Quadratic Convergence of SQP-Like Methods for Convex-Composite Optimization

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Joint work with
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ICCOPT Berlin 2019
August 5–8
This research is partially funded by the US National Science Foundation under DMS1514559

Convex-Composite Optimization

$$\min_{x \in \mathbb{R}^n} f(x) := h(c(x)) \qquad (\mathbf{P})$$

$h : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is closed, proper, convex
$c : \mathbb{R}^n \to \mathbb{R}^m$ is $\mathcal{C}^2$-smooth

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$c : \mathbb{R}^n \to \mathbb{R}^m$ is $\mathcal{C}^2$-smooth — The Data

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used to induce solution properties

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used to induce solution properties

70's
Fletcher, Powel, Osborne

80-90's
Burke, Ferris, Fletcher, Kawasaki, Masden, Poliquin, Powel, Osborne, Rockafellar, Womersley, Wright, Yuan

Recent (15-19's)
Aravkin, Bell, B, Chang, Cui, Duchi, Davis, Drusvyatskiy, Hoheisel, Hong, Lewis, Ioffe, Mordukhovich, Pang, Ruan

Examples: 70 - 90's

Non-linear least-squares: $f(x) = \|c(x)\|_2^2$

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where $C \subset \mathbb{R}^m$ is non-empty, closed, convex, and $\operatorname{dist}(y \mid C) := \inf \{\|y - z\| \mid z \in C\}$.

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Here $c(x) := (\varphi(x), \hat{c}(x))$ and $h(\mu, y) := \mu + \alpha \operatorname{dist}(y \mid C)$

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Non-linear programming: $\min \varphi(x) + \delta_C(\hat{c}(x))$.
Here $c(x) := (\varphi(x), \hat{c}(x))$ and $h(\mu, y) := \mu + \delta_C(y)$, where
$\delta_C(y) = 0$ if $y \in C$ and $+\infty$ otherwise.

More Recent Examples

Optimal Value Composition:

$$h(c) := \min \left\{ b^\top y \mid Ay \leq c \right\}$$

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Piecewise linear-quadratic (PLQ) penalties:
(Rockfellar-Wets (97))
Quadratic support functions with $U \subset \mathbb{R}^k$ non-empty, closed and convex polyhedron.

PLQ penalties in practice

Application	Objective	PLQs
Regression	$\|Ax-b\|^2$	L_2
Robust regression	$\rho_H(Ax-b)$	Huber
Quantile regression	$Q(Ax-b)$	Asym. L_1
Lasso	$\|Ax-b\|^2+\lambda\|x\|_1$	L_2 + L_1
Robust lasso	$\rho_H(Ax-b)+\lambda\|x\|_1$	Huber + L_1
SVM	$\frac{1}{2}\|w\|^2+H(\mathbf{1}-Ax)$	L_1 + hinge loss
SVR	$\rho_V(Ax-b)$	Vapnik loss
Kalman smoother	$\|Gx-w\|^2_{Q^{-1}}+\|Hx-z\|^2_{R^{-1}}$	L_2 + L_2
Robust trend smoothing	$\|Gx-w\|_1+\rho_H(Hx-z)$	L_1 + Huber

The Convex-Composite Lagrangian

$$\mathbf{P} \qquad \min_{x\in\mathbb{R}^n} h(c(x))$$

- The Lagrangian for **P**: (B. (87))

$$L(x,y) := \langle y, c(x)\rangle - h^*(y)$$

- The conjugate of h given by the support function for epi(h),

$$h^*(y) := \sup_x [\langle y, x\rangle - h(x)] \qquad = \sup_{(x,\mu)\in \mathrm{epi}(h)} \langle (y,-1),(x,\mu)\rangle$$

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$$L(x,y) := \langle y, c(x)\rangle - h^*(y) \qquad \begin{cases} \text{(Primal)} & \inf_x \sup_y \; L(x,y) \\ \\ \text{(Dual)} & \sup_y \inf_x \; L(x,y) \end{cases}$$

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Algorithms

$$\mathbf{P}_k \qquad \min_x h\left(c(x^k)+\nabla c(x^k)[x-x^k]\right)+\frac{1}{2}(x-x^k)^\top H_k(x-x^k),$$

- H_k approximates the Hessian of a Lagrangian for $\mathbf{P}$ at (x^k, y^k)
- Newton's method: $H_k := \nabla^2_{xx} L(x^k, y^k) = \sum_{k=1}^m y_i^k \nabla^2_{xx} c_i(x^k)$
- $\mathbf{P}_k$ may or may not be convex depending on whether $H_k \succeq 0$.
- In the context of NLP, this reduces to SQP (sequential quadratic programming)

Convergence of Convex-Composite Newton's Method

Robinson (72):

Assumed $h = \delta_K$ with $K := \{0\}^s \times \mathbb{R}^{m-s}_-$ (NLP case).

