

A Generalization of the Littlewood–Richardson Rule and the Robinson–Schensted–Knuth Correspondence

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1. INTRODUCTION

In this paper the intertwining number of representations of the symmetric group S_n corresponding to arbitrary skew diagrams, is computed. The answer is obtained in terms of the combinatorial notion of a picture which is essentially contained in [1]. The proof is based on the combinatorial result on pictures generalizing the well-known Robinson–Schensted–Knuth correspondence.

Let us give a more detailed account of the contents of this paper. In Section 2 we introduce the combinatorial terminology which will be used in the sequel. The main notions introduced here are those of a skew diagram (we call it simply a diagram), a partition diagram and a picture (i.e., a bijection between two diagrams satisfying some conditions). Our terminology is strongly influenced by [2].

In Section 3 the classical Littlewood–Richardson rule is formulated in terms of pictures (Proposition 1). This formulation, which I had learned from [1], is the cornerstone of the present paper.

Section 4 contains the main results of the paper (Theorems 1 and 2). We assign to each diagram κ with $|\kappa| = n$ the representation $\{\kappa\}$ of the symmetric group S_n . Choose diagrams κ_1 and κ_2 . Theorem 1 claims that the intertwining number of representations $\{\kappa_1\}$ and $\{\kappa_2\}$ is equal to the number of pictures $f: \kappa_1 \simeq \kappa_2$. With the account of the Littlewood–Richardson rule this Theorem follows immediately from combinatorial Theorem 2 which claims that there exists a natural bijection between pictures $f: \kappa_1 \simeq \kappa_2$ and pairs of pictures $(f_1: \nu \simeq \kappa_1, f_2: \nu \simeq \kappa_2)$, where ν runs all partition diagrams.

Theorem 2 is proved in Sections 5–8. In Section 5 we consider its particular case and show that in this case the bijection in Theorem 2 is a reformulation of the well-known Robinson–Schensted correspondence R . Its excellent exposition is given in [3] which will be our basic reference; the

more complete and modern exposition may be found in [2]. In Section 6 we reduce the general case of Theorem 2 to the certain statement on R (roughly speaking one must verify that R maps pictures into pictures). The explicit construction of R , translating [3] into our terminology, is given in Section 7. The proof of Theorem 2 is completed in Section 8.

In Sections 9–12 we give some applications of Theorems 1 and 2. I hope they clarify the relationships between combinatorics and the representation theory of symmetric groups. In Section 9 we give a simple combinatorial proof of the well-known result expressing the Young product of representations $\{\kappa\}$ in terms of diagrams. In Section 10 we prove Proposition 5 allowing one to decompose the Young product of representations $\{\kappa\}$ into the direct sum of representations corresponding to connected diagrams; it seems to be new. As a corollary we obtain the amusing decomposition of the regular representation of S_n .

In Section 11 we discuss the well-known property of commutativity of the Young product. We show that this property follows immediately from the equality $\{\kappa\} = \{\kappa^s\}$, where κ^s is obtained from κ by a central symmetry about some center. We give the combinatorial proof of this equality which is of independent interest. It is based on Theorem 2 and one remarkable property of the Robinson–Schensted correspondence due to Schützenberger.

In conclusive Section 12 we show that the Knuth correspondence [4] generalizing the Robinson–Schensted correspondence is in turn a particular case of Theorem 2.

2. COMBINATORIAL TERMINOLOGY

As usual, we denote by \mathbb{N} the set of positive integer numbers, by $[1, n]$ the subset $\{1, 2, \dots, n\} \subset \mathbb{N}$ and by $|\kappa|$ the number of elements of a finite set κ .

Define the partial ordering " \leq_p " on $\mathbb{N} \times \mathbb{N}$ by

$$(i, j) \leq_p (i', j') \Leftrightarrow i \leq i', j \leq j'.$$

Call a diagram any finite subset $\kappa \subset \mathbb{N} \times \mathbb{N}$, which is convex with respect to " \leq_p ", i.e. $a, b \in \kappa$ and $a \leq_p c \leq_p b$ imply $c \in \kappa$. A diagram containing the point $(1, 1)$, is called a partition diagram (some authors call our diagrams skew diagrams or skew Young diagrams; our partition diagrams are called Young or Ferrers diagrams). We identify the set of partition diagrams with the set of partitions $\lambda = (l_1, \dots, l_r)$ ($l_i \in \mathbb{N}$, $l_1 \geq l_2 \geq \dots \geq l_r$); the diagram corresponding to λ is $\{(i, j) \in \mathbb{N} \times \mathbb{N} \mid i \leq r, j \leq l_i\}$ and it is also denoted by λ . It is easy to verify that if $\lambda \supset \mu$ are two partition diagrams then their set-theoretic difference $\lambda \setminus \mu$ is a diagram; conversely, each diagram may be represented in such a form.

Let κ be a diagram. A subset κ' of κ is called regular if $x \not\prec_p y$ for each $x \in \kappa'$, $y \in \kappa \setminus \kappa'$; in particular, a point $x \in \kappa$ is regular if it is a maximal element of κ with respect to " \leq_p ". Evidently, if $\kappa' \subset \kappa$ is regular then κ' and $\kappa \setminus \kappa'$ are diagrams. We say that κ is connected if it cannot be divided into the disjoint union of two non-empty regular subsets. Clearly, each diagram κ is uniquely up to an ordering represented as the disjoint union of connected diagrams; they are called the connected components of κ .

