# Topologically protected edge states via highly oscillatory potentials.

Alexis Drouot – Joint work with Charles Fefferman and Michael Weinstein

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# Floquet–Bloch theory

We consider periodic operators:

$$\mathcal{P} = f(D_x) + V(x), \quad x \in \mathbb{R}, \quad f, V \in C^\infty(\mathbb{R}, \mathbb{R}),$$
$$V(x+1) = V(x), \quad f(\xi) \geq |\xi|^2 \text{ for } |\xi| \gg 1.$$

By periodicity, spaces of quasi-periodic functions

$$L^2_\xi(\mathbb{R}) \stackrel{\text{def}}{=} \{u \in L^2_{\text{loc}}(\mathbb{R}) : u(x+1) = e^{i\xi} u(x)\}$$

are invariant. Hence $L^2_\xi(\mathbb{R})$ admits a basis of eigenvectors of $\mathcal{P}$ with eigenvalues

$$\lambda_0(\xi) \leq \lambda_1(\xi) \leq \ldots$$

The spectrum of $\mathcal{P}$ on $L^2(\mathbb{R})$ is absolutely continuous, equal to

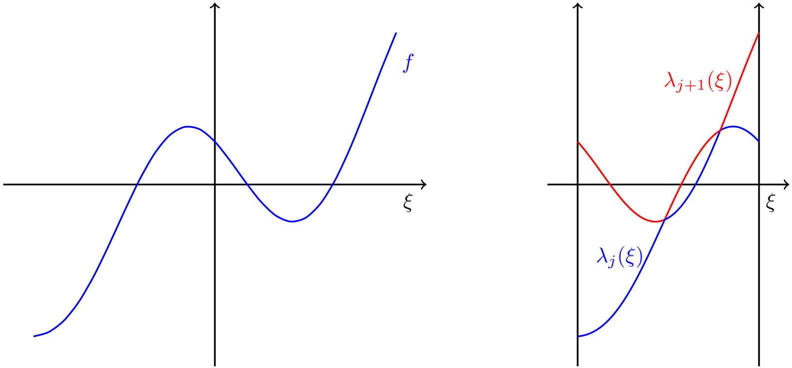
$$\{\lambda_j(\xi) : \xi \in [0, 2\pi), \ j \in \mathbb{N}\}.$$

# Example 1: $V \equiv 0$, $\mathcal{P} = f(D_x)$

In the case $V \equiv 0$, the eigenvalues of $\mathcal{P}$ on $L^2_\xi(\mathbb{R})$ are

$$\{f(\xi + 2\pi\ell) : \ell \in \mathbb{Z}\}.$$

We can then plot dispersion curves of $\mathcal{P}$ using the multi-valued function $\xi \mapsto f(\xi \mod 2\pi)$.

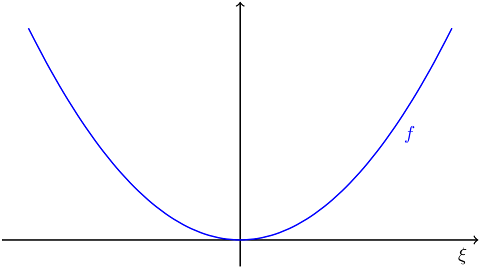


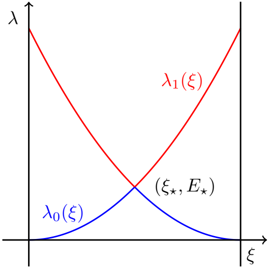
# Example 2: dimer models

Assume that $f$ and $V$ admit additional symmetries:

$$f(\xi) = f(-\xi), \quad V(x + 1/2) = V(x).$$

Basic example: $f(\xi) = \xi^2$, $V \equiv 0$.




There is a linear crossing: a Dirac point appears. Dirac points correspond to conical intersections of dispersion surfaces.

# Dirac points

Mathematically, Dirac points are pairs $(\xi_\star, E_\star)$ such that there exists $j, \nu$ with

$$\begin{aligned} \lambda_j(\xi) &= E_\star + \nu|\xi - \xi_\star| + O(\xi - \xi_\star)^2 \\ \lambda_{j+1}(\xi) &= E_\star - \nu|\xi - \xi_\star| + O(\xi - \xi_\star)^2. \end{aligned} \tag{1}$$

Their theoretical existence was postulated by Hamilton. Solutions of $D_t u = \mathcal{P} u$ supported at $t = 0$ near $\xi_\star$ are expected to approximately evolve according to

$$D_t u = (E_\star + \nu|D_x|)u. \tag{2}$$

Tremendous amount of work in the physics litterature.

Mathematical work: [Berry '80s], [Gérard '90], [Colin de Verdière '91], [Fefferman–Weinstein '12] (genericity of Dirac points, rigorous formulation of (**??**) as a matrix Dirac equation), [Lee '14] (point scatterers), [Fefferman–Lee-Thorp–Weinstein '16, '17] (perturbative results, tight binding regimes), [Berkolaiko–Comech '16] (symmetry-theoretic approach), [Kuchment '16] (survey),...

