SAMPLE QUESTION SOLUTIONS

1. Use the Two Phase Simplex Algorithm to solve the following LPs stating their solution, the solution to their duals, and their optimal values. (Solution methods other than the Two Phase Simplex Algorithm will be given zero credit)

   (a)

$$\begin{array}{llllllll} \text{maximize} & x_1 & + & x_2 & + & 3x_3 & & \\ \text{subject to} & x_1 & - & x_2 & - & 2x_3 & \leq & -2 \\ & x_1 & + & 2x_2 & + & 2x_3 & \leq & 2 \\ & 0 & \leq & x_1, & x_2, & x_3. & & \end{array}$$

   Solution: Primal solution $x = (0, 0, 1)^T$, Dual solution $y = (0, 3/2)^T$. But primal degeneracy implies that there are multiple dual solutions. However, there is no dual pivot! Recall that when there is no dual pivot there is a direction in which the dual objective can move off to infinity when the problem is non-degenerate. In the degenerate case it indicates a direction in the feasible region in which one can go off to infinity without changing the value of the objective. In this case the direction is $d = (1, 1)^T$ and can be read off of the coefficients in the degenerate row above the dual variables in the objective row. The complete set of dual optimal solutions is therefore unbounded and given by

$$\left\{ y \mid y = (0, 3/2)^T + t(1, 1)^T \text{ with } t \geq 0 \right\}.$$

   You can check this by plugging this expression in to the dual objective and constraints, e.g. $-2t + 2(3/2 + t) = 3$ for all $t \geq 0$.

   (b)

$$\begin{array}{llllll} \text{maximize} & x_1 & + & 4x_2 & & \\ \text{subject to} & x_1 & + & x_2 & \leq & 1 \\ & x_1 & - & x_2 & \leq & -2 \\ & 0 & \leq & x_1, & x_2. & \end{array}$$

   Solution: The LP is unbounded since the optimal value in the auxiliary problem is non-zero.

2. Use the Dual Simplex Algorithm to solve the following LP stating the solution, the solution to the duals, and the optimal value. (Solution methods other than the Dual Simplex Algorithm will be given zero credit)

$$\begin{array}{llllllllll} \text{maximize} & -2x_1 & - & 2x_2 & - & x_3 & - & 5x_4 & & \\ \text{subject to} & 2x_1 & - & x_2 & + & x_3 & - & x_4 & \leq & 4 \\ & & & x_2 & + & 2x_3 & - & x_4 & \leq & 5 \\ & x_1 & - & x_2 & - & x_3 & - & x_4 & \leq & -3 \\ & 0 & \leq & x_1, & x_2, & x_3, & x_4. & & & \end{array}$$

   Solution: $x = (0, 1, 2, 0)^T$ and $y = (0, 1, 3)^T$.

3. Formulate a dual for the following LPs.

   (a)

$$\begin{array}{llll} \text{minimize} & c^T x & & \\ \text{subject to} & Ax & \leq & 0 \\ & Bx & = & 0, \end{array}$$

   where $c \in I\!R^n$, $A \in I\!R^{s \times n}$, and $B \in I\!R^{t \times n}$.

   Solution: $\min 0^T u + 0^T v$ subject to $A^T u + B^T v + c = 0$ and $0 \leq u$.

(b)

$$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 + 10x_3 \\ \text{subject to} & x_1 + x_2 - x_3 = 12 \\ & x_1 - x_2 + x_3 \le 8 \\ & 0 \le x_2 \le 10 \end{array}$$

where $c \in I\!R^n$, $A \in I\!R^{s\times n}$, and $B \in I\!R^{t\times n}$.

Solution:

$$\begin{array}{lrcrcrcr} \text{minimize} & 12y_1 & + & 8y_2 & + & 10x_3 & & \\ \text{subject to} & y_1 & + & y_2 & & & = & 2 \\ & y_1 & - & y_2 & + & y_3 & \ge & -3 \\ & -y_1 & + & y_2 & & & = & 10 \\ & 0 \le & y_2, & y_3 & & & & \end{array}$$

4. Use the Geometric Duality Theorem to determine if the vector $x = (0, 5, 0, 1, 1)^T$ solves the LP

$$\begin{array}{lrcrcrcrcrcr} \text{maximize} & & & x_2 & & & + & 5x_4 & + & 5x_5 & & \\ \text{subject to} & x_1 & + & 2x_2 & - & x_3 & + & x_4 & & & \le & 11 \\ & 3x_1 & + & x_2 & + & 4x_3 & + & x_4 & + & x_5 & \le & 10 \\ & 2x_1 & - & x_2 & + & 2x_3 & + & x_4 & + & 2x_5 & \le & -2 \\ & x_1 & & & & & + & x_4 & + & 3x_5 & \le & 4 \\ & 0 & \le & x_1, & x_2, & x_3, & x_4, & x_5 & & & & \end{array}$$

Solution: This x cannot be the solution sinc ethe application of the Geometric Duality Theorem indicates that $y_4 = -3/5$ which cannot be the case.

5. Solutions to the sensitivity analysis questions are available on the course website.