Microlocal analysis of the bulk-edge correspondence.

Alexis Drouot, Columbia University

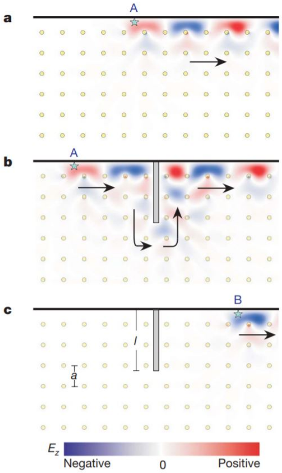
October 16th, MSRI

Plan of the talk

- **Physical motivation**
  - Waves in 2D materials guided along 1D interfaces
  - Robustness against defects
- **Mathematical framework for the bulk-edge correspondence**
  - Spectral theory for periodic Schrödinger operators
  - Bulk index: a Chern number **[Thouless–Kohmoto–Nightingale–den Nijs '82]**
  - Edge index: a conductivity / trace formula **[Kellendonk–Richter–Schulz-Baldes '02]**
- **Bulk-edge correspondence: a microlocal approach**
  - Edge index invariance
  - Semiclassical deformation
  - Trace asymptotics
- **Future perspectives**

Photonic experiments

[Haldane–Raghu '08, Wang–Chong–Joannopoulos–Soljacic '09] Periodic medium altered by a magnetic field, with a physical boundary. Waves are emitted from $\star$.

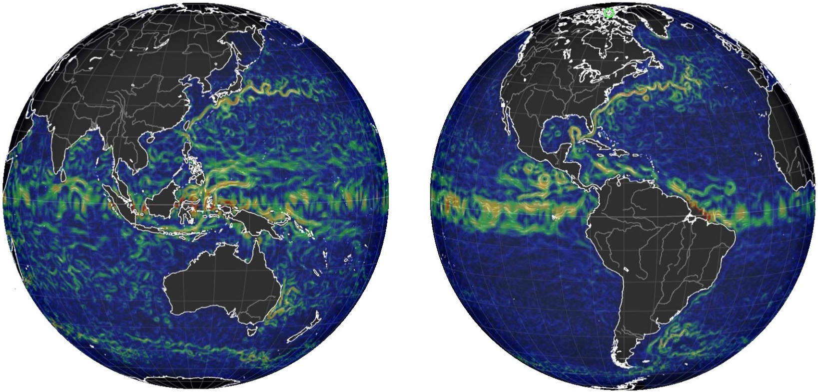


a. Source (A) emits a signal. It propagates rightward.

b. Source (A) emits a signal. It hits a defect (located to the right of A), moves around it, and keeps propagating rightward.

c. Source (B) emits a signal. It propagates rightward. The defect does not affect propagation.

Equatorial waves



Eastward-propagating currents, `earth.nullschool.net`, Feb. 20th 2019.

Theoretical analysis demonstrate their **topological character** **[Delplace–Martson–Venaille '17, Tauber–Delplace–Venaille '18, Faure '19].**

Also models quantum waves in molecules **[Faure–Zhilinskii '00-'02]**.

Crash course in quantum mechanics

A particle evolving in $\mathbb{R}^2$ is described by a probability density:

$$|\psi(x)|^2 dx, \quad \psi \in L^2(\mathbb{R}^2), \; \|\psi\|_{L^2} = 1.$$

Its time-evolution is unitary: there is a selfadjoint operator P with

$$\psi(t) = e^{-itP}\psi(0) \quad \Leftrightarrow \quad i\frac{\partial \psi(t)}{\partial t} = P\psi(t).$$

Example: average position of the particle at time t:

$$\mathbb{E}_t[\mathbf{x}] = \langle \psi(t), \mathbf{x} \cdot \psi(t) \rangle_{L^2} = \left\langle \psi(0), \mathbf{e}^{\mathbf{itP}} \mathbf{x} \mathbf{e}^{-\mathbf{itP}} \cdot \psi(0) \right\rangle_{L^2}$$

$$\Rightarrow \quad \frac{\partial \mathbb{E}_t[\mathbf{x}]}{\partial t} = \langle \psi(t), \mathbf{i}[\mathbf{P}, \mathbf{x}] \cdot \psi(t) \rangle_{L^2} = \mathbb{E}_t\big[\mathbf{i}[\mathbf{P}, \mathbf{x}]\big].$$

This is the Heisenberg picture: the time-evolution of an observable is

$$\frac{\partial \mathbf{A}(\mathbf{t})}{\partial t} = \mathbf{i}[\mathbf{P}, \mathbf{A}].$$

Perfect materials in condensed matter physics

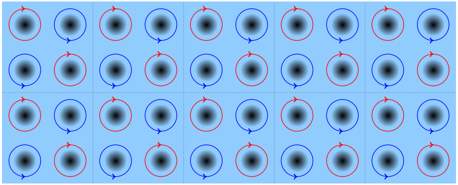
Perfect materials are described by ($\mathbb{Z}^2$-)periodic operators acting on $L^2(\mathbb{R}^2)$.

Example: magnetic Schrödinger operators

$$\begin{aligned} P_+ &\stackrel{\text{def}}{=} \big(D_x + A_+(x)\big)^2 + V_+(x) \quad \text{and} \\ P_- &\stackrel{\text{def}}{=} \big(D_x + A_-(x)\big)^2 + V_-(x). \end{aligned} \tag{1}$$

In (1), $x \in \mathbb{R}^2$ and:

- $V_+,\ V_-,\ A_+,\ A_- \in C_b^\infty(\mathbb{R}^2)$ are periodic w.r.t $\mathbb{Z}^2$.
- Magnetic field: $\partial_1 A_{\pm,2} - \partial_2 A_{\pm,1}$, pointing in direction of e_3. It has vanishing flux across a unit cell.



