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A ROBUST TRUST REGION METHOD FOR CONSTRAINED NONLINEAR PROGRAMMING PROBLEMS*

JAMES V. BURKE†

Abstract. Most of the published work on trust region algorithms for constrained optimization is derived from the original work of Fletcher on trust region algorithms for nondifferentiable exact penalty functions. These methods are restricted to applications where a reasonable estimate of the magnitude of an optimal Kuhn–Tucker multiplier vector can be given. More recently an effort has been made to extend the trust region methodology to the sequential quadratic programming (SQP) algorithm of Wilson, Han, and Powell. All of these extensions to the Wilson–Han–Powell SQP algorithm consider only the equality-constrained case and require strong global regularity hypotheses. This paper presents a general framework for trust region algorithms for constrained problems that does not require such regularity hypotheses and allows very general constraints. The approach is modeled on the one given by Powell for convex composite optimization problems and is driven by linear subproblems that yield viable estimates for the value of an exact penalty parameter. These results are applied to the Wilson–Han–Powell SQP algorithm and Fletcher's $S\ell_1QP$ algorithm. Local convergence results are also given.

Key words. trust regions, constrained optimization, exact penalty functions

AMS(MOS) subject classifications. 90C30, 65K05

1. Introduction. Consider the constrained nonlinear programming problem

$$\begin{aligned}\mathcal{P}: &\text{minimize } f(x)\\ &\text{subject to } x \in \Omega,\end{aligned}$$

where $\Omega := \{x \in X : g(x) \in C\}$, $X \in \mathbb{R}^n$ and $C \subset \mathbb{R}^m$ are nonempty closed convex sets, and $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are Frechet differentiable on an open set U containing X where the Frechet derivatives $f' : \mathbb{R}^n \to \mathbb{R}^n$ and $g' : \mathbb{R}^n \to \mathbb{R}^{m\times n}$ are bounded and continuous on X.

If $C = \mathbb{R}^s_- \times \{0\}_{\mathbb{R}^{m-s}}$ and

$$X := \{x \in \mathbb{R}^n : \underline{z}_i \le x_i \le \overline{z}_i, \quad i = 1, \cdots, n\}$$

where $\overline{z}_i$, $\underline{z}_i \in \mathbb{R} \cup \{\pm\infty\}$ for each $i = 1, \cdots, n$ with $\underline{z}_i \le \overline{z}_i$, $\underline{z}_i \ne +\infty$, and $\overline{z}_i \ne -\infty$ for $i = 1, \cdots, n$, then $\mathcal{P}$ is said to be in standard form. In general, the set X is considered to be some "simple" set of constraints so that the inclusion $x \in X$ is easily maintained.

In this paper we describe a framework for the development of robust trust region methods for solving $\mathcal{P}$. By "robust" we mean that the global convergence theory for these methods does not require assumptions concerning the regularity or the feasibility of $\mathcal{P}$. This is accomplished by designing the algorithm to locate stationary points for the problem

$$\begin{aligned}\widehat{\mathcal{P}}: &\text{minimize } f(x)\\ &\text{subject to } x \in \arg\min\{\overline{\varphi}(x) : x \in \mathbb{R}^n\},\end{aligned}$$

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†Department of Mathematics, GN-50, University of Washington, Seattle, Washington 98195.
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where

$$\arg\min\{\overline{\varphi}(x) : x \in \mathbb{R}^n\} := \{\overline{x} \in \mathbb{R}^n : \overline{\varphi}(\overline{x}) = \min\{\overline{\varphi}(x) : x \in \mathbb{R}^n\}\},$$

and

$$\overline{\varphi}(x) := \operatorname{dist}(g(x)|C) + \psi(x|X) \tag{1.1}$$

with

$$\operatorname{dist}(y|C) := \inf\{\|y - z\| : z \in C\} \tag{1.2}$$

and

$$\psi(x|X) := \begin{cases} 0 & \text{if } x \in X, \\ +\infty & \text{if } x \notin X \end{cases} \tag{1.3}$$

(here and throughout, the symbols $\|\cdot\|$ denote a given norm on $\mathbb{R}^n$ or $\mathbb{R}^m$). Clearly, if $\mathcal{P}$ is feasible, then $\mathcal{P}$ and $\widehat{\mathcal{P}}$ are equivalent. On the other hand, if $\mathcal{P}$ is not feasible, then further information about $\mathcal{P}$ can be obtained by studying $\widehat{\mathcal{P}}$. In [1], Burke introduces a notion of stationarity for $\widehat{\mathcal{P}}$ which will be reviewed in the next section. Burke [1] also discusses an algorithm for locating points that are stationary for $\widehat{\mathcal{P}}$. This algorithm extends the well-known SQP method of Wilson [28], Han [13], and Powell [17]. The plan of this paper is to extend the techniques of [1] to the trust region framework and then to apply these results to both the $S\ell_1QP$ algorithm of Fletcher [11], [12] and to a trust region implementation of the Wilson–Han–Powell SQP method.

Many other authors have considered trust region algorithms for constrained optimization. One can broadly classify this work into three categories: (1) methods for linear constraints, (2) methods for nonlinear equality constraints, and (3) exact penalization methods. The first class of methods is studied in Conn, Gould, and Toint [9]; Toint [26]; Moré [14]; and Burke, Moré, and Toraldo [5]. This class of methods corresponds to the case of $\mathcal{P}$ with the functional constraint $g(x) \in C$ absent, and is based on projected gradient techniques. The second class of methods concentrates on the instance of $\mathcal{P}$ where $C = \{0\}_{\mathbb{R}^m}$ and $X = \mathbb{R}^n$ and these methods can be viewed as extensions to the Wilson–Han–Powell SQP method. These methods are studied in Celis, Dennis, and Tapia [7]; Vardi [27]; Byrd, Schnabel, and Shultz [6]; and Powell and Yuan [18]. All of these papers require $g'(x)$ to be of full rank on $\mathbb{R}^n$. Under this hypothesis, the method of Celis, Dennis, and Tapia [7] has recently been provided with a convergence theory by El-Alem [10]. The methods of Vardi [27] and Byrd, Schnabel, and Shultz [6] obtain the feasibility of the modified constraint region by including an additional parameter $\eta \in [0, 1]$ in the constraint

$$\eta g(x) + g'(x)s = 0. \tag{1.4}$$

Unfortunately, there are many examples that defeat this trick. For instance, if one takes

$$g(x) := \begin{bmatrix} 1 - e^x \\ x \end{bmatrix} \tag{1.5}$$

with $g : \mathbb{R} \to \mathbb{R}^2$, then $\eta = 0, s = 0$ is the unique solution to (1.4) for all $x \in \mathbb{R}$. The difficulty here is that $g'(x)$ never has full rank. The method introduced in §5 has no difficulty with this example. The method proposed by Powell and Yuan [18] has a flavor that is similar to the approach suggested here for the case $C = \{0\}_{\mathbb{R}^m}$ and $X = \mathbb{R}^n$, but there remain fundamental differences.

There is a large body of work directly associated with the third class of algorithms, exact penalization methods [11], [12], [16], [29], [30], etc. Most of this literature is couched in the language of trust region algorithms for convex composite optimization and is based on the original work of Fletcher. In the context of problem $\mathcal{P}$ all of these methods implicitly require knowledge of an upper bound on the norm of some Kuhn–Tucker multiplier at a Kuhn–Tucker solution to $\mathcal{P}$. They also require that the procedure be initiated close enough to this Kuhn–Tucker solution. One of the fruits of this investigation is a modification of these methods that eliminates the need for hypotheses of this type in the global convergence theory.

We now describe the plan of the paper. In §2, we present the basic algorithm. In §3, the stationarity conditions for $\widehat{\mathcal{P}}$ given in [1] are recalled. In §4 the basic properties of the objects employed in the description of the algorithm are given and the convergence analysis is presented in §5. The application of these results to SQP and $S\ell_1QP$ are given in §§6 and 7, respectively.

The notation that we employ is standard. Nonetheless, a partial listing is given for the readers convenience. Given $x, y \in \mathbb{R}^k$ the inner product is denoted by

$$\langle x, y\rangle := x^T y := \sum_{i=1}^{k} x^i y^i,$$

where $x := (x^1, x^2, \cdots, x^k)^T$ and $y := (y^1, y^2, \cdots, y^k)^T$. If X and Y are subsets of $\mathbb{R}^k$, then

$$\alpha X + \beta Y := \{\alpha x + \beta y : x \in X, y \in Y\}.$$

The polar of X is defined as

$$X^0 := \{w \in \mathbb{R}^k : \langle w, x\rangle \leq 1 \text{ for all } x \in X\}.$$

If X is convex, that is, $\lambda x + (1-\lambda)y \in X$ for all $x, y \in X$ and $\lambda \in [0, 1]$, then the recession cone of X is defined as

$$\operatorname{rec}(X) := \{y \in \mathbb{R}^k : X + y \subset \operatorname{cl}(X)\}$$

where $\operatorname{cl}(X)$ is the closure of X. The normal cone to X at any point $\overline{x} \in X$ is defined by

$$N(\overline{x}|X) := \{w \in \mathbb{R}^k : \langle w, x - \overline{x}\rangle \leq 0 \text{ for all } x \in X\}.$$

The tangent cone to X at $\overline{x}$ is the polar of the normal cone,

$$T(\overline{x}|X) := N(\overline{x}|X)^0.$$

The support and convex indicator functions for X are given, respectively, by

$$\psi^*(w|X) := \sup\{\langle w, x\rangle : x \in X\}$$

and

$$\psi(x|X) := \begin{cases} +\infty, & \text{if } x \notin X, \\ 0, & \text{if } x \in X. \end{cases}$$

A norm on $\mathbb{R}^k$ is denoted by $\|x\|$ and its unit ball is designated by

$$\mathbb{B} := \{x : \|x\| \le 1\}.$$

The dual norm to $\|x\|$ is given by

$$\|x\|_0 := \psi^*(x|\mathbb{B})$$

and consequently the dual unit ball is $\mathbb{B}^0$. The two-norm plays a special role and it is denoted by $\|x\|_2 := (\langle x, x\rangle)^{1/2}$. The distance function for the set X associated with the norms $\|\cdot\|$ and $\|\cdot\|_0$ are given by

$$\operatorname{dist}(y|X) := \inf\{\|y - x\| : x \in X\}$$

and

$$\operatorname{dist}_0(y|X) := \inf\{\|y - x\|_0 : x \in X\},$$

respectively. Given $g : \mathbb{R}^n \to \mathbb{R}^m$ the Frechet derivative of g at a point $x \in \mathbb{R}^n$, if it exists, is the linear mapping $g'(x) : \mathbb{R}^n \to \mathbb{R}^m$ (if it exists, it is unique) for which

$$g(y) = g(x) + g'(x)(y - x) + o(\|y - x\|), \quad \text{where } \lim_{y\to x} \frac{o(\|y - x\|)}{\|y - x\|} = 0.$$

Since $g'(x)$ is a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$, it has a matrix representation in $\mathbb{R}^{m\times n}$, with respect to the standard basis. This representation is called the Jacobian of g at x. In this presentation, we identify $g'(x)$ with its Jacobian. Also, for a set $X \subset \mathbb{R}^n$ and a mapping $f : \mathbb{R}^n \to \mathbb{R}$ we define

$$\arg\min\{f(x) : x \in X\} := \{\overline{x} \in X : f(\overline{x}) := \min\{f(x) : x \in X\}\}.$$

The set $\arg\max\{f(x) : x \in X\}$ is defined similarly.