# Physical motivation

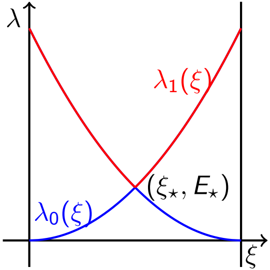
Mathematically speaking a material is an insulator at energy $\leq E$ if the corresponding operator has a spectral gap around $E$.
In dimer models Dirac points come from the existence of two symmetries:

$$x \mapsto -x, \ x \mapsto x + 1/2$$

Break the second symmetry by adding $\delta \cos(2\pi x)$: $\mathcal{P} = f(D_x)$ becomes

$$f(D_x) + \delta \cos(2\pi x).$$

An energy gap of size $\delta$ opens near the Dirac energy. The material becomes an insulator at energy $\leq E_\star$.



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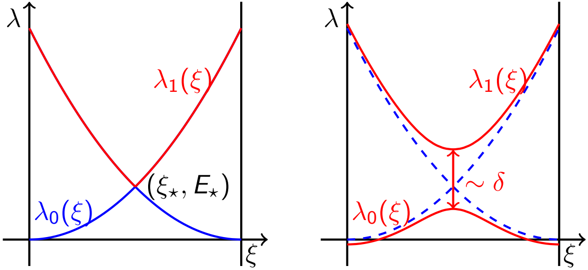
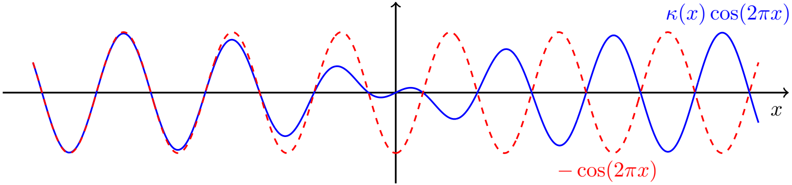


# Introducing phase defects

We study periodic structures with a phase defect. The typical potential is

$$\delta\kappa(\delta x)\cos(2\pi x) \quad : \quad \kappa(x) = \pm 1 \text{ for } x \text{ near } \pm\infty.$$

The potential "behaves" like $\cos(2\pi x)$ at both ends but acquires a phase defect when going from $-\infty$ to $+\infty$. The periodic structure is "stretched" in the middle.




We set $P = f(D_x) + \delta\kappa(\delta x)\cos(2\pi x)$. The essential spectrum is characterized by the asymptotic operators:

$$P_{\pm\delta} = f(D_x) \pm \delta\cos(2\pi x), \quad x \text{ near } \pm\infty$$

Hence $P$ has spectral gaps near Dirac energies of $f(D_x)$.

# Existing results

Recall that $D_x^2$ has a Dirac point at $(\pi, \pi^2)$.

### Theorem [Fefferman–Lee-Thorp–Weinstein '14]

*For $\delta$ sufficiently small, the operator $D_x^2 + \delta\kappa(\delta x)\cos(2\pi x)$ has an eigenvalue of energy $\pi^2 + O(\delta^2)$.*
*The corresponding eigenstate takes the form*

$$u(x) = \alpha_+(\delta x)e^{i\pi x} + \alpha_-(\delta x)e^{-i\pi x} + \dots$$

*where the vector $\alpha = (\alpha_-, \alpha_+)$ solves the Dirac equation*

$$\mathcal{D}\alpha = 0, \quad \mathcal{D} \overset{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} D_y + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \kappa(y)$$

**Comments:**

- $u \in L^2$ because $(-1, 1)$ is an essential spectrum gap of $\mathcal{D}$.
- This mode is topologically protected: it persists under arbitrarily large perturbations of $\kappa$ on compact sets.
- This supports the bulk/edge correspondence.

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**Comments:**

- The theorem still holds when $D_x^2$ is replaced by $D_x^2$ plus an even cosine series and $\cos(2\pi x)$ is replaced by an odd cosine series.
- This theorem is the basis for deeper results on topologically protected modes in honeycomb lattices.

# The multiscale analysis of [F–L-T–W '14, '16]

We derive formally this result with multiscale analysis. We look for an "linear" combination of Dirac eigenstates with slowly varying coefficients:

$$u(x,y) = \alpha_+(y)e^{i\pi x} + \alpha_-(y)e^{-i\pi x} + \delta v(x,y) + \dots, \quad y = \delta x.$$

In the variables $(x,y) \in \mathbb{S}^1 \times \mathbb{R}$:

$$D_x^2 + \delta\kappa(\delta x)\cos(2\pi x) \mapsto (D_x + \delta D_y)^2 + \delta\kappa(y)\cos(2\pi x).$$

Plug $u$ in RHS and group terms of order $1, \delta, \dots$:

$$\sum_{\pm} \alpha_\pm(y)(D_x^2 - \pi^2)e^{\pm i\pi x} = 0$$

$$(D_x^2 - \pi^2)v + \sum_{\pm} 2D_y\alpha_\pm(y) \cdot D_x e^{\pm i\pi x} + \alpha_\pm(y)\kappa(y)\cos(2\pi x)e^{\pm i\pi x} = 0.$$

