcal{T})$ and that $\ker(\psi) = H^0(G, \mathcal{D})$. Proposition 2.8 then implies the equality of the R-rank and R-corank for these two R-modules.

A similar argument implies that the R-rank of $\mathcal{T}_G$ is equal to the R-corank of $\mathcal{D}_G$. These modules are the maximal *quotients* on which G acts trivially. Consider the map $\phi' : \mathcal{T}^t \longrightarrow \mathcal{T}$ defined by $\phi'(x_1, ..., x_t) = \sum_{i=1}^t (g_i - 1)x_i$ for all $x \in \mathcal{T}^t$. The induced map $\psi' : \mathcal{D}^t \longrightarrow \mathcal{D}$ is again given by the same formula. It is easy to see that $\mathrm{coker}(\phi') = \mathcal{T}_G$ and $\mathrm{coker}(\psi') = \mathcal{D}_G$. The stated equality follows from proposition 2.8.

Remark 3.10.2. One can apply remark 2.1.2 to obtain a useful consequence if we assume that $\mathcal{T}$ is a free R-module. Then $\mathcal{T}^t$ is also a free R-module. Let ϕ be the map defined above. If $\mathcal{P}$ is a prime ideal of R, then $\phi_{\mathcal{P}}$ is defined by the same formula as ϕ. It follows that $\mathrm{rank}_{R/\mathcal{P}}\big((\mathcal{T}/\mathcal{P}\mathcal{T})^G\big) \geq \mathrm{rank}_R\big(\mathcal{T}^G\big)$ for *every* prime ideal $\mathcal{P}$ of R. According to proposition 2.1.1, equality holds on a nonempty Zariski-open subset of $\mathrm{Spec}(R)$. Also, note that if $(\mathcal{T}/\mathcal{P}\mathcal{T})^G = 0$ for some prime ideal $\mathcal{P}$, then it follows that $\mathcal{T}^G = 0$.

4 Coranks.

In this section we will prove theorems concerning Euler-Poincaré characteristics, lower bounds on the R-coranks of H^1 and H^2, and the relationship between the R-coranks of Ш^1 and Ш^2. Assume that R is a finite, integral extension of Λ. If X is a finitely generated R-module, then $\mathrm{rank}_\Lambda(X) = \mathrm{rank}_R(X)\mathrm{rank}_\Lambda(R)$. Hence we can derive the formulas for ranks or coranks by considering the various R-modules as Λ-modules. This simplifies the arguments since the prime ideals of height 1 in Λ are principal. Thus, we will formulate all the results for a discrete, cofinitely generated Λ-module $\mathcal{D}$ which has a Λ-linear action of the appropiate Galois groups. Proposition 3.2 implies that the Galois cohomology groups $H^i(K_\Sigma/K, \mathcal{D})$ and $H^i(K_v, \mathcal{D})$ are also cofinitely generated Λ-modules. Thus, we can consider their Λ-coranks.

A. Euler-Poincaré characteristics. We assume that $\mathcal{D}$ has a Λ-linear action of $\mathrm{Gal}(K_\Sigma/K)$. We will prove the following result.

Proposition 4.1. *Let r_2 denote the number of complex primes of K, $m = \mathrm{corank}_\Lambda(\mathcal{D})$, and $m_v^- = \mathrm{corank}_\Lambda(\mathcal{D}/\mathcal{D}^{G_{K_v}})$ for each real prime v of K. Then*

$$\sum_{i=0}^{2} (-1)^i \mathrm{corank}_\Lambda\big(H^i(K_\Sigma/K, \mathcal{D})\big) = -\delta_\Lambda(K, \mathcal{D})$$

where $\delta_\Lambda(K,\mathcal{D}) = r_2 m + \sum_{v \text{ real}} m_v^-$.

For $i \geq 3$, we have $H^i(K_\Sigma/K,\mathcal{D}) = 0$ except possibly when $p=2$. In fact, the global-to-local restriction maps induces an isomorphism for $i \geq 3$

$$H^i(K_\Sigma/K,\mathcal{D}) \cong \prod_{v|\infty} H^i(K_v,\mathcal{D})$$

(See [NSW], (8.6.13, *ii*).) This justifies our remark in the introduction that $\text{Ш}^i(K,\Sigma,\mathcal{D}) = 0$ for $i \geq 3$. The right-hand side is trivial if p is an odd prime. But suppose that $p=2$. In that case, if $v|\infty$, then $H^i(K_v,\mathcal{D})$ is of exponent 2 and hence can be regarded as a module over $\Lambda/(2)$ for any such v. Thus, if Λ has characteristic 0, then $H^i(K_\Sigma/K,\mathcal{D})$ is a cotorsion Λ-module for $i \geq 3$. However, if Λ is a formal power series ring over $\mathbb{F}_2$, then $H^i(K_\Sigma/K,\mathcal{D})$ can have positive Λ-rank.

We will also state a formula for a local Euler-Poincaré characteristic for every non-archimedean prime v of K. The cofinitely generated Λ-module $\mathcal{D}$ is just assumed to have a Λ-linear action of G_{K_v}.

Proposition 4.2. *Let v be any non-archimedean prime of K. Let $m = \mathrm{corank}_\Lambda(\mathcal{D})$.*

(a) If v lies over p, then $\sum_{i=0}^{2}(-1)^i \mathrm{corank}_\Lambda\big(H^i(K_v,\mathcal{D})\big) = -m[K_v:\mathbb{Q}_p]$.

(b) If v does not lie over p, then $\sum_{i=0}^{2}(-1)^i \mathrm{corank}_\Lambda\big(H^i(K_v,\mathcal{D})\big) = 0$.

Both of these propositions will be proved by a specialization argument, reducing to the case where the Krull dimension of Λ is 1. That case is then rather easy, derived from the Poitou-Tate formula for the Euler-Poincaré characteristic of a finite Galois module. The Euler-Poincaré characteristic is additive for an exact sequence $0 \to \mathcal{D}_1 \to \mathcal{D}_2 \to \mathcal{D}_3 \to 0$. For any $\mathcal{D}$, we let $\mathcal{D}_{\Lambda-\mathrm{div}}$ denote its maximal Λ-divisible Λ-submodule. Then $\mathcal{D}/\mathcal{D}_{\Lambda-\mathrm{div}}$ is Λ-cotorsion. Also, the Euler-Poincaré characteristic for a Λ-cotorsion module is 0. Thus, we can assume for the proof that $\mathcal{D}$ is Λ-divisible. The proofs of the two propositions are virtually the same and so we will just give the proof of proposition 4.1.

Proof. If the Krull dimension of Λ is 1, then either $\Lambda = \mathbb{Z}_p$ or $\Lambda = \mathbb{F}_p[[T]]$. In the first case, the result is known. One determines the $\mathbb{Z}_p$-corank by reducing to the case of the finite modules $\mathcal{D}[p^n]$, $n \geq 0$. In the second case, the argument would be similar, reducing to the case of the finite modules $\mathcal{D}[T^n]$, $n \geq 0$. If the Krull dimension is at least 2, then there are infinitely many prime ideals P of height 1 such that (Λ/P) is also a formal power series ring, but with Krull dimension reduced by 1. By remark 2.1.3, we can choose such a P so that $\mathrm{corank}_{\Lambda/P}(\mathcal{D}[P]) = \mathrm{corank}_\Lambda(\mathcal{D})$ and $\mathrm{corank}_{\Lambda/P}(\mathcal{D}[P]^{G_{K_v}}) =$

$\mathrm{corank}_\Lambda(\mathcal{D}^{G_{K_v}})$ for all archimedean primes v of K. Then $\delta_{\Lambda/P}(K, \mathcal{D}[P]) = \delta_\Lambda(K, \mathcal{D})$ for all such P. By proposition 3.5 and remark 2.1.3, we can also assume that P has the property that $\mathrm{corank}_{\Lambda/P}\big(H^i(G, \mathcal{D}[P])\big) = \mathrm{corank}_\Lambda\big(H^i(G, \mathcal{D})\big)$ for $i = 0, 1$, and 2. Choosing a P with all of these properties reduces the proof of proposition 4.1 to the corresponding result for $\mathcal{D}[P]$ considered as a module over the formal power series ring (Λ/P). By induction, we are done. ■

B. Lower bound on the Λ-corank of $H^1(K_\Sigma/K, \mathcal{D})$. We will derive a lower bound in terms of various local and global H^0's. First we do this for the Λ-corank of $H^2(K_\Sigma/K, \mathcal{D})$. Then applying proposition 4.1 gives a lower bound for the Λ-corank of $H^1(K_\Sigma/K, \mathcal{D})$. The theorems of Poitou-Tate determine the cokernel of the map

$$\gamma : H^2(K_\Sigma/K, \mathcal{D}) \to P^2(K, \Sigma, \mathcal{D})$$

where $P^2(K, \Sigma, \mathcal{D}) = \prod_{v \in \Sigma} H^2(K_v, \mathcal{D})$. Usually these theorems are stated for finite Galois modules. See [NSW], (8.6.13, *i*) for a complete statement in this case. But $\mathcal{D}$ is a direct limit of the finite Galois modules $\mathcal{D}[\mathfrak{m}^n]$ as $n \to \infty$, and one can therefore extend these theorems easily. In particular, we have

$$\mathrm{coker}(\gamma) \cong H^0(K_\Sigma/K, \mathcal{T}^*)^\wedge, \tag{6}$$

where $\mathcal{T}^* = \mathrm{Hom}(\mathcal{D}, \mu_{p^\infty})$. This module is the inverse limit of the finite Galois modules $\mathrm{Hom}(\mathcal{D}[\mathfrak{m}_\Lambda^n], \mu_{p^\infty})$ as $n \to \infty$. One can also extend Tate's local duality theorem ([NSW], (7.2.6)), usually stated for finite Galois modules, to $\mathcal{D}$ obtaining, for example, the isomorphisms $H^2(K_v, \mathcal{D}) \cong H^0(K_v, \mathcal{T}^*)^\wedge$ for every non-archimedean prime v of K. When Λ has characteristic 2, it is also necessary to consider the real archimedean primes since $H^2(K_v, \mathcal{D})$ could then have a positive Λ-corank. If v is such a prime, then the Pontryagin dual of $H^2(K_v, \mathcal{D})$ is $\widehat{H}^0(K_v, \mathcal{T}^*) = (\mathcal{T}^*)^{G_{K_v}}/(1 + \sigma_v)\mathcal{T}^*$, where σ_v is the nontrivial element of G_v.

We will use the following abbreviations for various ranks and coranks over Λ. For $i \geq 0$, let $h_i(K_\Sigma/K, \mathcal{D}) = \mathrm{corank}_\Lambda\big(H^i(K_\Sigma/K, \mathcal{D})\big)$. If $i = 0$, we will usually write K in place of K_Σ/K since the group is then just the G_K-invariant elements. We let $h_0(K, \mathcal{T}^*) = \mathrm{rank}_\Lambda\big(H^0(K, \mathcal{T}^*)\big)$ and $h_0(K_v, \mathcal{T}^*) = \mathrm{rank}_\Lambda\big(H^0(K_v, \mathcal{T}^*)\big)$. If v is archimedean, we will let $\widehat{h}_0(K_v, \mathcal{T}^*)$ denote the Λ-rank of $\widehat{H}^0(K_v, \mathcal{T}^*)$. With this notation, we get the following lower bound for $\mathrm{corank}_\Lambda\big(H^2(K_\Sigma/K, \mathcal{D})\big)$:

$$h_2(K_\Sigma/K, \mathcal{D}) \;\geq\; \sum_{v \mid \infty} \widehat{h}_0(K_v, \mathcal{T}^*) \;+\; \sum_{v \in \Sigma, v \nmid \infty} h_0(K_v, \mathcal{T}^*) \;-\; h_0(K, \mathcal{T}^*) \tag{7}$$

Equality occurs precisely when $\text{Ш}^2(K, \Sigma, \mathcal{D}) = \ker(\gamma)$ has Λ-corank equal to 0.

The terms in the quantity $\delta_\Lambda(K,\mathcal{D})$ (defined in proposition 4.1) are mostly Λ-ranks of H^0's. For a complex prime v, one obviously has $m = h_0(K_v,\mathcal{T}^*)$. For a real prime v, one sees easily that $m_v^- = h_0(K_v,\mathcal{T}^*)$ if the characteristic of Λ is not 2. This is not necessarily so if Λ has characteristic 2. However, in all cases, one has the following result.