2. The model algorithm. As in [1] our approach is based on a type of "linearization" of the constraint region Ω. Given $x \in X$, $0 \le \rho_1 \le \rho_2$, and $\theta \in [0, 1]$ we define

$$L\Omega(x, \rho_1, \rho_2, \theta) := \{s \in [X - x] \cap \rho_2\mathbb{B}^n |\ g(x) + g'(x)s \in C + \nu(x, \rho_1, \theta)\mathbb{B}^m\}, \tag{2.1}$$

where $\mathbb{B}^n$ and $\mathbb{B}^m$ are the closed unit balls of the norms that are given for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and for $\tau \ge 0$

$$\nu(x, \tau, \theta) := \varphi(x, 0) + \theta[\varphi(x, \tau) - \varphi(x, 0)], \tag{2.2}$$

and

$$\varphi(x, \tau) := \inf\{\operatorname{dist}(g(x) + g'(x)s|C) | s \in [X - x] \cap \tau\mathbb{B}^n\} \tag{2.3}$$

(henceforth the symbol $\mathbb{B}$ is used to denote the unit ball of either $\mathbb{R}^n$ or $\mathbb{R}^m$ unless some ambiguity is possible). We refer to the multifunction $L\Omega : X \times T \times [0, 1] \rightrightarrows \mathbb{R}^n$, where

$$T := \{(\rho_1, \rho_2) : 0 \le \rho_1 \le \rho_2\},$$

as a "linearization" of Ω. Such linearizations are well studied in the literature [2], [19], [21], [22]. Given $(x,(\rho_1,\rho_2),\theta)\in X\times T\times[0,1]$, the set $L\Omega(x,\rho_1,\rho_2,\theta)$ is a nonempty compact convex subset of $\mathbb{R}^n$. Moreover, if

$$(\rho_1,\rho_2,\theta)\in \text{int}\,(T\times[0,1]) \quad \text{and} \quad \varphi(x,\rho_1)\neq\varphi(x,0),$$

then $\text{int}\,[L\Omega(x,\rho_1,\rho_2,\theta)]\neq\emptyset$ (the notation $\text{int}\,(S)$ means the interior of the set S). This observation is significant since we use $L\Omega(x,\rho_1,\rho_2,\theta)$ as the constraint region for our convex programming subproblems. The condition $\text{int}\,[L\Omega(x,\rho_1,\rho_2,\theta)]\neq\emptyset$ implies that the Slater constraint qualification [25] is satisfied and so these convex programming subproblems have Kuhn–Tucker multipliers [23] at their solution.

The condition $\varphi(x,\rho_1)\neq\varphi(x,0)$ is of particular significance in the construction of the multifunction $L\Omega$. In [3] it is shown that if $\tau>0$, then $\varphi(x,\tau)=\varphi(x,0)$ if and only if x is a stationary point for the function $\overline{\varphi}$ defined in (1.1) (see §3). Moreover, given $(x,\rho_1,\rho_2,\theta)\in X\times T\times[0,1]$, it is shown that the inequality

$$\overline{\varphi}'(x;s)\leq\theta[\varphi(x,\rho_1)-\varphi(x,0)]\leq 0$$

holds for every $s\in L\Omega(x,\rho_1,\rho_2,\theta)$ where

$$\overline{\varphi}'(x;s):=\lim_{t\downarrow 0}\frac{\overline{\varphi}(x+ts)-\overline{\varphi}(x)}{t}$$

is the usual directional derivative of $\overline{\varphi}$ at x in the direction s. Consequently, if $\theta\neq 0$ and x is not a stationary point for $\overline{\varphi}$, then $L\Omega(x,\rho_1,\rho_2,\theta)$ is contained in the set of directions of strict descent for $\overline{\varphi}$ at x. This relationship supports the goal of locating stationary points for $\widehat{\mathcal{P}}$.

If $\mathcal{P}$ is in standard form and the norms chosen for $\mathbb{R}^n$ and $\mathbb{R}^m$ are polyhedral, then $L\Omega(x,\rho_1,\rho_2,\theta)$ is always a polyhedral convex set and the computation of the value $\varphi(x,\rho_1)$ reduces to solving a linear program. Thus, in this case, the set $L\Omega(x,\rho_1,\rho_2,\theta)$ can be specified in finite time.

In order to develop a local convergence theory, it is important that the set $L\Omega(x,\rho_1,\rho_2,\theta)$ closely resemble the constraint region in the standard SQP algorithm whenever possible. For example, if $x\in X$ is such that $\varphi(x,\rho_1)=0$, we would like to set $\theta=1$, since then

$$L\Omega(x,\rho_1,\rho_2,1):=\{s\in[X-x]\cap\rho_2\mathbb{B}|\ g(x)+g'(x)s\in C\}.$$

If $\mathcal{P}$ is in standard form and the norms on $\mathbb{R}^n$ and $\mathbb{R}^m$ are polyhedral, then this is indeed possible. However, in general, such a choice of θ is not theoretically sound. Nonetheless, we can choose θ as a function of x so that $\theta(x)\to 1$ as $[\varphi(x,\rho_1)-\varphi(x,0)]\to 0$. Specifically, given $\theta_0>0$ we consider functions $\theta: X\to[\theta_0,1]$ such that if any one of the sets $C, X, \mathbb{B}^n$, or $\mathbb{B}^m$ is not polyhedral, then

$$\theta(x)=1 \quad \text{only if } \varphi(x,\rho_1)=\varphi(x,0). \tag{2.4}$$

Two examples of such functions are

$$\theta_1(x):=\theta_0 \quad \text{for all } x\in X \tag{2.5}$$

and

$$\theta_2(x):=\max\{\theta_0,1+[\varphi(x,\rho_1)-\varphi(x,0)]\}. \tag{2.6}$$

The structure of the trust region algorithms that we discuss is standard, and is modeled on the one given by Powell [16] for convex composite functions. There are also similarities to Fletcher's $S\ell_1QP$ method [11], [12]. In particular, the acceptance of the trial step s_k at the kth iteration depends on the quadratic approximation

$$(2.7)\quad P_\alpha(s;x,H) := f(x)+\nabla f(x)^T s+\tfrac{1}{2}s^T Hs+\alpha\ \mathrm{dist}(g(x)+g'(x)s|C)+\psi(x+s|X)$$

to the exact penalty function

$$(2.8)\qquad P_\alpha(x) := f(x)+\alpha\overline{\varphi}(x)$$

for $\mathcal{P}$. As usual, the matrix $H \in \mathbb{R}^{n\times n}$ is intended to approximate the Hessian of the Lagrangian. The trial step s_k is chosen so that the reduction in $P_\alpha(s;x,H)$ is comparable to that which could be obtained by choosing the step that optimizes a linear model of $\mathcal{P}$ at x_k. In the case of constrained optimization, a typical linear model considered by Powell [16] is $P_\alpha(s;x,0)$ for some prespecified $\alpha > 0$. The linear model that we use is given by

$$\begin{aligned} LP(x) :\ &\text{minimize } f(x)+f'(x)s \\ &\text{subject to } s \in L\Omega(x,\rho_1,\rho_2,\theta(x)) \end{aligned}$$

for a fixed choice of $(\rho_1,\rho_2) \in \mathrm{int}\,(T)$.

The subproblems $LP(x)$ are also used to obtain updates for the penalty parameter α_k. The update rule is similar to the one proposed by Han in [13];

$$(2.9)\qquad \alpha_k := \max\{\|y_k\|_0+\varepsilon,\alpha_{k-1}+4\varepsilon\},$$

where $y_k \in \mathbb{R}^m$ is any Kuhn–Tucker multiplier vector for the constraint

$$(2.10)\qquad g(x_k)+g'(x_k)s \in C+\nu(x_k,\rho_1,\theta(x_k))\mathbb{B}$$

in $LP(x_k)$, where $\|\cdot\|_0$ is the norm dual to the norm $\|\cdot\|$ (i.e., $\|y\|_0 := \sup\{z^T y : z \in \mathbb{B}\}$) and $\varepsilon > 0$. From Burke [1], [3], this set of Kuhn–Tucker multipliers is given by

$$KTM(x) := \{y|(s,y,w,z) \in KT(x) \text{ for some } s,w,z \in \mathbb{R}^m\},$$

where

$$KT(x) := \left\{ (s,y,w,z) \;\middle|\; \begin{array}{l} s \in L\Omega(x,\rho_1,\rho_2,\theta(x)), w \in N(x+s|X), \\ z \in N(s|\rho_2\mathbb{B}), y \in N(g(x)+g'(x)s|C+\nu(x,\rho_1,\theta(x))\mathbb{B}), \\ 0 = f'(x)^T+g'(x)^T y+w+z \end{array} \right\}$$

is the multifunction of Kuhn–Tucker solutions to $LP(x)$. In general, the α_k's can be updated by any rule such that $\|y_k\|_0 \le \alpha_k$ for all $k = 1,2,\cdots$, and α_k is updated infinitely many times if and only if $\sup\{\|y_k\|_0 : k = 1,2,\cdots\} = +\infty$. Having α_k, the trial step s_k is accepted if

$$(2.11)\qquad P_{\alpha_k}(x_k+s_k)-P_{\alpha_k}(x_k) \le \beta_1\Delta P_{\alpha_k}(s_k;x_k,H_k),$$

where $0 < \beta_1 < 1$ and

$$(2.12)\qquad \Delta P_{\alpha_k}(s_k;x_k,H_k) := P_{\alpha_k}(s_k;x_k,H_k)-P_{\alpha_k}(x_k).$$

A detailed description of the algorithm follows.