The second equation has a solution iff the second term is $L_x^2(\mathbb{S}^1)$-orthogonal to $e^{\pm i\pi x}$. Thus we must have

$$\left\langle \sum_{\pm} 2D_y\alpha_\pm(y) \cdot D_x e^{\pm i\pi x} + \alpha_\pm(y)\kappa(y)\cos(2\pi x)e^{\pm i\pi x}, e^{\pm i\pi x} \right\rangle_{L_x^2(\mathbb{S}^1)} = 0$$

# The multiscale analysis of [F–L-T–W '14, '16]

This yields the Dirac equation

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} D_y \alpha_+ \\ D_y \alpha_- \end{bmatrix} + \kappa(y) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_+ \\ \alpha_- \end{bmatrix} = 0.$$

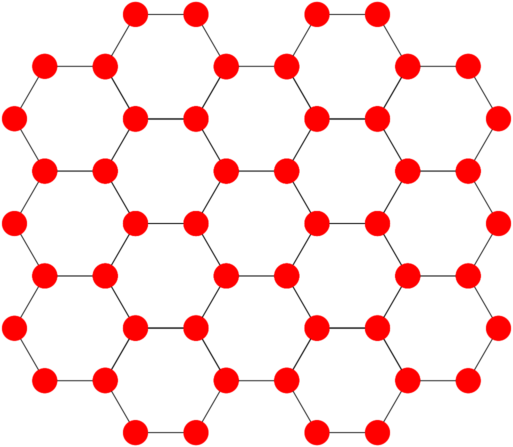
This way we construct a quasimode with energy in a spectral gap, hence there is an eigenvector with energy nearby. It is quite hard to show that

$$\alpha_+(\delta x) e^{i\pi x} + \alpha_-(\delta x) e^{-i\pi x} + \dots \tag{3}$$

is indeed an eigenstate. Selfadjoint principles only show that (**??**) is near a linear combination of eigenstates of $P$ with energy near $\pi^2$.

# Honeycomb lattices

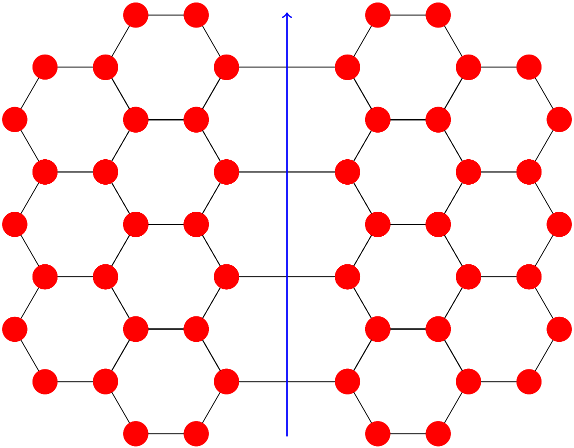
The model is a potential well at each vertex of a hexagonal lattice



Such structures generically admit Dirac points ([Fefferman–Weinstein '12], [Lee '14], [Berkolaiko–Comech '16], [Fefferman–Lee-Thorp–Weinstein '17]).

# Perturbation along an edge

An edge perturbation of honeycomb lattices is obtained by fixing (say) a rational edge and stretching adiabatically the system along this edge



[Fefferman–Lee-Thorp–Weinstein '16] studies the existence of states located along the edge with energy Dirac energies.

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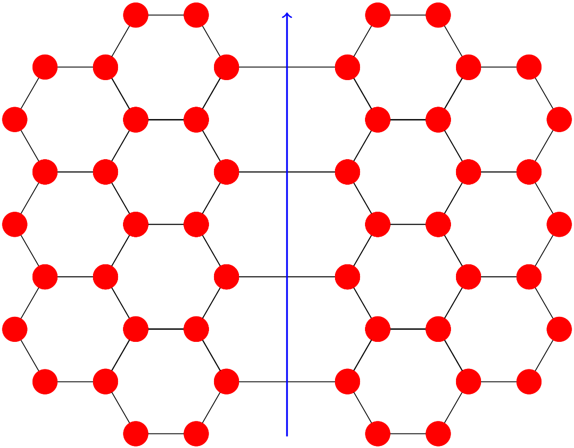
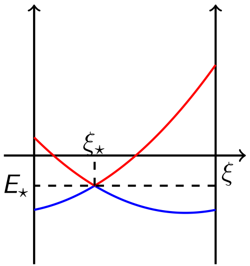


Such states accounts for the insulator/conductor characteristics of the material, depending on the direction of propagation.

# The no-fold condition of [F–L-T–W '16]

The existence of an eigenstate depends whether "stretching" the periodic structure along the edge opens a spectral gap "in the edge direction".