Let v be a partition diagram with $|v| = n$. Call a plane partition of shape v any morphism $\varphi: (v, \leq_p) \rightarrow (\mathbb{N}, \leq)$; this means that φ is a function $v \rightarrow \mathbb{N}$ such that $a, b \in v$, $a \leq_p b \Rightarrow \varphi(a) \leq \varphi(b)$. Let $\pi = (p_1, p_2, \dots)$ be a sequence ($p_k \in \mathbb{Z}_+$); we say that φ is of type π , if $|\varphi^{-1}(k)| = p_k$ for each $k \in \mathbb{N}$ (we write also $\pi = (p_1, \dots, p_r)$ if $p_{r+1} = p_{r+2} = \dots = 0$). A plane partition φ is called row-strict, if it is strictly monotonous on each row of v (similarly for columns), a Young tableau if it is injective and a standard tableau if $\text{Im } \varphi = [1, n]$.

Define the linear ordering " \leq_J " on $\mathbb{N} \times \mathbb{N}$ by

$$(i, j) \leq_J (i', j') \Leftrightarrow \text{either } i < i' \text{ or } i = i', j \geq j'.$$

Let κ be a diagram. We say that a map $f: \kappa \rightarrow \mathbb{N} \times \mathbb{N}$ satisfies (J) if f is a morphism of the relation " \leq_p " to the relation " \leq_J ", i.e., $a, b \in \kappa$, $a \leq_p b$ imply that $f(a) \leq_J f(b)$. Call a picture any bijection $f: \kappa_1 \simeq \kappa_2$ between two diagrams such that both f and the inverse bijection $f^{-1}: \kappa_2 \simeq \kappa_1$ satisfy (J). Trivially, the bijection inverse to a picture is a picture itself. Denote by $\mathcal{P}(\kappa_1, \kappa_2)$ the set of pictures $f: \kappa_1 \simeq \kappa_2$.

The notion of a picture in a less symmetric form is contained in [1]. It plays the fundamental role in this work.

3. LITTLEWOOD-RICHARDSON RULE

It is well known that irreducible complex representations of S_n are naturally numerated by the partition diagrams λ with $|\lambda| = n$ (see, e.g., [5]); denote by $\{\lambda\}$ the representation corresponding to λ . For each partition diagrams λ, μ, ν with $|\mu| + |\nu| = |\lambda| = n$ denote by $\{\mu\} \cdot \{\nu\}$ the induced representation

$$\text{Ind}_{S_{|\mu|} \times S_{|\nu|}}^{S_n} (\{\mu\} \otimes \{\nu\})$$

of S_n and by $g_{\mu\nu}^\lambda = \langle \{\lambda\}, \{\mu\} \cdot \{\nu\} \rangle$ the multiplicity of $\{\lambda\}$ in $\{\mu\} \cdot \{\nu\}$ (the representation $\{\mu\} \cdot \{\nu\}$ is called the Young product of $\{\mu\}$ and $\{\nu\}$).

PROPOSITION 1. (Littlewood-Richardson rule). *The coefficient $g_{\mu\nu}^\lambda$ may be non-zero only if $\lambda \supset \mu$. In this case it is equal to the number of pictures $f: \nu \simeq \lambda \setminus \mu$, i.e., $g_{\mu\nu}^\lambda = |\mathcal{P}(v, \lambda \setminus \mu)|$.*

One can easily deduce this proposition from the classical formulation of the Littlewood-Richardson rule (see [5, Chap. VI, Theorem V]). It is done in Appendix 2 to [6]; also obtained is the new proof of the rule just in the present formulation.

4. MAIN RESULTS

Let us assign to each diagram κ the representation $\{\kappa\}$ of the group $S_{|\kappa|}$. For this we represent κ as $\lambda \setminus \mu$, where $\lambda \supset \mu$ are partition diagrams, and put

$$\{\kappa\} = \sum_v g_{\mu v}^\lambda \cdot \{v\};$$

according to Proposition 1 this assignment is well defined. Using the Frobenius reciprocity one may also define $\{\lambda \setminus \mu\}$ by the formula

$$\text{Res}_{S_k \times S_l}^{S_n} \{\lambda\} = \sum_{|\mu|=k} \{\mu\} \otimes \{\lambda \setminus \mu\}$$

(here $k + l = n$ and λ is a partition diagram with $|\lambda| = n$).

In these terms Proposition 1 means that the multiplicity $\langle \{v\}, \{\kappa\} \rangle$ of the irreducible representation $\{v\}$ in the representation $\{\kappa\}$ equals $|\mathcal{P}(v, \kappa)|$. Our main result is the following generalization.

THEOREM 1. *For every two diagrams κ_1 and κ_2 the intertwining number $\langle \{\kappa_1\}, \{\kappa_2\} \rangle$ equals $|\mathcal{P}(\kappa_1, \kappa_2)|$.*

By definition,

$$\langle \{\kappa_1\}, \{\kappa_2\} \rangle = \sum_v \langle \{v\}, \{\kappa_1\} \rangle \cdot \langle \{v\}, \{\kappa_2\} \rangle$$

(the sum runs all partition diagrams v). So Theorem 1 follows from Proposition 1 and the combinatorial

THEOREM 2. *For every two diagrams κ_1, κ_2 there exists a natural bijection*

$$R: \mathcal{P}(\kappa_1, \kappa_2) \simeq \bigsqcup_v (\mathcal{P}(v, \kappa_1) \times \mathcal{P}(v, \kappa_2)) \quad (*)$$

between pictures $f: \kappa_1 \simeq \kappa_2$ and the ordered pairs of pictures $(f_1: v \simeq \kappa_1, f_2: v \simeq \kappa_2)$, where v runs over all partition diagrams.

