Semiclassical Dirac operators for topological insulators

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In honor of Hart Smith

Overview

Goal. Study solutions to

$$(hD_t + \not{D})\Psi_t = 0, \qquad 0 < h \ll 1, \qquad D_t \overset{\text{def}}{=} \frac{1}{i}\partial_t \tag{E}$$

and $\not{D}$ is a semiclassical Dirac operator on $\mathbb{R}^2$: $\not{D} = \not{\partial}^w$ with

$$\not{\partial}(x,\xi) = \sum_{j=1}^{3} p_j(x,\xi)\sigma_j = \begin{bmatrix} p_3(x,\xi) & p_1(x,\xi) - ip_2(x,\xi) \\ p_1(x,\xi) + ip_2(x,\xi) & -p_3(x,\xi) \end{bmatrix}.$$

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$$\not{D} = \begin{bmatrix} \kappa(x) & \big(hD_1 - A_1(x)\big) - i\big(hD_2 - A_2(x)\big) \\ \big(hD_1 - A_1(x)\big) + i\big(hD_2 - A_2(x)\big) & -\kappa(x) \end{bmatrix}.$$

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Motivation. Explain transport phenomena central to topological insulators.

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Tools. Local symplectic geometry, Fourier integral operators, WKB analysis.

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Brief explanation. Key objects: eigenvalues of $\not{\partial}(x,\xi)$:

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- For a given $x_0 \notin \gamma$, $\pm\left|p(x_0,\xi)\right|$ are distinct for all ξ. Yields eigenbundles $\mathcal{E}_-(x_0) \to \mathbb{R}^2_\xi$. Topology encoded in a number $c_1(x_0) \in \{\pm 1/2\}$.
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Bulk-edge correspondence. γ supports 1 traveling state: asymmetric, topologically protected transport.

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Questions

Can we identify this state?
What happens if forcing propagation in the opposite direction?

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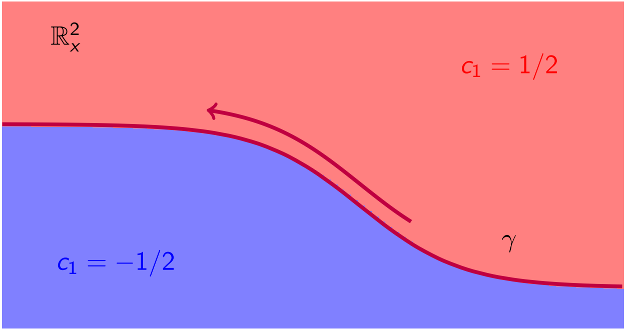

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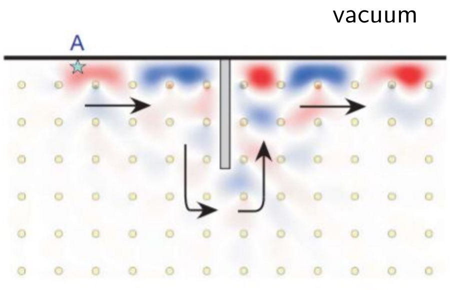

Photonic experiment [Wang et al '09]

Brief semiclassical analysis

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$$\frac{d(x_t,\xi_t)}{dt} = H_{\pm|p|}(x_t,\xi_t) = \pm H_{|p|}(x_t,\xi_t),$$

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> No asymmetric transport where the eigenvalues of $\not{\partial}$ are distinct
> $\Rightarrow$ **We must study (E) at the crossing.**

Technical setup

Goal: Study the propagation of singularities for

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Wavepackets

$\Psi : \mathbb{R}^2 \to \mathbb{C}^2$ *is a wavepacket concentrated at* (x_0, ξ_0) *if*

$$\Psi(x) \;=\; e^{\frac{i}{h}\xi_0(x-x_0)} \cdot \frac{1}{\sqrt{h}} a\left(\frac{x-x_0}{\sqrt{h}}\right) \qquad \textit{(up to a phase)}$$

for some $a \in \mathcal{S}(\mathbb{R}^2, \mathbb{C}^2)$. *If* $a(x) = \alpha(x)u$ *for some* $u \in \mathbb{C}^2$, $\alpha \in \mathcal{S}(\mathbb{R}^2, \mathbb{C})$, *we say that* Ψ *is oriented along* $\mathbb{C}u$.