Initialization: Choose $x_0 \in X, H_0 \in \mathbb{R}^{n\times n}, \alpha_{-1} > 0, \varepsilon > 0$, $t_0 \in (0,1)$, $0 < \gamma_1 \leq \gamma_2 < 1 \leq \gamma_3, 0 < \beta_1 \leq \beta_2 < \beta_3 \leq 1$. Set $k = 0$.

Step 1. If $KTM(x_k) = \emptyset$, set $\alpha_k := \alpha_{k-1}$; otherwise choose $y_k \in KTM(x_k)$ and set

$$\alpha_k := \begin{cases} \alpha_{k-1}, & \text{if } \alpha_{k-1} \geq \|y_k\|_0 + \varepsilon, \\ \max\{\|y_k\|_0 + \varepsilon, \alpha_{k-1} + 4\varepsilon\}, & \text{otherwise.} \end{cases}$$

Step 2. Choose $\widehat{s}_k, s_k \in [X - x_k] \cap t_k\rho_2\mathbb{B}$ with $\Delta P_{\alpha_k}(s_k; X_k, H_k) < 0$. If no such s_k exists, then stop.

Step 3. Set $r_k := [P_{\alpha_k}(x_k + s_k) - P_{\alpha_k}(x_k)][\Delta P_{\alpha_k}(s_k; x_k, H_k)]^{-1}$ and $\widehat{r}_k := [P_{\alpha_k}(x_k + \widehat{s}_k) - P_{\alpha_k}(x_k)][\Delta P_{\alpha_k}(s_k; x_k, H_k)]^{-1}$. If $r_k \leq \widehat{r}_k$, reset $r_k := \widehat{r}_k$ and $s_k := \widehat{s}_k$. If $r_k \geq \beta_3$, choose $t_{k+1} \in [t_k, \min\{1, \gamma_3 t_k\}]$; if $\beta_2 \leq r_k < \beta_3$, set $t_{k+1} := t_k$; if $r_k < \beta_2$, choose $t_{k+1} \in [\gamma_1 t_k, \gamma_2 t_k]$.

Step 4. If $r_k < \beta_1$, set $x_{k+1} := x_k, \alpha_{k+1} := \alpha_k$, $k := k+1$, and return to Step 2.

Step 5. Choose $H_{k+1} \in \mathbb{R}^{n\times n}$, set $x_{k+1} := x_k + s_k$, $k := k+1$, and return to Step 1.

Remarks. (1) The alternate trial step $\widehat{s}_k$ in Step 2 of the algorithm is introduced to facilitate the discussion of second-order corrections in §§6 and 7. It will be shown that one may always take $s_k := t_k\widetilde{s}_k$ where $\widetilde{s}_k$ solves $LP(x_k)$ and then set $\widehat{s}_k := s_k$.

(2) The updating formula for the penalty parameter depends upon the knowledge of a dual solution to $LP(x_k)$. This linear subproblem has a fixed trust region radius that could be adjusted finitely many times without affecting the global convergence behavior of the procedure. Nonetheless, it would seem to be more natural, if not more efficient, to let the trust region radius of this subproblem be the same as in the choice of trial step s_k. Unfortunately, our proof theory does not allow such a variation. In particular, if the trust region radius in $LP(x_k)$ is allowed to vary, then we are unable to provide a satisfactory analysis of the cases where the sequence $\{t_k\}$ is not bounded away from zero.

(3) The function $\theta(x)$ is introduced primarily for considerations associated with local convergence and to simplify adjustments in the trust region radius. The ability to adjust the trust region radius in this way follows from the inclusion

$$L\Omega(x, t\rho_1, t\rho_2, \theta) \subset L\Omega(x, \rho_1, t\rho_2, t\theta),$$

to be established in Proposition 4.1. In the polyhedral case the function $\theta(x)$ can also be used to reduce the effort required to obtain y_k in Step 1 whenever $\varphi(x_k, \rho_1) \neq \varphi(x_k, 0) \neq 0$. This is done by implicitly defining $\theta(x)$ in terms of the algorithm used to evaluate $\varphi(x, \rho_1)$ and any other function $\widehat{\theta}(x)$ satisfying (2.4). The algorithm for evaluating $\varphi(x, \rho_1)$ should produce a sequence $\{(\lambda_i, \widehat{s}_k)\} \in \mathbb{R} \times ([X - x] \cap \rho_1\mathbb{B})$ such that

$$\operatorname{dist}(g(x) + g'(x)\widehat{s}_k | C) \downarrow \varphi(x, \rho_1)$$

and

$$\lambda_i \uparrow \varphi(x, \rho_1).$$

One then terminates the procedure when

$$\operatorname{dist}[g(x) + g'(x)s_k | C] - \varphi(x, 0) \leq \widehat{\theta}(x)[\lambda_i - \varphi(x, 0)]$$

(which must occur after a finite number of iterations i if $\varphi(x,\rho) \neq \varphi(x,0)$) and define

$$\theta(x) := \begin{cases} \frac{\text{dist}\,[g(x)+g'(x)\widehat{s}_i|C]-\varphi(x,0)}{\varphi(x,\rho_1)-\varphi(x,0)}, & \text{if } \varphi(x,\rho_1) \neq \varphi(x,0), \\ 1, & \text{otherwise.} \end{cases}$$

In this case

$$\widehat{\theta}(x) \leq \theta(x) \leq 1$$

and

$$\theta(x)[\varphi(x,\rho_1) - \varphi(x)] = \text{dist}[g(x) + g'(x)\widehat{s}_k|C] - \varphi(x,0)$$

so that $\varphi(x,\rho_1)$ need not be computed except when $\varphi(x,\rho_1) = \varphi(x,0) \neq 0$.

(4) There are many ways to update the penalty parameter α_k in order to guarantee the existence of a trial step s_k so that $\Delta P_{\alpha_k}(s_k; x_k, H_k) < 0$, however, not all of these methods guarantee the inequality

$$\alpha_k \geq \text{dist}_0(0|KTM(x_k)).$$

Our proof of convergence requires this inequality since we need to invoke Proposition 4.2(2) when $\{\alpha_k\}$ is bounded.

(5) As described above the sequence of penalty parameters $\{\alpha_i\}$ is necessarily nondecreasing. However, one can employ a clever device proposed by Sahba [24] for reducing the penalty parameter on certain iterations. Specifically, at the end of the kth iteration one evaluates

$$\widetilde{\varphi}_k := \min\{\widetilde{\varphi}_{k-1}, \varphi(x_k, 0)\}.$$

If $\widetilde{\varphi}_k \leq \widetilde{\varphi}_{k-1} - \widetilde{\varepsilon}$ for some prespecified $\widetilde{\varepsilon} > 0$, then one resets α_{i+1} to any positive real number, say,

$$\alpha_{i+1} := \|y_i\|_0 + \varepsilon.$$

Clearly this reinitialization of α_i can only occur a finite number of times. Hence the convergence analysis remains unaltered.

(6) In the case where C and X are polyhedral and the norms on $\mathbb{R}^n$ and $\mathbb{R}^n$ are polyhedral, then $LP(x)$ is a linear program and the evaluation of $\varphi(x,\rho)$ reduces to solving a linear program.

We now proceed to the analysis of the algorithm. The first step in this process is to describe the first-order necessary conditions for optimality in $\widehat{\mathcal{P}}$.

3. Stationarity conditions for $\mathcal{P}$. We say that a point $x \in X$ is a stationary point for $\mathcal{P}$ if it is a stationary point for $\widehat{\mathcal{P}}$. By this we mean that $\overline{x}$ satisfies first-order necessary conditions for optimality in both of the problems

$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\ \overline{\varphi}(x) \tag{3.1}$$

and

$$\begin{aligned} &\text{minimize } f(x) \\ &\text{subject to } x \in X \quad \text{and} \quad g(x) \in C + \overline{\varphi}(\overline{x})\mathbb{B}. \end{aligned} \tag{3.2}$$

It is shown in [1, §2] that these conditions can be expressed in terms of the multifunctions

$$M_1(x) := \left\{ \begin{pmatrix} y \\ w \end{pmatrix} \middle| \begin{array}{l} y \in N(g(x)|C + \overline{\varphi}(x)\mathbb{B}), w \in N(x|X), \\ 0 = f'(x)^T + g'(x)^T y + w \end{array} \right\}$$

and

$$M_0(x) := \left\{ \begin{pmatrix} y \\ w \end{pmatrix} \middle| \begin{array}{l} y \in N(g(x)|C + \overline{\varphi}(x)\mathbb{B}), w \in N(x|X), \\ 0 = g'(x)^T y + w, \end{array} \right\}$$

where $x \in X$.

THEOREM 3.1. (Burke [1, §2].) *Let $\overline{x} \in X$.*

(1) *If $\overline{x}$ is a stationary point for $\overline{\varphi}$, then either $M_0(\overline{x}) \neq \{0\}$ or $g(\overline{x}) \in C$, or both. Moreover, if $\overline{\varphi}(\overline{x}) \neq 0$, then $M_0(\overline{x}) \neq \{0\}$ if and only if $\varphi(\overline{x}, \rho) = \varphi(x, 0)$ for every $\rho > 0$.*

(2) *If $\overline{x}$ is a stationary point for* (3.2)*, then either $M_1(x) \neq \emptyset$ or $M_0(x) \neq \{0\}$, or both.*

In Clarke's [8] terminology the sets $M_1(x)$ and $M_0(x)$ are called the normal and abnormal multipliers for (3.2) at $x \in X$. We will call $M_1(x)$ the set of Kuhn–Tucker multipliers for (3.2) at $x \in X$ and $M_0(x)$ the set of Fritz John multipliers for (3.2) at $\overline{x} \in X$. If $\overline{x}$ is such that $\overline{\varphi}(\overline{x}) = 0$ and $\mathcal{P}$ is in standard form, then $M_1(\overline{x})$ is precisely the set of Kuhn–Tucker multipliers for $\mathcal{P}$ that one normally encounters in mathematical programming. A point $\overline{x} \in X$ is called a Kuhn–Tucker point for $\mathcal{P}$ if $\overline{\varphi}(\overline{x}) = 0$ and $M_1(\overline{x}) \neq 0$; it is called a Fritz John point for $\mathcal{P}$ if $\overline{\varphi}(\overline{x}) = 0$ and $M_0(\overline{x}) \neq \{0\}$; and it is called a nonfeasible stationary point for $\mathcal{P}$ if $\overline{\varphi}(\overline{x}) \neq 0$ and $M_0(\overline{x}) \neq \{0\}$. Any point that is either a Kuhn–Tucker point, a Fritz John point, or a nonfeasible stationary point for $\mathcal{P}$ is simply called a stationary point for $\mathcal{P}$.