Proposition 4.3. *Let* $b_\Lambda^1(K,\Sigma,\mathcal{D}) = h_0(K,\mathcal{D}) + \sum_{v\in\Sigma} h_0(K_v,\mathcal{T}^*) - h_0(K,\mathcal{T}^*)$. *Then we have the inequality* $h_1(K_\Sigma/K,\mathcal{D}) \geq b_\Lambda^1(K,\Sigma,\mathcal{D})$. *Equality holds if and only if* $\text{Ш}^2(K,\Sigma,\mathcal{D})$ *is* Λ*-cotorsion.*

Of course, one can similarly define all the quantities in terms of R-ranks and coranks. The corresponding lower bound will be denoted by $b_R^1(K,\Sigma,\mathcal{D})$.

Proof. Assume first that either p is odd or, if $p=2$, that Λ has characteristic 0. Note that the sum is over all $v\in\Sigma$, finite and infinite. The contribution to this sum from the infinite primes is just $\delta_\Lambda(K,\mathcal{D})$. Indeed, each complex prime contributes an m. To check the contribution when v is a real prime, let σ_v be a generator of G_v. Let $\beta_v = 1+\sigma_v$, the norm map. Then $(\mathcal{T}^*)^{G_{K_v}}/\beta_v\mathcal{T}^*$ has exponent 2 and is therefore a torsion Λ-module. Hence $(\mathcal{T}^*)^{G_{K_v}}$ and $\beta_v\mathcal{T}^*$ have the same Λ-ranks. Since σ_v acts by inversion on μ_{p^∞}, $\beta_v\mathcal{T}^*$ is the Pontryagin dual of $\mathcal{D}/\mathcal{D}^{G_{K_v}}$ as a Λ-module. Thus, the contribution from v will be m_v^-. It follows that the contribution from the infinite primes is just $\delta_\Lambda(K,\mathcal{D})$ and so the stated inequality then follows from proposition 4.1 together with (7). The fact that $\widehat{h}_0(K_v,\mathcal{T}^*)=0$ implies that equality holds if and only if it holds in (7) and that is equivalent to the vanishing of the Λ-corank of $\text{Ш}^2(K,\Sigma,\mathcal{D})$.

Now assume that Λ has characteristic 2. For the complex primes and finite primes, everything is the same as before. If v is a real prime, then it is still true that $\beta_v\mathcal{T}^*$ is the Pontryagin dual of $\mathcal{D}/\mathcal{D}^{G_{K_v}}$ as a Λ-module. Thus, the Λ-rank of $\beta_v\mathcal{T}^*$ is m_v^-. It follows that $h_0(K_v,\mathcal{T}^*) = m_v^- + \widehat{h}_0(K_v,\mathcal{T}^*)$. Using that observation, the inequality in proposition 4.3 follows from proposition 4.1 and (7). Equality is again equivalent to the validity of hypothesis L. ■

Remark 4.3.1. One can express all the quantities in the inequality in terms of discrete Λ-modules. Let $\mathcal{D}^* = \mathcal{T}^*\otimes_\Lambda \widehat{\Lambda}$. Then, $h_0(K_v,\mathcal{T}^*) = \text{corank}_\Lambda\big(H^0(K_v,\mathcal{D}^*)\big)$ for each $v\in\Sigma$ and $h_0(K,\mathcal{T}^*) = \text{corank}_\Lambda\big(H^0(K,\mathcal{D}^*)\big)$. These equalities follow from proposition 3.10 or remark 3.10.1. Note that G_{K_v} is topologically finitely generated and that the action of $\text{Gal}(K_\Sigma/K)$ on $\mathcal{T}^*$ factors through a quotient group G satisfying that property.

In theorem 1, we assume that $H^0(K_{v_o},\mathcal{T}^*)=0$ for at least one non-archimedean $v_o\in\Sigma$. Since $\mathcal{T}^*$ is torsion-free Λ-module in that theorem, an equivalent assumption would be that $h_0(K_v,\mathcal{T}^*)=0$ for some such v_o. Note that this assumption obviously implies that $H^0(K,\mathcal{T}^*)=0$ or, equivalently, that $h_0(K,\mathcal{T}^*)=0$.

C. The coranks of Ш^1 and Ш^2. Another part of the Poitou-Tate duality theorems gives a perfect pairing between Ш^2 for a finite Galois module A and Ш^1 for the *"Kummer dual"* $A^* = \mathrm{Hom}(A, \mu_N)$, where $N = |A|$. See [NSW], (8.6.8). Taking direct and inverse limits gives a perfect pairing between Ш$^2(K, \Sigma, \mathcal{D})$ and Ш$^1(K, \Sigma, \mathcal{T}^*)$. As discussed in the introduction, both groups might be zero in important cases. We prefer to consider Ш^1 for a discrete module $\mathcal{D}^*$, but this may often be nonzero even if Ш$^1(K, \Sigma, \mathcal{T}^*) = 0$. We can only prove a relationship between the Λ-coranks. It is not even quite clear how one should define $\mathcal{D}^*$. We have some freedom because the Λ-corank of Ш^i is not changed by a Λ-isogeny of the coefficient module, as we show below. For the purpose of the following proposition, we define $\mathcal{D}^* = \mathcal{T}^* \otimes_\Lambda \widehat{\Lambda}$, although this may differ from $\mathcal{D}^*$, as defined in the introduction, by a Λ-isogeny.

Proposition 4.4. *The Λ-coranks of* Ш$^2(K, \Sigma, \mathcal{D})$ *and* Ш$^1(K, \Sigma, \mathcal{D}^*)$ *are equal.*

We will use the following lemma which is the analogue of proposition 3.5 for Ш^i.

Lemma 4.4.1. *Suppose that $\mathcal{D}$ is a cofinitely generated Λ-module. Let $i \geq 1$. Then, for almost all $P \in \mathrm{Spec}_{ht=1}(\Lambda)$, both the kernel and the cokernel of the map*

$$Ш^i(K, \Sigma, \mathcal{D}[P]) \longrightarrow Ш^i(K, \Sigma, \mathcal{D})[P]$$

will be cotorsion as (Λ/P)-modules. Hence Ш$^i(G, \mathcal{D}[P])$ *and* Ш$^i(G, \mathcal{D})[P]$ *will have the same (Λ/P)-coranks.*

Proof. Applying proposition 3.5 to the global and local cohomology groups shows that the kernels and cokernels of the maps

$$H^i(K_\Sigma/K, \mathcal{D}[P]) \longrightarrow H^i(K_\Sigma/K, \mathcal{D})[P], \quad P^i(K, \Sigma, \mathcal{D}[P]) \longrightarrow P^i(K, \Sigma, \mathcal{D})[P]$$

are Λ-cotorsion for all but finitely many P's of height 1. A straightforward application of the snake lemma implies the result. One uses the fact that the kernels of both maps and the cokernel of the first map are Λ-cotorsion. ■

Now we show that the Λ-corank of Ш^i is unchanged by Λ-isogenies. Assume that $\mathcal{D}_1$ and $\mathcal{D}_2$ are cofinitely generated Λ-modules with a Λ-linear action of $\mathrm{Gal}(K_\Sigma/K)$ and that $\phi : \mathcal{D}_1 \to \mathcal{D}_2$ is a $\mathrm{Gal}(K_\Sigma/K)$-equivariant Λ-isogeny. Then ϕ induces maps on both the global and local cohomology groups and one has a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Ш^i(K, \Sigma, \mathcal{D}_1) \arrow[r] \arrow[d, dashed, "\alpha"] & H^i(K_\Sigma/K, D_1) \arrow[r] \arrow[d, "\kappa"] & P^i(K, \Sigma, \mathcal{D}_1) \arrow[d, "\lambda"] \\
0 \arrow[r] & Ш^i(K, \Sigma, \mathcal{D}_2) \arrow[r] & H^i(K_\Sigma/K, D_2) \arrow[r, "\sigma"] & P^i(K, \Sigma, \mathcal{D}_2)
\end{tikzcd}$$

The maps κ and λ are Λ-isogenies. It is clear that the image of $Ш^i(K,\Sigma,\mathcal{D}_1)$ under the map κ is contained in the kernel of σ and so the map α corresponding to the dashed arrow making the diagram commutative does exist. The fact that κ and λ are Λ-isogenies implies that α is a Λ-isogeny.

Let $s_2 = \mathrm{corank}_\Lambda\big(Ш^2(K,\Sigma,\mathcal{D})\big)$, $s_1^* = \mathrm{corank}_\Lambda\big(Ш^1(K,\Sigma,\mathcal{D}^*)\big)$. We prove the equality by induction. If the Krull dimension of Λ is 1, then proposition 4.4 is, as before, rather straightforward to derive from the Poitou-Tate duality theorems for finite Galois modules. In that case, let $\mathcal{V} = \mathcal{T} \otimes_\Lambda \mathcal{L}$, $\mathcal{V}^* = \mathcal{T}^* \otimes_\Lambda \mathcal{L}$, where $\mathcal{L}$ is the fraction field for Λ. Thus, $\mathcal{L} = \mathbb{Q}_p$ or $\mathcal{L} = \mathbb{F}_p((T))$. One then verifies that $s_2 = \dim_{\mathcal{L}}\big(Ш^2(K,\Sigma,\mathcal{V})\big)$ and $s_1^* = \dim_{\mathcal{L}}\big(Ш^1(K,\Sigma,\mathcal{V}^*)\big)$. Also, the duality theorem asserts that $Ш^2(K,\Sigma,\mathcal{V})$ and $Ш^1(K,\Sigma,\mathcal{V}^*)$ are dual vector spaces, and so the equality $s_2 = s_1^*$ follows. If the Krull dimension d of Λ is at least 2, we reduce to the case of Krull dimension $d-1$ by using remark 2.1.3 and the above lemma. These imply that $s_2 = \mathrm{corank}_{\Lambda/P}\big(Ш^2(K,\Sigma,\mathcal{D}[P])\big)$ and $s_1^* = \mathrm{corank}_{\Lambda/P}\big(Ш^1(K,\Sigma,\mathcal{D}^*[P])\big)$ for all but finitely many P of height 1. We may assume, inductively, that $s_2 = \mathrm{corank}_{\Lambda/P}\big(Ш^1(K,\Sigma,\mathcal{D}[P]^*)\big)$. We can also assume that $\mathcal{D}$ is Λ-divisible, replacing $\mathcal{D}$ by its maximal Λ-divisible submodule if necessary. This doesn't change s_2. Then $\mathcal{T}^*$ will be a torsion-free Λ-module. Also, $\mathcal{D}^*$ and s_1^* will be unchanged.

To prove that $s_1^* = s_2$, it is now enough to show that $\mathcal{D}^*[P]$ is (Λ/P)-isogenous to $\mathcal{D}[P]^*$. Now $\mathrm{Hom}(\mathcal{D}[P],\mu_{p^\infty})$ is isomorphic to $\mathcal{T}^*/P\mathcal{T}^*$ and so, by definition,

$$\mathcal{D}[P]^* \cong (\mathcal{T}^*/P\mathcal{T}^*) \otimes_{\Lambda/P} (\widehat{\Lambda/P})$$

According to (3), we therefore have a surjective map $\mathcal{D}[P]^* \to \mathcal{D}^*[P]$. The remark following (3) implies that this map is actually a (Λ/P)-isogeny. ∎

The most interesting case is as described in the introduction. A somewhat different proof of proposition 4.4 works nicely in that case, which we will sketch here. Assume that $\mathcal{T}$ is a free R-module and that $\mathcal{D} = \mathcal{T} \otimes_R \widehat{R}$. As above, let $\mathcal{T}^* = \mathrm{Hom}(\mathcal{D},\mu_{p^\infty})$. We now take $\mathcal{D}^* = \mathcal{T}^* \otimes \widehat{R}$. Then, one can verify that $\mathcal{D}^*$ is canonically isomorphic to $\mathrm{Hom}(\mathcal{T},\mu_{p^\infty})$. Hence the theorems of Poitou and Tate can be applied to the dual pair $\mathcal{D}^*$ and $\mathcal{T}$.

One can define $Ш^2(K,\Sigma,\mathcal{T})$ for the compact R-module $\mathcal{T}$ as the kernel of the homomorphism

$$\gamma_{cpt} : H^2_{cts}(K_\Sigma,\mathcal{T}) \longrightarrow P^2_{cts}(K,\Sigma,\mathcal{T})$$

where $P^2(K,\Sigma,\mathcal{T}) = \prod_{v\in\Sigma} H^2_{cts}(K_v,\mathcal{T})$. The cokernel of γ_{cpt} is isomorphic to $H^0(K,\mathcal{D}^*)^\wedge$. If one applies proposition 3.10 to all the global and local terms, one deduces that the R-rank of $\ker(\gamma_{cpt})$ is equal to the R-corank of $\ker(\gamma)$. That is,

$$\mathrm{rank}_R\big(Ш^2(K,\Sigma,\mathcal{T})\big) \;=\; \mathrm{corank}_R\big(Ш^2(K,\Sigma,\mathcal{D})\big)$$

Now $\text{Ш}^1(K,\Sigma,\mathcal{D}^*)$ is isomorphic to the Pontryagin dual of $\text{Ш}^2(K,\Sigma,\mathcal{T})$ as an R-module and so its R-corank must indeed be equal to the R-corank of $\text{Ш}^2(K,\Sigma,\mathcal{D})$.