Denote the set in the right-hand side of (*) by $\mathcal{P}^{(2)}(\kappa_1, \kappa_2)$. The proof of Theorem 2 is outlined in Sections 5-8.

5. THE CRUCIAL SPECIAL CASE

At first consider the particular case of Theorem 2 when $\kappa_1 = \kappa_2 = i^n = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid i + j = n + 1\}$ ($n \in \mathbb{N}$ is fixed). Since any two points of i^n are incomparable with respect to " \leq_p " any map $f: i^n \rightarrow \mathbb{N} \times \mathbb{N}$ satisfies (J). It follows that any bijection $f: i^n \simeq i^n$ is a picture, so $\mathcal{P}(i^n, i^n)$ is the set of permutations of i^n . On the other hand, the set $\mathcal{P}^{(2)}(i^n, i^n)$ consists of pairs (f_1, f_2) , where f_i is a bijection $v \simeq i^n$ satisfying (J). Now consider the isomorphism $I: (i^n, \leq_j) \simeq ([1, n], \leq)$ (more simply, $I(i, j) = i$). Identifying i^n with $[1, n]$ via this isomorphism, one identifies $\mathcal{P}(i^n, i^n)$ with S_n and $\mathcal{P}^{(2)}(i^n, i^n)$ with the set $S_n^{(2)}$ of pairs (φ_1, φ_2) of standard tableaux of the same shape v ($|v| = n$). So Theorem 2 in this case means that there exists a natural bijection

$$R: S_n \simeq S_n^{(2)}.$$

This bijection is well known; it is called the Robinson-Schensted correspondence. We see that Theorem 2 gives a generalization of the Robinson-Schensted correspondence.

6. REFORMULATION OF THEOREM 2

Now consider the general case of Theorem 2. Choose diagrams κ_1, κ_2 with $|\kappa_1| = |\kappa_2| = n$ and denote by $S(\kappa_1, \kappa_2)$ the set of all bijections $f: \kappa_1 \simeq \kappa_2$ and by $S^{(2)}(\kappa_1, \kappa_2)$ the set of pairs $(f_1: v \simeq \kappa_1, f_2: v \simeq \kappa_2)$, where v is a partition diagram and f_1, f_2 satisfy (J). For each diagram κ with $|\kappa| = n$ denote by $I(\kappa)$ the isomorphism of ordered sets $([1, n], \leq) \rightarrow (\kappa, \leq_j)$. We identify S_n with $S(\kappa_1, \kappa_2)$ via the map $\sigma \mapsto I(\kappa_2) \circ \sigma \circ I(\kappa_1)^{-1}$; similarly, we identify $S_n^{(2)}$ with $S^{(2)}(\kappa_1, \kappa_2)$ via the map $(\varphi_1, \varphi_2) \mapsto (I(\kappa_1) \circ \varphi_1, I(\kappa_2) \circ \varphi_2)$. Using these identifications, the Robinson-Schensted correspondence $R: S_n \simeq S_n^{(2)}$ gives rise to the bijection

$$R: S(\kappa_1, \kappa_2) \simeq S^{(2)}(\kappa_1, \kappa_2).$$

(more precisely, we use the correspondence $S_n \simeq S_n^{(2)}$ differing from the one described in [3] by the transposition of Young tableaux). By definitions, $\mathcal{P}(\kappa_1, \kappa_2)$ and $\mathcal{P}^{(2)}(\kappa_1, \kappa_2)$ are the subsets of $S(\kappa_1, \kappa_2)$ and $S^{(2)}(\kappa_1, \kappa_2)$, respectively. To prove Theorem 2 it suffices to verify that $R(\mathcal{P}(\kappa_1, \kappa_2)) = \mathcal{P}^{(2)}(\kappa_1, \kappa_2)$ so R gives the desired bijection. In other words, we must prove that R transforms the condition on $f \in S(\kappa_1, \kappa_2)$ that f and f^{-1} satisfy (J), into the condition on $(f_1, f_2) \in S^{(2)}(\kappa_1, \kappa_2)$ that f_1^{-1} and f_2^{-1} satisfy (J).

7. EXPLICIT DESCRIPTION OF R

Now we give the explicit description of the correspondence $R: S(\kappa_1, \kappa_2) \simeq S^{(2)}(\kappa_1, \kappa_2)$, translating the statements from [3] into our notation. Let us introduce two algorithms.

Algorithm I (Insertion)

Data. A partition diagram ν , an injection $f: \nu \rightarrow \mathbb{N} \times \mathbb{N}$ satisfying (J) and a point $a \in \mathbb{N} \times \mathbb{N} \setminus \text{Im} f$. Algorithm constructs the partition diagram $\bar{\nu}$ obtained from ν by the addition of one point, and the injection $\bar{f}: \bar{\nu} \rightarrow \mathbb{N} \times \mathbb{N}$ satisfying (J) and such that $\text{Im} \bar{f} = \text{Im} f \cup \{a\}$.