Wavepackets are normalized in L^2 and of order $h^{-1/2}$ in L^∞.

Technical setup

Transversality assumption

p_1, p_2, p_3 vanish transversely along $\Gamma = \{p_1 = p_2 = p_3 = 0\}$:

$$p_1 = p_2 = p_3 = 0 \ \Rightarrow \ dp_1, dp_2, dp_3 \text{ are linearly independent.}$$

At least one Poisson bracket $\{p_j, p_k\}$ is $\neq 0$. Otherwise the kernels of dp_j are equal to the span of $H_{p_1}, H_{p_2}, H_{p_3}$ and dp_j are multiple of each other.

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Flow on Γ

Given $(x_0, \xi_0) \in \Gamma$ we define (x_t, ξ_t) by:

$$\frac{d(x_t, \xi_t)}{dt} = V_{\not\partial}(x_t, \xi_t), \qquad V_{\not\partial} \overset{\text{def}}{=} -\frac{\sum \varepsilon_{jk\ell}\{p_j, p_k\} H_{p_\ell}}{\left(2\sum \{p_j, p_k\}^2\right)^{1/2}} = -2i \frac{Tr\left[\{\not\partial, \not\partial\} H_{\not\partial}\right]}{Tr\left[\{\not\partial, \not\partial\}^2\right]}.$$

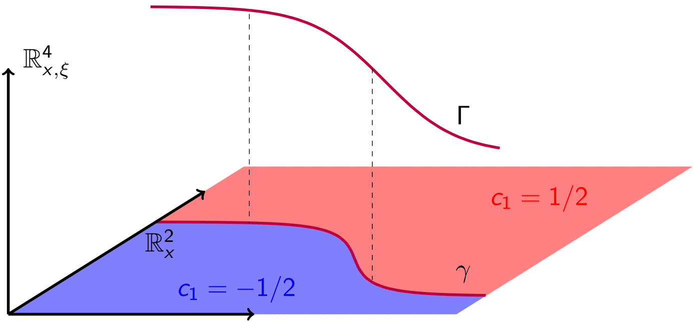
$V_{\not\partial}$ is tangent to Γ; invariant under symplectic changes of variables; invariant under conjugation of $\not\partial$ by a $SU(2)$ map.

Technical setup

Topological assumption

The projection $\pi : \Gamma \subset \mathbb{R}^4_{x,\xi} \times \mathbb{R}^2_{\xi} \to \gamma \subset \mathbb{R}^2_x$ *is a diffeomorphism.*

(OK if $\pi|_\Gamma$ is a local diffeomorphism near (x_0, ξ_0) under small time assumption)



Main result

Theorem [D'22]

There exists $T > 0$ with the following property. Let Ψ_0 be a wavepacket concentrated at $(x_0, \xi_0) \in \Gamma$ and Ψ_t the solution to

$$(hD_t + \not{D})\Psi_t = 0. \tag{E}$$

There is a wavepacket Φ_t concentrated at $(x_t, \xi_t) = e^{tV_{\not\partial}}(x_0, \xi_0)$ such that:

$$\Psi_t = \Phi_t + O_{L^\infty}(h^{-1/4}) + O_{L^2}(h^{1/2}), \qquad t \in (0, T). \tag{R}$$

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- In L^∞, $\Psi_t \simeq \Phi_t$, generally of order $h^{-1/2}$;
- In L^2, $\Psi_t \simeq \Phi_t + O_{L^\infty}(h^{-1/4})$: partial loss of coherence.
- Coherent transport occurs along Γ at speed $V_{\not\partial}$: undirectional and independent of the initial profile.

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- If Ψ_0 is appropriately oriented, then (R) holds for all $T < \infty$ and has no $O_{L^\infty}(h^{-1/4})$ remainder.
- If Ψ_0 has the orthogonal orientation then $\Phi_t = 0$.
- Under reversibility and spreading assumptions on an emerging flow, (R) holds for all $T < \infty$. (Happens when a certain quantity λ is constant)

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Few results on this system. Recent work [Bal '22]: long-time statement for

$$\not{\partial}(x, \xi) = \sum_{k,\ell=1}^{2} a_{k\ell}(x)\xi_k\sigma_\ell + \kappa(x)\sigma_3, \qquad \|\nabla\kappa(x)\| = 1.$$

In this case $\Gamma \subset \mathbb{R}^2_x \times \{0\}^2$ (eliminates need for microlocal techniques) and $\lambda = 1$ (the emerging flow is spreading and reversible).