5 Local Galois cohomology groups.

Suppose that v is a prime of K and that p is any prime number. We assume that $\mathcal{D}$ is a cofinitely generated Λ-module with a Λ-linear action of G_{K_v}. Let $\mathcal{T}^* = \mathrm{Hom}(\mathcal{D},\mu_{p^\infty})$. We will consider first the local H^2 and then various properties for the local H^1. Most results will be for non-archimedean primes. We discuss the archimedean primes at the end of this section.

A. The structure of $H^2(K_v,\mathcal{D})$. If v is non-archimedean, then it is known that the p-cohomological dimension of G_{K_v} is equal to 2. (See theorem (7.1.8) in [NSW].) Proposition 3.3 therefore has the following immediate consequence:

Proposition 5.1. *Let v be a non-archimedean prime of K. If $\mathcal{D}$ is Λ-divisible, then $H^2(K_v,\mathcal{D})$ is Λ-divisible. If $\mathcal{D}$ is Λ-coreflexive, then $H^2(K_v,\mathcal{D})$ is Λ-coreflexive.*

The fact that the Λ-module $H^2(K_v,\mathcal{D})$ is coreflexive when $\mathcal{D}$ is coreflexive can also be seen as follows. The Pontryagin dual of $H^2(K_v,\mathcal{D})$ is $H^0(K_v,\mathcal{T}^*) = (\mathcal{T}^*)^{G_{K_v}}$. Since the Λ-module $\mathcal{T}^*$ is reflexive, it follows that $(\mathcal{T}^*)^{G_{K_v}}$ is also reflexive, as observed in section 2, part **C**.

Remark 5.1.1. It is not difficult to give an example where $H^2(K_v,\mathcal{D})$ fails to be Λ-cofree even if $\mathcal{D}$ is assumed to be Λ-cofree. This is based on the example described in remark 2.6.3. We will use the same notation. There we exhibited a reflexive, but non-free, Λ-submodule Y of $X = \Lambda^r$ for some r assuming that the Krull dimension of Λ is at least 3. Suppose that $\Lambda = \mathbb{Z}_p[[T_1,T_2]]$. Recall that Y was the kernel of a Λ-module homomorphism $X \to Z$ where Z was torsion-free and of rank 1. Arbitrarily choosing an injective Λ-module homomorphism $Z \to X$, we can regard Y as the kernel of a Λ-module homomorphism $\tau : X \to X$. Choose a basis for the Λ-module X. We will identify τ with the corresponding matrix. Multiplying τ by an element of Λ, if necessary, we can assume that τ has entries in $\mathfrak{m}_\Lambda$. The kernel will still be Y. Thus, $\sigma = 1+\tau$ will be an invertible matrix over Λ. The closed subgroup $\overline{<\sigma>}$ of $GL_r(\Lambda)$ generated topologically by σ will be a pro-p group, either isomorphic to $\mathbb{Z}_p$ or to a finite cyclic group of p-power order. In either case, we can easily define a continuous, surjective homomorphism $G_{K_v} \to \overline{<\sigma>}$. Thus, G_{K_v} acts Λ-linearly on X. If we let $\mathcal{D} = \mathrm{Hom}(X,\mu_{p^\infty})$, then $\mathcal{D}$ has the desired properties. Note that this example arises from a representation of G_{K_v} over Λ of rank r. It is also easy to arrange for this representation to be the restriction to G_{K_v} of such a representation of $\mathrm{Gal}(K_\Sigma/K)$ if $v \in \Sigma$.

The next result holds for any prime of K, archimedean or non-archimedean.

Proposition 5.2. *Let v be any prime of K. Let $\mathcal{D}$ be a cofinitely generated Λ-module. Assume that $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is Λ-reflexive. For almost all $P \in \mathrm{Spec}_{ht=1}(\Lambda)$, the map*

$$H^2(K_v, \mathcal{D}[P]) \to H^2(K_v, \mathcal{D})$$

is injective.

Proof. First assume that v is non-archimedean. We take P to be a prime ideal of height 1 in Λ. To prove injectivity of the map in question, we consider the adjoint map on the Pontryagin duals: $H^0(K_v, \mathcal{T}^*) \longrightarrow H^0(K_v, \mathcal{T}^*/P\mathcal{T}^*)$. If we let $X = \mathcal{T}^*$, then we must prove that the map $X^{G_{K_v}} \longrightarrow (X/PX)^{G_{K_v}}$ is surjective for all but finitely many P's. Let $Y = X^{G_{K_v}}$, the Pontryagin dual of $H^2(K_v, \mathcal{D})$.

According to proposition 3.5, both the kernel and cokernel of the map in question will be (Λ/P)-cotorsion for all but finitely many P's. Therefore, the same will be true for the adjoint map $Y/PY \to (X/PX)^{G_{K_v}}$. Let $Z = X/Y$. By assumption, Z is a reflexive Λ-module. Now we have an exact sequence of (Λ/P)-modules:

$$0 \to Y/PY \to X/PX \to Z/PZ \to 0 \tag{8}$$

and the image of $(X/PX)^{G_{K_v}}$ in Z/PZ is (Λ/P)-torsion. Since Z/PZ is a torsion-free (Λ/P)-module, it is clear that this image must be trivial, i.e. the map $Y/PY \to (X/PX)^{G_{K_v}}$ is surjective as we needed to prove.

Suppose now that v is a real prime of K. We again must prove the surjectivity of the adjoint map: $\widehat{H}^0(K_v, \mathcal{T}^*) \longrightarrow \widehat{H}^0(K_v, \mathcal{T}^*/P\mathcal{T}^*)$, involving the modified H^0's. But these Λ-modules are quotients of the Λ-modules $H^0(K_v, \mathcal{T}^*)$ and $H^0(K_v, \mathcal{T}^*/P\mathcal{T}^*)$ considered above. It follows that the adjoint maps will again be surjective for all but finitely many $P \in \mathrm{Spec}_{ht=1}(\Lambda)$. ■

Remark 5.2.1. The assumption that $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is a reflexive Λ-module is important. In the notation of the above proof, let's assume that $X = \mathcal{T}^*$ is itself reflexive, but that $Z = X/Y$ is not. Thus, the Krull dimension of Λ is at least 2. Let $\widetilde{Z}$ be the reflexive hull of the torsion-free Λ-module Z. Then $U = \widetilde{Z}/Z$ is nonzero. Corollary 2.5.1 asserts that there are infinitely many prime ideals $P = (\pi)$ of Λ such that $U[P] = U$. Since U is pseudo-null as a Λ-module, U is then a torsion (Λ/P)-module. Multiplication by π induces an isomorphism $U = \widetilde{Z}/Z \to \pi\widetilde{Z}/\pi Z$ which is a (Λ/P)-submodule of Z/PZ. Also, $Z/\pi\widetilde{Z}$ is a submodule of the (Λ/P)-module $\widetilde{Z}/\pi\widetilde{Z}$, which is torsion-free by proposition 2.6. Thus, the maximal torsion (Λ/P)-submodule of Z/PZ is isomorphic to U. Let $Z' = \pi\widetilde{Z}$ and let X' be the inverse image in X of Z' under the surjective map $X \to Z$. Then $Z'/PZ \cong U$ and we have an exact sequence derived from (8)

$$0 \to Y/PY \to X'/PX \to Z'/PZ \to 0$$

Since X/PX is a torsion-free (Λ/P)-module (by proposition 2.6) and the image of Y/PY is contained in $(X/PX)^{G_{K_v}}$, it follows that $X'/PX \subset (X/PX)^{G_{K_v}}$. Furthermore, excluding only finitely many P's, we can assume that the (Λ/P)-ranks of Y/PY and $(X/PX)^{G_{K_v}}$ are both equal to $\mathrm{rank}_\Lambda(Y)$. Then we have $X'/PX = (X/PX)^{G_{K_v}}$. It follows that the map $Y \to (X/PX)^{G_{K_v}}$ will not be surjective for such P's. The cokernel will be isomorphic to U. These considerations imply the following statement.

If $\mathcal{T}^*$ is reflexive, but $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is not reflexive as Λ-modules, then there exist infinitely many prime ideals $P \in \mathrm{Spec}_{ht=1}(\Lambda)$ such that the map $H^2(K_v, \mathcal{D}[P]) \to H^2(K_v, \mathcal{D})$ has a nonzero kernel.

The kernel of the map will be isomorphic to $\widehat{U}$ for infinitely many P's.

B. Almost divisibility of $H^1(K_v, \mathcal{D})$. Proposition 3.7 has the following consequence.

Proposition 5.3. *Suppose that v is a non-archimedean prime. If $\mathcal{D}$ is Λ-coreflexive and $H^2(K_v, \mathcal{D}) = 0$, then $H^1(K_v, \mathcal{D})$ is an almost divisible Λ-module.*

Here is a more general result. It follows from proposition 5.2 together with proposition 3.6.

Proposition 5.4. *Suppose that v is any prime of K. Assume that $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is reflexive as a Λ-module. If $\mathcal{D}$ is an almost divisible Λ-module, then $H^1(K_v, \mathcal{D})$ is an almost divisible Λ-module.*

Remark 5.2.1 makes it clear that the assumption concerning $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is crucial. The following proposition makes this more precise when $\mathcal{D}$ is Λ-coreflexive and v is non-archimedean.

Proposition 5.5. *Let v be a non-archimedean prime. Assume that $\mathcal{D}$ is Λ-coreflexive. Then the maximal pseudo-null Λ-submodule of $H^1(K_v, \mathcal{D})^\wedge$ is isomorphic to $\widetilde{Z}/Z$, where $\widetilde{Z}$ denotes the reflexive hull of the Λ-module $Z = \mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$.*

Proof. Let $U = \widetilde{Z}/Z$. Let U' denote the maximal pseudo-null Λ-submodule of $H^1(K_v, \mathcal{D})^\wedge$. There is nothing to prove unless Λ has Krull dimension at least 2. Applying corollary 2.5.1 to the pseudo-null Λ-module $U \times U'$, we see that there exist prime ideals $P = (\pi)$ of Λ such that $\pi U = 0$ and $\pi U' = 0$. We can also assume that P is not an associated prime for the Λ-torsion submodule of $H^1(K_v, \mathcal{D})^\wedge$. It follows that $H^1(K_v, \mathcal{D})^\wedge[P] = U'$. We therefore have an isomorphism

$$H^1(K_v, \mathcal{D})/\pi H^1(K_v, \mathcal{D}) \cong \ker\left(H^2(K_v, \mathcal{D}[P]) \longrightarrow H^2(K_v, \mathcal{D})[P]\right)$$

since $\mathcal{D}$ is assumed to be Λ-divisible. The choice of P implies that the first group is precisely the Pontryagin dual of U' and, as explained in remark 5.2.1, the second group is the Pontryagin dual of U. Thus, indeed, $U \cong U'$. ■

C. Divisibility of $H^1(K_v, \mathcal{D})$. It is rather common for $H^1(K_v, \mathcal{D})$ to be a divisible Λ-module. Proposition 3.3 gives sufficient conditions. The assumption that $H^2(K_v, \alpha) = 0$ for a G_{K_v}-irreducible subquotient α of the $\mathbb{F}_p$-representation space $\mathcal{D}[\mathfrak{m}_\Lambda]$ means that $H^0(K_v, \mathrm{Hom}(\alpha, \mu_p)) = 0$, or, equivalently, that $\alpha \not\cong \mu_p$. Thus, we need just assume that μ_p is not a subquotient of $\mathcal{D}[\mathfrak{m}_\Lambda]$ for the action of G_{K_v} to apply that proposition.

Proposition 5.6. *Suppose that v is non-archimedean. Assume that μ_p is not a G_{K_v}-subquotient of $\mathcal{D}[\mathfrak{m}_\Lambda]$ and that $\mathcal{D}$ is Λ-divisible. Then $H^1(K_v, \mathcal{D})$ is Λ-divisible.*

Even if μ_p is a subquotient of $\mathcal{D}[\mathfrak{m}_\Lambda]$, one can prove divisibility under other assumptions. Here is one such result.