For each $j \in \mathbb{N}$ let $C_j = \{x \in \nu \mid pr_2 x = j\}$ be the j 'th column of ν and $n_j = \max_{x \in C_j} pr_1 x$ be its length. According to (J) one has

$$f((1, j)) <_J f((2, j)) <_J \cdots <_J f((n_j, j)).$$

Put $a_0 = a$ and define successively the points $x_1, a_1, x_2, a_2, \dots, x_l, a_l$ by the following rule. For $j \geq 1$ x_j is the point of C_j with the minimal value of pr_1 such that $a_{j-1} <_J f(x_j)$; then a_j is defined as $f(x_j)$. This process finishes when $f(x) <_J a_l$ for all $x \in C_{l+1}$; put also $x_{l+1} = (n_{l+1} + 1, l + 1)$.

Now put $\bar{\nu} = \nu \cup \{x_{l+1}\}$ and define the map $\bar{f}: \bar{\nu} \rightarrow \mathbb{N} \times \mathbb{N}$, setting $\bar{f} = f$ on $\nu \setminus \{x_1, \dots, x_l\}$ and $\bar{f}(x_j) = a_{j-1}$ for $j = 1, 2, \dots, l + 1$. Clearly, \bar{f} is injective and $\text{Im} \bar{f} = \text{Im} f \cup \{a\}$. It is easy to see that $pr_1 x_1 \geq pr_1 x_2 \geq \cdots \geq pr_1 x_{l+1}$; it follows that $\bar{\nu}$ is a partition diagram. The fact that \bar{f} satisfies (J), is verified directly (cf. [3]).

Algorithm D (Deletion)

Data. A partition diagram $\bar{\nu}$, an injection $\bar{f}: \bar{\nu} \rightarrow \mathbb{N} \times \mathbb{N}$ satisfying (J) and a regular point $x \in \bar{\nu}$. Algorithm constructs the injection $f: \bar{\nu} \setminus \{x\} \rightarrow \mathbb{N} \times \mathbb{N}$ satisfying (J) and such that $\text{Im} f \subset \text{Im} \bar{f}$.

Let $x \in C_{l+1}$, i.e., $pr_2 x = l + 1$ ($l \geq 0$). Put $x_{l+1} = x$ and define successively the points $a_l, x_l, a_{l-1}, x_{l-1}, \dots, a_0$ by the following rule: $a_j = \bar{f}(x_{j+1})$ and x_j is the point of C_j with the maximal value of pr_1 such that $\bar{f}(x_j) <_J a_j$. It is easy to verify that this sequence is well defined, i.e., for $1 \leq j \leq l$ there exists $x'_j \in C_j$ such that $\bar{f}(x'_j) <_J a_j$ (one may put $x'_j = (pr_1 x_{j+1}, j)$).

Define the map $f: \bar{\nu} \setminus \{x\} \rightarrow \mathbb{N} \times \mathbb{N}$ setting $f = \bar{f}$ on $\bar{\nu} \setminus \{x_1, \dots, x_{l+1}\}$ and $f(x_j) = a_j$ for $1 \leq j \leq l$. The properties of f claimed above are verified directly (cf. [3]).

Algorithms I and D are clearly inverse to each other in the obvious sense. Using them we define now the correspondence $R: S(\kappa_1, \kappa_2) \simeq S^{(2)}(\kappa_1, \kappa_2)$. Let $f \in S(\kappa_1, \kappa_2)$, i.e., f is a bijection $\kappa_1 \simeq \kappa_2$. For each $k = 1, 2, \dots, n$ put $b_k = I(\kappa_1)(k)$ and $c_k = f(b_k)$ (thus, $\{b_1, \dots, b_n\} = \kappa_1$ and $b_1 <_J b_2 <_J \cdots <_J b_n$).

Define partition diagrams $v_1 \subset v_2 \subset \dots \subset v_n$ with $|v_k| = k$ and bijections $f^{(k)}: v_k \simeq \{c_1, \dots, c_k\}$ by induction as follows: $v_1 = \{(1, 1)\}$, $f^{(1)}((1, 1)) = c_1$, and $f^{(k)}$ for $k > 1$ is obtained by Algorithm I applied to the map $f^{(k-1)}$ and the point $a = c_k$. Clearly, all $f^{(k)}$ satisfy (J). Put $v = v_n$ and $f_2 = f^{(n)}$ so v is a partition diagram and f_2 is a bijection $v \simeq \kappa_2$ satisfying (J). Let $\{d_k\} = v_k \setminus v_{k-1}$ ($k = 1, \dots, n$). Define the bijection $f_1: v \simeq \kappa_1$ by $f_1(d_k) = b_k$ ($k = 1, \dots, n$). The properties of Algorithm I imply that d_k is a regular point of v_k ; it immediately results that f_1 satisfies (J). The correspondence R by definition assigns to f the pair $(f_1, f_2) \in S^{(2)}(\kappa_1, \kappa_2)$.

Let us describe now the inverse correspondence $R^{-1}: S^{(2)}(\kappa_1, \kappa_2) \simeq S(\kappa_1, \kappa_2)$. Let $(f_1, f_2) \in S^{(2)}(\kappa_1, \kappa_2)$, i.e., $f_1: v \simeq \kappa_1$ and $f_2: v \simeq \kappa_2$ are bijections satisfying (J). Define points $b_1, \dots, b_n \in \kappa_1$ as above and put $d_k = f_1^{-1}(b_k)$, $v_k = \{d_1, d_2, \dots, d_k\}$. It is easy to see that all v_k are partition diagrams and d_k is a regular point of v_k . Define the maps $f^{(k)}: v_k \rightarrow \mathbb{N} \times \mathbb{N}$ by the downward induction: $f^{(n)} = f_2$ and $f^{(k)}$ for $k < n$ is obtained by Algorithm D, applied to the map $f^{(k+1)}$ and the point $x = d_{k+1}$. Put $\{c_k\} = \text{Im } f^{(k)} \setminus \text{Im } f^{(k-1)}$ ($k = 1, \dots, n$) and define the bijection $f: \kappa_1 \simeq \kappa_2$ by $f(b_k) = c_k$. Since Algorithms I and D are inverse to each other, it follows that the correspondence $(f_1, f_2) \mapsto f$ is actually the inverse to R .

