A quantitative description of Hawking radiation.

Drouot Alexis

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then the state dynamics must satisfy

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- If you want to study the dynamics of quantum fields, **you must study the backward propagation given by $U(0, t)$.**
- This reduces the analysis of quantum fields to (a) a PDE problem and (b) a (possibly difficult) computation.

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$$\Delta_r = r^2 \left(1 - \frac{\Lambda r^2}{3}\right) - 2M_0 r, \quad \Lambda, M > 0$$
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- This metric can be extended beyond the horizons $r = r_+$ and $r = r_-$.
- The surface gravities of the black hole and cosmological horizons are characteristic parameters given by:

$$\kappa_\pm = \frac{|\Delta_r'(r_\pm)|}{2r_\pm^2}.$$

Collapsing star in SdS

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- A collapsing star is a timelike submanifold

$$\mathcal{B} = \{(t, x, \omega) : x = z(t)\}$$

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- We want to study quantum fields in this space. We need an evolution equation for particles.

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- We consider spin-0 particles with mass m in the Schwarzschild–de Sitter spacetime. The equation is given by

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- We will focus only on (a) in this talk.

Asymptotic of scalar fields

Theorem [D '17]

Consider u_0, u_1 *smooth with compact support, and* u *solution of*

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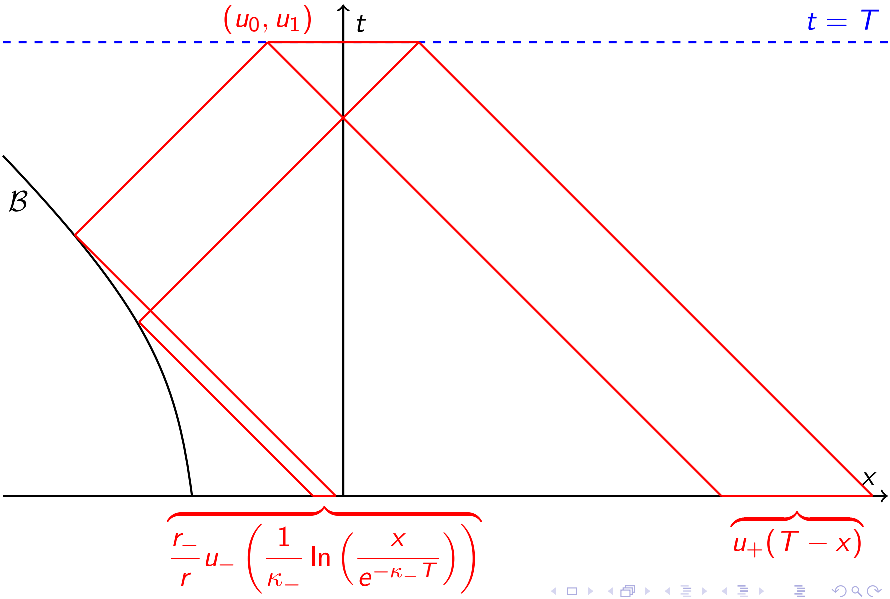
There exist scattering fields (see later) u_-, u_+ smooth and exponentially decaying; and $c_0 > 0$ such that for t near 0,

$$u(0, x, \omega) = \frac{r_-}{r} u_- \left(\frac{1}{\kappa_-} \ln \left(\frac{x}{e^{-\kappa_- T}} \right), \omega \right) + u_+(T - x, \omega) + O_{H^{1/2}}(e^{-c_0 T}).$$

(κ_- is the surface gravity of the black-hole.)

Pictorial representation



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- The fields u_- and u_+ are Freidlander's radiation fields; they do not depend on $\mathcal{B}$.
- Thus the result gives exponential convergence to equilibrium. The rate c_0 can be computed explicitly: it depends only on κ_-, κ_+ and the first resonance of the K–G equation on the black-hole background.

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- Thanks to the theorem:

$$\mathbb{E}^{\mathbb{H}_0, 2\pi/\kappa_+}(U(0,T)(u_0,u_1))$$

$$= \mathbb{E}^{D_x^2, 2\pi/\kappa_+}(u_+, D_x u_+) \cdot \mathbb{E}^{D_x^2, 2\pi/\kappa_-}(u_-, D_x u_-) \cdot \left(1 + O\left(e^{-c_0 T}\right)\right).$$

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- As time goes, this state splits to two Bose–Einstein states with respect to the asymptotic Hamiltonians D_x^2.
- The first one sees no change in temperature while the second one acquires the black-hole temperature $\kappa_-/(2\pi)$.