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Systems with crossings:

- Symmetric systems [Braam–Duistermaat '95, Colin de Verdière '03, Nolan–Uhlmann '06]
- Landau–Zener transtion [Haggedorn, Joye '90s, Colin de Verdière, Fermanian Kammerer, Gérard '00s, Lasser, Gamble, Hari, . . .]

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- **Wavepackets.** [Smith '98, Smith–Tataru '05, Smith '06, Smith–Sogge '07, dots]: systematic use of wavepacket superpositions to produce parametrices and prove L^p bounds.

Example

Magnetic Dirac operator:

$$\not{D} = \begin{bmatrix} \kappa(x) & \nabla_A^* \\ \nabla_A & -\kappa(x) \end{bmatrix}, \qquad \nabla_A = \partial_1 - A_1(x) + i(\partial_2 - A_2(x)),$$

$$\Gamma = \{(x,\xi) : \kappa(x) = 0,\ \xi = A(x)\}, \qquad \gamma = \{x : \kappa(x) = 0\}$$

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Magnetic field: $B = \partial_2 A_1 - \partial_1 A_2$. Transversality and (local) topological assumptions: $\nabla\kappa(x) \neq 0$ when $\kappa(x) = 0$.

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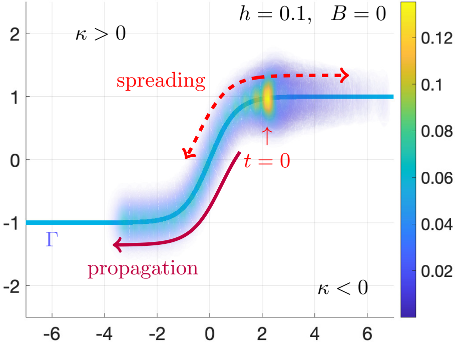

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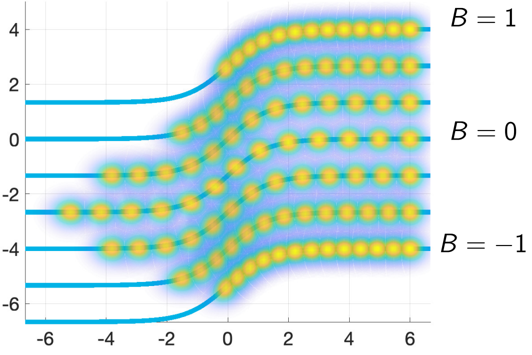

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Wavepackets travel along $\gamma = \kappa^{-1}(0)$ at speed:

$$\frac{\|\nabla\kappa(x)\|}{\sqrt{\|\nabla\kappa(x)\|^2 + B(x)^2}}. \qquad \text{(V)}$$

[Bal–Becker–D '22]: by-hand construction;
[D' 22]: semiclassical framework discussed here.

(V) remains true on curved backgrounds (replace $\|\cdot\|$ by the Riemannian norm)

Ideas of proof

We can present a complete proof under the assumption:

Linear assumption

The symbols p_1, p_2, p_3 *(hence* $\not{\partial}$*) are linear in* (x, ξ). *(LA)*

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Three steps:

1. Reduce (via $SU(2)$-conjugation and symplectomorphism) the symbol $\not{\partial}$ to
$$\not{\partial}_0 = \begin{bmatrix} \xi_1 & \xi_2 - ix_2 \\ \xi_2 + ix_2 & -\xi_1 \end{bmatrix}, \qquad \text{up to multiplicative constant.}$$
2. Produce explicit solutions to $(hD_t + \not{D}_0)\Psi_t = 0$ and prove quantitative estimates.
3. Deduce quantitative estimates for $(hD_t + \not{D})\Psi_t = 0$.