Proposition 5.7. *Suppose that v is non-archimedean. Assume that $\mathcal{D}$ is Λ-coreflexive. Let $\mathcal{D}^* = \mathcal{T}^* \otimes_\Lambda \widehat{\Lambda}$. Assume that $H^0(K_v, \mathcal{D}^*)$ is a co-pseudo-null Λ-module. Then $H^1(K_v, \mathcal{D})$ is a divisible Λ-module.*

Note that the assumption about $H^0(K_v, \mathcal{D}^*)$ implies that $H^0(K_v, \mathcal{T}^*) = 0$ according to proposition 3.10, and hence that $H^2(K_v, \mathcal{D}) = 0$. Therefore, we already know that $H^1(K_v, \mathcal{D})$ is an almost divisible Λ-module.

Proof. Let $P = (\pi)$ be any prime ideal of Λ of height 1. Since $H^2(K_v, \mathcal{D}) = 0$, we must show that $H^2(K_v, \mathcal{D}[P]) = 0$ in order to conclude that $H^1(K_v, \mathcal{D})$ is divisible by π. (See (6) for $I = P$, $i = 2$.) Now $\mathcal{D}[P]$ is (Λ/P)-divisible and hence so is $H^2(K_v, \mathcal{D}[P])$. It therefore suffices to prove that its (Λ/P)-corank is 0. The Pontryagin dual of this group is $(\mathcal{T}^*/P\mathcal{T}^*)^{G_{K_v}}$. By proposition 3.10, the rank of this (Λ/P)-module is equal to the corank of the (Λ/P)-module $\big((\mathcal{T}^*/P\mathcal{T}^*) \otimes_{\Lambda/P} (\widehat{\Lambda/P})\big)^{G_{K_v}}$. As pointed out at the end of section 2, part **E**, the map $(\mathcal{T}^*/P\mathcal{T}^*) \otimes_{\Lambda/P} (\widehat{\Lambda/P}) \to \mathcal{D}^*[P]$ is a (Λ/P)-isogeny and so the submodules of G_{K_v}-invariant elements have the same (Λ/P)-coranks. Finally, note that $\mathcal{D}^*[P]^{G_{K_v}} = (\mathcal{D}^*)^{G_{K_v}}[P]$. The (Λ/P)-corank of this module is equal to 0 because the Pontryagin dual of the Λ-module $(\mathcal{D}^*)^{G_{K_v}}$ has no associated prime ideals of height 1. ∎

D. Coreflexivity of $H^1(K_v, \mathcal{D})$. Proposition 3.3 immediately gives one simple sufficient condition for coreflexivity.

Proposition 5.8. *Suppose that v is non-archimedean and that μ_p is not a G_{K_v}-subquotient of $\mathcal{D}[\mathfrak{m}_\Lambda]$. If $\mathcal{D}$ is Λ-coreflexive, then $H^1(K_v, \mathcal{D})$ is also Λ-coreflexive.*

A more subtle result is the following.

Proposition 5.9. *Suppose that v is non-archimedean. Assume that $\mathcal{D}$ is Λ-cofree. Let $\mathcal{D}^* = \mathcal{T}^* \otimes_\Lambda \widehat{\Lambda}$. Assume that every associated prime ideal for the Λ-module $H^0(K_v, \mathcal{D}^*)^\wedge$ has height at least 3. Then $H^1(K_v, \mathcal{D})$ is a coreflexive Λ-module.*

Proof. Let d denote the Krull dimension of Λ. Let $P \in \mathrm{Spec}_{ht=1}(\Lambda)$ be fixed. We will denote Λ/P by R' and $\mathcal{D}[P]$ by $\mathcal{D}'$. Thus, $\mathcal{D}'$ is a cofree R'-module. Since P is a principal ideal, the ring R' is a complete intersection and is therefore a Cohen-Macaulay ring. (See section 2.3 in [B-H].) It follows that R' contains a subring Λ' such that: *(i)* Λ' is isomorphic to a formal power series ring and *(ii)* R' is a free, finitely generated Λ'-module. The Krull dimension of Λ' is $d-1$. Note that $\mathcal{D}[P]$ is cofree and hence coreflexive as a Λ'-module. We will apply proposition 5.7 to this Λ'-module. For that purpose, the role of $\mathcal{T}^*$ is played by $\mathcal{T}'^* = \mathcal{T}^*/P\mathcal{T}^*$ and $\mathcal{D}^*$ by $\mathcal{D}'^* = \mathcal{T}'^* \otimes_{\Lambda'} \widehat{\Lambda}'$.

Since $\mathcal{T}^*$ is Λ-free, the discussion at the beginning of section 2, part **E**, shows that $\mathcal{T}'^* \otimes_{R'} \widehat{R}'$ is isomorphic to $\mathcal{D}^*[P]$ as an R'-module. Since R' is free as a Λ'-module, $\mathcal{T}'^* \otimes_{R'} \widehat{R}'$ is isomorphic to $\mathcal{T}'^* \otimes_{\Lambda'} \widehat{\Lambda}'$ and so $\mathcal{D}'^*$ and $\mathcal{D}^*[P]$ are isomorphic. The isomorphisms are G_{K_v}-equivariant. The assumption about $H^0(K_v, \mathcal{D}^*)$ implies that $H^0(K_v, \mathcal{D}'^*) = H^0(K_v, \mathcal{D}^*)[P]$ is co-pseudo-null as an R'-module, and hence as a Λ'-module. Therefore, proposition 5.7 implies that $H^1(K_v, \mathcal{D}')$ is Λ'-divisible, and hence R'-divisible. That is, $H^1(K_v, \mathcal{D}[P])$ is a divisible (Λ/P)-module.

Now we have a surjective homomorphism $H^1(K_v, \mathcal{D}[P]) \to H^1(K_v, \mathcal{D})[P]$. Therefore, for all $P \in \mathrm{Spec}_{ht=1}(\Lambda)$, the (Λ/P)-module $H^1(K_v, \mathcal{D})[P]$ is also divisible. Corollary 2.6.1 implies that $H^1(K_v, \mathcal{D})$ is indeed coreflexive as a Λ-module. ∎

Remark 5.9.2. An example that we have in mind in propositions 5.7 and 5.9 arises from classical Iwasawa theory over the local field K_v. Suppose that $K_{\infty,v}/K_v$ is a $\mathbb{Z}_p^m$-extension where $m \geq 1$. Let $\Lambda = \mathbb{Z}_p[[\mathrm{Gal}(K_{\infty,v}/K_v]]$. If $v \nmid p$, then one can only have $m = 1$, but if $v|p$, then m could be as large as $[K_v : \mathbb{Q}_p] + 1$. If $D = V/T$ is a G_{K_v}-module isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^n$, let $\mathcal{D} = \mathrm{Ind}_{K_{\infty,v}/K_v}(D)$. There is a comparison theorem just as stated in the introduction, but for a local field. We have that $H^0(K_v, \mathcal{D}^*)$ is isomorphic as a Λ-module to $H^0(K_{\infty,v}, D^*) = D^*(K_{\infty,v})$. This module has finite $\mathbb{Z}_p$-corank and is often even finite.

Assume first that $D^*(K_{\infty,v})$ is finite. Then the only associated prime ideal will be $\mathfrak{m}_\Lambda$. In that case, propositions 5.7 and 5.9 imply that $H^1(K_{\infty,v}, D)$ is a divisible Λ-module for $m \geq 1$ and even coreflexive for $m \geq 2$. If $D^*(K_{\infty,v})$ is infinite, then $H^1(K_{\infty,v}, D)$ is Λ-divisible if $m \geq 2$ and Λ-coreflexive if $m \geq 3$.

This is not a new result. See lemma 5.4 in [O-V], which even applies to non-abelian p-adic Lie extensions of a local field. Also, for the case $m = 1$, more precise results can be found in section 3 of [Gr89].

E. Cofreeness of $H^1(K_v, \mathcal{D})$. We can prove cofreeness under suitable assumptions. Let $\mathbb{Z}/p\mathbb{Z}$ denote the one-dimension $\mathbb{F}_p$-vector space with trivial Galois action.

Proposition 5.10. *Suppose that v is a non-archimedean prime and that neither $\mathbb{Z}/p\mathbb{Z}$ nor μ_p are G_{K_v}-subquotients of $\mathcal{D}[\mathfrak{m}_\Lambda]$. If $\mathcal{D}$ is a cofree Λ-module, then $H^1(K_v, \mathcal{D})$ is also a cofree*

Λ-*module.*

Proof. We can apply proposition 3.4 to conclude that the map

$$H^1(K_v, \mathcal{D}[\mathfrak{m}_\Lambda]) \longrightarrow H^1(K_v, \mathcal{D})[\mathfrak{m}_\Lambda]$$

is an isomorphism. The hypothesis about $\mathbb{Z}/p\mathbb{Z}$ nor μ_p means that for every G_{K_v}-irreducible subquotient α of the $\mathbb{F}_p$-representation space $\mathcal{D}[\mathfrak{m}_\Lambda]$, we have $H^0(K_v, \alpha) = H^2(K_v, \alpha) = 0$. Hence, by corollary 3.1.1, it follows that $H^0(K_v, \mathcal{D}) = 0$ and $H^2(K_v, \mathcal{D}) = 0$. We can then apply proposition 4.2 to determine the Λ-corank of $H^1(G_{K_v}, \mathcal{D})$, which will be either equal to 0 if $v \nmid p$ or equal to $[K_v : \mathbb{Q}_p]\mathrm{corank}_\Lambda(\mathcal{D})$ if $v|p$. However, we also have $H^0(K_v, \mathcal{D}[\mathfrak{m}_\Lambda]) = 0$ and $H^2(K_v, \mathcal{D}[\mathfrak{m}_\Lambda]) = 0$. The Euler-Poincaré characteristic formula for the finite G_{K_v}-module $\mathcal{D}[\mathfrak{m}_\Lambda]$ determines the $\mathbb{F}_p$-dimension of $H^1(K_v, \mathcal{D}[\mathfrak{m}_\Lambda])$. It will either equal 0 if $v\ nmidp$ or equal $[K_v : \mathbb{Q}_p]\dim_{\mathbb{F}_p}(\mathcal{D}[\mathfrak{m}_\Lambda])$ if $v|p$.

The hypothesis that $\mathcal{D}$ is a cofree Λ-module implies that $\mathrm{corank}_\Lambda(\mathcal{D}) = \dim_{\mathbb{F}_p}(\mathcal{D}[\mathfrak{m}_\Lambda])$. Thus, the above observations show that

$$\mathrm{corank}_\Lambda\big(H^1(G_{K_v}, \mathcal{D})\big) = \dim_{\mathbb{F}_p}\big(H^1(K_v, \mathcal{D}[\mathfrak{m}_\Lambda])\big) = \dim_{\mathbb{F}_p}\big(H^1(K_v, \mathcal{D})[\mathfrak{m}_\Lambda]\big)$$

We now use Nakayama's lemma. Let $r = \mathrm{corank}_\Lambda\big(H^1(G_{K_v}, \mathcal{D})\big)$. Let X be the Pontryagin dual of $H^1(G_{K_v}, \mathcal{D})$. Then X is a finitely generated Λ-module of rank r and the minimum number of generators of X is $\dim_{\mathbb{F}_p}(X/\mathfrak{m}_\Lambda X)$, which is also equal to r. Thus, there is a surjective Λ-module homomorphism $\Lambda^r \to X$. Comparing ranks, it is clear that this map is an isomorphism. Thus, X is free and so $H^1(G_{K_v}, \mathcal{D})$ is indeed cofree as a Λ-module. ■

Remark 5.10.1. If $v \nmid p$, then one could just assume that $\mathcal{D}$ is Λ-divisible. The assumption about $\mathbb{Z}/p\mathbb{Z}$ and μ_p implies that $H^i(G_{K_v}, \mathcal{D}) = 0$ for $i = 0$ and $i = 2$. Proposition 4.2 then implies that $H^1(G_{K_v}, \mathcal{D})$ is Λ-cotorsion. By proposition 5.6, $H^1(G_{K_v}, \mathcal{D})$ is also Λ-divisible and so we have $H^1(G_{K_v}, \mathcal{D}) = 0$, which is trivially Λ-cofree.

It is worthwhile to point out that the above proof applies with virtually no change if one assumes that $\mathcal{D}$ is a cofree R-module over a complete Noetherian local domain R. One concludes that, for any non-archimedean v, $H^1(G_{K_v}, \mathcal{D})$ is a cofree R-module under the same hypothesis about $\mathbb{Z}/p\mathbb{Z}$ nor μ_p.

