Problem Set 5. (due on Friday, March 8th)

A. Let $G = A \times B$, where A and B are groups. Define a map $\varphi : G \to B$ by

$$\varphi\big(\ (a,\ b)\ \big) \ = \ b$$

for all elements $(a,\ b) \in G$. Prove that φ is a surjective group homomorphism. Determine the kernel of φ.

B. Let $G = A \times A$, where A is a nonabelian group. Consider

$$H \ = \ \{\ (a,\ a) \ \big|\ a \in A\} \ \ .$$

Prove that H is a subgroup of G, but that H is not a normal subgroup of G. Prove that H is isomorphic to A. Does G have any normal subgroups which are isomorphic to A?

C. Suppose that G is a finite group and that M and N are normal subgroups of G. Suppose also $M \cap N = \{e\}$, where e is the identity element of G. Suppose also that $|G| = |N| \cdot |M|$. Consider the map $\varphi : G \to (G/M) \times (G/N)$ defined as follows:

$$\varphi(g) \ = \ \big(gM,\ gN\big)$$

for all $g \in G$. Prove that φ is an isomorphism from the group G to the group $(G/M)\times(G/N)$.

THERE ARE MORE PROBLEMS ON THE BACK

D. Let σ be the following element in S_9:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 5 & 1 & 8 & 9 & 7 & 6 \end{pmatrix} .$$

(a) Find the cycle decomposition of σ.

(b) Let $H = \langle \sigma \rangle$, the cyclic subgroup of S_9 generated by σ. Determine $|H|$ and $[S_9 : H]$.

(c) Does there exist an element $\tau \in S_9$ such that $\tau\sigma\tau^{-1} = \tau^3$? If so, find such a τ. If not, explain why.

(d) Does there exist an element $\tau \in S_9$ such that $\tau\sigma\tau^{-1} = \tau^2$? If so, find such a τ. If not, explain why.

(e) Determine the cardinality of the conjugacy class of σ in S_9.

E: Suppose that G is a group of order 35. We will prove in class that G must have at least one normal subgroup N of order 7. You may use that fact in this problem. Prove that if H is any subgroup of G such that $|H| = 7$, then $H = N$. (Thus, it follows that G has exactly one subgroup of order 7.)

F. Suppose that G is a finite, abelian group. Let $n = |G|$. Suppose that $k \in \mathbb{Z}$ and that $gcd(k,\ n) = 1$. Consider the map $\varphi : G \to G$ defined by

$$\varphi(g) = g^k$$

for all $g \in G$. Prove that φ is an automorphism of the group G.