We conclude this section by recalling certain elementary facts concerning the distance function $\operatorname{dist}(y|C)$, the support function $\psi^*(y|C)$, and normal cones that are used in our study. For the proofs of these facts we refer the reader to [3] and [23].

LEMMA 3.2. *Let K be a nonempty closed convex subset of $\mathbb{R}^q$.*

(1) *The distance function*

$$\operatorname{dist}(y|K) := \inf\{\|y - z\| : z \in K\}$$

is convex on $\mathbb{R}^q$ with convex subdifferential

$$\partial \operatorname{dist}(x|K) := \begin{cases} \mathbb{B}^0 \cap N(x|K), & \text{if } y \in K, \\ (bdry\mathbb{B}^0) \cap N(x|K + \operatorname{dist}(y|K)\mathbb{B}), & \text{if } y \notin K. \end{cases}$$

Consequently, $\operatorname{dist}(\cdot|K)$ *is globally Lipschitz continuous on $\mathbb{R}^q$ with Lipschitz constant of* 1.

(2) *If $x \in K$, then $w \in N(x|K)$ if and only if*

$$\langle w, x \rangle = \psi^*(w|K).$$

(3) *For any $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^q$, it is always the case that*

$$\langle w, x \rangle - \psi^*(w|K) \leq \|w\|_0 \operatorname{dist}(x|K).$$

4. The linear subproblem $LP(x)$.

We begin this section with a description of the properties of the linearization $L\Omega$.

PROPOSITION 4.1. *Let $x_1, x_2 \in X$, $0 \leq \rho_1 \leq \rho_2$, $0 \leq \overline{\rho}_1 \leq \overline{\rho}_2$, and $\theta_1, \theta_2, t, \sigma \in [0, 1]$, and suppose that $M > 0$ is a bound for f' and g' on X.*

(1) *If $s \in [X - x_1] \cap \rho_1\mathbb{B}$, then*

$$\operatorname{dist}[s|[X - x_2] \cap \rho_1\mathbb{B}] \leq 2\|x_1 - x_2\|.$$

(2) $|\varphi(x_1,\rho_1)-\varphi(x_1,0)|\le M\rho_1$.

(3) *$\varphi(x_1,\cdot):\mathbb{R}_+\to\mathbb{R}$ is a convex function.*

(4) $|\varphi(x_1,\rho_1)-\varphi(x_2,\rho_1)|\le 3M\|x_1-x_2\|+\rho_1\|g'(x_1)-g'(x_2)\|$.

(5) *$\nu(x_1,\cdot,\theta_1):\mathbb{R}_+\to\mathbb{R}$ is a convex function and so*

$$\nu(x_1,t\rho_1,\theta_1)\le\nu(x_1,\rho_1,t\theta_1).$$

(6) $|\nu(x_1,\rho_1,\theta_1)-\nu(x_2,\rho_1,\theta_2)|\le 5M\|x_1-x_2\|+M\rho_1|\theta_1-\theta_2|+\rho_1\|g'(x_1)-g'(x_2)\|$.

(7) $L\Omega(x_1,t\rho_1,t\rho_2,\theta_1)\subset L\Omega(x_1,\rho_1,t\rho_2,t\theta_1)$.

(8) $tL\Omega(x,\rho_1,\rho_2,\theta_1)+(1-t)L\Omega(x,\rho_1,\overline{\rho}_2,\theta_2)\subset L\Omega(x,\rho_1,t\rho_2+(1-t)\overline{\rho}_2,t\theta_1+(1-t)\theta_2)$.

(9) *The multifunction $L\Omega$ is upper semicontinuous on $X\times T\times[0,1]$.*

(10) *If $x\in X$ is such that $M_0(x)=\{0\}$ and $(\rho_1,\rho_2,\theta)\in\operatorname{int}[T\times[0,1]]$, then the multifunction $L\Omega$ is continuous near (x,ρ_1,ρ_2,θ) relative to $X\times T\times[0,1]$.*

Proof. (1) If $x_2=x_1+s$ we are done since $0\in(X-x_2)\cap\rho_1\mathbb{B}$ and $\|s-0\|=\|x_1-x_2\|$. If $x_2\ne x_1+s$ choose $\lambda>0$ so that $\lambda\|x_1+s-x_2\|=\rho_1$. If $\lambda\ge 1$, then $\widehat{s}:=(x_1+s)-x_2\in[X-x_2]\cap\rho_1\mathbb{B}$ and $\|s-\widehat{s}\|=\|x_1-x_2\|$, from which the result follows. If $\lambda<1$, then again $\widehat{s}:=\lambda[x_1+s-x_2]\in[X-x_2]\cap\rho_1\mathbb{B}$, since $x_2+\lambda[x_1+s-x_2]=\lambda(x_1+s)+(1-\lambda)x_2\in X$. Moreover,

$$\begin{aligned}\|s-\widehat{s}\|&\le\|x_1-x_2\|+(1-\lambda)\|x_1+s-x_2\|\\&=\|x_1-x_2\|+\|x_1+s-x_2\|-\rho_1\\&\le 2\|x_1-x_2\|+\|s\|-\rho_1\\&\le 2\|x_1-x_2\|.\end{aligned}$$

(2) Let $s\in[X-x_1]\cap\rho_1\mathbb{B}$ be such that

$$\varphi(x_1,\rho_1)=\operatorname{dist}(g(x_1)+g'(x_1)s|C).$$

Then, by Lemma 3.2,

$$|\varphi(x_1,\rho_1)-\varphi(x_1,0)|=\operatorname{dist}(g(x_1)+g'(x_1)s|C)-\operatorname{dist}(g(x_1)|C)\le\|g'(x_1)\|\rho_1.$$

(3) This follows immediately from the fact that

$$\lambda[X-x_1]\cap\rho_1\mathbb{B}+(1-\lambda)[X-x_1]\cap\rho_2\mathbb{B}\subset[X-x_1]\cap(\lambda\rho_1+(1-\lambda)\rho_2)\mathbb{B}.$$

(4) Let $s_2\in[X-x_2]\cap\rho_1\mathbb{B}$ be such that

$$\varphi(x_2,\rho_1)=\operatorname{dist}[g(x_2)+g'(x_2)s_2|C]$$

and let $\widehat{s}_2\in[X-x_1]\cap\rho_1\mathbb{B}$ be such that

$$\|s_2-\widehat{s}_2\|=\operatorname{dist}[s_2|[X-x_1]\cap\rho_1\mathbb{B}].$$

Then, by (1) of the proof and Lemma 3.2(1),

$$\begin{aligned}\varphi(x_1,\rho_1)&\le\operatorname{dist}[g(x_1)+g'(x_1)\widehat{s}_2|C]\\&\le\|g(x_1)-g(x_2)\|+\|g'(x_1)-g'(x_2)\|\|\widehat{s}_2\|\\&\qquad+\|g'(x_2)\|\|s_2-\widehat{s}_2\|+\varphi(x_2,\rho_1)\\&\le 3M\|x_1-x_2\|+\rho_1\|g'(x_1)-g'(x_2)\|+\varphi(x_2,\rho_1).\end{aligned}$$

The result now follows by symmetry.

(5) This follows immediately from (3).

(6) This follows immediately from Lemma 3.2(1) and parts (2) and (4) above.

(7) This follows directly from the inequality $\nu(x_1, t_1\rho_1, \theta_1) \le \nu(x_1, \rho_1, t_1\theta_1)$ in part (5).

(8) This follows from part (5), the convexity of the sets C and X, and the fact that $\eta_1\mathbb{B} + \eta_2\mathbb{B} = (\eta_1 + \eta_2)\mathbb{B}$ for every $\eta_1, \eta_2 \ge 0$.

(9) This follows directly by continuity.

(10) This is established in Burke [1, Thm. 9.3]. □

Let $0 < \rho_1 < \rho_2$ be fixed throughout the remainder of the paper. Also let $\theta : X \to [\theta_0, 1]$ be given so that (2.4) is satisfied unless all of the sets $X, C, \mathbb{B}^n$, and $\mathbb{B}^m$ are polyhedral. Moreover, we assume that θ is chosen so that there are constants $K_1, K_2 \ge 0$ such that

$$|\theta(x) - \theta(y)| \le K_1\|x - y\| + K_2\|g'(x) - g'(y)\| \quad \text{for all } x, y \in X. \tag{4.1}$$

The functions θ_1 and θ_2 given in (2.5) and (2.6), respectively, satisfy (4.1). The fact that (2.6) satisfies (4.1) is an easy consequence of Proposition 4.1(4).

Now, given $x \in X$, recall the structure of the linear subproblems discussed in §2:

$$LP(x) : \text{minimize}\,\{f(x) + f'(x)s : s \in L\Omega(x, \rho_1, \rho_2, \theta(x))\}.$$

As has been observed, the subproblem $LP(x)$ is always well defined and finite valued since $L\Omega(x, \rho_1, \rho_2, \theta(x))$ is a nonempty convex compact subset of $\mathbb{R}^n$ for all $x \in X$. In conjunction with $LP(x)$, we also need to consider the value function for $LP(x)$,

$$\ell(x) := \min\{f(x) + f'(x)s | s \in L\Omega(x, \rho_1, \rho_2, \theta(x))\},$$

the multifunction of Kuhn–Tucker solutions to $LP(x)$, $KT(x)$, and the multifunction of Kuhn–Tucker multipliers for the functional constraint $g(x) + g'(x)s \in C + \nu(x, \rho_1, \theta(x))\mathbb{B}$, $KTM(x)$. The properties of these objects that are important for our study are given in the following proposition.

