PROBLEM SET 4 (due on Friday, March 1st)

A. Suppose that G is a group and that H is a subgroup of G such that $[G : H] = 2$. Suppose that $a, b \in G$, but $a \notin H$ and $b \notin H$. Prove that $ab \in H$.

B: This problem concerns the group $G = \mathbb{Q}/\mathbb{Z}$.

(a) Prove that every element of G has finite order.

(b) Prove that every finite subgroup of G is a cyclic group.

(c) Give a specific example of a proper subgroup H of G which is not finite.

(d) Prove that no proper subgroup of G can have finite index.

C: Suppose that G is a group and that N and M are normal subgroups of G.

TRUE OR FALSE: *If* $G/M \cong G/N$*, then* $M \cong N$*.*

If this statement is true, give a proof. If it is false, give a specific counterexample.

D: If G is an abelian group, then every subgroup of G is a normal subgroup. Is the converse of that fact true? If true, give a proof. If false, give a counterexample.

E: Suppose that G is a finite group and that N is a normal subgroup of G. Suppose also that G/N has an element of order m, where m is a positive integer. Carefully prove that G has an element of order m.

F: Suppose that A and B are groups. Let $G = A \times B$. Let e be the identity element of A and let f be the identity element of B. Then $(e,\ f)$ is the identity element in G. Let

$$H = \{\ (a,\ f) \ \big|\ a \in A\ \}\ .$$

Prove that H is a normal subgroup of G. Furthermore, prove that $H \cong A$ and that $G/H \cong B$.