F. Local assumptions *(a)* and *(b)*. Assume now that we are in the situation described in the introduction. Thus, $\mathcal{T}$ is a free R-module of rank n, $\mathcal{D} = \mathcal{D}_R$ is R-cofree, and $\mathcal{T}^*$ is R-free. We have several comments about the important assumption that $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is also R-free. For many results proven in this section, it suffices to assume that $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is Λ-reflexive, but we don't know how to verify such an assumption in itself. Freeness is more accessible.

As a first observation, note that if $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is a free R-module, then it follows that $\mathcal{T}^* \cong (\mathcal{T}^*)^{G_{K_v}} \oplus (\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}})$ as R-modules. Hence, $(\mathcal{T}^*)^{G_{K_v}}$ is a projective R-module and therefore must also be free. Let $r = \mathrm{rank}_R\big((\mathcal{T}^*)^{G_{K_v}}\big)$. It follows, furthermore, that the image of $(\mathcal{T}^*)^{G_{K_v}}$ in $\mathcal{T}^*/\mathfrak{m}\mathcal{T}^*$ will have dimension r over the residue field $k = R/\mathfrak{m}$. Conversely, if $(\mathcal{T}^*)^{G_{K_v}}$ is free of rank r and its image in $\mathcal{T}^*/\mathfrak{m}\mathcal{T}^*$ has dimension r, then $(\mathcal{T}^*)^{G_{K_v}}$ will be a direct summand of $\mathcal{T}^*$ and the complementary summand, which is isomorphic to $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ will also be R-free.

An important case to consider is $\mathcal{D} = \mathrm{Ind}_{K_\infty/K}(D)$, where K_∞/K is a $\mathbb{Z}_p^m$-extension and $D = V/T$ is a Galois module isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^n$ for some $n \geq 1$. In this case, T is a free $\mathbb{Z}_p$-module of rank n and $\mathcal{T}^* \cong T^* \otimes_{\mathbb{Z}_p} \Lambda$, a free Λ-module of rank n. We take $R = \Lambda$. The action of $\mathrm{Gal}(K_\Sigma/K)$ on Λ in the tensor product is given by a homomorphism $\kappa : \mathrm{Gal}(K_\Sigma/K) \to \Gamma \to \Lambda^\times$ as described in the introduction. Now if v is a prime of K which splits completely in K_∞/K, including, in particular, all archimedean primes, then $\kappa|_{G_{K_v}}$ is trivial. Thus, the action of G_{K_v} on $\mathcal{T}^*$ is via the first factor T^* in the tensor product. One sees easily that $(\mathcal{T}^*)^{G_{K_v}} \cong (T^*)^{G_{K_v}} \otimes_{\mathbb{Z}_p} \Lambda$ and $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}} \cong (T^*/(T^*)^{G_{K_v}}) \otimes_{\mathbb{Z}_p} \Lambda$. Since $T^*/(T^*)^{G_{K_v}}$ is a torsion-free $\mathbb{Z}_p$-module, it is $\mathbb{Z}_p$-free. This implies that $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is indeed a free Λ-module and hence local assumption *(a)* is satisfied if v splits completely. If v doesn't split completely in K_∞/K, then one can use remark 3.10.2 to verify that $(\mathcal{T}^*)^{G_{K_v}} = 0$.

In some cases, assumption *(a)* can be verify by considering just the residual representation $\overline{\rho}$. We illustrate this when $n = 2$. Thus, $\overline{\rho}$ is a 2-dimensional representation over the residue field k. There is nothing to show unless $\mathrm{rank}_R\big((\mathcal{T}^*)^{G_{K_v}}\big) = 1$ and so we assume this is the case. Suppose that $\overline{\rho}|_{G_{K_v}}$ is reducible and that the two $k^\times$-valued characters that occur are distinct. Then the same is true for $\overline{\rho}^*$ and so it follows that the k-subspace $(\mathcal{T}^*/\mathfrak{m}\mathcal{T}^*)^{G_{K_v}}$ of $\mathcal{T}^*/\mathfrak{m}\mathcal{T}^*$ has dimension 1 and that the action of G_{K_v} on the corresponding quotient is by a nontrivial character $\overline{\eta} : G_{K_v} \to k^\times$. One deduces easily that there exists a finite cyclic subgroup Δ of G_{K_v} such that $p \nmid |\Delta|$ and $\overline{\eta}|_\Delta$ is still nontrivial. Considering just the action of Δ on $\mathcal{T}^*$, we see that we have a direct sum decomposition

$$\mathcal{T}^* = (\mathcal{T}^*)^{\eta_o} \oplus (\mathcal{T}^*)^{\eta}$$

as R-modules, where η_o is the trivial character and η is a "lifting" of $\overline{\eta}$, both characters of Δ having values in $R^\times$. Since $(\mathcal{T}^*)^{G_{K_v}} \subseteq (\mathcal{T}^*)^\Delta = (\mathcal{T}^*)^{\eta_o}$ and $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is a torsion-free R-module, it follows that $(\mathcal{T}^*)^{G_{K_v}} = (\mathcal{T}^*)^{\eta_o}$, which is indeed a direct summand, verifying assumption *(a)*.

Note that if G_v acts on $\mathcal{T}^*$ through a finite quotient group Δ whose order is not divisible by p, then one has $(\mathcal{T}^*)^{G_{K_v}} = (\mathcal{T}^*)^\Delta$, which is again obviously a direct summand of $\mathcal{T}^*$. The idempotent e_o for the trivial character η_o of Δ is in the group ring $\mathbb{Z}_p[\Delta]$. One has $(\mathcal{T}^*)^{G_{K_v}} = e_o\mathcal{T}^*$ and the complementary direct summand is $(1 - e_o)\mathcal{T}^*$. In particular, assumption *(a)* is satisfied for archimedean primes if p is odd - an unimportant case because the groups $\widehat{H}^i(K_v, \mathcal{D})$ are then trivial.

Now suppose that v is a real prime of K and that $p = 2$. Otherwise, the corresponding cohomology groups are all trivial. Let σ_v denote the nontrivial element of G_{K_v}. Note that $\sigma_v(\zeta) = \zeta^{-1}$ for $\zeta \in \mu_{p^\infty}$. First assume that R has characteristic 0. Let $\alpha_v = \sigma_v - 1$ which we consider as an R-module endomorphism of $\mathcal{T}^*$. Thus $\ker(\alpha_v) = (\mathcal{T}^*)^{G_{K_v}}$ and so assumption *(a)* is equivalent to the statement that $\mathrm{im}(\alpha_v) = \alpha_v(\mathcal{T}^*)$ is R-free.

Let $\beta_v = \sigma_v + 1$ be the norm map on $\mathcal{T}^*$. The Pontryagin dual of $H^1(K_v, \mathcal{D})$ is $H^1(K_v, \mathcal{T}^*) = \ker(\beta_v)/\mathrm{im}(\alpha_v)$, a consequence of the local duality theorem but also easily verified directly from the definitions of these groups. Assume now that R is a finite, integral extension of Λ and is reflexive. Then $\mathcal{T}^*$ is a reflexive Λ-module. Since $\mathcal{T}^*/\ker(\beta_v)$ is a torsion-free Λ-module, it follows that $\ker(\beta_v)$ is reflexive and that $\mathrm{im}(\alpha_v)$ is reflexive if and only if $\ker(\beta_v)/\mathrm{im}(\alpha_v)$ has no nonzero pseudo-null Λ-submodules. That is, $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is a reflexive Λ-module if and only if $H^1(K_v, \mathcal{D})$ is an almost divisible Λ-module. Since this group has exponent 2, one can simply take $\theta = 2$ in the definition of almost divisibility, which then simply means that $H^1(K_v, \mathcal{D})$ is divisible when considered as a $\Lambda/(2)$-module.

It is easy to give an example where assumption *(a)* is not satisfied. Suppose that $R = \Lambda = \mathbb{Z}_2[[S]]$ and that $\mathcal{T}^* \cong \Lambda^2$. Suppose that σ_v acts on $\mathcal{T}^*$ by the matrix $\begin{bmatrix} -1 & S \\ 0 & 1 \end{bmatrix}$. Then $\mathrm{im}(\alpha_v)$ is isomorphic to $\mathfrak{m}_\Lambda$ and is not reflexive. Note that $H^1(K_v, \mathcal{D}) \cong \Lambda/\mathfrak{m}_\Lambda \cong \mathbb{F}_2$ in this example. We have just specified the action of G_{K_v}, but it is not hard to contrive a global representation ρ over Λ where G_{K_v} acts in this way.

Now assume that R has characteristic 2. Then $\alpha_v = \sigma_v - 1 = \beta_v$ and α_v^2 is the zero-map. We have $H^1(K_v, \mathcal{T}^*) \cong \ker(\alpha_v)/\mathrm{im}(\alpha_v)$. Using the notation from the introduction, we have

$$n_v^- = \mathrm{corank}_R(\mathcal{D}/\mathcal{D}^{G_{K_v}}) = \mathrm{rank}_R\big(\mathrm{im}(\alpha_v)\big)$$

since $\mathrm{im}(\alpha_v)$ is the orthogonal complement of $\mathcal{D}^{G_{K_v}}$ under the pairing $\mathcal{D} \times \mathcal{T}^* \to \mu_2$. If we define $n_v^+ = \mathrm{corank}_R\big(\mathcal{D}^{G_{K_v}}\big)$, then $n = n_v^+ + n_v^-$. Since $\mathrm{im}(\alpha_v) \subseteq \ker(\alpha_v)$, it follows that $n_v^- \le n_v^+$ and $H^1(K_v, \mathcal{T}^*)$ has R-rank equal to $n_v^+ - n_v^-$. Almost anything could occur subject to these constraints. One could simply define α_v so that $\mathrm{im}(\alpha_v) \subseteq \ker(\alpha_v)$. It could be any R-submodule of $\mathcal{T}^*$ which has a generating set of n elements and has R-rank at most $n/2$. This submodule could certainly fail to be R-free or Λ-reflexive. Note that $(1 + \alpha_v)^2$ is the identity map and so we can define an action of G_{K_v} on $\mathcal{T}^*$ (and hence on $\mathcal{D}$) by letting $\sigma_v = 1 + \alpha_v$.

Finally, we will discuss the verification of assumption *(b)*. Suppose that v_o is a non-archimedean prime in Σ. Since $\mathcal{T}^*$ is a torsion-free R-module, so is $(\mathcal{T}^*)^{G_{K_{v_o}}}$. Hence

$(\mathcal{T}^*)^{G_{K_{v_o}}} = 0$ if and only if its rank over R is equal to 0. According to remark 3.10.2, we have the inequality

$$\operatorname{rank}_R\big((\mathcal{T}^*)^{G_{K_{v_o}}}\big) \ \le\ \operatorname{rank}_{R_{\mathcal{P}}}\big((\mathcal{T}^*/\mathcal{P}\mathcal{T}^*)^{G_{K_{v_o}}}\big)$$

for every prime ideal $\mathcal{P}$ of R. Therefore, it suffices to find just one $\mathcal{P}$ such that $(\mathcal{T}^*/\mathcal{P}\mathcal{T}^*)^{G_{K_{v_o}}}$ has $(R/\mathcal{P})$-rank equal to 0, or equivalently, such that $\mathcal{D}^*[\mathcal{P}]^{G_{K_{v_o}}}$ has $(R/\mathcal{P})$-corank equal to 0. For example, this may occur for $\mathcal{P} = \mathfrak{m}$. In that case, one would have $(\mathcal{D}^*)^{G_{K_{v_o}}} = 0$. If the Krull dimension d of R is at least 2, then there are infinitely many prime ideals $\mathcal{P}$ of R of height $d-1$. Then $R/\mathcal{P}$ has Krull dimension 1. If $\mathcal{D}^{G_{K_{v_o}}}$ is indeed R-cotorsion, then remark 2.1.3 implies that $\mathcal{D}[\mathcal{P}]^{G_{K_{v_o}}} = \mathcal{D}^{G_{K_{v_o}}}[\mathcal{P}]$ is finite for infinitely many such $\mathcal{P}$'s. Exhibiting one such $\mathcal{P}$ is sufficient to verify assumption *(b)*.