PROPOSITION 4.2. (1) *Both $KT(x)$ and $KTM(x)$ are nonempty as long as $x \in X$ and $M_0(x) = \{0\}$. Moreover, both KT and KTM are upper semicontinuous on X.*

(2) *If $\alpha > \text{dist}_0(0|KTM(x))$ for all $x \in S \subset X$, then there exist nonnegative constants K_3, K_4, and K_5 such that*

$$|\ell(x) - \ell(y)| \le K_3\|x - y\| + K_4\|f'(x) - f'(y)\| + K_5\|g'(x) - g'(y)\| \quad \textit{for all } y \in S. \tag{4.2}$$

Proof. (1) The first statement follows from Burke [1, Thm. 4.4] and the second follows from Proposition 4.1(9) and Burke [1, Prop. 6.1].

(2) Consider the exact penalty function

$$\widehat{P}_\alpha(s, x) := f(x) + f'(x)s + \alpha\ \text{dist}(g(x) + g'(x)s | C + \nu(x, \rho_1, \theta(x))\mathbb{B}) + \psi(s | [X - x] \cap \rho_2\mathbb{B})$$

for $LP(x)$. From the hypothesis on α we obtain from Burke [3, Thms. 10.3 and 10.7] that the solution sets of the two convex programs $LP(x)$ and

$$LP_\alpha(x) : \text{minimize}\,\{\widehat{P}_\alpha(s; x) | s \in \mathbb{R}^n\}$$

coincide on S with

$$\ell(x) = \min\{\widehat{P}_\alpha(s; x) | s \in \mathbb{R}^n\}.$$

Let s be a solution to $LP(y)$, $\widehat{s} \in [X - x] \cap \rho_2\mathbb{B}$ satisfy

$$\|s - \widehat{s}\| = \operatorname{dist}[s|[X - x] \cap \rho_2\mathbb{B}],$$

and $z \in \mathbb{B}$ satisfy

$$\operatorname{dist}(g(y) + g'(y)s + \nu(y, \rho_1, \theta(y))z|C) = \operatorname{dist}(g(y) + g'(y)s|C + \nu(y, \rho_1, \theta(y))\mathbb{B}).$$

Then, by Lemma 3.2 and parts (1) and (6) of Proposition 4.1, we have

$$\begin{aligned}
\ell(x) - \ell(y) &= \ell(x) - \widehat{P}_\alpha(s; y) \le \widehat{P}_\alpha(\widehat{s}; x) - \widehat{P}_\alpha(s; y) \\
&\le \rho_2\|f'(x) - f'(y)\| + M\|s - \widehat{s}\| + M\|x - y\| \\
&\quad + \alpha[\operatorname{dist}[g(x) + g'(x)\widehat{s} + \nu(x, \rho_1, \theta(x))z|C] \\
&\qquad - \operatorname{dist}[g(y) + g'(y)s + \nu(y, \rho_1, \theta(y))z|C]] \\
&\le \rho_2\|f'(x) - f'(y)\| + 3M\|x - y\| \\
&\quad + \alpha\|g(x) + g'(x)\widehat{s} + \nu(x, \rho_1, \theta(x))z \\
&\qquad - (g(y) + g'(y)s + \nu(y, \rho_1, \theta(y))z)\| \\
&\le \rho_2\|f'(x) - f'(y)\| + 3M\|x - y\| \\
&\quad + \alpha[M\|x - y\| + \rho_2\|g'(x) - g'(y)\| \\
&\qquad + M\|s - \widehat{s}\| + |\nu(x, \rho_1, \theta(x)) - \nu(y, \rho_1, \theta(y))|] \\
&\le \rho_2\|f'(x) - f'(y)\| + 3M\|x - y\| \\
&\quad + \alpha[3M\|x - y\| + \rho_2\|g'(x) - g'(x)\| + 5M\|x - y\| \\
&\qquad + \rho_1 M|\theta(y) - \theta(y)| + \rho_1\|g'(x) - g'(y)\|] \\
&\le \rho_2\|f'(x) - f'(y)\| + (3 + 8\alpha + \rho_1 K_1\alpha)M\|x - y\| \\
&\quad + \alpha(\rho_2 + \rho_1 + \rho_1 K_2 M)\|g'(x) - g'(y)\|.
\end{aligned}$$

The result now follows by symmetry. □

Remark. For each $x \in X$ the function given by $\widehat{\ell}(x, t) := \min\{f(x) + f'(x)s | s \in L\Omega(x, \rho_1, t\rho_2, t\theta(x))\}$ is convex in t on $\mathbb{R}_+$. This follows from Proposition 4.1(8). Although this property has interesting consequences, we do not directly make use of it in our study.

The subproblems $LP(x)$ can also be used to characterize stationarity in $\mathcal{P}$ and to obtain descent directions for P_α for an appropriate choice of α.

PROPOSITION 4.3. *Let $x \in X$.*

(1) *Suppose that $KT(x)$ is nonempty and choose*

$$\alpha \ge \operatorname{dist}_0(0|KTM(x)) + \varepsilon \tag{4.3}$$

for some $\varepsilon \ge 0$. Then

$$\Delta_\alpha(x) := \ell(x) - f(x) + \alpha\theta(x)[\varphi(x, \rho_1) - \varphi(x, 0)] \le \varepsilon\theta(x)[\varphi(x, \rho_1) - \varphi(x, 0)]. \tag{4.4}$$

Moreover, if $\Delta_\alpha(x) = 0$, then x is a stationary point for $\mathcal{P}$. If both $\Delta_\alpha(x) = 0$ and $\varphi(x, 0) = 0$, then x is a Kuhn–Tucker point for $\mathcal{P}$.

(2) *If x is a Kuhn–Tucker point for $\mathcal{P}$, then*

$$\Delta_\alpha(x) = 0 \quad \textit{for all } \alpha \ge \operatorname{dist}_0(0|KTM(x)).$$

(3) *If* $(s, y, w, z) \in KT(x)$, *then*

$$P'_\alpha(s; x) \leq \Delta_\alpha(x). \tag{4.5}$$

Remarks. (1) If $x \in X$ is a stationary point for $\mathcal{P}$ that is not a Kuhn–Tucker point, then it is still possible that $KT(x)$ is nonempty and $\Delta_\alpha(x) < 0$ where α satisfies (4.3). This is illustrated by considering the example

$$\min\{x : x^3 \leq 0, -25 \leq x\}$$

at the point $x = 0$. This is an attractive feature of the subproblem $LP(x)$, since even if one is at such a stationary point for $\mathcal{P}$ it may still be possible to obtain descent directions for $\mathcal{P}$.

(2) Observe in (4.5) that if α is chosen with

$$\alpha \geq \|y\|_0 + \varepsilon, \tag{4.6}$$

then (4.3) is satisfied and so

$$P'_\alpha(x; s) \leq \Delta_\alpha(x) \leq 0 \tag{4.7}$$

with $P'_\alpha(x; s) = 0$ only if x is stationary for $\mathcal{P}$.

Proof. We begin by establishing statements (1) and (3) of the proposition. Let $(s, y, w, z) \in KT(x)$ and in the case of (1) we also assume that (s, y, w, z) is chosen so that

$$\|y\|_0 = \text{dist}_0(0|KTM(x)).$$

By Lemma 3.2, we have

$$\begin{aligned} -\langle y, g'(x)s\rangle &= \langle y, g(x)\rangle - \langle y, g(x) + g'(x)s\rangle \\ &= \langle y, g(x)\rangle - \psi^*(y|C + \nu(x, \rho_1, \theta(x))\mathbb{B}) \\ &\leq \|y\|_0 \ \text{dist}(g(x)|C + \nu(x, \rho_1, \theta(x))\mathbb{B}) \\ &= \|y\|_0\theta(x)[\varphi(x, 0) - \varphi(x, \rho_1)] \\ &\leq (\alpha - \varepsilon)\theta(x)[\varphi(x, 0) - \varphi(x, \rho_1)], \end{aligned} \tag{4.8}$$

$$\begin{aligned} -\langle w, s\rangle &= \langle w, x\rangle - \langle w, x + s\rangle \\ &= \langle w, x\rangle - \psi^*(w|X) \\ &\leq \|w\|_0 \ \text{dist}(x|X) \\ &\leq 0, \end{aligned} \tag{4.9}$$

and

$$-\langle z, s\rangle = -\psi^*(z|\rho_2\mathbb{B}) = -\rho_2\|z\|_0. \tag{4.10}$$

Since

$$f'(x)s = -[\langle y, g'(x)s\rangle + \langle w, s\rangle + \langle z, s\rangle], \tag{4.11}$$

these relations yield the inequality

$$f'(x)s \leq (\alpha - \varepsilon)\theta(x)[\varphi(x,0) - \varphi(x,\rho_1)] - \rho_2\|z\| \\ \leq (\alpha - \varepsilon)\theta(x)[\varphi(x,0) - \varphi(x,\rho_1)], \tag{4.12}$$

from which inequality (4.4) immediately follows. Now if $\Delta_\alpha(x) = 0$, then (4.4) implies $\varphi(x, \rho_1) = \varphi(x, 0)$. Thus, by Theorem 3.1, we may as well assume that $\varphi(x, 0) = 0$. In this case $f'(x)s = 0$ and so (4.12) implies that $z = 0$. But then, by (4.11),

$$0 = \langle y, g'(x)s \rangle + \langle w, x \rangle,$$

while (4.8) and (4.9) imply

$$0 \leq \langle y, g'(x)s \rangle$$

and

$$0 \leq \langle w, s \rangle,$$

respectively. Hence $0 = \langle y, g'(x)s \rangle = \langle w, s \rangle$. Consequently, again by (4.8) and (4.9),

$$\langle y, g(x) \rangle = \psi^*(y|C)$$

and

$$\langle w, x \rangle = \psi^*(w|X),$$

and so, by Lemma 3.2, $y \in N(g(x)|C)$ and $w \in N(x|X)$ with

$$0 = f'(x)^T + g'(x)^T y + w.$$

Therefore, x is a Kuhn–Tucker point for $\mathcal{P}$.