We shall need the following important property of R , which is a reformulation of Theorem B from [3, 5.1.4].

PROPOSITION 2 (Schützenberger). *If $R(f) = (f_1, f_2)$ then $R(f^{-1}) = (f_2, f_1)$.*

8. END OF THE PROOF

Now we are able to prove Theorem 2.

BASIC LEMMA. (a) *If in data of Algorithm I the map f is a picture, $\text{Im } f \cup \{a\}$ is a diagram and a is its regular point then \bar{f} is a picture.*

(b) *If in data of Algorithm D the map \bar{f} is a picture then $\text{Im } \bar{f} \setminus \text{Im } f$ is the regular point of $\text{Im } \bar{f}$ and f is a picture.*

Part (a) is proved in [6, Appendix 2]; part (b) is proved similarly.

The next Lemma follows immediately from definitions.

LEMMA 1. *Let $f \in S(\kappa_1, \kappa_2)$. In notation of Section 7 f^{-1} satisfies (J) iff for each $k = 1, 2, \dots, n$ the set $\{c_1, \dots, c_k\}$ is a diagram and c_k is its regular point.*

Proof of Theorem 2. We shall use the notation of Section 7. Let $f \in \mathcal{P}(\kappa_1, \kappa_2)$ and $(f_1, f_2) = R(f)$. Using Lemma 1 and Part (a) of the Basic

Lemma several times one obtains that all $f^{(k)}$ are pictures. In particular, $f_2 = f^{(n)}$ is a picture. Now we apply this statement to the picture f^{-1} instead of f and use Proposition 2. One obtains that f_1 is also a picture, hence, $(f_1, f_2) \in \mathcal{P}^{(2)}(\kappa_1, \kappa_2)$.

Conversely, let $(f_1, f_2) \in \mathcal{P}^{(2)}(\kappa_1, \kappa_2)$ and $f = R^{-1}((f_1, f_2))$, i.e., f is the bijection $\kappa_1 \simeq \kappa_2$ corresponding to (f_1, f_2) via the construction of Section 7. Using Part (b) of the Basic Lemma several times and then Lemma 1 one obtains that f^{-1} satisfies (J). Now we apply this statement to the pair (f_2, f_1) instead of (f_1, f_2) and use Proposition 2. One obtains that f satisfies (J), i.e., f is a picture. Therefore, R maps $\mathcal{P}(\kappa_1, \kappa_2)$ onto $\mathcal{P}^{(2)}(\kappa_1, \kappa_2)$, and Theorem 2 is done. As indicated above, Theorem 1 is its immediate consequence.

In the remainder of this paper we give some applications of Theorems 1 and 2.

9. YOUNG PRODUCT IN TERMS OF DIAGRAMS

We shall give a simple combinatorial proof of the well-known fact that the representation $\{\kappa\}$ is isomorphic to the Young product of representations corresponding to connected components of $\{\kappa\}$.

First of all note that according to Proposition 1 the representation $\{\kappa\}$ depends only on the shape of κ , i.e., $\{\kappa\} = \{\kappa'\}$ if κ' is a shift of κ by some vector $v \in \mathbb{Z} \times \mathbb{Z}$. We shall write $\kappa \sim \kappa'$ if diagrams κ and κ' have the same shape and sometimes identify such diagrams. Let $\kappa, \kappa_1, \kappa_2$ be diagrams; we say that κ is a product of κ_1 and κ_2 if $\kappa = \kappa'_1 \sqcup \kappa'_2$, where $\kappa'_1 \sim \kappa_1, \kappa'_2 \sim \kappa_2$ and for each $(i_1, j_1) \in \kappa'_1, (i_2, j_2) \in \kappa'_2$ one has $i_1 < i_2, j_1 > j_2$. Since κ is not determined uniquely by κ_1 and κ_2 , we shall write $\kappa \in \kappa_1 \kappa_2$ instead of $\kappa = \kappa_1 \kappa_2$ (cf. [2]).

PROPOSITION 3. *If $\kappa \in \kappa_1 \kappa_2$ then the Young product $\{\kappa_1\} \cdot \{\kappa_2\}$ (see Section 3) is isomorphic to $\{\kappa\}$.*

We deduce this proposition from the next combinatorial

PROPOSITION 4. *Let $\kappa_1, \kappa_2, \kappa$ and κ^0 be diagrams and $\kappa \in \kappa_1 \kappa_2$. There exists a natural bijection between the set $\mathcal{P}(\kappa^0, \kappa)$ of pictures $f: \kappa^0 \simeq \kappa$ and the set of triples (κ', f_1, f_2) , where κ' is regular subset of $\kappa^0, f_1 \in \mathcal{P}(\kappa^0 \setminus \kappa', \kappa_1)$ and $f_2 \in \mathcal{P}(\kappa', \kappa_2)$.*

Proof. Let $\kappa = \kappa'_1 \sqcup \kappa'_2$ be the decomposition described above, and $t_1: \kappa_1 \simeq \kappa'_1, t_2: \kappa_2 \simeq \kappa'_2$ be shifts. Assign to a triple (κ', f_1, f_2) the map $f: \kappa^0 \rightarrow \kappa$ which is equal to $t_1 \circ f_1$ on $\kappa^0 \setminus \kappa'$ and to $t_2 \circ f_2$ on κ' ; it follows immediately from definitions that it is the desired bijection.