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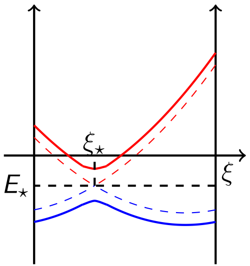

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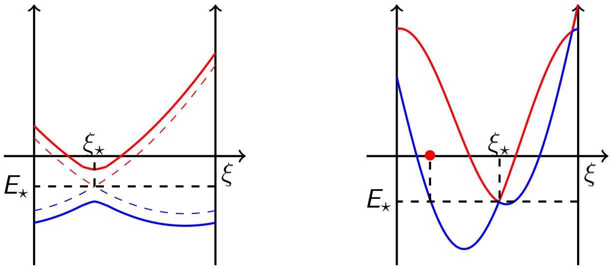

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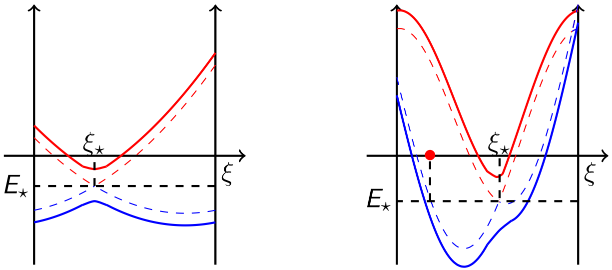

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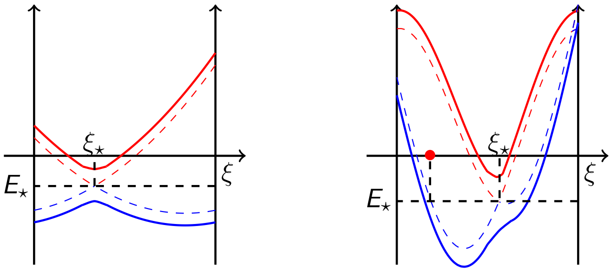


In the right picture, dispersion surfaces "fold over" the energy $E_\star$. [Fefferman–Lee-Thorp–Weinstein '16] conjectured that if the no-fold condition fails topologically protected resonances appear.

We will study 1D operators modeling the picture on the right. We first define resonances.

# Resonances of periodic systems [Gérard '90]

Resonances are poles of the meromorphic continuation of the resolvent.
If $T$ is periodic, we write

$$T(\xi) = T : \ L^2_\xi(\mathbb{R}) \to L^2_\xi(\mathbb{R}).$$

$T(\xi)$ has compact resolvent. By Floquet–Bloch theory,

$$T = \int_0^{\oplus 2\pi} T(\xi) d\xi, \quad \Im\lambda > 0 \ \Rightarrow \ (T-\lambda)^{-1} = \int_0^{\oplus 2\pi} (T(\xi)-\lambda)^{-1} d\xi.$$

Since $(T(\xi)-\lambda)^{-1}$ is periodic we can change the contour $[0, 2\pi]$ to the unit circle:

$$\Im\lambda > 0 \ \Rightarrow \ (T-\lambda)^{-1} = \oint_{\mathbb{S}^1} (\mathcal{T}(z)-\lambda)^{-1} \frac{dz}{iz}, \ \mathcal{T}(e^{i\xi}) = T(\xi).$$

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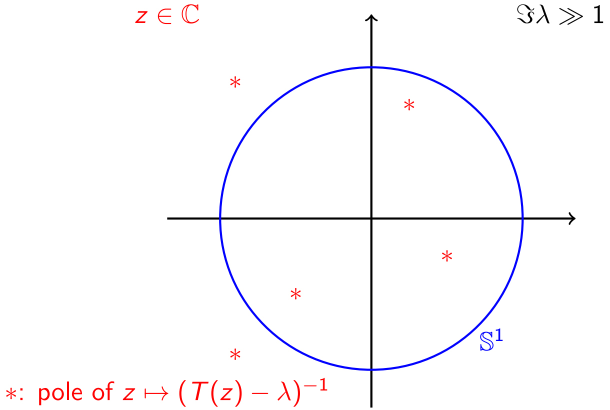
$z \mapsto (\mathcal{T}(z)-\lambda)^{-1}$ has complex poles; as $\lambda$ approaches $\mathbb{R}$, these poles converge to points on $\mathbb{S}^1$. Except when poles end up pinching $\mathbb{S}^1$, we can deform $\mathbb{S}^1$ to avoid them. Pinching points correspond to extrema of dispersion hypersurfaces and induce resonances.



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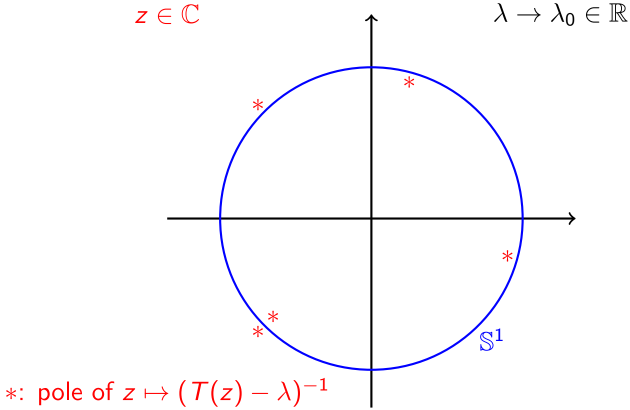