To obtain (4.5) we simply observe that

$$\begin{aligned} P'_\alpha(x; s) &\leq f'(x)s + \alpha[\operatorname{dist}(g(x) + g'(x)s|C) - \operatorname{dist}(g(x)|C)] \\ &\leq f'(x)s + \alpha[\nu(x, \rho_1, \theta(x)) - \varphi(x, 0)] \\ &= \Delta_\alpha(x). \end{aligned}$$

(2) If $(s, y, w, z) \in KT(x)$, then, as in the proof of part (1), $(x, y, w) \in M_1(x)$. Hence $\alpha \geq \operatorname{dist}_0(0|KTM(x)) \geq \operatorname{dist}_0(0|M_c(x))$ where

$$M_c(x) := \{y : (y, w) \in M_1(x) \text{ for some } w \in \mathbb{R}^n\}.$$

Hence, by Burke [3, Thm. 10.7], $P'_\alpha(x; s) \geq 0$ for all $s \in \mathbb{R}^n$. Thus $\Delta_\alpha(x) = 0$ by (4.4) and (4.5). $\square$

From Proposition 4.2 we know that $KTM(x_k) \neq \emptyset$ as long as $M_0(x_k) = \{0\}$. Proposition 4.3 shows that if $KT(x_k) \neq \emptyset$ and $\Delta_{\alpha_k}(x_k) = 0$, then x_k is a stationary point for $\mathcal{P}$. This proposition also assures us of the existence of an element $s \in L\Omega(x_k, \rho_1, t_k\rho_2, t_k\theta(x_k))$ for which $\Delta P_{\alpha_k}(s; x_k, H_k) < 0$ whenever $\Delta_{\alpha_k}(x_k) < 0$. Therefore, in Step 2 of the algorithm in §2 one can always locate an $s_k \in [X - x_k] \cap t_k\rho_2\mathbb{B}$ (note that s_k need not be in $L\Omega(x_k, \rho_1, t_k\rho_2, t_k\theta(x_k))$) for which $\Delta P_{\alpha_k}(s_k; x_k, H_k) < 0$ as long as x_k is not a stationary point for $\mathcal{P}$. If $M_0(x_k) \neq \{0\}$, it may still be possible to obtain s_k such that $\Delta P_{\alpha_k}(s_k; x_k, H_k) < 0$ as noted in remark (1) after Proposition 4.3. This is an attractive feature of the algorithm and it explains why we set $\alpha_{k+1} := \alpha_k$ if $KTM(x_k) = \emptyset$. Particular choices of the trial step s_k are studied in §§6 and 7.

5. Convergence. The convergence theory presented in this section is modeled on that given in Powell [16, §4]. Consequently, we require the following assumption (see Powell [16, Thm. 2]).

ASSUMPTION 5.1. For every $\delta > 0$ there exist constants $\kappa_1, \kappa_2 > 0$ such that the inequality

$$\Delta P_{\alpha_k}(s_k, x_k, H_k) \leq -\kappa_1 \min\{\kappa_2, t_k\} \tag{5.1}$$

holds whenever $\Delta_{\alpha_k}(x_k) \leq -\delta$.

Inequality (5.1) is used to guarantee that the reduction in $P_{\alpha_k}(s; x_k, H_k)$ induced by s_k is comparable to the reduction that one would expect to obtain by use of the linear model $LP(x_k)$ alone. The following proposition indicates a way to choose s_k which assures the validity of inequality (5.1) when the sequence $\{H_k\}$ is bounded.

PROPOSITION 5.1. *Let $x \in X, H \in \mathbb{R}^{n\times n}$, $\alpha > 0$, and $\bar{t} \in (0,1)$. If $s \in [X - x] \cap \rho_2\mathbb{B}$ solves $LP(x)$, then there exists $\hat{t} \in [0, \bar{t}]$ such that*

$$\Delta P_\alpha(\hat{t}s; x, H) \leq \frac{1}{2}\Delta_\alpha(x) \min\left\{\frac{|\Delta_\alpha(x)|}{(\sigma\rho_2)^2\|H\|_2}, \bar{t}\right\}, \tag{5.2}$$

where $\sigma > 0$ is chosen so that $\|z\|_2 \leq \sigma\|z\|$ for $z \in \mathbb{R}^n$.

Proof. For $\lambda \in [0, \bar{t}]$ observe that

$$\begin{aligned}
\Delta P_\alpha(\lambda s; x, H) &\leq \lambda f'(x)s + \frac{\lambda^2}{2}(\sigma\rho_2)^2\|H\|_2 \\
&\qquad + \alpha\lambda[\operatorname{dist}[g(x) + g'(x)s|C] - \operatorname{dist}(g(x)|C)] \\
&\leq \lambda f'(x)s + \frac{\lambda^2}{2}(\sigma\rho_2)^2\|H\|_2 \\
&\qquad + \alpha\lambda[\nu(x, \rho_1, \theta(x)) - \varphi(x, 0)] \\
&= \lambda\Delta_\alpha(x) + \frac{\lambda^2}{2}(\sigma\rho_2)^2\|H\|_2.
\end{aligned}$$

If we now let

$$\hat{t} \in \arg\min\left\{\lambda\Delta_\alpha(x) + \frac{\lambda^2}{2}(\sigma\rho_2)^2 : \lambda \in [0, \bar{t}]\right\},$$

then it is straightforward to show that (5.1) is satisfied (see, for example, the proof of [16, Lemma 5, p. 20]). □

The following technical lemma greatly facilitates the discussion of convergence.

LEMMA 5.2. *Let $x \in X$, $H \in \mathbb{R}^{n\times n}$, $0 < \beta_1 < \beta_2 < 1$, and $\alpha, \kappa_1, \kappa_2 > 0$, and choose $\bar{t} > 0$ so that*

$$\kappa_1(1 - \beta_2)\min\{\kappa_2, t\} \geq (1 + \alpha)t\rho_2\omega_x(t\rho_2) + \tfrac{1}{2}(t\rho_2)^2\|H\|_2$$

for all $t \in [0, \bar{t}]$ where

$$\omega_x(t\rho_2) := \max\{\|f'(y) - f'(x)\|, \|g'(y) - g'(x)\| : y \in x + t\rho_2\mathbb{B}\}. \tag{5.3}$$

Then for every $t \in [0, \bar{t}]$ and $s \in [X - x] \cap t\rho_2\mathbb{B}$ for which

$$\Delta P_\alpha(s; x, H) \leq -\kappa_1 \min\{\kappa_2, t\}, \tag{5.4}$$

one has

$$(5.5)\qquad [P_\alpha(x+s) - P_\alpha(x)] \le \beta_1 \Delta P_\alpha(s;x,H).$$

Proof. By Lemma 3.2(1), we have

$$\begin{aligned} P_\alpha(x+s) - P_\alpha(x) &\le f(x) + f'(x)s + \alpha\ \mathrm{dist}(g(x)+g'(x)s|C) - P_\alpha(x) \\ &\qquad + \|f(x+s) - [f(x)+f'(x)s]\| \\ &\qquad + \alpha\|g(x+s) - [g(x)+g'(x)s]\| \\ &\le \Delta P_\alpha(s;x,H) + \tfrac{1}{2}(t\rho_2)^2\|H\|_2 \\ &\qquad + (1+\alpha)t\rho_2\omega_x(t\rho_2) \\ &\le \Delta P_\alpha(s;x,H) + \kappa_1(1-\beta_1)\min\{\kappa_2, t\} \\ &\le \beta_1\Delta P_\alpha(s;x,H) \end{aligned}$$

for every $t \in [0,\bar{t}]$ and $s \in [X - x] \cap t\rho_2\mathbb{B}$ satisfying (5.4). □

By combining Proposition 5.1 and Lemma 5.2 we see that unless $\Delta_{\alpha_k}(x_k) = 0$, one can always choose s_k in Step 2 of the algorithm of §2 so that $\Delta P_{\alpha_k}(s_k; x_k, H_k) < 0$ and Assumption 5.1 is satisfied. Furthermore, the procedure cannot jam at x_k with $x_{k+i} = x_k$ for all $i = 1, 2, \cdots$.

The main result is now given. The proof of this result is based on the approach of Powell in [16, Thm. 2].

THEOREM 5.3. *Let $\{x_k\}$ be a sequence generated by the algorithm of* §2 *for which Assumption* 5.1 *is satisfied.*

Furthermore, assume that f' and g' are bounded and uniformly continuous on $S := [\overline{co}\{x_i\} + \rho_2\mathbb{B}] \cap X$ and that the sequence $\{H_i\}$ is also bounded. Then at least one of the following must occur:

(1) $\Delta_{\alpha_k}(x_k) = 0$ *for some k and the procedure terminates,*

(2) $\alpha_k \uparrow +\infty$,

(3) $P_{\alpha_k}(x_k) \downarrow -\infty$,

(4) $\Delta_{\alpha_k}(x_k) \to 0$.

Proof. We will assume that none of (1)–(4) occur and derive a contradiction. First note that by Proposition 5.1 the sequence $\{x_k\}$ is infinite. Also observe that since α_k is bounded the updating strategy of Step 1 assures us that α_k remains constant for all k sufficiently large. Thus we may assume that $\alpha_k = \alpha$ for all $k = 1, 2, \cdots$. Now since $\Delta_\alpha(x_k) \nrightarrow 0$ there is a constant $\delta > 0$ and a subsequence $J \subset \mathbb{N}$ such that $\sup\{\Delta_\alpha(x_k) : k \in J\} < -2\delta < 0$. Consequently, by Assumption 5.1, there are constants $\kappa_1, \kappa_2 > 0$ such that (5.1) holds for all $k \in J$. Via Lemma 5.2, the uniform continuity of f' and g' now yield the existence of a $\bar{t} > 0$ such that

$$r_k \ge \beta_1 \quad \text{and} \quad x_{k+1} = x_k + s_k$$

whenever $t_k \le \bar{t}$. Suppose there is a $\xi > 0$ and a subsequence $\widehat{J}$ of J such that

$$\inf[t_k | k \in \widehat{J}] > \xi.$$

Then for each $k \in \widehat{J}$ let $\sigma(k)$ be the first integer greater than or equal to k for which $x_{\sigma(k)+1} = x_{\sigma(k)} + s_{\sigma(k)}$ and consider the subsequence $\widehat{J}_\sigma := \{\sigma(k) | k \in \widehat{J}\}$. Observe that for each $k \in \widehat{J}_\sigma$ we have $t_k \ge \min\{\gamma_1\bar{t}, \gamma_1\xi\}$. Consequently, $P_\alpha(x_{k+1}) \le$

$P_\alpha(x_k) - \kappa_1\beta_1 \min\{\kappa_2, \gamma_1\bar{t}, \gamma_1\xi\}$ for each $k \in \widehat{J}_\alpha$. But then $P_\alpha(x_k) \downarrow -\infty$, which is a contradiction. Therefore, we can assume that $t_k \leq \bar{t}$ for all $k \in J$ and $\lim_J t_k = 0$.