Symplectic reduction of $\not\partial = \sum p_j \sigma_j$

Key object: $$M_{\not\partial} \overset{\text{def}}{=} \frac{1}{2i}\{\not\partial, \not\partial\} = \begin{bmatrix} \{p_1, p_2\} & \{p_2, p_3\} - i\{p_3, p_1\} \\ \{p_2, p_3\} + i\{p_3, p_1\} & -\{p_1, p_2\} \end{bmatrix}.$$

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$$U \cdot \not\partial \circ S \cdot U^* = \lambda \cdot \widetilde{\not\partial} \circ S = \lambda \cdot \begin{bmatrix} \xi_1 & \xi_2 - ix_2 \\ \xi_2 + ix_2 & -\xi_1 \end{bmatrix} = \lambda \cdot \not\partial_0.$$

Symplectic reduction of $\not\partial = \sum p_j \sigma_j$

Linear reduction theorem

Under (LA), there exists $U \in SU(2)$, $\lambda > 0$ *and* S *symplectic* 4×4 *such that*

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Linear reduction theorem (quantized)

Under (LA), there exists a Fourier integral $\mathcal{F}$ operator such that:

$$\mathcal{F}^{-1} \cdot \not D \cdot \mathcal{F} = \lambda \cdot \begin{bmatrix} \xi_1 & \xi_2 - ix_2 \\ \xi_2 + ix_2 & -\xi_1 \end{bmatrix} = \lambda \cdot \not D_0.$$

Related results [Braam–Duistermaat '95, Colin de Verdiére '03-'04] with quite different proofs.

Hermite reduction

$\not{D}_0$-equation:
$$\left(hD_t + \lambda \begin{bmatrix} hD_1 & hD_2 - ix_2 \\ hD_2 + ix_2 & -hD_1 \end{bmatrix}\right)\Psi_t = 0 \qquad \text{(M)}$$

$$\Leftrightarrow \qquad \left(h\partial_t + \lambda \begin{bmatrix} h\partial_1 & \mathfrak{a} \\ \mathfrak{a}^* & h\partial_1 \end{bmatrix}\right)\Psi_t = 0, \qquad \begin{cases} \mathfrak{a}^* = x_2 - h\partial_2 \\ \mathfrak{a} = x_2 + h\partial_2 \end{cases}.$$

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$$\begin{cases} \mathfrak{a} g_{n+1} = \sqrt{2nh} g_n \\ \mathfrak{a}^* g_n = \sqrt{2nh} g_{n+1} \end{cases} \Rightarrow \begin{bmatrix} 0 & \mathfrak{a} \\ \mathfrak{a}^* & 0 \end{bmatrix} G_n = \begin{bmatrix} 0 & \sqrt{2nh} \\ \sqrt{2nh} & 0 \end{bmatrix} G_n, \qquad G_n \overset{\text{def}}{=} \begin{bmatrix} g_n \\ g_{n+1} \end{bmatrix}.$$

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Decompose $\Psi_t(x) = \sum_n \psi_t^{(n)}(x_1) \otimes G_n(x_2)$ then (M) decouples as:

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Emerging new semiclassical parameter: $\sqrt{h}$. Study separately $n = 0$, $n > 0$.

Propagating mode

$$n = 0 : \qquad \left(\sqrt{h}\partial_t + \lambda \begin{bmatrix} \sqrt{h}\partial_1 & 0 \\ 0 & -\sqrt{h}\partial_1 \end{bmatrix} \right) \psi_t^{(0)} = 0.$$

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Rightwards (top) and leftwards (bottom) modes. But

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$$g_0 = 0 \quad \Rightarrow \quad G_0 = \begin{bmatrix} 0 \\ g_1 \end{bmatrix}.$$

So these induce **only leftwards modes:**

$$\begin{aligned} \Psi_t(x) &= \sum_n \psi_t^{(n)}(x_1) \otimes G_n(x_2) \\ &= \psi_0^{(0)}(\lambda t + x_1) g_1(x_2) + \sum_{n>0} \psi_t^{(n)}(x_1) \otimes G_n(x_2). \end{aligned}$$

Dispersive modes

$$n > 0 : \quad \left(\sqrt{h}\partial_t + \lambda \begin{bmatrix} \sqrt{h}\partial_1 & \sqrt{2n} \\ \sqrt{2n} & \sqrt{h}\partial_1 \end{bmatrix} \right) \psi_t = 0$$

(dropped n)

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Explicit solution:

$$\psi_t(x_1) = \int_{\mathbb{R}} \exp\left(\frac{i}{\sqrt{h}}\left(\xi_1 x_1 - i\lambda t \begin{bmatrix} \xi_1 & \sqrt{2n} \\ \sqrt{2n} & -\xi_1 \end{bmatrix}\right)\right) \widehat{\psi_0}(\xi_1) d\xi_1 \qquad \text{(I)}$$