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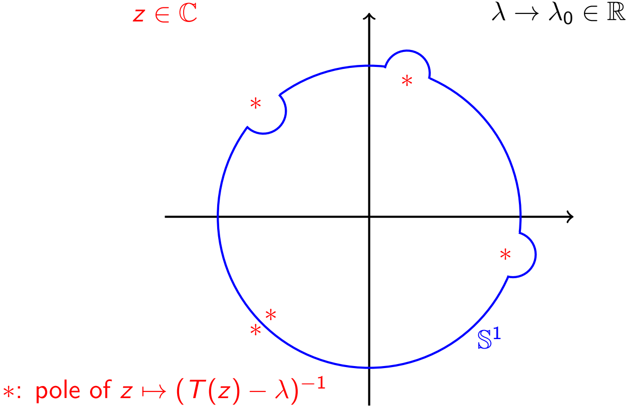

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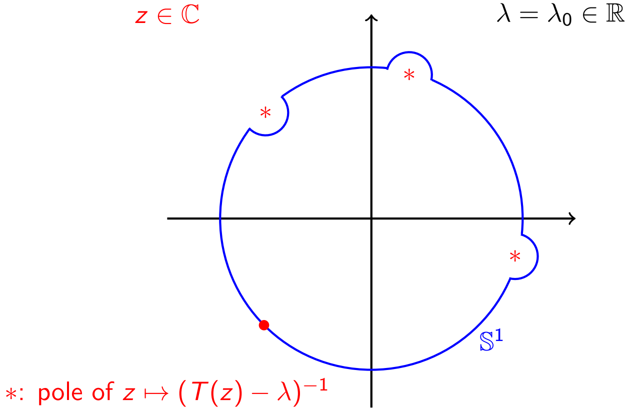


# Edge-perturbed periodic systems

The black-box approach of Sjöstrand–Zworski provides a meromorphic continuation of the resolvent of $P = f(D_x) + \delta\kappa(\delta x)\cos(2\pi x)$.
Asymptotic operators: $P_{\pm\delta} = f(D_x) \pm \delta\cos(2\pi x)$ for $x$ near $\pm\infty$.

This motivates an ad-hoc parametrix for $P - \lambda$:

$$Q(\lambda) = \frac{1-\kappa}{2}(P_{-\delta} - \lambda)^{-1} + \frac{1+\kappa}{2}(P_{\delta} - \lambda)^{-1}.$$

We observe that $(P-\lambda)Q(\lambda) = \mathrm{Id} + K(\lambda)$ with

$$K(\lambda) \overset{\text{def}}{=} \delta A_\delta \left((P_{-\delta} - \lambda)^{-1} - (P_\delta - \lambda)^{-1}\right), \quad A_\delta \text{ of lower order.}$$

This provides the meromorphic continuation of $(P-\lambda)^{-1}$:

$$(P-\lambda)^{-1} = Q(\lambda)(\mathrm{Id} + K(\lambda))^{-1}.$$

At distance $\sim\delta$ from Dirac energies, resonances are poles of $(\mathrm{Id} + K(\lambda))^{-1}$; the key operator is the resolvent difference $(P_{-\delta} - \lambda)^{-1} - (P_\delta - \lambda)^{-1}$.

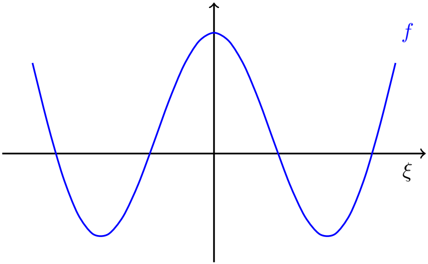
# Assumptions

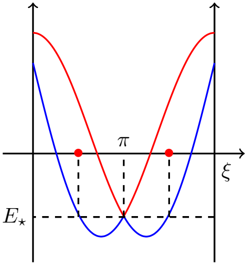
Let $P$ be of the form

$$P = f(D_x) + \delta\kappa(\delta x)\cos(2\pi x), \quad f \text{ analytic and even.}$$

Set $f'(\pi) = E_\star$; WLOG $f'(\pi) = 1$. We assume that

1. $f(\xi) = E_\star$ implies $f'(\xi) \neq 0$;
2. The only Dirac point of $P$ at energy $E_\star$ is $(\pi, E_\star)$.



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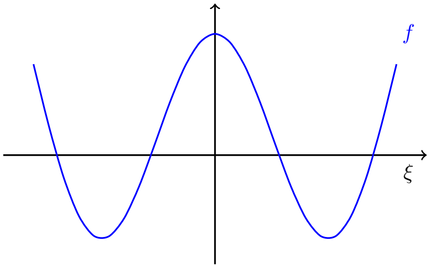


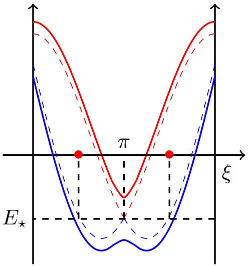

# Result $(P = f(D_x) + \delta\kappa(\delta x)\cos(2\pi x))$

The spectrum of the Dirac operator $\mathcal{D}$ is $(-\infty, -1] \cup [1, \infty) \cup \{\mu_j\}$ where $-1 < -\mu_n \leq ... \leq -\mu_1 < \mu_0 = 0 < \mu_1 \leq ... \leq \mu_n < 1$.

