Math 554 Final Exam December 1, 2014

You are to work on these problems individually, **no collaboration**. You may use the course notes and *Horn and Johnson* as your **only** resources, i.e., no other source materials can be used, especially, the web.

Due Friday, December 5, 4pm.

(1) Consider the finite dimensional vector space X over $\mathbb{C}$ given as

$$X = \text{Span}\{1, \mathrm{e}^x, x\mathrm{e}^x, \frac{x^2}{2}\mathrm{e}^x\} \ .$$

(a) (5 points) Show that $\{1, \mathrm{e}^x, x\mathrm{e}^x, \frac{x^2}{2}\mathrm{e}^x\}$ is a basis for X.

(b) (5 points) The differential operator D maps X to X. Give the matrix representation M of D in the basis of part (a).

(c) (5 points) What are the eigenvalues and eigenvectors of M and D?

(2) Let $A \in \mathbb{R}^{m\times n}$, $W \in \mathbb{R}^{m\times m}$, and $V \in \mathbb{R}^{n\times n}$ with W and V symmetric.

(a) (5 points) Show that V is positive definite on $\ker A$, i.e.,

$$u^TVu > 0 \quad \text{whenever} \quad u \neq 0 \text{ and } u \in \ker A,$$

if and only if there is a $\kappa > 0$ such that the matrix $V + \kappa A^TA$ is positive definite.

(b) (5 points) Suppose V is positive semidefinite on $\ker A$, i.e.,

$$u^TVu \geq 0 \quad \text{whenever} \quad u \in \ker A.$$

Show that the matrix $M := \begin{bmatrix} V & A^T \\ A & 0 \end{bmatrix}$ is nonsingular if and only if V is positive definite on $\ker A$ and the rank of A is m.

(c) (5 points) Show that the matrix

$$T := \begin{bmatrix} V & A^T \\ A & W \end{bmatrix}$$

is positive definite if and only if the matrices V and $W - AV^{-1}A^T$ are positive definite.

(3) (10 points) Given $A \in \mathbb{C}^{n\times n}$ let $\lambda(A) \in \mathbb{C}^n$ be the vector of eigenvalues of A including multiplicities with the components lexicographically ordered largest to smallest, i.e. for $\lambda, \zeta \in \mathbb{C}$, $\lambda \geq \zeta$ if and only if either $\mathcal{R}e\lambda \geq \mathcal{R}e\zeta$, or $\mathcal{R}e\lambda = \mathcal{R}e\zeta$ and $\mathcal{I}m\lambda \geq \mathcal{I}m\zeta$. Show that $A \in \mathbb{C}^{n\times n}$ is normal if and only if $\|A\|_F = \|\lambda(A)\|_2$.

(4) Consider the monic polynomial

$$p(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_{n-1}\lambda^{n-1} + \lambda^n,$$

where $a_j \in \mathbb{C},\ j = 0, 1, \ldots, n-1$. The *companion matrix* associated with p is the $n \times n$ matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & 0 & \ldots & 0 & -a_1 \\ 0 & 1 & 0 & \ldots & 0 & -a_2 \\ 0 & 0 & 1 & \ldots & 0 & -a_3 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -a_{n-1} \end{bmatrix}.$$

Let e_j denote the jth standard basis vector in $\mathbb{C}^n$, i.e. the jth component of e_j is 1 and all others are zero.

(a) (5 points) Show that p is the characteristic polynomial of A.
(Hint: Expand on the last column.)

(b) (5 points) Show that

$$\begin{aligned} A\mathrm{e}_k &= \mathrm{e}_{k+1} &&= A^k e_1 \qquad k = 1, \ldots, n-1, \\ A\mathrm{e}_n &= (A^n - p(A))e_1 &&= A^n e_1. \end{aligned}$$

(c) (5 points) Show that p is the minimal polynomial of A (hence A is nonderogatory).

(5) (15 points) Let $A \in \mathbb{C}^{n\times n}$ and $\epsilon > 0$. Show that the three sets $\mathcal{A}, \mathcal{B}, \mathcal{C}$ defined below are equal.

$$\begin{aligned} \mathcal{A} &= \{\, \lambda \in \mathbb{C} \mid \lambda \in \Lambda(X),\ \|A - X\| \le \epsilon \,\}, \\ \mathcal{B} &= \left\{\, \lambda \in \mathbb{C} \;\middle|\; \|(A - \lambda I)^{-1}\| \ge \epsilon^{-1} \text{ or } (A - \lambda I) \text{ is singular.} \,\right\}, \\ \mathcal{C} &= \{\, \lambda \in \mathbb{C} \mid \sigma_{\min}(A - \lambda I) \le \epsilon \,\}, \end{aligned}$$

where the we have used the operator 2-norm and $\sigma_{\min}(A - \lambda I)$ is the smallest singular value of $(A - \lambda I)$.

Hint: To show that $\mathcal{A} \subset \mathcal{C}$, use a unit eigenvector associated with $\lambda \in \mathcal{A}$ and the fact that

$$v^* T^* T v \ge \sigma^2_{\min}(T) \quad \forall\, T \in \mathbb{C}^{n\times n} \text{ and } v \in \mathbb{C}^n \text{ with } \|v\| = 1.$$

To show that $\mathcal{C} \subset \mathcal{A}$, let $\lambda \in \mathcal{C}$ and let u and v be unit left and right singular vectors for $(A - \lambda I)$ associated with the singular value $\sigma_{\min}(A - \lambda I)$, respectively. Then use the SVD to construct a matrix X for which

$$A - X = \sigma_{\min}(A - \lambda I) u v^*.$$

(6) (10 points) Let $f : [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and satisfy the *generalized Lipschitz condition*

$$|f(t, x) - f(t, y)| \le \kappa(t)|x - y| \quad \forall\ t \in [0, a] \text{ and } x, y \in \mathbb{R}^n,$$

where $\kappa(t)$ is non-negative and continuous on $(0, a]$, but possibly unbounded at $t = 0$. Show that if $\int_0^a \kappa(t)dt < \infty$, then the IVP $x' = f(t, x),\ x(0) = x_0$, has at most one solution on $[0, a]$.