Practice Questions for the Final

A. Let σ be the following element in S_{10}:

$$\sigma \ = \ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 5 & 2 & 4 & 8 & 7 & 9 & 1 & 6 \end{pmatrix} \ .$$

(a) Find the cycle decomposition of σ.

(b) Does there exist an element $\tau \in S_9$ such that $\tau\sigma\tau^{-1} = \sigma^4$? If so, find such a τ. If not, explain why.

(c) Does there exist an element $\tau \in S_9$ such that $\tau\sigma\tau^{-1} = \sigma^6$? If so, find such a τ. If not, explain why.

B. Consider the element $\sigma = (1\ 3)(2\ 4)$ in S_4. Let $C(\sigma)$ denote the centralizer of σ in S_4. Determine $C(\sigma)$. (Hint: Problem 5 on the handout about Conjugacy might be helpful.)

C. Suppose that G is a group. Suppose that N is a normal subgroup of G and that $|N| = 2$. Prove that $N \subseteq Z(G)$.

D. Suppose that G is a group and that M and N are normal subgroups of G. Assume also that $M \cap N = \{e\}$, where e is the identity element in G. Suppose that $m \in M$ and $n \in N$. Prove that $mn = nm$.

E. Let $A = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. For each of the following groups G, determine if G has a subgroup isomorphic to A. Justify your answers fully.

$$G = S_3, \qquad G = S_4 \;, \qquad G = Q_8 \;,$$

$$G = D_4 \;, \qquad G = \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \;, \qquad G = \mathbf{Z}/48\mathbf{Z} \;.$$

F: Recall that $\mathbb{R}$ is a group under $+$ and that $\mathbb{Z}$ is a subgroup of $\mathbb{R}$.

(a) Explain why $\mathbb{Z}$ is a normal subgroup of $\mathbb{R}$.

(b) Show that $\mathbb{R}/\mathbb{Z}$ contains infinitely many elements of finite order.

(c) How many elements in $\mathbb{R}/\mathbb{Z}$ have order 7? How many elements have order 49?

(d) Show that $\mathbb{R}/\mathbb{Z}$ contains infinitely many elements of infinite order.

G. In this problem, suppose that G and G' are groups and that $\varphi : G \to G'$ is a homomorphism. Suppose that $a \in G$ and that $|a| = m$, where $m \geq 1$.

(a) Prove that $|\varphi(a)|$ divides m.

(b) Let $N = Ker(\varphi)$. Suppose that N is finite and that $gcd(m, \ |N|) = 1$. Prove that $|\varphi(a)| = m$.

(c) Give a specific example where $|a| = 25$ and $|\varphi(a)| = 5$. Justify your answer. (Note: You must specify G, G', φ, and a in your example.)