## Theorem [Drouot–Fefferman–Weinstein, in progress]

*Fix $\mu_n < \mu < 1$. For $\delta$ sufficiently small, $P$ continues meromorphically to $\mathbb{D}(E_\star, \mu\delta)$ and has exactly $2n+1$ resonances in this disk, given by*

$$\lambda_j = E_\star + \delta\mu_j + o(\delta).$$

*If in addition the no-fold condition is satisfied ($f(\xi) = 0$ iff $\xi = \pm\pi$) then these resonances are eigenvalues and the corresponding eigenstates are*

$$\alpha_{+,j}(\delta x)e^{i\pi x} + \alpha_{-,j}(\delta x)e^{-i\pi x} + ...$$

*where $(\alpha_{+,j}, \alpha_{-,j})$ are the eigenvectors of $\mathcal{D}$ at energy $\mu_j$.*

**Comments**

- This is some progress towards the F–L-T–W conjecture.

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**Comments**

- When the no-fold condition is satisfied, it characterizes all the eigenstates of $P$ in the gap. When $f(D_x) = D_x^2$ this improves the F–L-T–W theorem; and it proves the bulk-edge correspondence.

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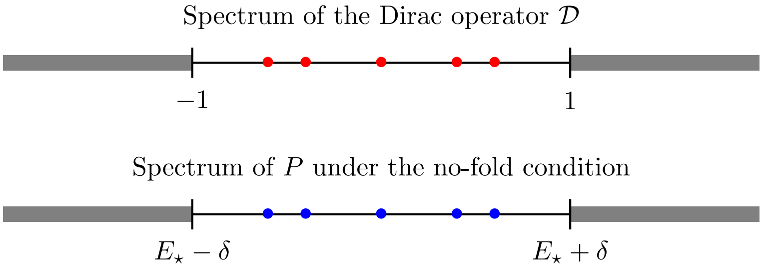
*where $(\alpha_{+,j}, \alpha_{-,j})$ are the eigenvectors of $\mathcal{D}$ at energy $\mu_j$.*

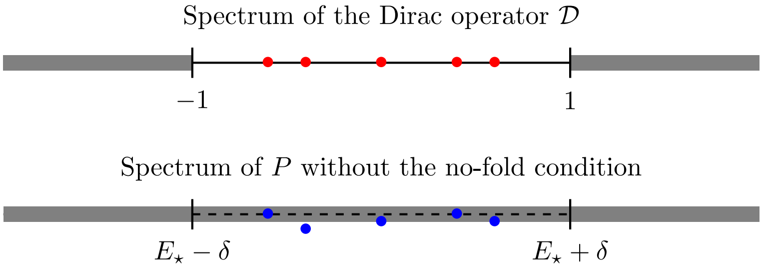
**Comments**

- However when the no-fold condition fails we cannot show that the resonances in $\mathbb{D}(E_\star, \mu\delta)$ are "true" resonances ($\Im\lambda_j < 0$). Classical perturbation theory seems to give only $\Im\lambda_j = O(\delta^\infty)$!

# Pictorial representation



# Pictorial representation



# Highly oscillatory potentials

## Theorem [Drouot–Fefferman–Weinstein, in progress]

*$E_\star + \mu\delta + o(\delta)$ is a resonance of $P$ if and only if $\mu^2 - 1$ is an eigenvalue of*

$$D_x^2 + V\left(x, \frac{x}{\delta}\right) \tag{4}$$

*where $V \in C_0^\infty(\mathbb{R} \times \mathbb{S}^1)$ is a $2 \times 2$ matrix potential with*

$$V\left(x, \frac{x}{\delta}\right) \rightharpoonup \mathcal{V} \overset{\text{def}}{=} \begin{bmatrix} \kappa^2 - 1 & -i\kappa' \\ i\kappa' & \kappa^2 - 1 \end{bmatrix}.$$

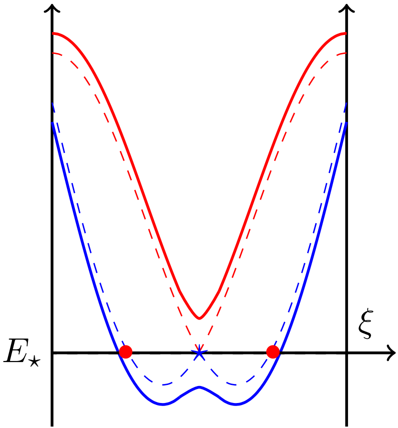
The resonances of $V(x, x/\delta)$ were completely described in [Drouot '15] (full expansion, derivation of effective potentials, ...) following work of [Duchêne–Vukićević–Weinstein '14]. They converge to those of $\mathcal{V}$.