By Proposition 4.1(4), Proposition 4.2(2), and (4.1), the uniform continuity of f' and g' imply the uniform continuity of Δ_α on S. Hence there is an $\bar{\varepsilon} > 0$ such that

$$|\Delta_\alpha(x_i) - \Delta_\alpha(x_j)| \leq \delta$$

whenever $\|x_i - x_j\| \leq \varepsilon$, $i, j \in \mathbb{N}$. Given $k \in J$ let $v(k)$ be the first integer greater than k for which one of

$$\|x_{v(k)} - x_k\| \leq \bar{\varepsilon} \tag{5.6}$$

and

$$t_{v(k)} \leq \bar{t} \tag{5.7}$$

is violated. If (5.6) is violated, then

$$P_\alpha(x_{s+1}) \leq P_\alpha(x_s) - \kappa_1\beta_1 \min\{\kappa_2, t_s\}$$

and

$$t_{s+1} \geq t_s$$

for $s = k, \cdots, v(k) - 1$. Hence

$$P_\alpha(x_{v(k)}) \leq P_\alpha(x_k) - \kappa_1\beta_1 \min\{\kappa_2, \bar{\varepsilon}/\rho_2\}$$

since

$$\sum_{k}^{\nu(k)-1} t_k\rho_2 \geq \|x_{v(k)} - x_k\| \geq \bar{\varepsilon}.$$

If (5.7) is violated, then

$$P_\alpha(x_{v(k)}) \leq P_\alpha(x_{v(k)-1}) - \kappa_1\beta_1 \min\{\kappa_2, \gamma_3^{-1}\bar{t}\}.$$

In either case we have

$$P_\alpha(x_{v(k)}) \leq P_\alpha(x_k) - \kappa_1\beta_1 \min\{\kappa_2, \bar{\varepsilon}/\rho_2, \gamma_2^{-1}\bar{t}\},$$

which implies that $P_\alpha(x_k) \downarrow -\infty$. This is the contradiction that establishes the result. □

COROLLARY 5.4. *Let $\{x_k\}, \{H_k\}, f'$, and g' be as in Theorem* 5.3.

(1) *If $\alpha_k \uparrow +\infty$, then every cluster point $\bar{x}$ of the subsequence $J := \{i : \alpha_{i+1} > \alpha_i\}$ satisfies $M_0(\bar{x}) \neq \{0\}$ and so is either a Fritz John point or a nonfeasible stationary point for $\mathcal{P}$.*

(2) *If $\alpha := \sup\{\alpha_k\} < \infty$, then every cluster point $\bar{x}$ of $\{x_i\}$ is a stationary point for $\mathcal{P}$. Moreover, if $\varphi(x, 0) = 0$, then $\bar{x}$ is a Kuhn–Tucker point for $\mathcal{P}$.*

Proof. (1) Suppose to the contrary that $M_0(\bar{x}) = \{0\}$. Since $\alpha_k \uparrow \infty$, the multifunction

$$LM_1(x) := \{(y, w, z) | (s, y, w, z) \in KT(x) \text{ for some } s \in \mathbb{R}^n\}$$

is locally unbounded at $\overline{x}$. By Burke [1, Thm. 6.3] it must be the case that $\theta(\overline{x}) = 1$ and so $\varphi(\overline{x}, \rho_1) = \varphi(\overline{x}, 0)$ by (2.4). But then $\varphi(\overline{x}, 0) = 0$ since $M_0(\overline{x}) = \{0\}$. Furthermore, by Burke [3, Prop. 3.7],

$$LM_0(\overline{x}) := \text{rec}[LM_1(\overline{x})] \neq \{0\}$$

since $LM_1(x)$ is locally unbounded at $\overline{x}$. But then by Burke [1, Thm. 4.3], $M_0(\overline{x}) \neq \{0\}$, a contradiction.

(2) Since the α_k's are bounded they eventually equal α. Moreover, for all k sufficiently large, $P_{\alpha_k}(x_k) \geq P_\alpha(\overline{x})$. Therefore, by Theorem 5.3, $\Delta_\alpha(\overline{x}) = 0$. Consequently, by (4.4), $\varphi(\overline{x}, \rho_1) = \varphi(\overline{x}, 0)$. Thus we can assume that $\varphi(\overline{x}, 0) = 0$. We now show that $\overline{x}$ is a Kuhn–Tucker point for $LP(\overline{x})$.

Let $\{(s_k, y_k, w_k, z_k)\}$ be such that $(s_k, y_k, w_k, z_k) \in KT(x_k)$ and $\alpha_k \geq \|y_k\|_0$ for all $k = 1, 2, \cdots$. Let $J \in \mathbb{N}$ be a subsequence for which $x_k \xrightarrow{J} \overline{x}$. If $\{(w_k, z_k)\}_J$ is bounded, then $\overline{x}$ is a Kuhn–Tucker point for $LP(\overline{x})$ since $KT(x)$ is upper semicontinuous. If $\{(w_k, z_k)\}_J$ is unbounded, we can assume that J is such that $(w_k, z_k)/(\|w_k\|_0 + \|z_k\|_0) \xrightarrow{J} (\widehat{w}_k, \widehat{z}_k) \neq (0, 0)$ and $s_k \xrightarrow{J} \widehat{s}$. Then $\widehat{w} \in N(\overline{x} + \widehat{s}|X)$, $\widehat{z} \in N(\widehat{s}|\rho_2\mathbb{B})$, and $0 = \widehat{w} + \widehat{z}$ since $(s_k, y_k, w_k, z_k) \in KT(x_k)$ for all $k \in J$. Hence, by Lemma 3.2,

$$\begin{aligned}
0 &= -[\langle \widehat{w}, \widehat{s} \rangle + \langle \widehat{z}, \widehat{s} \rangle] \\
&= \langle \widehat{w}, \overline{x} \rangle - \langle \widehat{w}, \overline{x} + \widehat{s} \rangle - \rho_2 \|\widehat{z}\|_0 \\
&= \langle \widehat{w}, \overline{x} \rangle - \psi^*(\widehat{w}|X) - \rho_2 \|\widehat{z}\|_0 \\
&\leq \|\widehat{w}\|_0 \ \text{dist}(\overline{x}|X) - \rho_2 \|\widehat{z}\|_0 \\
&= -\rho_2 \|\widehat{z}\|_0 \leq 0,
\end{aligned}$$

but then $\widehat{z} = \widehat{w} = 0$, which is a contradiction.

Since $\overline{x}$ is a Kuhn–Tucker point for $LP(\overline{x})$ at which $\Delta_\alpha(\overline{x}) = 0$ for all $\alpha \geq \text{dist}_0(0|KTM(\overline{x})) + \varepsilon$, the result follows from Proposition 4.3. □

6. Application to $S\ell_1 QP$. In this section we assume that $\mathcal{P}$ is given in standard form, the norms chosen for $\mathbb{R}^n$ and $\mathbb{R}^m$ are polyhedral, and the function $\theta : X \to [\theta_0, 1]$ of §4 is such that $\theta(x) = 1$ whenever $\varphi(x, \rho_1) = 0$. We now consider an instance of the algorithm of §2 wherein the choice of trial step s_k is based on the $S_{l_1}QP$ algorithm of Fletcher. The procedure incorporates the second-order correction technique due to Fletcher [11], [12] in order to avoid the Marotos effect.

Initialization. Choose $x_0 \in X$, $H_0 \in \mathbb{R}^{n \times n}$, $\alpha_{-1} > 0$, $\varepsilon > 0$, and $t_0 \in (0, 1)$. Set $k := 0$ and choose $\sigma > 0$ so that $\|x\|_2 \leq \sigma \|x\|$ for all $x \in \mathbb{R}^n$.

Step 1. Choose $(\widetilde{s}_k, \widetilde{y}_k, \widetilde{w}_k, \widetilde{z}_k) \in KT(x_k)$. If $\widetilde{s}_k = 0$, then stop; otherwise set

$$\alpha_k := \begin{cases} \alpha_{k-1}, & \text{if } \alpha_{k-1} \geq \|\widetilde{y}_k\|_0 + \varepsilon, \\ \max\{\|\widetilde{y}_k\|_0 + \varepsilon, \alpha_{k-1} + 4\varepsilon\}, & \text{otherwise,} \end{cases}$$

and

$$\widetilde{t}_k := \arg\min \left\{ \lambda \Delta_{\alpha_k}(x_k) + \frac{\lambda^2}{2} \widetilde{s}_k^T H_k \widetilde{s}_k : 0 \leq \lambda \leq k_k \right\}.$$

Step 2. Let s_k be a stationary point of the subproblem

$$QP_1(x_k, t_k) : \min P_{\alpha_k}(s; x_k, H_k) \quad \text{subject to } s \in t_k \rho_2 \mathbb{B} \cap S_k$$

for which

$$P_{\alpha_k}(s_k; x_k, H_k) \le P_{\alpha_k}(\tilde{t}_k \tilde{s}_k; x_k, H_k), \tag{6.1}$$

where S_k is any subspace of $\mathbb{R}^n$ containing $\tilde{s}_k$.

Step 3. Set $r_k := [P_{\alpha_k}(x_k + s_k) - P_{\alpha_k}(x_k)][\Delta P_{\alpha_k}(s_k; x_k, H_k)]^{-1}$. If $r_k > 0.75$, go to Step 9.

Step 4. Let $\hat{s}_k$ be a stationary point for the problem

$$\widehat{QP}(x_k, t_k) : \min \widehat{P}_k(s) \quad \text{subject to } s \in [X - x_k] \cap t_k\rho_2\mathbb{B} \cap \widehat{S}_k,$$

where

$$\widehat{P}_k(s) := f(x_k) + f'(x_k)s + \tfrac{1}{2}s^T H_k s + \alpha_k \operatorname{dist}(g(x_k+s_k) - g'(x_k)s_k + g'(x_k)s | C)$$

and $\widehat{S}_k$ is any subspace of $\mathbb{R}^n$ containing S_k.
Set

$$r_k^e := r_k + \frac{\widehat{P}_k(0) - \widehat{P}_k(\hat{s}_k)}{\Delta P_{\alpha_k}(s_k; x_k, H_k)}.$$

If $r_k < .25$, go to Step 6.

Step 5. If $r_k^e \in [0.9, 1.1]$, set $t_{k+1} := 2t_k$ and go to Step 11; otherwise go to Step 10.