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- This work provides the first rates of convergence. The previous proofs were not fully constructive.
- We take full advantage of recent decay results for waves in black hole spacetimes. For the dS black-holes, see Bachelot–Motet-Bachelot ′93, Sa-Barreto–Zworski ′97 (resonances), Bony–Häfner ′07 (exponential decay), Dafermos–Rodnianski ′07 (polynomial decay), Melrose–Sa-Barreto–Vasy ′08, Vasy ′13 (geometric methods), Dyatlov ′11 −′ 12 (rotating black holes), Hintz–Vasy ′14− (non-linear results),...

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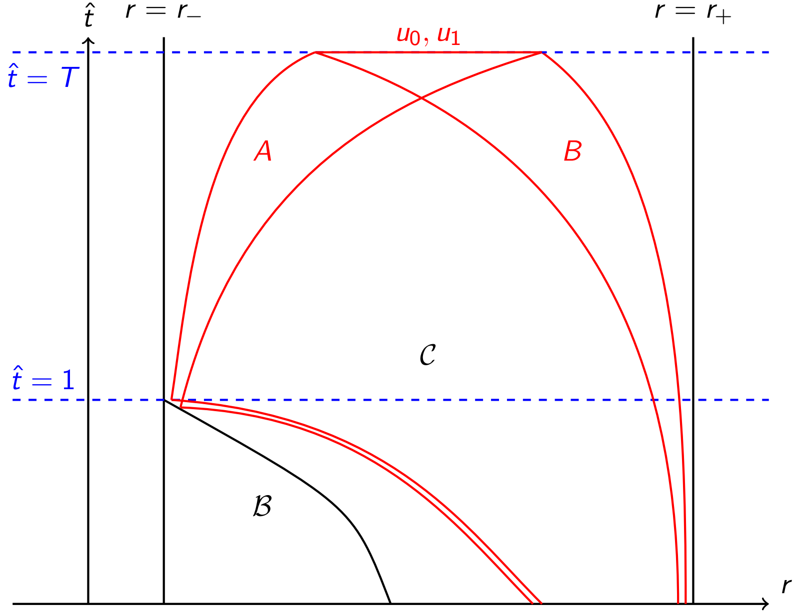
- We are studying the backward propagation. Nothing can cross the horizons. Hence the propagation takes place in compact smooth slices.
- After possibly rescaling, in $\hat{\mathcal{S}}$ the collapsing star is given by

$$\mathcal{B} = \{(t, z(\hat{t}), \omega)\}, \quad z(\hat{t}) = r_- - \alpha(\hat{t} - 1) + O(\hat{t} - 1)^2.$$

Propagation in $\hat{\mathcal{S}}$



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- Now we study two separate problems: propagation for $t \in [1, T]$ (before reflection) and propagation for $t \in [0, 1]$ (after reflection).

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- This strategy is due to Friedlander '80s (in the more complicated Euclidean scattering). For related perspectives in MGR, see Gérard–Georgescu–Häfner '14-'17, Nicolas '17, Dafermos–Rodnianski–Shlapentokh-Rothman '17.

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Theorem

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Let $\tilde{u}(\hat{t}, r, \omega) = u(-\hat{t} - 2F(r) + T, r, \omega)$ *(the time-reversed solution).*

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Then for some $\nu > 0$,

- $v_\pm(x, \omega) = 0$ *for* $x \leq 0$ *and* $v_\pm(x, \omega) = O(e^{-\nu x})$ *for large* x.

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$$\begin{cases} (\Box_g + m^2)u = 0 \\ (u, \partial_t u)(\hat{t} = T) = (u_0, u_1) \in C_0^\infty \end{cases}$$

Let $\tilde{u}(\hat{t}, r, \omega) = u(-\hat{t} - 2F(r) + T, r, \omega)$ (the time-reversed solution). Set $v_\pm$ be the traces of $\tilde{u}$ on the horizons $r_\pm$:

$$v_\pm(x, \omega) = \tilde{u}(x, r_\pm, \omega).$$

Then for some $\nu > 0$,

- $v_\pm(x, \omega) = 0$ *for* $x \leq 0$ *and* $v_\pm(x, \omega) = O(e^{-\nu x})$ *for large* x.
- $u(\hat{t}, r, \omega) - (v_+ + v_-)(T - \hat{t} - 2F(r), \omega) = O(e^{-\nu T})$ *as* $T \to +\infty$.