The Dirac operator comes from

$$D_x^2 + \mathcal{V}(x) - (\mu^2 - 1) = (\mathcal{D} - \mu)(\mathcal{D} + \mu).$$

# Principle of proof (WLOG $E_\star = 0$)

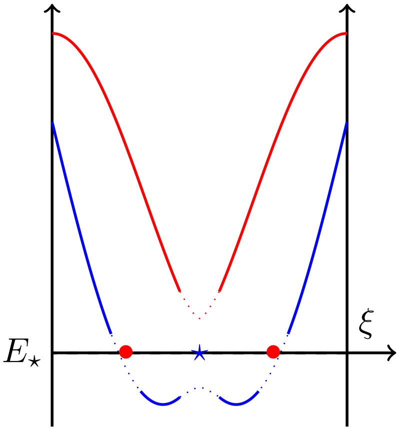
Goal: study the resolvent difference $(P_\delta(\xi) - \lambda)^{-1} - (P_{-\delta}(\xi) - \lambda)^{-1}$ after projection in energy/momenta in three cases:




(whole dispersion curves)

# Principle of proof (WLOG $E_\star = 0$)

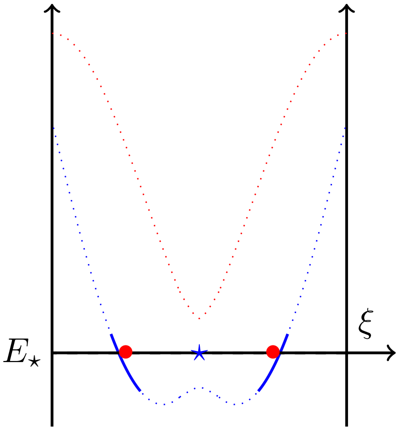
Goal: study the resolvent difference $(P_\delta(\xi) - \lambda)^{-1} - (P_{-\delta}(\xi) - \lambda)^{-1}$ after projection in energy/momenta in three cases:




**Case I:** Away from problems.

# Principle of proof (WLOG $E_\star = 0$)

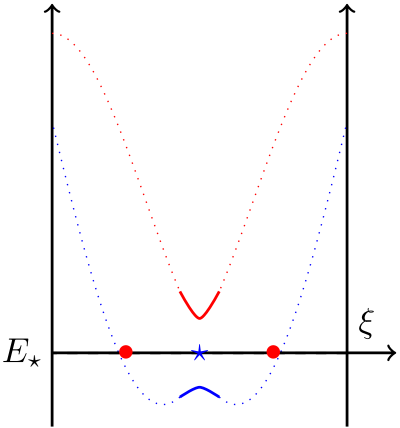
Goal: study the resolvent difference $(P_\delta(\xi) - \lambda)^{-1} - (P_{-\delta}(\xi) - \lambda)^{-1}$ after projection in energy/momenta in three cases:




**Case II:** Near resonant momenta.

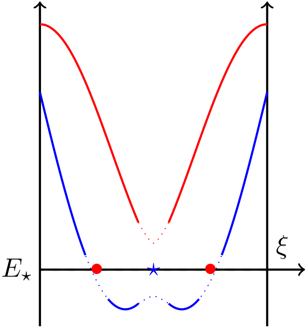
# Principle of proof (WLOG $E_\star = 0$)

Goal: study the resolvent difference $(P_\delta(\xi) - \lambda)^{-1} - (P_{-\delta}(\xi) - \lambda)^{-1}$ after projection in energy/momenta in three cases:




**Case III:** Near Dirac momenta.

# I. Momenta with energies away from 0



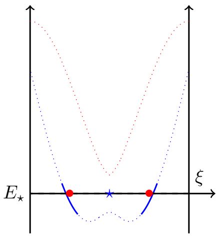

Look at the resolvent difference:

$$(P_\delta(\xi) - \lambda)^{-1} - (P_{-\delta}(\xi) - \lambda)^{-1}$$

$$= -2 \sum_{k=0}^{\infty} (P_0(\xi) - \lambda)^{-1} \left( \delta \cos(2\pi x)(P_0(\xi) - \lambda)^{-1} \right)^{2k+1} = O(\delta)$$

because $(P_0(\xi) - \lambda)^{-1}$ is sufficiently small when $\lambda \in \mathbb{D}(0, \delta)$ and momenta/energy are in that range. Integrate over such $\xi$ to deduce that this terms have negligible contributions.

# II. Near resonant momenta/energy




Use a Cauchy formula, then the resolvent difference formula:

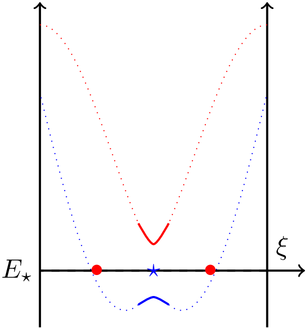
$$\frac{\pi_\delta(\xi)}{\lambda_{j,\delta}(\xi)-\lambda} - \frac{\pi_{-\delta}(\xi)}{\lambda_{j,-\delta}(\xi)-\lambda}$$

$$= \oint_0 \left((P_\delta(\xi)-z)^{-1} - (P_{-\delta}(\xi)-z)^{-1}\right) \frac{dz}{2\pi i(z-\lambda)}$$

$$= -2\sum_{k=0}^{\infty} \oint_0 (P_0(\xi)-\lambda)^{-1} \left(\delta\cos(2\pi x)(P_0(\xi)-\lambda)^{-1}\right)^{2k+1} \frac{dz}{2\pi i(z-\lambda)}.$$

Absence of resonances implies good bounds for complex-valued $\xi$.
Integrate over such $\xi$ to obtain negligible (complex) terms.

