Propositions about Isomorphisms

Definition. Suppose that A and B are groups. A map $\varphi : A \to B$ is called an *isomorphism* if φ is a bijection and has the property that $\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$ for all $a_1,\ a_2 \in A$. If such an isomorphism exists, we say that A *is isomorphic to* B.

In the following propositions, we will always assume that φ is an isomorphism from a group A to a group B. Let e_A and e_B denote the identity element of A and B, respectively.

1. We have $\varphi(e_A) = e_B$. Furthermore, if $a \in A$, then $\varphi(a^{-1}) = \varphi(a)^{-1}$.

2. If $a \in A$ and $k \in \mathbb{Z}$, then $\varphi(a^k) = \varphi(a)^k$.

3. If C is a subgroup of A, then $D = \varphi(C)$ is a subgroup of B. Furthermore, the groups C and D are isomorphic.

4. Suppose $a \in A$. Then $\varphi(\langle a \rangle) = \langle \varphi(a) \rangle$. Furthermore, $\varphi(a)$ has the same order as a.

5. Assume that A and B are cyclic groups, that $|A| = |B|$, that a is a generator of A, and that b is a generator of B. Then there exists an isomorphism $\varphi : A \to B$ such that $\varphi(a) = b$. For any $k \in \mathbb{Z}$, we have $\varphi(a^k) = b^k$.

Automorphisms

Definition. Suppose that A is a group. An isomorphism $\varphi : A \to A$ is called an *automorphism of* A.

6. Assume that A is a finite cyclic group. Let $n = |A|$. Suppose that $r \in \mathbb{Z}$ and that $gcd(r,\ n) = 1$. Define a map $\varphi : A \to A$ by $\varphi(x) = x^r$ for all $x \in A$. The map φ is an automorphism of A.

7. Assume that A is any group. Let a be a fixed element of A. Define a map $\varphi : A \to A$ by

$$\varphi(x) \ = \ axa^{-1}$$

for all $x \in A$. The map φ is an automorphism of A. (This type of automorphism of a group A is called an *inner automorphism of* A.)

Propositions about Conjugacy

Definition. Suppose that G is a group. Suppose that $x,\ y \in G$. We say that x and y are *conjugate in* G if there exists an element $a \in G$ such that $y = axa^{-1}$. We will write $x \sim_{G} y$ if x and y are conjugate in G.

1. The relation $\sim_{G}$ is an equivalence relation on the set G. Each equivalence class under this equivalence relation is called a *conjugacy class* in G.

2. If x and y are conjugate in G, then $|x| = |y|$.

3. A group G is abelian if and only if each conjugacy class consists of exactly one element.

4. An element $z \in G$ is in the center $Z(G)$ of G if and only if the set $\{z\}$ is a conjugacy class in G.