Proof of Proposition 3. It suffices to verify that $\langle \{\lambda\}, \{\kappa_1\} \cdot \{\kappa_2\} \rangle = \langle \{\{\lambda\}, \{\kappa\}\} \rangle$ for each partition diagram λ . The Frobenius reciprocity and definition of $\{\lambda\mu\}$ imply that

$$\langle \{\lambda\}, \{\kappa_1\} \cdot \{\kappa_2\} \rangle = \sum \langle \{\mu\}, \{\kappa_1\} \rangle \cdot \langle \{\lambda\backslash\mu\}, \{\kappa_2\} \rangle$$

(the sum is over partition diagrams $\mu \subset \lambda$). By Theorem 1 the right-hand side of this equality is the number of triples (μ, f_1, f_2) , where μ is a partition diagram contained in λ , $f_1 \in \mathcal{P}(\mu, \kappa_1)$, $f_2 \in \mathcal{P}(\lambda\backslash\mu, \kappa_2)$. Clearly, regular subsets of λ are just $\lambda\backslash\mu$ so by Proposition 4 the number of triples (μ, f_1, f_2) is equal to the number of pictures $f: \lambda \simeq \kappa$. By Theorem 1, it equals $\langle \{\lambda\}, \{\kappa\} \rangle$. Q.E.D.

COROLLARY. *The representation $\{v^n\}$ (see Section 5) is regular representation of S_n .*

According to this Corollary and Theorem 1, for each diagram κ with $|\kappa| = n$ the dimension of $\{\kappa\}$ equals the number of bijections $f: \kappa \simeq v^n$ satisfying (J) . In particular, if λ is a partition diagram then $\dim\{\lambda\}$ is equal to the number of standard tableaux of shape λ .

10. DECOMPOSITION OF YOUNG PRODUCTS

The next proposition allows one to decompose any representation $\{\kappa\}$ into the direct sum of representations corresponding to connected diagrams.

PROPOSITION 5. *Let κ_1 and κ_2 be diagrams, x_1 be the maximal element of κ_1 and x_2 be the minimal element of κ_2 with respect to the relation " \leq ". Suppose $x_2 = x_1 + (1, -1)$. Denote by κ_2^{\uparrow} and κ_2^{\rightarrow} the shifts of κ_2 by vectors $(-1, 0)$ and $(0, 1)$, respectively. Then*

$$\{\kappa_1\} \cdot \{\kappa_2\} = \{\kappa_1 \cup \kappa_2\} = \{\kappa_1 \cup \kappa_2^{\uparrow}\} \oplus \{\kappa_1 \cup \kappa_2^{\rightarrow}\}$$

(see Fig. 1).

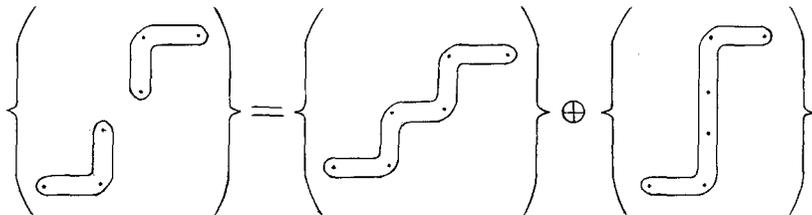


FIGURE 1

Proof. Let κ be an arbitrary diagram. According to Theorem 1, one must verify that

$$|\mathcal{P}(\kappa_1 \cup \kappa_2, \kappa)| = |\mathcal{P}(\kappa_1 \cup \kappa_2^\uparrow, \kappa)| + |\mathcal{P}(\kappa_1 \cup \kappa_2^\rightarrow, \kappa)|. \quad (*)$$

One has $\mathcal{P}(\kappa_1 \cup \kappa_2, \kappa) = \mathcal{P}^\uparrow \sqcup \mathcal{P}^\rightarrow$, where \mathcal{P}^\uparrow (resp. \mathcal{P}^\rightarrow) consists of pictures $f: \kappa_1 \cup \kappa_2 \cong \kappa$ such that $f(x_2) <_J f(x_1)$ (resp. $f(x_1) <_J f(x_2)$). Let $f \in \mathcal{P}^\uparrow$; define the bijection $f^\uparrow: \kappa_1 \cup \kappa_2^\uparrow \cong \kappa$ by

$$f^\uparrow|_{\kappa_1} = f, \quad f^\uparrow(x) = f(x + (1, 0)) \quad \text{for } x \in \kappa_2^\uparrow.$$

It immediately follows from definition of a picture that the correspondence $f \mapsto f^\uparrow$ is a bijection between \mathcal{P}^\uparrow and $\mathcal{P}(\kappa_1 \cup \kappa_2^\uparrow, \kappa)$. Similarly, one can construct the bijection between \mathcal{P}^\rightarrow and $\mathcal{P}(\kappa_1 \cup \kappa_2^\rightarrow, \kappa)$; it proves (*). Q.E.D.