# III. Near Dirac momentum/energy



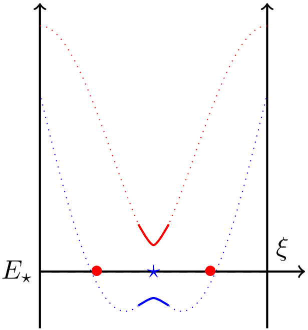

These terms contribute. In fact look at

$$\pi_\delta(\xi) P_\delta(\xi) \pi_\delta(\xi) \sim \pi_0(\xi) P_\delta(\xi) \pi_0(\xi).$$

In the above, $\pi_\delta(\xi)$ project onto the 2D vector space of Bloch modes and $\pi_0(\xi)$ projects on $\mathbb{C}e^{i(\xi-2\pi)x} \oplus \mathbb{C}e^{i\xi x}$. The matrix of $\pi_0(\xi) P_\delta(\xi) \pi_0(\xi)$ is

$$\begin{bmatrix} f(\xi - 2\pi) & \delta \\ \delta & f(\xi) \end{bmatrix} \sim \begin{bmatrix} -(\xi - \pi) & \delta \\ \delta & \xi - \pi \end{bmatrix} \text{ (WLOG } f'(\pi) = 1)$$

# III. Near Dirac momentum/energy

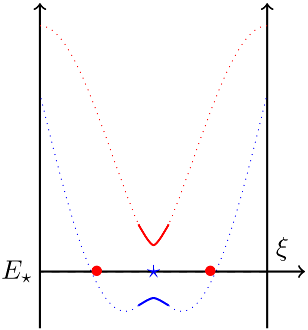



Hence $(P_\delta(\xi) - \lambda)^{-1} - (P_{-\delta}(\xi) - \lambda)^{-1}$

$$\sim \begin{bmatrix} -(\xi-\pi)-\lambda & \delta \\ \delta & \xi-\pi-\lambda \end{bmatrix}^{-1} - \begin{bmatrix} -(\xi-\pi)-\lambda & -\delta \\ \delta & \xi-\pi-\lambda \end{bmatrix}^{-1}$$

$$= \frac{2\delta}{(\xi-\pi)^2 - \lambda^2 + \delta^2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

# III. Near Dirac momentum/energy




We deduce

$$(P_\delta(\xi) - \lambda)^{-1} - (P_{-\delta}(\xi) - \lambda)^{-1} \sim \frac{2\delta e^{i(\xi-2\pi)x} \otimes e^{i\xi x}}{(\xi - \pi)^2 - \lambda^2 + \delta^2} + \text{s.t.}$$

# Connection to the Laplacian resolvent

We showed: $(P_\delta(\xi)-\lambda)^{-1}-(P_{-\delta}(\xi)-\lambda)^{-1}\sim \dfrac{2\delta e^{i(\xi-2\pi)x}\otimes e^{i\xi x}}{(\xi-\pi)^2-\lambda^2+\delta^2}+\text{s.t.}$

Hence using $-\lambda^2+\delta^2=-z^2\delta^2$ we get

$$(P_\delta-\lambda)^{-1}-(P_{-\delta}-\lambda)^{-1}\sim\int_{|\xi-\pi|\le\delta^{1/3}}\frac{2\delta e^{i(\xi-2\pi)x}\otimes e^{i\xi x}}{(\xi-\pi)^2-\delta^2z^2}+\text{s.t.}$$

The RHS has kernel

$$(x,x')\mapsto\int_{|\xi-\pi|\le\delta^{1/3}}\frac{2\delta e^{i(\xi-2\pi)x-i\xi x'}}{(\xi-\pi)^2-\delta^2z^2}d\xi.$$

Rescale using $\xi-\pi\mapsto\delta\xi$, $x\mapsto x/\delta$, $x'\mapsto x'/\delta$ to get

$$e^{i\pi x/\delta}\cdot\int_{|\xi|\le\delta^{-2/3}}\frac{2e^{i\xi\cdot(x-x')}}{\xi^2-z^2}d\xi\cdot e^{i\pi x'/\delta}.$$

This is how the resolvent of the Laplacian appears. Further algebraic manipulations show that the problem reduces to the analysis of $D_x^2+V(x,x/\delta)$. [Drouot '15] yields the Theorem.

# Remaining problems/projects

- Show that these resonances have generically non-zero imaginary part.
- Study the dynamics of near-resonant states.
- Extend the analysis to the hexagonal lattice, Lieb lattice,...

**Thanks for your attention!**