6 Global Galois cohomology groups.

Assume that $\mathcal{D}$ is a cofinitely generated Λ-module and that $\mathrm{Gal}(K_\Sigma/K)$ acts Λ-linearly on $\mathcal{D}$, where Σ is a finite set of primes of K containing all primes above p and ∞. Let $\mathcal{T}^* = \mathrm{Hom}(\mathcal{D}, \mu_{p^\infty})$. This section will contain the proof of theorem 1. It will be a consequence of somewhat more general theorems. The heart of the matter is to study $H^2(K_\Sigma/K, \mathcal{D})$ and certain Λ-submodules obtained by requiring local triviality at some of the primes in Σ. The almost divisibility assertion in theorem 1 for $H^1(K_\Sigma/K, \mathcal{D})$ will follow easily.

A. The structure of $H^2(K_\Sigma/K, \mathcal{D})$ and certain submodules. Assume first that p is an odd prime. It is then known that $\mathrm{Gal}(K_\Sigma/K)$ has p-cohomological dimension 2 and so propositions 3.3 has the following immediate consequence.

Proposition 6.1. *Assume that p is an odd prime. If $\mathcal{D}$ is Λ-divisible, then $H^2(K_\Sigma/K, \mathcal{D})$ is Λ-divisible. If $\mathcal{D}$ is Λ-coreflexive, then $H^2(K_\Sigma/K, \mathcal{D})$ is Λ-coreflexive.*

We will prove a more general result. The arguments depend on the fundamental commutative diagram below. We assume that $\mathcal{D}$ is a cofinitely generated, divisible Λ-module. Suppose that Σ' is any subset of Σ. We make the following definition:

$$H^i_{\Sigma'}(K_\Sigma/K, \mathcal{D}) = \ker\big(H^i(K_\Sigma/K, \mathcal{D}) \to \prod_{v \in \Sigma'} H^i(K_v, \mathcal{D})\big)$$

for $i \ge 1$. Since $H^i_{\Sigma'}(K_\Sigma/K, \mathcal{D})$ is clearly a Λ-submodule of $H^i(K_\Sigma/K, \mathcal{D})$, it is also cofinitely generated. Note that if $\Sigma' = \Sigma$, then $H^i_{\Sigma'}(K_\Sigma/K, \mathcal{D}) = \text{Ш}^i(K, \Sigma, \mathcal{D})$. However, we will now assume from here on that there is at least one non-archimedean prime v_o in Σ which is not in Σ'. Thus Σ' will be a proper subset of Σ. We will also always make the assumption that $\mathcal{D}$ is a cofinitely generated, divisible Λ-module. Here is the fundamental diagram, where we take P to be any prime ideal of Λ of height 1.

$$\begin{tikzcd}
0 \arrow[r] & H^2_{\Sigma'}(K_\Sigma/K, \mathcal{D}[P]) \arrow[r] \arrow[d, dashed, "\delta"] & H^2(K_\Sigma/K, \mathcal{D}[P]) \arrow[r, "\sigma"] \arrow[d, "\kappa"] & \prod_{v\in\Sigma'} H^2(K_v, \mathcal{D}[P]) \arrow[r] \arrow[d, "\lambda"] & 0 \\
0 \arrow[r] & H^2_{\Sigma'}(K_\Sigma/K, \mathcal{D}) \arrow[r] \arrow[d, "\varphi"] & H^2(K_\Sigma/K, \mathcal{D}) \arrow[r] \arrow[d, "\chi"] & \prod_{v\in\Sigma'} H^2(K_v, \mathcal{D}) \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & H^2_{\Sigma'}(K_\Sigma/K, \mathcal{D}) \arrow[r] \arrow[d, "\epsilon"] & H^2(K_\Sigma/K, \mathcal{D}) \arrow[r] \arrow[d] & \prod_{v\in\Sigma'} H^2(K_v, \mathcal{D}) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & H^3(K_\Sigma/K, \mathcal{D}[P]) \arrow[r, "\tau"] & \prod_{v\in\Sigma'} H^3(K_v, \mathcal{D}[P]) \arrow[r] & 0
\end{tikzcd}$$

The 2nd and 3rd columns of maps in this diagram are induced by the exact sequence

$$0 \longrightarrow \mathcal{D}[P] \longrightarrow \mathcal{D} \xrightarrow{\pi} \mathcal{D} \longrightarrow 0$$

where we have chosen a generator π for P. Thus, those columns are certainly exact. The maps φ, χ and ψ are all just multiplication by π. As for the rows, the exactness of the last row is part of the Poitou-Tate theorems. (See [NSW], (8.6.13).) For the other rows, the only issue is the surjectivity of the global-to-local maps. This follows from the following general lemma since we are assuming that $\Sigma - \Sigma'$ contains at least one non-archimedean prime v_o.

Lemma 6.2. *Let v_o be any non-archimedean prime in Σ. Then the map*

$$H^2(K_\Sigma/K, \mathcal{D}) \longrightarrow \prod_{v\in\Sigma, v\neq v_o} H^2(K_v, \mathcal{D})$$

is surjective.

Proof. First consider the case where $\mathcal{D}$, and hence $\mathcal{T}^* = \mathrm{Hom}(\mathcal{D}, \mu_{p^\infty})$, are just finite $\mathrm{Gal}(K_\Sigma/K)$-modules. One has an exact sequence

$$H^2(K_\Sigma/K, \mathcal{D}) \xrightarrow{\gamma} P^2(K, \Sigma, \mathcal{D}) \xrightarrow{\alpha} H^0(K_\Sigma/K, \mathcal{T}^*)^\wedge,$$

where $P^2(K, \Sigma, \mathcal{D}) = \prod_{v\in\Sigma} H^2(K_v, \mathcal{D})$. The map γ is just the global-to-local restriction map. Let $\mathcal{G}$ denote its image. Let $\mathcal{H}_{v_o}$ denote the factor $H^2(K_{v_o}, \mathcal{D})$ in the product $P^2(K, \Sigma, \mathcal{D})$.

The assertion to be proved is that $\mathcal{G}\mathcal{H}_{v_o} = P^2(K,\Sigma,\mathcal{D})$. The map α is the adjoint of the "diagonal" map

$$\beta : H^0(K_\Sigma/K,\mathcal{T}^*) \to P^0(K,\Sigma,\mathcal{T}^*)$$

where $P^0(K,\Sigma,\mathcal{T}^*) = \prod_{v|\infty} \widehat{H}^0(K_v,\mathcal{T}^*) \times \prod_{v\in\Sigma, v\nmid\infty} H^0(K_v,\mathcal{T}^*)$. Since $\mathcal{G}$ is the kernel of the map α, its orthogonal complement is the image of β. The orthogonal complement of $\mathcal{H}_{v_o}$ is just the kernel of the natural projection map $\pi_{v_o} : P^0(K,\Sigma,\mathcal{T}^*) \to H^0(K_{v_o},\mathcal{T}^*)$. The assertion means that the intersection of these orthogonal complements is trivial. Since v_o is non-archimedean, the map $H^0(K_\Sigma/K,\mathcal{T}^*) \to H^0(K_{v_o},\mathcal{T}^*)$ is injective. That is, the composite map $\pi_{v_o} \circ \beta$ is injective. This implies that $\mathrm{im}(\beta') \cap \ker(\pi_{v_o}) = 0$ which proves the assertion. In general, $\mathcal{D}$ is a union of finite Galois modules, e.g. $\mathcal{D} = \bigcup_{n\geq 0} \mathcal{D}[\mathfrak{m}_\Lambda^n]$, and the surjectivity therefore follows in general. ■

It remains to discuss the maps δ and ϵ. Under the assumptions that we are making, the equality $\mathrm{im}(\varphi) = \ker(\epsilon)$ is established. It amounts to proving Λ-divisibility.

Proposition 6.3. *If $\mathcal{D}$ is a divisible Λ-module, then $H^2_{\Sigma'}(K_\Sigma/K,\mathcal{D})$ is a divisible Λ-module.*

Proof. We must show that φ is surjective. Applying the snake lemma to the 2nd and 3rd rows gives an exact sequence

$$\ker(\chi) \xrightarrow{a} \ker(\psi) \longrightarrow \mathrm{coker}(\varphi) \longrightarrow \mathrm{coker}(\chi) \xrightarrow{b} \mathrm{coker}(\psi)$$

Since σ is surjective, it follows that the map a is surjective too. Now τ is injective and so it follows that the map b is also injective. The exact sequence then implies that $\mathrm{coker}(\varphi) = 0$ as we want. ■

Finally, we consider the map δ in the fundamental diagram. The first two rows in that diagram can be rewritten as follows. We use the letters d, k and l for the vertical maps corresponding to δ, κ, and λ.

$$\begin{tikzcd}
0 \arrow[r] & H^2_{\Sigma'}(K_\Sigma/K,\mathcal{D}[P]) \arrow[r] \arrow[d, "d"] & H^2(K_\Sigma/K,\mathcal{D}[P]) \arrow[r, "\sigma"] \arrow[d, "k"] & \prod_{v\in\Sigma'} H^2(K_v,\mathcal{D}[P]) \arrow[r] \arrow[d, "l"] & 0 \\
0 \arrow[r] & H^2_{\Sigma'}(K_\Sigma/K,\mathcal{D})[P] \arrow[r] & H^2(K_\Sigma/K,\mathcal{D})[P] \arrow[r] & \prod_{v\in\Sigma'} H^2(K_v,\mathcal{D})[P] &
\end{tikzcd}$$

The maps k and l are surjective. Since k is surjective, the snake lemma gives us an exact sequence $\ker(l) \longrightarrow \mathrm{coker}(d) \longrightarrow 0$. We can now apply proposition 5.2 to deduce that d is at least sometimes surjective. If so, the first column of maps in the fundamental diagram will then be exact.

Proposition 6.4. *Assume that $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is a reflexive Λ-module for all $v \in \Sigma'$. Then, for almost all $P \in \mathrm{Spec}_{ht=1}(\Lambda)$, we have* $\mathrm{im}(\delta) = \ker(\varphi)$.

Proof. The assumption concerning $\mathcal{T}^*$ implies that $\ker(l) = 0$ for almost all prime ideals of Λ of height 1. It would then follow that $\mathrm{coker}(d) = 0$ and so d is indeed surjective for those P's. ■

We can apply this proposition to obtain the following important result.

Proposition 6.5. *Assume that $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is a reflexive Λ-module for all $v \in \Sigma'$. If $\mathcal{D}$ is a coreflexive Λ-module, then $H^2_{\Sigma'}(K_\Sigma/K, \mathcal{D})$ is also a coreflexive Λ-module.*

Proof. Excluding just finitely many prime ideals $P \in \mathrm{Spec}_{ht=1}(\Lambda)$, the stated assumptions imply the following statements: The map d will be surjective and $\mathcal{D}[P]$ will be a cofinitely generated, divisible (Λ/P)-module. Proposition 6.3 implies that $H^2_{\Sigma'}(K_\Sigma/K, \mathcal{D}[P])$ is (Λ/P)-divisible for all those P's. Therefore, its image $H^2_{\Sigma'}(K_\Sigma/K, \mathcal{D})[P]$ under the map d will also be (Λ/P)-divisible. Corollary 2.6.1 implies that $H^2_{\Sigma'}(K_\Sigma/K, \mathcal{D})$ is coreflexive. ■

The assumption about $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ in theorem 1 is due primarily to our need for that assumption in propositions 6.5. Since we assume that R is a cofree Λ-module, the assumption that $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is R-free implies that this module is also Λ-reflexive. The other local assumption in theorem 1 is made for the following simple reason. If $(\mathcal{T}^*)^{G_{K_{v_o}}} = 0$ for some non-archimedean prime $v_o \in \Sigma$, then we have $H^2(K_{v_o}, \mathcal{D}) = 0$. If we then let $\Sigma' = \Sigma - \{v_o\}$, it is clear that $\mathrm{Ш}^2(K, \Sigma, \mathcal{D}) = H^2_{\Sigma'}(K_\Sigma/K, \mathcal{D})$. We then can apply the above propositions to get the following result.

Proposition 6.6. *Assume that $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is Λ-reflexive for all $v \in \Sigma$ and that $(\mathcal{T}^*)^{G_{K_{v_o}}} = 0$ for at least one non-archimedean prime $v_o \in \Sigma$. If $\mathcal{D}$ is Λ-divisible, then $\mathrm{Ш}^2(K, \Sigma, \mathcal{D})$ is Λ-divisible. If $\mathcal{D}$ is Λ-coreflexive, then $\mathrm{Ш}^2(K, \Sigma, \mathcal{D})$ is Λ-coreflexive.*

Thus, all but the final statement is theorem 1 has been proven.