Semiclassical description of the blueshift effect

- Near the black holes, asymptotically backwards waves look like

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where $u_{-}(x, \omega) = 0$ for $x \leq 0$ and decays exponentially for $x \geq 0$.

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$$\begin{aligned} u_-(T - 2F(r), \omega) &\sim u_- \left(T + \frac{1}{\kappa_-} \ln(r - r_-), \omega \right) \\ &= u_- \left(\ln \left(\frac{r - r_-}{h} \right), \omega \right). \end{aligned}$$

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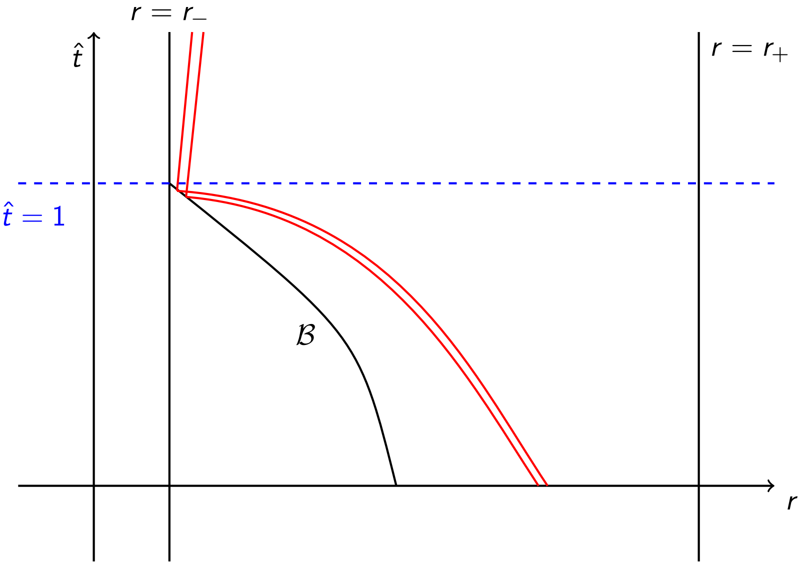
- The semiclassical wavefront set of the h-dependent distribution

$$u_- \left(\ln \left(\frac{r - r_-}{h} \right), \omega \right)$$

satisfies $\mathrm{WF}_h \subset \{(r_-, \omega, \xi, 0)\}$. This gives a semiclassical description of the blueshift effect.

Study of the reflection



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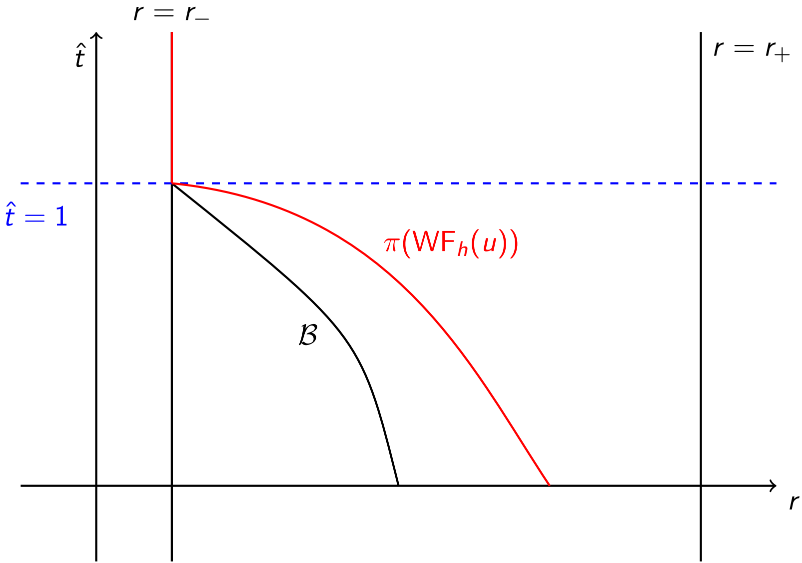