Step 6. If $r_k^e \notin [0.75, 1.25]$, go to Step 7. Set

$$\hat{r}_k := \frac{P_{\alpha_k}(x_k + \hat{s}_k) - P_{\alpha_k}(x_k)}{\Delta P_{\alpha_k}(s_k; x_k, H_k)}.$$

If $\hat{r}_k > 0.75$, set $s_k := \hat{s}_k, r_k := \hat{r}_k$, and go to Step 9.
If $\hat{r}_k \ge 0.25$, set $s_k := \hat{s}_k, r_k := \hat{r}_k$, and go to Step 10.
If $\hat{r}_k \ge r_k$, set $s_k := \hat{s}_k$ and $r_k := \hat{r}_k$.

Step 7. Choose $t_{k+1} \in [0.1t_k, 0.5t_k]$. If $r_k > 0.05$, go to Step 11.

Step 8. Set $x_{k+1} := x_k, k := k+1$, and go to Step 2.

Step 9. If $\|s_k\| < t_k\rho_2$, go to Step 10. If $r_k > 0.9$, then $t_{k+1} := 4t_k$; otherwise $t_{k+1} := 2t_k$. Go to Step 11.

Step 10. Set $t_{k+1} := t_k$.

Step 11. Set $x_{k+1} := x_k + s_k$.

Step 12. Choose $H_{k+1} \in \mathbb{R}^{n\times n}$, set $t_{k+1} := \min\{t_{k+1}, 1\}$, $k := k+1$, and go to Step 1.

Remarks. (1) The vector $\tilde{s}_k$ in Step 1 is often called the Cauchy step since it naturally corresponds to the best step obtainable from first-order information. The vector $\tilde{s}_k$ is used in (6.1) in order to assure the validity of inequality (5.2). In this way, Assumption 5.1 is satisfied.

(2) Except for the possibility of increasing t_k when $0.25 \le r_k \le 0.75$, this algorithm is an instance of the algorithm of §2. However, it is easily verified that this slight change in the implementation does not nullify the validity of Theorem 5.3 and Corollary 5.4.

The remarks above demonstrate that the results of §5 provide a global convergence theory for the algorithm in this section. Let us now concentrate on the local convergence. These results are obtained by appealing to the work of Yuan [30]. To this end we assume that $X = \mathbb{R}^n$ and we set

$$H_k := \nabla^2_{xx} L(x_k, y_k) = \nabla^2 f(x_k) + \sum_{i=1}^{m} y_{k+1}^{(i)} \nabla^2 g_i(x_k) \tag{6.2}$$

in Step 12, where $L(x, y) := f(x)+y^T g(x)$ is the Lagrangian for $\mathcal{P}$ and y_k is a multiplier estimate. For example, the multiplier estimate may be chosen as the solution to a least squares problem based on the optimality conditions. If $\{x_k\}$ is the sequence generated by the algorithm of this section, then we also assume the existence of a Kuhn–Tucker point $\overline{x}$ of $\mathcal{P}$ to which the sequence $\{x_k\}$ converges and at which the following hypotheses are satisfied:

(H1) (linear independence of the active constraint gradients). The gradients $\{g_i'(\overline{x}) : i \in A(\overline{x}) \cup \{s+1, \cdots, m\}\}$ are linearly independent where

$$A(x) := \{i \in \{1, \cdots, s\} | g_i(x) \geq 0\}.$$

(H2) (strict complementary slackness). The unique Kuhn–Tucker multiplier vector $\overline{y} \in \mathbb{R}^n$ is such that $\overline{y}^{(i)} > 0$ for each $i \in A(\overline{x})$.

(H3) (second-order sufficiency condition). For each

$$s \in \{d \in \mathbb{R}^n : f'(\overline{x})d = 0 \text{ and } g'(\overline{x})d \in T(g(\overline{x})|C)\}$$

with $s \neq 0$, one has

$$s^T \left[\nabla^2 f(\overline{x}) + \sum_{i=1}^{m} \overline{y}^{(i)} \nabla^2 g_i(\overline{x}) \right] s > 0.$$

THEOREM 6.1. *Let $\{x_k\}$ be a sequence generated by the algorithm of* §6 *with $S_k = \widehat{S}_k = \mathbb{R}^n$ for all $k = 0, 1, 2$. Assume that $x_k \to \overline{x}$ and that hypotheses* (H1)–(H3) *hold at $\overline{x}$. Furthermore, assume that $H_k := \nabla^2_x L(x_k, y_k)$ and that s_k and $\widehat{s}_k$ solve $QP_1(x_k, t_k)$ and $\widehat{QP}(x_k, t_k)$, respectively, for each $k = 0, 1, \cdots$ with $\{y_k\}$ chosen so that y_k converges to $\overline{y}$, the unique Kuhn–Tucker multiplier for $\mathcal{P}$ at $\overline{x}$. Then $x_k \to \overline{x}$ superlinearly, and if y_k is chosen to be the value of y that minimizes*

$$\|\nabla f(x_k) + g'(x_k)^T y\|_2,$$

then x_k converges to $\overline{x}$ quadratically.

Proof. The hypothesis (H1) implies that $M_0(\overline{x}) = \{0\}$, consequently, by Theorem 5.3, α_k is constant for all k sufficiently large. Therefore the algorithm is eventually an instance of the algorithm studied by Yuan in [30] and so the result follows from [30, Thm. 2.5, Cor. 2.6]. □

Remark. The assumption about the choice of multipliers $\{y_k\}$ is satisfied if, for example, one chooses the y_k's to be solutions to the least squares problems

$$\min\{\tfrac{1}{2}\|\nabla_x L(x_k, y)\|^2 : y \in \mathbb{R}^m\}.$$

7. Application to SQP. In this section we again assume that $\mathcal{P}$ is given in standard form with $X = \mathbb{R}^n$, that the norms chosen for $\mathbb{R}^n$ and $\mathbb{R}^m$ are polyhedral, and that the function $\theta : X \to [\theta_0, 1]$ of §4 is such that $\theta(x) = 1$ whenever $\varphi(x, \rho_1) = 0$.

We consider an instance of the algorithm of §2 wherein the choice of trial step s_k is based on a modification to the Wilson–Han–Powell SQP subproblem. The algorithm is identical to the algorithm of §6 except that the subproblem $QP_1(x_k, t_k)$ in Step 2 is replaced by Step 2′.

Step 2′. Let s_k be a stationary point of the subproblem

$$QP_2(x_k, t_k) : \min f(x_k) + f'(x_k)s + \tfrac{1}{2}s^T H_k s \quad \text{subject to } s_k \in L\Omega(x_k, \rho_1, t_k\rho_2, t_k\theta(x_k)) \cap S_k$$

for which

$$f(x_k) + f'(x_k)s_k + \frac{1}{2}s_k^T H_k s_k \leq f(x_k) + f'(x_k)(\tilde{t}_k\tilde{s}_k) + \frac{\tilde{t}_k^2}{2}\tilde{s}_k^T H_k \tilde{s}_k, \tag{7.1}$$

where S_k is any subspace of $\mathbb{R}^n$ containing $\tilde{s}_k$.

Remarks. (1) Observe that from Proposition 4.1(7), we have

$$L\Omega(x_k, t_k\rho_1, t_k\rho_2, \theta(x_k)) \subset L\Omega(x_k, \rho_1, t_k\rho_2, t_k\theta(x_k)).$$

Hence the subproblems $QP_2(x_k, t_k)$ are always well defined. The subspaces S_k (and $\widehat{S}_k$) are introduced to reduce the dimensionality of the feasible region for the subproblems $QP_2(x_k, t_k)$ ($\widehat{QP}(x_k, t_k)$). For example, when $\varphi(x, \rho_1) = 0$, a typical choice for S_k would be the span of $\tilde{s}_k$ and the solution to the Wilson–Han–Powell SQP subproblem when this subproblem has a solution, e.g., see Celis, Dennis, and Tapia [7].

(2) Inequality (7.1) plays a role similar to that of inequality (6.1) in that it guarantees that Assumption 5.1 holds. Consequently, the results of §5 provide a global convergence theory for this modification to the algorithm of §6.

The local convergence theory for the algorithm when Step 2′ is used instead of Step 2 is not yet well understood. However, we conjecture that if the hypotheses (H1)–(H3) hold, then the subproblems QP_1 and QP_2 should produce identical trial steps s_k when x_k is sufficiently close to $\bar{x}$. If this is indeed true, then Theorem 6.1 remains valid when Step 2′ is used instead of Step 2. The resolution of this conjecture is the topic of ongoing research.

In lieu of establishing this conjecture, one can obtain a preliminary local convergence result by assuming that the trust region radius in the modified algorithm is eventually inactive. In this case, a local convergence result is easily obtained by appealing to results in Robinson [20].

THEOREM 7.1. *Let $\{x_k\}$ be a sequence generated by the algorithm in* §6 *with Step* 2 *replaced by Step* 2′ *and $S_k = \widehat{S}_k = \mathbb{R}^n$ for all $k = 0, 1, 2$. Assume that $x_k \to \bar{x}$ where $\bar{x}$ satisfies the assumptions* (H1)–(H3). *Furthermore, assume that $H_k := \nabla_x^2 L(x_k, y_{k-1})$ and that s_k and $\hat{s}_k$ solve $QP_2(x_k, t_k)$ and $\widehat{QP}(x_k, t_k)$, respectively, for all $k \geq k_0$ for some $k_0 \in \mathbb{N}$, with each y_k chosen as a Kuhn–Tucker multiplier vector associated with the constraint*

$$g(x_k) + g'(x_k)s_k \in C + \nu(x_k, \rho_1, t_k\theta(x_k))\mathbb{B}$$

in $QP_2(x_k, t_k)$. If the trust region radius in QP_2 is eventually inactive, then $x_i \to \bar{x}$ quadratically.

Proof. Since the trust region constraint in the subproblems QP_2 is eventually inactive, the subproblems QP_2 reduce to the standard subproblems employed in the

Wilson–Han–Powell SQP method. Thus quadratic convergence follows from Robinson [20, Thm. 3.1]. □

Before closing, we wish to emphasize that the assumption that the trust region constraint is locally inactive is very strong. A more complete convergence result would establish conditions under which this hypothesis is valid. Until a clearer picture of the convergence properties of this procedure is established, the usefulness of Step 2′ remains in doubt. Nonetheless, we introduce this alternative to Step 2 since we conjecture that the resulting algorithm possesses convergence properties similar to those described in Theorem 6.1. The resolution of this conjecture is the subject of ongoing research.

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