> *Established quadratic convergence in the NLP case under linear independence of the active constraint gradients, strict complementarity, and strong second-order sufficiency.*

Robinson (80):

> *Introduced the revolutionary notion of* **generalized equations** *which, among many other consequences, re-established quadratic convergence for NLP. The generalized equations approach is much more powerful as it allows access to a very rich sensitivity theory including metric regularity properties of solution mappings.*

Convergence of Convex-Composite Newton's Method

Womersley (85):

Assumed h is finite-valued piecewise linear convex.

Established quadratic convergence under NLP-like conditions: LICQ, strict complementarity, and strong second-order sufficiency.

B-Ferris (95):

Assumed h is finite-valued closed, proper, convex.

Established quadratic convergence when $C := \arg\min h$ *is a set of weak sharp minima for* h*, and*

$$\arg\min f = \{x \mid c(x) \in C\}.$$

Only first-order information on c required.

Cibulka-Dontchev-Kruger (16):

Assumed h is piecewise linear convex (not nec.ly finite-valued).

Established super-linear convergence under the Dennis-Moré conditions using generalized equations.

The Program

A long standing open problem:

Establish second-order rates using the rich history of second-order ideas for convex-composite functions?

B(87), Kawazaki(88), Ioffe(88), B-Poliquin(92), Rochafellar-Wets(92), Nguyen(17-19)

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Inspiration: R. Cibulka, A. Dontchev, and A. Kruger (2017) arXiv:1701.02078.

Key new ingredient is *partial smoothness* (Lewis (02)).

PLQ Functions

$h : \mathbb{R}^m \to \overline{\mathbb{R}}$ is called piecewise linear-quadratic (PLQ) if $\operatorname{dom} h \neq \emptyset$ and, for $\mathcal{K} \geq 1$,

$$\operatorname{dom} h = \bigcup_{k=1}^{\mathcal{K}} C_k,$$

where the sets C_k are convex polyhedrons,

$$C_k = \{c \mid \langle a_{kj}, c\rangle \leq \alpha_{kj},\ \text{for all } j \in \{1, \ldots, s_k\}\},$$

and relative to which $h(c)$ is given by an expression of the form

$$h(c) = \frac{1}{2}\langle c, Q_k c\rangle + \langle b_k, c\rangle + \beta_k \quad \forall\ c \in C_k$$

with $\beta_k \in \mathbb{R}$, $b_k \in \mathbb{R}^n$, and $Q_k \in \mathbb{S}^m$.

Variational Analysis of PLQ-Composite Functions

Assume $f := h \circ c$ with h convex PLQ and c in $\mathcal{C}^2(\mathbb{R}^n)$.

Active Set: For $c \in \operatorname{dom} h$, the active set at c is

$$\mathcal{K}(c) := \{k \mid c \in C_k\}.$$

Basic Constraint Qualification: (BCQ)

$$\ker \nabla c(\bar{x})^\top \cap N_{\operatorname{dom} h}(c(\bar{x})) = \{0\}$$

Subdifferential: Under the BCQ

$$\partial f(x) = c'(x)^T \partial h(c(x)).$$

Directional Derivative: Under BCQ

$$f'(x;d) = \lim_{t \downarrow 0} \frac{f(x+td) - f(x)}{t} = h'(c(x); c'(x)d)$$

with

$$h'(\bar{c}; w) = \langle Q_k \bar{c} + b_k, w \rangle \quad \forall\ k \in \mathcal{K}(\bar{c}) \text{ and } w \in T_{C_k}(\bar{c}).$$

Directions of Non-Ascent and Multipliers

Directions of non-ascent:

$$\begin{aligned} D(x) &:= \left\{ d \in \mathbb{R}^n \;\middle|\; f'(x:d) \le 0 \right\} \\ &= \left\{ d \in \mathbb{R}^n \;\middle|\; h'(c(x); \nabla c(x) d) \le 0 \right\} \qquad \text{(BCQ)} \end{aligned}$$