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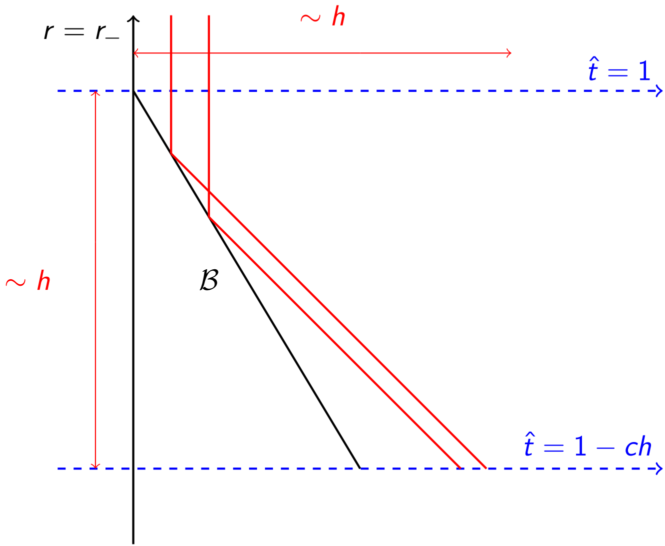
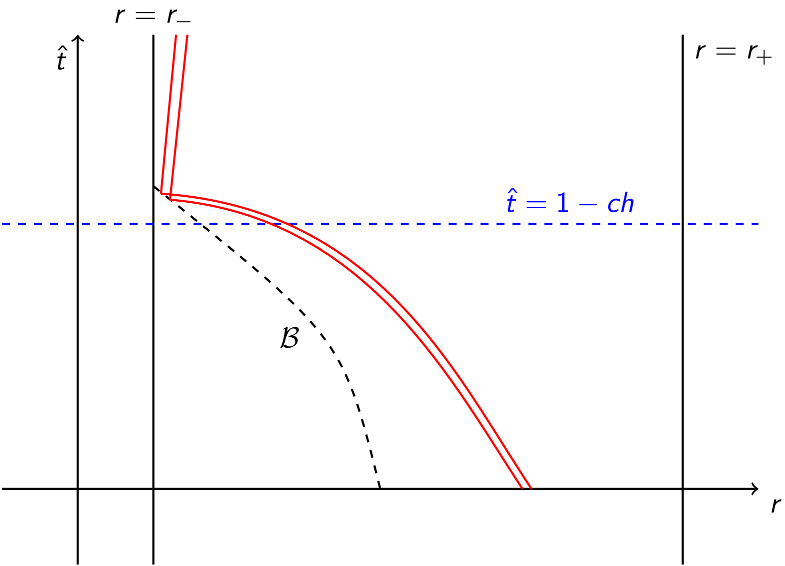
  The angular part does not matter because the reflecting data is only supported near radial frequencies.
- This gives a good enough approximation of u after reflection for times in $[1 - ch, 1]$ for any fixed $c > 0$.

Zoom in a box of size $O(h)$ near $r = r_-$ and $\hat{t} = 1$



Global study of the reflection



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- As the initial data is localized in frequencies $\sim h^{-1}$, we can construct a WKB approximate solution for $(\Box + m^2)u_{\mathrm{WB}} = 0$.
- The trace of the approximate solution is $O(h)$ on $\mathcal{B}$.
- By Hörmander's hyperbolic energy estimates, u (the solution with boundary) is well approximated by this explicit WKB parametrix for $t \in [0, 1 - ch]$, with error of order $O(h) = O(e^{-\kappa_- T})$.

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- Going back to $\mathcal{S}_*$, we get the theorem:

Theorem [D '17]

If u solves

$$\begin{cases} (\Box_g + m^2)u = 0 \\ (u, \partial_t u)(T) = (u_0, u_1) \in C_0^\infty, \qquad u|_{\mathcal{B}} = 0 \end{cases}$$

then there exist u_-, u_+ smooth and exponentially decaying; and $c_0 > 0$ such that for t near 0, in $\mathcal{S}_*$

$$u(0, x, \omega) = \frac{r_-}{r} u_- \left(\frac{1}{\kappa_-} \ln \left(\frac{x}{e^{-\kappa_- T}} \right), \omega \right) \text{ WKB part from BH}$$

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- This describes the PDE part of the problem. A delicate calculation remains to derive Hawking's radiation from here.

Extensions to non-symmetric backgrounds

- The simplest class consists of metric of the form

$$g = g_0 + \varepsilon\eta,$$

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- It is more technical because the WKB phases and amplitudes are no longer explicit; and because the angular propagation kicks in.

Asymptotic of scalar fields

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Consider u_0, u_1 *smooth with compact support, and* u *solution of*

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- *a solves the transport equation $g(\nabla a, \nabla\phi) + \Box\phi = 0$.*

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Thank you!