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(I) is an oscillatory integral:

- Large parameter $h^{-1/2}$,
- Amplitude independent of h when Ψ_0 is a wavepacket:

$$\widehat{\psi_0}(\xi_1) = \int_{\mathbb{R}} e^{-i\frac{\xi_1}{\sqrt{h}}x_1}\psi_0(x_1)\frac{dx_1}{2\pi\sqrt{h}} = \int_{\mathbb{R}} e^{-i\xi_1 x_1} a_0(x_1)\frac{dx_1}{2\pi}$$

- Matrix phase with eigenvalues

$$\phi_{t,x_1}(\xi_1) = \xi_1 x_1 \pm \lambda t\sqrt{\xi_1^2 + 2n}.$$

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Note $\phi''_{t,x_1}(\xi_1) > 0$. Van der Corput: expect dispersion: (I) $= O_{L^\infty}(h^{1/4})$.

Dispersive modes

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Since $G_n = O_{L^\infty}(h^{-1/2})$, end up with:

$$\begin{aligned}\Psi_t(x) &= \psi_0^{(0)}(\lambda t + x_1)g_1(x_2) + \sum_{n>0}\psi_t^{(n)}(x_1)\otimes G_n(x_2) \\ &= \underbrace{\psi_0^{(0)}(\lambda t + x_1)g_1(x_2)}_{\textit{propagating}} + \underbrace{O_{L^\infty}(h^{-1/4})}_{\textit{dispersive}}\end{aligned}$$

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$$n > 0 : \quad \left(\sqrt{h}\partial_t + \lambda \begin{bmatrix} \sqrt{h}\partial_1 & \sqrt{2n} \\ \sqrt{2n} & \sqrt{h}\partial_1 \end{bmatrix}\right) \psi_t = 0 \qquad \text{(dropped } n\text{)}$$

Explicit solution:

$$\begin{aligned} \psi_t(x_1) &= \int_{\mathbb{R}} \exp\left(\frac{i}{\sqrt{h}}\left(\xi_1 x_1 - i\lambda t \begin{bmatrix} \xi_1 & \sqrt{2n} \\ \sqrt{2n} & -\xi_1 \end{bmatrix}\right)\right) \widehat{\psi_0}(\xi_1) d\xi_1 \\ &= O_{L^\infty}(h^{1/4}). \end{aligned} \tag{I}$$

Since $G_n = O_{L^\infty}(h^{-1/2})$, end up with:

$$\begin{aligned} \Psi_t(x) &= \psi_0^{(0)}(\lambda t + x_1) g_1(x_2) + \sum_{n>0} \psi_t^{(n)}(x_1) \otimes G_n(x_2) \\ &= \underbrace{\psi_0^{(0)}(\lambda t + x_1) g_1(x_2)}_{\text{propagating}} + \underbrace{O_{L^\infty}(h^{-1/4})}_{\text{dispersive}} \end{aligned}$$

Apply back the Fourier integral operator for estimates on the initial operator (uses the topological assumption). **This proves the theorem for linear symbols.**

General dependence of p_1, p_2, p_3 in (x, ξ)

- Reduction theorem: $$\not{\partial} \sim \left(\lambda(x_1) + \mu(x_1)\xi_1\right) \cdot \begin{bmatrix} \xi_1 & \xi_2 - ix_2 \\ \xi_2 + ix_2 & -\xi_1 \end{bmatrix}.$$

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$$\partial_t \phi \pm \lambda(x_1)\sqrt{(\partial_1 \phi)^2 + 2n} = 0$$

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  Hence $\partial_{\xi_1}^2\phi(t, x_1, \xi_1) > 0$ for small times: still dispersion. Longer times: we do not know, it depends on spreading / reversibility of the eikonal flow.

Conclusion and ongoing work

We studied Dirac waves for

$$\not{D} = \not{\partial}^W, \qquad \not{\partial} = \begin{bmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{bmatrix}.$$

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Generic Hermitian 2×2 symbol. Rather different dynamics: expect Landau–Zener transitions [Quan – ongoing].

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Happy birthday, Hart!