The Multiplier Set:

$$M(\bar{x}) := \ker \nabla c(\bar{x})^\top \cap \partial h(c(\bar{x})) = \left\{ y \;\middle|\; \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} \partial_x L(\bar{x}, y) \\ \partial_y(-L)(\bar{x}, y) \end{pmatrix} \right\}$$

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Strict Criticality (SC):

$$\ker \nabla c(\bar{x})^\top \cap \operatorname{ri}\left(\partial h(c(\bar{x}))\right) = \{\bar{y}\}$$

Implied by "strict complementarity and LICQ".
Under SC, $D(\bar{x})$ is a subspace on which $h'(c(x); \nabla c(x) d) = 0$.

The Second Directional Derivative

The PLQ second directional derivative:
(Rockafellar-Wets (97))

$$0 \le h''(\bar{c}; w) := \lim_{t \searrow 0} \frac{h(\bar{c} + tw) - h(\bar{c}) - th'(\bar{c}; w)}{\frac{1}{2}t^2}$$
$$= \begin{cases} \langle w, Q_k w \rangle & \text{when } w \in T_{C_k}(\bar{c}), \\ \infty & \text{when } w \notin T_{\operatorname{dom} h}(\bar{c}). \end{cases}$$

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and $h''(\bar{c}; \cdot)$ is PLQ, but not necessarily convex.

Moreover, there exists a neighborhood V of $\bar{c}$ such that

$$h(c) = h(\bar{c}) + h'(\bar{c}; c - \bar{c}) + \frac{1}{2}h''(\bar{c}; c - \bar{c}) \text{ for } c \in V \cap \operatorname{dom} h.$$

PLQ-Composite 2nd-Order Nec. and Suff. Conditions

(Rockafellar-Wets (97))

Let $\bar{x} \in \operatorname{dom} f$ such that f satisfies BCQ at $\bar{x}$.

(1) (Nec.) If f has a local minimum at $\bar{x}$, then
$0 \in \nabla c(\bar{x})^\top \partial h(c(\bar{x}))$ and, $\forall\ d \in D(\bar{x})$,

$$h''(c(\bar{x}); \nabla c(\bar{x})d) + \max\left\{\left\langle d, \nabla^2_{xx} L(\bar{x}, y)d\right\rangle \mid y \in M(\bar{x})\right\} \geq 0 \ .$$

(2) (Suff.) If $0 \in \nabla c(\bar{x})^\top \partial h(c(\bar{x}))$ and, $\forall\ d \in D(\bar{x}) \setminus \{0\}$,

$$h''(c(\bar{x}); \nabla c(\bar{x})d) + \max\left\{\left\langle d, \nabla^2_{xx} L(\bar{x}, y)d\right\rangle \mid y \in M(\bar{x})\right\} > 0,$$

then $\bar{x}$ is a strong local minimizer of f,
that is, there exists $\varepsilon > 0, \mu > 0$ such that

$$f(x) \geq f(\bar{x}) + \frac{\mu}{2} \|x - \bar{x}\|_2^2 \quad \forall\ x \in B(\bar{x}, \varepsilon).$$

Convex-Composite Generalized Equations

Let $f := h \circ c$ be convex-composite, and define the set-valued mapping $g + G : \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^{n+m}$ by

$$g(x, y) = \begin{pmatrix} \nabla c(x)^\top y \\ -c(x) \end{pmatrix}, \quad G(x, y) = \begin{pmatrix} \{0\}^n \\ \partial h^*(y) \end{pmatrix}.$$

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The associated generalized equation for $\mathbf{P}$ is $0 \in g + G$.

For a fixed $(\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^m$, define the linearization mapping

$$\mathcal{G} : (x,y) \mapsto g(\bar{x},\bar{y}) + \nabla g(\bar{x},\bar{y}) \begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix} + G(x,y),$$

where $\nabla g(\bar{x},\bar{y}) = \begin{pmatrix} \nabla^2(\bar{y}c)(\bar{x}) & \nabla c(\bar{x})^\top \\ -\nabla c(\bar{x}) & 0 \end{pmatrix}.$

Newton's Method for Generalized Equations

- Let $f := h \circ c$ be convex-composite.
- For $(\hat{x}, \hat{y}) \in \mathbb{R}^n \times \mathbb{R}^m$ set $\widehat{H} := \nabla^2_{xx} L(\hat{x}, \hat{y})$.
- Assume f satisfies BCQ at $\hat{x}$.