It is interesting to consider the case where Σ' is as small as possible - just the set of archimedean primes of K. We will then denote $H^2_{\Sigma'}(K_\Sigma/K, \mathcal{D})$ by $H^2_\infty(K_\Sigma/K, \mathcal{D})$. For any real prime v of K, we let σ_v denote the nontrivial element of G_{K_v}. Then propositions 6.3 and 6.5 give the following result. The content is the same as proposition 6.1 when $p \neq 2$. Note that the assumption about $(1 + \sigma_v)\mathcal{D}$ is true when p is odd and is equivalent to the assumption that $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is reflexive when $p = 2$.

Proposition 6.7. *If $\mathcal{D}$ is a divisible Λ-module, then $H^2_\infty(K_\Sigma/K, \mathcal{D})$ is a divisible Λ-module. If $\mathcal{D}$ is a coreflexive Λ-module and if $(1+\sigma_v)\mathcal{D}$ is also coreflexive for every real prime v of K, then $H^2_\infty(K_\Sigma/K, \mathcal{D})$ is a coreflexive Λ-module.*

B. The cokernel of γ. The Poitou-Tate duality theorems have some interesting consequences concerning the cokernel of the map $\gamma: \; H^2(K_\Sigma/K, \mathcal{D}) \longrightarrow P^2(K, \Sigma, \mathcal{D})$, the map whose kernel is $\text{Ш}^2(K, \Sigma, \mathcal{D})$. According to (6), $\text{coker}(\gamma)^\wedge \cong (\mathcal{T}^*)^{\text{Gal}(K_\Sigma/K)}$ which is a Λ-submodule of $\mathcal{T}^*$. If $\mathcal{T}^*$ is Λ-reflexive, then so is $(\mathcal{T}^*)^{\text{Gal}(K_\Sigma/K)}$. (See part **C** in section 2.) Furthermore, proposition 3.10 implies that the Λ-rank of $(\mathcal{T}^*)^{\text{Gal}(K_\Sigma/K)}$ is equal to the Λ-corank of $H^0(K_\Sigma/K, \mathcal{D}^*)$. These remarks give us the following results.

Proposition 6.8. *If $\mathcal{D}$ is Λ-divisible, then* $\text{coker}(\gamma)$ *is also Λ-divisible. If $\mathcal{D}$ is Λ-coreflexive, then* $\text{coker}(\gamma)$ *is also Λ-coreflexive.*

Proposition 6.9. *Assume that $\mathcal{D}$ is Λ-divisible and that $H^0(K_\Sigma/K, \mathcal{D}^*)$ is Λ-cotorsion. Then γ is surjective. In general, $H^0(K_\Sigma/K, \mathcal{D}^*)$ and* $\text{coker}(\gamma)$ *have the same Λ-corank.*

One simple case where γ is surjective is if $H^0(K_\Sigma/K, \mathcal{D}^*[\mathfrak{m}_\Lambda]) = 0$. Then, of course, $H^0(K_\Sigma/K, \mathcal{D}^*)[\mathfrak{m}_\Lambda] = 0$, and Nakayama's lemma implies that $H^0(K_\Sigma/K, \mathcal{D}^*) = 0$. Another important case is if $\mathcal{D}$ is induced from some $D = V/T$ via a $\mathbb{Z}_p^m$-extension K_∞/K, where $m \geq 1$. Then $H^0(K_\Sigma/K, \mathcal{D}^*) = H^0(K_\Sigma/K_\infty, D^*) = D^*(K_\infty)$ has finite $\mathbb{Z}_p$-corank and so is clearly Λ-cotorsion since the Krull dimension of Λ is greater than 1. More generally, if assumption *(b)* in theorem 1 holds, then, as we pointed out in the introduction, it follows that $H^0(K_\Sigma/K, \mathcal{T}^*) = 0$ and hence that γ is surjective.

C. The structure of $H^1(K_\Sigma/K, \mathcal{D})$. We now complete the proof of theorem 1. The hypotheses are somewhat broader and so we state this as a proposition.

Proposition 6.10. *Assume that $\mathcal{D}$ is Λ-coreflexive, that $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is Λ-reflexive for all $v \in \Sigma$, that $(\mathcal{T}^*)^{G_{K_{v_o}}} = 0$ for some non-archimedean $v_o \in \Sigma$, and that $\text{Ш}^2(K, \Sigma, \mathcal{D}) = 0$. Then $H^1(K_\Sigma/K, \mathcal{D})$ is an almost divisible Λ-module.*

Proof. The assertion will follow from proposition 3.6 if we show that κ is an injective map for almost all $P \in \text{Spec}_{ht=1}(\Lambda)$. We have an exact sequence

$$0 \longrightarrow \ker(\delta) \longrightarrow \ker(\kappa) \longrightarrow \ker(\lambda)$$

Proposition 5.2 implies that $\ker(\lambda) = 0$ for almost all $P \in \text{Spec}_{ht=1}(\Lambda)$. Thus it suffices to prove the same statement for $\ker(\delta)$.

If $\Sigma' = \Sigma - \{v_o\}$, then we have $H^2_{\Sigma'}(K_\Sigma/K, \mathcal{D}) = \text{Ш}^2(K, \Sigma, \mathcal{D}) = 0$. Hence, $\ker(\delta) = H^2_{\Sigma'}(K_\Sigma/K, \mathcal{D}[P])$. Now $\mathcal{D}[P]$ is a divisible (Λ/P)-module for all $P \in \text{Spec}_{ht=1}(\Lambda)$ and hence proposition 6.3 implies that $H^2_{\Sigma'}(K_\Sigma/K, \mathcal{D}[P])$ is also (Λ/P)-divisible. Therefore, it suffices to prove that $\ker(\delta)$ has (Λ/P)-corank equal to 0 for almost all $P \in \mathrm{Spec}_{ht=1}(\Lambda)$. It will then follow that $\ker(\delta) = 0$ and hence that κ is injective. Proposition 3.5 implies that the (Λ/P)-corank of $\ker(\kappa)$ is 0 for almost all $P \in \mathrm{Spec}_{ht=1}(\Lambda)$ and therefore the same must be true for the submodule $\ker(\delta) = 0$. This argument proves that, under the stated assumptions, $H^1(K_\Sigma/K, \mathcal{D})$ is indeed an almost divisible Λ-module. ■

It is worth pointing out that $H^1(K_\Sigma/K, \mathcal{D})$ is not necessarily a divisible Λ-module as the following proposition shows. It is not hard to find examples satisfying the hypotheses and where at least one of the local factors $H^1(K_v, \mathcal{D})$ for $v \in \Sigma'$ fails to be Λ-divisible.

Proposition 6.11. *Assume that $\mathcal{D}$ is Λ-divisible, that p is odd, that $H^2(K_v, \mathcal{D}) = 0$ for all nonarchimedean $v \in \Sigma'$, and that $H^2(K_\Sigma/K, \mathcal{D}) = 0$. Then the natural map*

$$H^1(K_\Sigma/K, \mathcal{D})/H^1(K_\Sigma/K, \mathcal{D})_{\Lambda-div} \longrightarrow \prod_{v \in \Sigma'} H^1(K_v, \mathcal{D})/H^1(K_v, \mathcal{D})_{\Lambda-div}$$

is surjective.

If A is a discrete Λ-module, then $A_{\Lambda-div}$ denotes the maximal Λ-divisible submodule of A. If $X = \widehat{A}$, then the Pontryagin dual of $A/A_{\Lambda-div}$ is isomorphic to the torsion Λ-submodule of X.

Proof. Referring to the two-row commutative diagram above, the map $\ker(k) \to \ker(l)$ is clearly surjective. That is, we have a surjective homomorphism

$$H^1(K_\Sigma/K, \mathcal{D})/PH^1(K_\Sigma/K, \mathcal{D}) \longrightarrow \prod_{v \in \Sigma'} H^1(K_v, \mathcal{D})/PH^1(K_v, \mathcal{D})$$

for *all* $P \in \mathrm{Spec}_{ht=1}(\Lambda)$. In general, suppose that A and B are two cofinitely generated, cotorsion Λ-modules and that $\psi : A \to B$ is a Λ-module homomorphism with the property that the induced map $A/PA \to B/PB$ is surjective for all $P \in \mathrm{Spec}_{ht=1}(\Lambda)$. This means that $\psi(A) + PB = B$ for all such P's. Let $C = \mathrm{coker}(\psi)$, which is also a cotorsion Λ-module. It follows that $\pi C = C$ for all irreducible elements of Λ. Thus C is a divisible Λ-module and so $C = 0$. This proves the proposition. ■

D. A discussion of hypothesis L. One natural way to verify hypothesis L for a given Galois module $\mathcal{D}$ is to show that the inequality in proposition 4.3, which gives a lower bound $b^1_\Lambda(K, \Sigma, \mathcal{D})$ on the Λ-corank of $H^1(K_\Sigma/K, \mathcal{D})$, is actually an equality. One can often verify this by specialization. For example, suppose that Λ is a formal power series over $\mathbb{Z}_p$ in m variables, where $m \geq 1$. Consider a cofree, cofinitely generated Λ-module $\mathcal{D}$ with Λ-corank n. Suppose that P is a prime ideal such that $\Lambda' = \Lambda/P$ is isomorphic to a formal power series ring over $\mathbb{Z}_p$ or $\mathbb{F}_p$ in m' variable, where $0 \le m' \le m$. (If $m' = 0$, we mean that $\Lambda' \cong \mathbb{Z}_p$ or $\mathbb{F}_p$. In the latter case, $P = \mathfrak{m}_\Lambda$.) Since Λ' is a regular local ring, remark 3.4.2 can be applied. If one can verify the equality $\mathrm{corank}_{\Lambda'}\big(H^1(K_\Sigma/K, \mathcal{D}[P])\big) = b^1_\Lambda(K, \Sigma, \mathcal{D})$ for one such prime ideal P, then hypothesis L for $\mathcal{D}$ would follow. Of course, it may happen $b^1_{\Lambda'}(K, \Sigma, \mathcal{D}[P]) > b^1_\Lambda(K, \Sigma, \mathcal{D})$, in which case, the equality would be impossible. However, remark 3.10.2 implies that there exists a nonzero ideal I of Λ such that $b^1_{\Lambda'}(K, \Sigma, \mathcal{D}[P]) = b^1_\Lambda(K, \Sigma, \mathcal{D})$ for all $P \notin V(I)$.

We will discuss various special cases and give examples where hypothesis L fails to be true. But it will be clear that these examples are rather special.

Elliptic curves. Suppose that E is an elliptic curve defined over K and that the Mordell-Weil group $E(K)$ has rank $r > [K : \mathbb{Q}]$. Let $s_K = r - [K : \mathbb{Q}]$. Let p be any prime number and let Σ be a finite set of primes of K containing all primes lying above p or ∞ and the primes where E has bad reduction. The Kummer map defines an injective homomorphism

$$E(K) \otimes_{\mathbb{Z}} (\mathbb{Q}_p/\mathbb{Z}_p) \to H^1(K_\Sigma/K, E[p^\infty])$$

It follows that $\mathrm{corank}_{\mathbb{Z}_p}\big(H^1(K_\Sigma/K, E[p^\infty])\big) \ \ge\ r$. In the notation of proposition 4.1, we have $\delta_{\mathbb{Z}_p}(K, E[p^\infty]) = [K : \mathbb{Q}]$. The Euler-Poincaré characteristic formula then implies that $\mathrm{corank}_{\mathbb{Z}_p}\big(H^2(K_\Sigma/K, E[p^\infty])\big) > 0$. But $H^2(K_v, E[p^\infty]) = 0$ for every non-archimedean prime v of K and is finite for the archimedean primes (trivial if $p > 2$). Hence it follows that $\mathrm{corank}_{\mathbb{Z}_p}\big(\text{Ш}^2(K, \Sigma, E[p^\infty])\big) > 0$. Thus hypothesis L fails if $R = \mathbb{Z}_p$ and $\mathcal{D} = E[p^\infty]$. This example corresponds to the representation ρ giving the action of $\mathrm{Gal}(K_\Sigma/K)$ on the Tate module $\mathcal{T} = T_p(E)$.

In this example, the Krull dimension of R is 1. However, one can simply extend scalars to obtain a *"constant"* deformation of $T_p(E)$ where R has arbitrary Krull dimension and hypothesis L still fails to be valid. For example, let $\mathcal{T} = T_p(E) \otimes_{\mathbb{Z}_p} \Lambda$, where Λ is a formal power series ring over $\mathbb{Z}_p$ in m variables. We assume that the Galois action on Λ is trivial. Define $\mathcal{D} = \mathcal{T} \otimes_\Lambda \widehat{\Lambda}$. If $m \ge 1$, there are infinitely many homomorphism $\phi : \Lambda \to \mathbb{Z}_p$ and one has $T_\phi \cong T_p(E)$, $D_\phi \cong E[p^\infty]$ for all such ϕ. It follows easily (by using lemma 4.4.1 for example) that $\mathrm{corank}_\Lambda\big(\text{Ш}^2(K, \Sigma, \mathcal{D})\big) = \mathrm{corank}_{\mathbb{Z}_p}\big(\text{Ш}^2(K, \Sigma, E[p^\infty])\big)$.