COROLLARY. *Call a skew-hook of length n any subset of $\mathbb{N} \times \mathbb{N}$ of the form $\{x_1, \dots, x_n\}$, where for $k = 2, \dots, n$ either $x_k = x_{k-1} + (1, 0)$ or $x_k = x_{k-1} + (0, -1)$; clearly, each skew-hook is a diagram. Then regular representation of S_n is the direct sum of representations $\{\kappa\}$, where κ runs all skew-hooks of various shape of length n (it is easy to see that the number of these skew-hooks is 2^{n-1}).*

For the proof it suffices to apply $(n - 1)$ times Proposition 5, each time to the case when κ_2 consists of one point.

11. COMMUTATIVITY OF YOUNG PRODUCT AND THE CENTRAL SYMMETRY

It is well known that Young product is commutative; one can easily prove it by means of the representation theory. We shall give the combinatorial proof of this fact which is of independent interest. The usual trick to prove the commutativity is to define some anti-involution, which in fact becomes identical. We apply this trick geometrically on the level of diagrams.

For each finite subset $\kappa \subset \mathbb{N} \times \mathbb{N}$ denote by κ^s the set obtained from κ by the central symmetry s about some point (since we are interested in the shape of κ^s only, the choice of the center does not matter). For each map $f: \kappa_1 \rightarrow \kappa_2$ denote by f^s the map $s \circ f \circ s: \kappa_1^s \rightarrow \kappa_2^s$. Clearly, $(\kappa^s)^s = \kappa$ and $(f^s)^s = f$. The following properties are also trivial.

PROPOSITION 6. *Let κ_1, κ_2 be diagrams.*

- (a) *The correspondence $f \mapsto f^s$ induces the bijection $\mathcal{P}(\kappa_1, \kappa_2) \cong \mathcal{P}(\kappa_1^s, \kappa_2^s)$.*
- (b) *If $\kappa \in \kappa_1 \kappa_2$ then $\kappa^s \in \kappa_2^s \kappa_1^s$.*

According to Part (b) of this Proposition and Proposition 3, the commutativity of the Young product follows from the next

PROPOSITION 7. *For each diagram κ one has $\{\kappa^s\} = \{\kappa\}$.*

By Proposition 1, the combinatorial version of Proposition 7 is the next

PROPOSITION 8. *For each diagram κ and partition diagram ν there exists the natural bijection*

$$S: \mathcal{P}(\nu, \kappa) \simeq \mathcal{P}(\nu, \kappa^s)$$

To define S we use the combinatorial algorithm which (as Algorithms I and D above) is the translation to our language of the algorithm S from [3].

Algorithm S

Data. A partition diagram ν and an injection $f: \nu \rightarrow \mathbb{N} \times \mathbb{N}$ satisfying (J). Algorithm constructs the regular point $x \in \nu$ and the injection $f': \nu \setminus \{x\} \rightarrow \mathbb{N} \times \mathbb{N}$ satisfying (J) and such that $\text{Im } f' = \text{Im } f \setminus \{b\}$, where b is the minimal point of $\text{Im } f$ with respect to " \leq_J " (notation $b = \min_J \text{Im } f$).

Define the sequence x_1, x_2, \dots, x_p of points of ν : set $x_1 = (1, 1)$ and let x_k for $k > 1$ be the point of $\{x_{k-1} + (0, 1), x_{k-1} + (1, 0)\}$ with the minimal value of f (with respect to " \leq_J "); the last point x_p is the regular point of ν . Put $x = x_p$ and define the injection $f': \nu \setminus \{x\} \rightarrow \mathbb{N} \times \mathbb{N}$ by $f' = f$ on $\nu \setminus \{x_1, \dots, x_p\}$ and $f'(x_k) = f(x_{k+1})$ for $k = 1, 2, \dots, p - 1$. The desired properties of f' , i.e., that f' satisfies (J) and $\text{Im } f' = \text{Im } f \setminus \{b\}$, are verified directly (cf. [3]).

Now let ν be a partition diagram and κ a finite subset of $\mathbb{N} \times \mathbb{N}$ (not necessarily a diagram). Using Algorithm S and induction on $|\nu|$ we shall define the map

$$\begin{aligned} S: \{ \text{bijections } \nu \simeq \kappa \text{ satisfying (J)} \} \\ \rightarrow \{ \text{bijections } \nu \simeq \kappa^s \text{ satisfying (J)} \} \end{aligned}$$

Let $f: \nu \simeq \kappa$ be a bijection satisfying (J) and $b = \min_J \kappa$. Let $f': \nu \setminus \{x\} \simeq \kappa \setminus \{b\}$ be the bijection obtained by applying Algorithm S to f . By inductive assumption there is defined the bijection $S(f'): \nu \setminus \{x\} \simeq (\kappa \setminus \{b\})^s = \kappa^s \setminus \{b^s\}$. Define $S(f): \nu \simeq \kappa^s$ by $S(f) = S(f')$ on $\nu \setminus \{x\}$ and $S(f)(x) = b^s$. One can easily verify that $S(f)$ satisfies (J) as desired.

We claim that S gives rise to the bijection desired in Proposition 8. It suffices to verify that S maps pictures into pictures and $S^2 = \text{Id}$. The direct proof of these statements is rather difficult. We shall deduce them from Theorem 2 and the following remarkable result of Schützenberger, relating S with the Robinson–Schensted correspondence R (cf. [3, 5.1.4, Theorem D]).