Then, $(\tilde{x}, \tilde{y}) \in \mathbb{R}^n \times \mathbb{R}^m$ satisfy the optimality conditions for

$$\min_{x \in \mathbb{R}^n} h(c(\hat{x}) + \nabla c(\hat{x})(x - \hat{x}) + \frac{1}{2}(x - \hat{x})^\top \widehat{H}(x - \hat{x})$$

if and only if $(\tilde{x}, \tilde{y})$ solves the Newton equations for $g + G$:

$$0 \in g(\hat{x}, \hat{y}) + \nabla g(\hat{x}, \hat{y}) \begin{pmatrix} x - \hat{x} \\ y - \hat{y} \end{pmatrix} + G(x, y).$$

Strong Metric Subregularity

A set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *strongly metrically subregular* at $\bar{u}$ for $\bar{v}$ if $(\bar{u}, \bar{v}) \in \operatorname{graph}(S)$ and there exists $\kappa \geq 0$ and a neighborhood U of $\bar{u}$ such that

$$\|u - \bar{u}\| \leq \kappa \operatorname{dist}(\bar{v} \mid S(u)) \text{ for all } u \in U.$$

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Theorem: (B-Engel(18)) $h : \mathbb{R}^m \to \overline{\mathbb{R}}$ convex PLQ and $f := h \circ c$ satisfies BCQ at $\bar{x} \in \operatorname{dom} f$. Then, the following are equivalent:

(1) The multiplier set $M(\bar{x}) := \ker \nabla c(\bar{x})^\top \cap \partial h(c(\bar{x}))$ is a singleton $\{\bar{y}\}$ and the second-order sufficient conditions are satisfied at $\bar{x}$.

(2) The mapping $g + G$ is strongly metrically subregular at $(\bar{x}, \bar{y})$ for 0 and $\bar{x}$ is a strong local minimizer of f.

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Corollary: The matrix secant method converges superlinearly if the Dennis-Móre condition holds.

Partial Smoothness: Lewis (02)

- $h : \mathbb{R}^m \to \overline{\mathbb{R}}$ is a closed and proper function.
- $\mathcal{M}$ a $\mathcal{C}^2$-smooth manifold and $\bar{c} \in \mathcal{M} \subset \mathbb{R}^m$.

The function h is *partly smooth* at $\bar{c}$ relative to $\mathcal{M}$ if $\mathcal{M}$ the following four properties hold:

(1) (Restricted Smoothness) The restriction $h|_{\mathcal{M}}$ is smooth around $\bar{c}$, in that there exists a neighborhood V of $\bar{c}$ and a $\mathcal{C}^2$-smooth function g defined on V such that $h = g$ on $V \cap \mathcal{M}$;

(2) (Existence of Subgradients) At every point $c \in \mathcal{M}$ close to $\bar{c}$, $\partial h(c) \neq \emptyset$;

(3) (Normals and Subgradients Parallel) $\text{par}\,\partial h(\bar{c}) = N_{\mathcal{M}}(\bar{c})$;

(4) (Subgradient Continuity) the subdifferential map ∂h is inner semicontinuous at $\bar{c}$ relative to $\mathcal{M}$.

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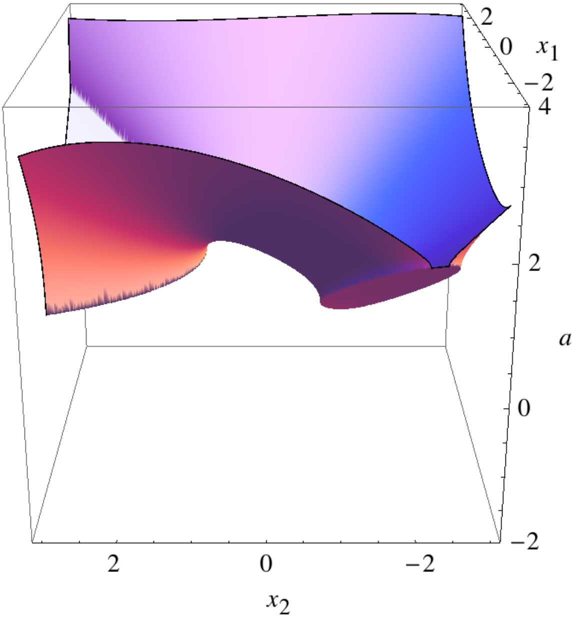
Generalizes classical notions of *nondegeneracy, strict complementarity, and active constraint identification.*