One natural deformation to consider is as described in the introduction. Suppose that K_∞/K is a $\mathbb{Z}_p^m$-extension, where $m \ge 1$, and let $\mathcal{D} = \mathrm{Ind}_{K_\infty/K}(E[p^\infty])$. It is known in certain cases that $\mathrm{rank}\big(E(K')\big)$ is unbounded as K' varies over the finite extensions of K contained in K_∞. One can find a discussion of this phenomenon in [M], [M-R], [Va], and [C-V], for example. To produce an example where Hypothesis L fails based on the above discussion, one would need $s_{K'} = \mathrm{rank}\big(E(K')\big) - [K' : \mathbb{Q}]$ to be unbounded above as K' varies. No such examples are known. It is hard to imagine that they could exist.

Suppose that $R = \Lambda \cong \mathbb{Z}_p[[T_1, ..., T_m]]$ and that $\mathcal{D}[P] \cong E[p^\infty]$ for some prime ideal P of Λ, as in the example in the previous paragraph. Note that both $b^1_\Lambda(K, \Sigma, \mathcal{D})$ and $b^1_{\mathbb{Z}_p}(K, \Sigma, E[p^\infty])$ equal $[K : \mathbb{Q}]$. Suppose further that $\mathrm{corank}_{\mathbb{Z}_p}\big(H^1(K_\Sigma/K, E[p^\infty])\big) = [K : \mathbb{Q}]$, i.e., that hypothesis L holds for $\mathcal{D}[P]$. Thus, by our initial remarks, hypothesis L would then hold for $\mathcal{D}$. One example where this happens is if $K = \mathbb{Q}$, $E(\mathbb{Q})$ has rank 1, and the p-primary subgroup of the Tate-Shafarevich group for $E/\mathbb{Q}$ is finite. (See [M-C] for a discussion of this case.)

As another example, suppose instead that $\mathcal{D}[P] \cong \mathrm{Ind}_{K_\infty^{cyc}/K}(E[p^\infty])$ for some prime ideal P of Λ, where K_∞^{cyc} denotes the cyclotomic $\mathbb{Z}_p$-extension of K. Assume also that E has ordinary reduction at all the primes of K lying above p. A conjecture of Mazur asserts that the p-Selmer group $\mathrm{Sel}_E(K_\infty^{cyc})$ for E over K_∞^{cyc} is a cotorsion module over $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty^{cyc}/K)]]$. Since $\text{Ш}^1(K_\infty^{cyc}, \Sigma, E[p^\infty]) \subseteq \mathrm{Sel}_E(K_\infty^{cyc})$, Mazur's conjecture would imply that $\text{Ш}^1(K_\infty^{cyc}, \Sigma, E[p^\infty])$ is also cotorsion. Now $E[p^\infty]^* \cong E[p^\infty]$ and so it would follow that conjecture L holds for $\mathcal{D}[P]$. It then would hold for $\mathcal{D}$. One special case is $\mathcal{D} = \mathrm{Ind}_{K_\infty/K}(E[p^\infty])$, where K_∞ is a $\mathbb{Z}_p^m$-extension of K containing K_∞^{cyc}.

A twist of $\mathbb{Q}_p/\mathbb{Z}_p$. Let K denote the maximal real subfield of $\mathbb{Q}(\mu_p)$. Assume that $p = 37$, an irregular prime. Let Σ be the set of primes of K lying above p and ∞. Let M_∞ be the maximal abelian pro-p-extension of K_∞^{cyc} which is unramified outside of Σ. Then it is known that $X = \mathrm{Gal}(M_\infty/K_\infty^{cyc}) \cong \mathbb{Z}_p$. The action of $\Gamma = \mathrm{Gal}(K_\infty^{cyc}/K)$ on X is given by a nontrivial homomorphism $\phi : \Gamma \to 1 + p\mathbb{Z}_p$. We define $\rho : \mathrm{Gal}(K_\Sigma/K) \to GL_1(\mathbb{Z}_p)$ to be the composition of ϕ with the restriction map $\mathrm{Gal}(K_\Sigma/K) \to \Gamma$. Thus the corresponding Galois module $\mathcal{D}$ is isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$ and $\mathrm{Gal}(K_\Sigma/K)$ acts via ρ. We denote this $\mathcal{D}$ by $(\mathbb{Q}_p/\mathbb{Z}_p)(\rho)$. Then we have

$$H^1(K_\Sigma/K, \mathcal{D}) \cong H^1(K_\Sigma/K_\infty^{cyc}, \mathcal{D})^\Gamma \cong \mathbb{Q}_p/\mathbb{Z}_p$$

The $\mathbb{Z}_p$-corank is 1. We have $\delta_{\mathbb{Z}_p}(K, \mathcal{D}) = 0$ and so it follows that $\mathrm{corank}_{\mathbb{Z}_p}\big(H^2(K_\Sigma/K, \mathcal{D})\big) > 0$. We again have $H^2(K_v, \mathcal{D}) = 0$ for all $v \in \Sigma$ and so, as in example 1, $\text{Ш}^2(K, \Sigma, \mathcal{D})$ fails to be a cotorsion module over $R = \mathbb{Z}_p$. Just as before, one can form a constant deformation of ρ over an arbitrary R to construct additional examples where hypothesis L also fails to hold. However, if instead one considers $\mathcal{D} = \mathrm{Ind}_{K_\infty^{cyc}/K}((\mathbb{Q}_p/\mathbb{Z}_p)(\rho))$, a cofree module over $R = \mathbb{Z}_p[[\Gamma]]$ of corank 1, then $H^1(K_\Sigma/K, \mathcal{D}) \cong \mathrm{Hom}(X, (\mathbb{Q}_p/\mathbb{Z}_p)(\rho))$, which is isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$ as a group and is a cotorsion R-module. Hypothesis L holds in this case.

Consider an arbitrary number field K. Let K_∞ denote the compositum of all $\mathbb{Z}_p$-extensions of K. Let $\Gamma = \mathrm{Gal}(K_\infty/K)$, which is isomorphic to $\mathbb{Z}_p^m$ for some $m \geq 1$. Let $\mathcal{D} = \mathrm{Ind}_{K_\infty/K}(D)$, where $D = \mu_{p^\infty}$. Thus D is simply the twist of $\mathbb{Q}_p/\mathbb{Z}_p$ by the cyclotomic character χ and $\mathcal{D}$ is a cofree module over $\Lambda = \mathbb{Z}_p[[\Gamma]]$ with corank 1. As we pointed out in the introduction, hypothesis L is true for $\mathcal{D}$ and $Ш^1(K, \Sigma, \mathcal{D}^*)$ is essential just the Pontryagin dual of the Galois group $Y = \mathrm{Gal}(L'_\infty/K_\infty)$. It is conjectured that Y is a pseudo-null module over Λ. Thus, $Ш^1(K, \Sigma, \mathcal{D}^*)$ should even be a co-pseudo-null Λ-module. However, this module can be nontrivial and it is conceivable that examples where hypothesis L fails can arise by specialization.

Suppose that P is a prime ideal of Λ which is an associated prime ideal for Y. Then $Ш^1(K, \Sigma, \mathcal{D}^*)[P]$ will have positive corank over Λ/P. Consider the map

$$Ш^1(K, \Sigma, \mathcal{D}^*[P]) \longrightarrow Ш^1(K, \Sigma, \mathcal{D}^*)[P]$$

Thus, for such a P, either the cokernel of this map or $Ш^1(K, \Sigma, \mathcal{D}^*[P])$ will have a positive (Λ/P)-corank. If it is the latter, then hypothesis L would fail to be true for the (Λ/P)-module $\mathcal{D}[P]$. Virtually nothing is known about the associated prime ideals of Y in general. One can construct examples where Y has an associated prime ideal P such that Λ/P is of characteristic 0 and has arbitrarily large Krull dimension. However, the construction is an imitation of classical genus theory and it is probably the cokernel of the above map which has positive (Λ/P)-corank. This example illustrates the subtlety of hypothesis L.

Characteristic p. Let R be a formal power series ring over $\mathbb{F}_p$ in any number of variables. Let Σ' be a finite set of primes of K containing the primes above p and ∞. Suppose that we have a representation $\rho : \mathrm{Gal}(K_{\Sigma'}/K) \to GL_n(R)$. Let $\mathcal{D}$ be the cofree R-module of corank n with Galois action given by ρ. We will make the following assumption: there exist infinitely many primes v of K such that *(i)* $\rho|_{G_v^{\Sigma'}}$ is trivial and *(ii)* $\mu_p \subset K_v$. Here $G_v^{\Sigma'}$ denotes the decomposition subgroup of $\mathrm{Gal}(K_{\Sigma'}/K)$ for any prime of $K_{\Sigma'}$ lying above v. For any prime v satisfying *(i)* and *(ii)*, it is clear that G_{K_v} acts trivially on $\mathcal{T}^* = \mathrm{Hom}(\mathcal{D}, \mu_p)$. Thus, the R-rank of $H_0(K_v, \mathcal{T}^*)$ is n and so the R-corank of $H^2(K_v, \mathcal{D})$ is equal to n. Suppose that $\Upsilon = \{v_1, ..., v_t\}$ is a set consisting of such primes. Let $\Sigma = \Sigma' \cup \Upsilon$. Then, by (7), we have the following inequality:

$$\mathrm{corank}_R\big(H^2(K_\Sigma/K, \mathcal{D})\big) \ \geq \ (t-1)n$$

If we assume that $H^0(K_{\Sigma'}/K, \mathcal{T}^*)$ is a torsion R-module, then the lower bound would be tn instead. It follows that the lower bound $b_R^1(K, \Sigma, \mathcal{D})$ on the R-corank of $H^1(K_\Sigma/K, \mathcal{D})$ is unbounded as $t \to \infty$.

Now let c' denote the R-corank of $\prod_{v \in \Sigma'} H^1(K_v, \mathcal{D})$. The definition of $H^1_{\Sigma'}(K_\Sigma/K, \mathcal{D})$ gives the following inequality:

$$\mathrm{corank}_R\big(H^1_{\Sigma'}(K_\Sigma/K, \mathcal{D})\big) \ \geq \ b_R^1(K, \Sigma, \mathcal{D}) - c'$$

We can make this corank positive by choosing a sufficiently large set Υ. We will assume that the primes in Υ do not lie over p. The elements of $H^1_{\Sigma'}(K_\Sigma/K, \mathcal{D})$ are locally trivial at all $v \in \Sigma'$, but could be nontrivial at the primes $v \in \Upsilon$. However, for each $v \in \Upsilon$, G_{K_v} acts trivially on $\mathcal{D}$. This module is just a vector space over $\mathbb{F}_p$ - a direct sum of copies of the trivial Galois module $\mathbb{Z}/p\mathbb{Z}$. Let L_v denote the maximal abelian extension of K_v such that $\mathrm{Gal}(L_v/K_v)$ has exponent p. Thus $[L_v : K_v] = p^2$. Every element of $H^1(K_v, \mathbb{Z}/p\mathbb{Z})$ becomes trivial when restricted to G_{L_v} and so the same thing is true for the elements of $H^1(K_v, \mathcal{D})$.

Choose a finite extension F of K such that, for each $v \in \Upsilon$ and for every prime η lying over v, the completion F_η contains L_v. We will also assume that F is chosen so that $F \cap K_\Sigma = K$. Such a choice is easily seen to be possible. Suppose that $\sigma \in H^1_{\Sigma'}(K_\Sigma/K, \mathcal{D})$. Let $\sigma|_F$ denote the image of σ under the restriction map $H^1(K_\Sigma/K, \mathcal{D}) \to H^1(F_{\Sigma_F}/F, \mathcal{D})$. Here Σ_F denotes the set of primes of F lying over those in Σ. This restriction map is easily seen to be injective. Then $\sigma|_F$ is locally trivial at *all* primes $\eta \in \Sigma_F$. That is, $\sigma|_F \in \text{Ш}^1(F, \Sigma_F, \mathcal{D})$. It follows that $\mathrm{corank}_R\big(\text{Ш}^1(F, \Sigma_F, \mathcal{D})\big)$ will be positive and so we do get examples where hypothesis L fails.

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