PROPOSITION 9. *Let κ_1, κ_2 be diagrams. Then the composition*

$$S^{(2)}(\kappa_1, \kappa_2) \xrightarrow{R^{-1}} S(\kappa_1, \kappa_2) \longrightarrow S(\kappa_1^s, \kappa_2^s) \xrightarrow{R} S^{(2)}(\kappa_1^s, \kappa_2^s),$$

where the middle map is $f \mapsto f^s$, acts by

$$(f_1, f_2) \mapsto (S(f_1), S(f_2)).$$

The equality $S^2 = \text{Id}$ follows immediately from this Proposition; to prove that S maps pictures into pictures it suffices to apply Theorem 2 and Proposition 6(a). This completes the proof of Proposition 8.

12. KNUTH CORRESPONDENCE

In conclusion we shall show that the Knuth correspondence [4], generalizing the Robinson-Schensted correspondence, is in turn a particular case of Theorem 2.

Denote by ι_n (ε_n) the diagram consisting of a single row (column) of length n . For each partition $\lambda = (l_1, \dots, l_r)$ denote by ι_λ (ε_λ) any diagram such that $\iota_\lambda \in \iota_{l_1} \cdots \iota_{l_r}$ ($\varepsilon_\lambda \in \varepsilon_{l_1} \cdot \varepsilon_{l_2} \cdots \varepsilon_{l_r}$)—see Section 9; although the shape of ι_λ and ε_λ is not determined uniquely, it does not matter in the sequel. We shall consider the particular case of Theorem 2 when each of κ_1 and κ_2 is either of the form ι_λ or of the form ε_λ .

PROPOSITION 10. *Let $\lambda = (l_1, \dots, l_r)$, $\mu = (m_1, \dots, m_s)$ be partitions, and ν be a partition diagram.*

(a) *There exists the natural bijection between the set $\mathcal{P}(\iota_\lambda, \iota_\mu)$ (resp. $\mathcal{P}(\iota_\lambda, \varepsilon_\mu)$) and the set of $r \times s$ -matrices $K = (k_{ij})$ with integer nonnegative entries (resp. with entries 0 and 1) such that $\sum_{j=1}^s k_{ij} = l_i$ for $i = 1, \dots, r$ and $\sum_{i=1}^r k_{ij} = m_j$ for $j = 1, \dots, s$.*

(b) *There exists the natural bijection between the set $\mathcal{P}(\nu, \iota_\lambda)$ (resp. $\mathcal{P}(\nu, \varepsilon_\lambda)$) and the set of all column-strict (resp. row-strict) plane partitions of shape ν and type λ .*

Proof. In case (a) assign to each picture $f: \iota_\lambda \rightarrow \iota_\mu$ (resp. $f: \iota_\lambda \rightarrow \varepsilon_\mu$) the matrix $K = (k_{ij})$ defined by $k_{ij} = |f(\iota_{l_i}) \cap \iota_{m_j}|$ (resp. $k_{ij} = |f(\iota_{l_i}) \cap \varepsilon_{m_j}|$). In case (b) assign to each picture $f: \nu \rightarrow \iota_\lambda$ (resp. $f: \nu \rightarrow \varepsilon_\lambda$) the map $\varphi: \nu \rightarrow \mathbb{N}$ defined by

$$\varphi(x) = i \quad \text{when } f(x) \in \iota_{l_i} \quad (\text{resp. } f(x) \in \varepsilon_{l_i}).$$

We claim that these correspondences $f \mapsto K$ and $f \mapsto \varphi$ are just the desired bijections. The direct proof of these facts is left to the reader.

According to this Proposition, one can rewrite Theorem 2 in case $\kappa_1 = \iota_\lambda$, $\kappa_2 = \iota_\mu$ as follows: there exists the natural bijection between the set of matrices $K = (k_{ij})$, where $k_{ij} \in \mathbb{Z}_+$, $\sum_j k_{ij} = l_i$ and $\sum_i k_{ij} = m_j$, and the set of pairs (φ_1, φ_2) of column-strict plane partitions of the same shape, φ_1 being of type λ and φ_2 of type μ . Similarly, in case $\kappa_1 = \iota_\lambda$, $\kappa_2 = \varepsilon_\mu$ one obtains the natural bijection between the set of matrices $K = (k_{ij})$ with entries 0 and 1 such that $\sum_j k_{ij} = l_i$, $\sum_i k_{ij} = m_j$, and the set of pairs (φ_1, φ_2) of plane partitions of the same shape, φ_1 being column-strict of type λ and φ_2 row-strict of type μ . These correspondences are due to Knuth [4].

Theorem 1 and Proposition 10 allow one to compute the intertwining numbers $\langle \{\iota_\lambda\}, \{\iota_\mu\} \rangle$ and $\langle \{\iota_\lambda\}, \{\varepsilon_\mu\} \rangle$. It is well known that $\{\iota_n\}$ is the identity representation of S_n and $\{\varepsilon_n\}$ is the signum character of S_n . Hence by Proposition 3 $\{\iota_\lambda\}$ is the representation of $S_{|\lambda|}$ induced by the identity representation of the subgroup $S_{l_1} \times \cdots \times S_{l_r}$ (similarly for $\{\varepsilon_\lambda\}$). Intertwining numbers between these representations play the crucial role in the classical approach to the representation theory of symmetric groups. Moreover, in the recent work by Stanley [7] the whole representation theory of S_n is derived directly from the Knuth correspondence.

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