Partial Smoothness



The Active Manifold

- $\mathcal{M}$ Active set: $\mathcal{K}(c) := \{k \in \mathbb{R}^m \mid c \in C_k,\ k \in \{1, 2, \ldots, \mathcal{K}\}\}$

- Active Manifold: $\mathcal{M}_{\bar{c}} := \operatorname{ri} \bigcap_{k \in \mathcal{K}(\bar{c})} C_k$

Lemma: Let $\bar{c} \in \operatorname{dom} f$ and assume $\operatorname{dom} h$ is given by an RWR. Then, for all $c \in \mathcal{M}_{\bar{c}}$ and $k \in \mathcal{K}(\bar{c})$,

$$\mathcal{K}(c) = \mathcal{K}(\bar{c}),\ \mathcal{M}_c = \mathcal{M}_{\bar{c}} \text{ and } I_k(c) = I_k(\bar{c}).$$

The Subdifferential of h

Given that a certain *nondegeneracy* condition holds (a property of the representation of $\operatorname{dom} h$), then $\partial h(c)$ has a *structure functional* representation (Osborne (01)).

The Subdifferential of h

Given that a certain *nondegeneracy* condition holds (a property of the representation of $\operatorname{dom} h$), then $\partial h(c)$ has a *structure functional* representation (Osborne (01)).

Lemma: Let $c \in \mathcal{M}_{\bar{c}}$ and suppose nondegeneracy holds. Then there is a polyhedral convex set $\mathcal{U}(c)$ and a matrix $\bar{A}$ such that, for every $y \in \partial h(c)$, there is a unique $\mu(c, y) \in \mathcal{U}(c)$ for which $y = \lambda_0(c) + \bar{A}\mu(c, y)$.
In particular,

$$\partial h(c) = \lambda_0(c) + \bar{A}\,\mathcal{U}(c).$$

Newton's Method Hypotheses

Let $f = h \circ c$ be PLQ convex composite, $\bar{x} \in \operatorname{dom} f$, $\bar{y} \in \partial h(c(\bar{x}))$, and set $\bar{c} := c(\bar{x})$.

Assumptions:

(a) c is $\mathcal{C}^3$-smooth,

(b) $\mathcal{M}_{\bar{c}}$ satisfies the nondegeneracy condition,

(c) f satisfies SC at $\bar{x}$ for $\bar{y}$,

(d) $\bar{x}$ satisfies the second-order sufficient conditions, i.e.,
$$h''(c(\bar{x}); \nabla c(\bar{x})d) + \left\langle d, \nabla^2_{xx} L(\bar{x}, \bar{y})d \right\rangle > 0 \quad \forall\, d \in \ker A^\top \nabla c(\bar{x}) \setminus \{0\},$$
where $M(\bar{x}) = \{\bar{y}\}$ and $D(\bar{x}) = \ker A^\top \nabla c(\bar{x})$.

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NLP Analogues:

(b) linear independence of the active constraint gradients,

(c) strict complementary slackness, and

(d) strong second-order sufficiency condition.

Convergence of Newton's Method

There exists a neighborhood $\mathcal{N}$ of $(\bar{x}, \bar{y})$ such that if $(x^0, y^0) \in \mathcal{N}$, then there exists a unique sequence $\{(x^k, y^k)\}$ satisfying the optimality conditions of $\mathbf{P_k}$ with $H_k := \nabla^2_{xx} L(x^k, y^k)$ such that, for all $k \in \mathbb{N}$,

(i) $c(x^{k-1}) + \nabla c(x^{k-1})[x^k - x^{k-1}] \in \mathcal{M}_{\bar{c}}$,

(ii) $y^k \in \operatorname{ri} \partial h(c(x^{k-1}) + \nabla c(x^{k-1})[x^k - x^{k-1}])$,

(iii) $H_{k-1}[x^k - x^{k-1}] + \nabla c(x^{k-1})^\top y^k = 0$,

(iv) x^{k+1} is a strong local minimizer of $\mathbf{P_k}$.

Moreover, the sequence (x^k, y^k) converges to $(\bar{x}, \bar{y})$ at a quadratic rate.