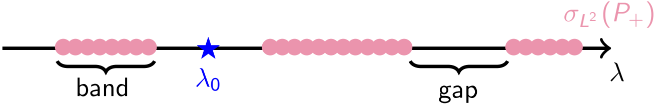
From now on, we fix P_+ and P_- in the form (1).

Periodic operators and Chern numbers

P_+ is $\mathbb{Z}^2$-periodic. It acts on eigenspaces of $\mathbb{Z}^2$-translations:

$$L^2_\xi(\mathbb{R}^2) \overset{\text{def}}{=} \left\{ u \in L^2_{\text{loc}}(\mathbb{R}^2),\ u(x+e_j) = e^{i\xi_j} \cdot u(x) \right\}, \quad \xi \in (\mathbb{T}^2)^* \overset{\text{def}}{=} \mathbb{R}^2/(2\pi\mathbb{Z})^2.$$

One recovers L^2-spectrum from L^2_ξ-spectra: $\sigma_{L^2}(P_+) \ = \ \cup_{\xi \in (\mathbb{T}^2)^*} \sigma_{L^2_\xi}(P_+)$.



Assume $\lambda_0 \notin \sigma_{L^2}(P_+)$. Then $\lambda_0 \notin \sigma_{L^2_\xi}(P_+)$ for all $\xi \in (\mathbb{T}^2)^*$.

Physical interpretation: there are no solutions to

$$\begin{cases} (P_+ - \lambda_0)u = 0 \\ u(x+e_j) = e^{i\xi_j} \cdot u(x) \end{cases}.$$

No plane-wave like propagation at energy λ_0. P_+ **models an insulator at energy** λ_0**.**

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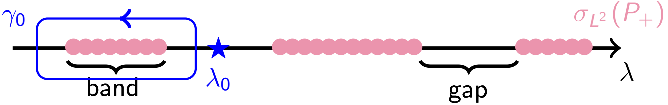

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Set: $$\Pi_{+,\xi} \overset{\text{def}}{=} \frac{1}{2i\pi} \oint_{\gamma_0} (z - P_+)^{-1}\big|_{L^2_\xi}\, dz$$

Eigenprojector of constant rank: it varies **smoothly** with $\xi \in (\mathbb{T}^2)^*$. It induces a **bundle** $\mathcal{E}_+ \to (\mathbb{T}^2)^*$, with fibers Range$(\Pi_{+,\xi})$.

Chern number: $$c_1(\mathcal{E}_+) \overset{\text{def}}{=} \frac{i}{2\pi} \int_{(\mathbb{T}^2)^*} \text{Tr}_{L^2_\xi}\Big(\Pi_{+,\xi}\left[\partial_{\xi_1}\Pi_{+,\xi}, \partial_{\xi_2}\Pi_{\xi,+}\right]\Big)\, d\xi.$$

[Thouless–Kohmoto–Nightingale–den Nijs '82]

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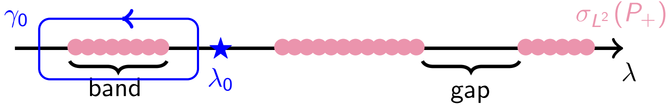
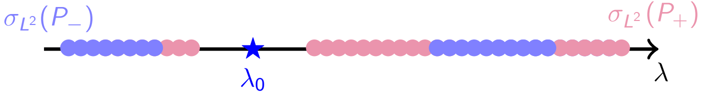


If $c_1(\mathcal{E}_+) \neq 0$, P_+ models a **topological insulator at energy λ_0.**

For the rest of the talk we fix $\boxed{\lambda_0 \notin \sigma_{L^2}(P_+) \bigcup \sigma_{L^2}(P_-).}$



Physically, P_+ and P_- model (potentially topological) insulators with Chern numbers $c_1(\mathcal{E}_+)$ and $c_1(\mathcal{E}_-)$.

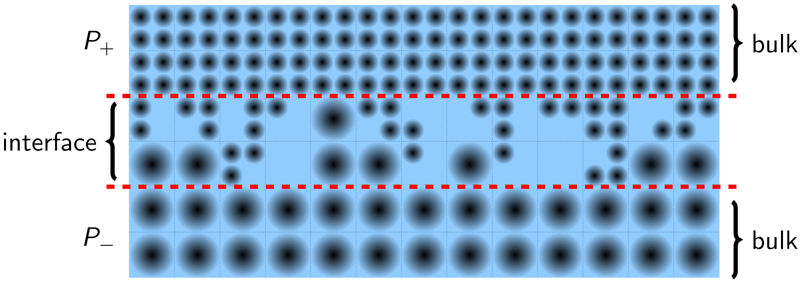
Interface operator

Setting: $P_\pm = (D_x + A_\pm(x))^2 + V_\pm(x)$ (potentially topological) insulators.

Goal: study interface effects between P_+ and P_-.

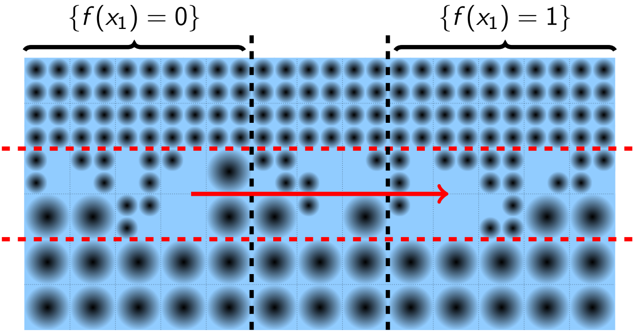
Let $A, V \in C_b^\infty(\mathbb{R}^2)$ and introduce

$$P = (D_x + A(x))^2 + V(x) \quad \text{such that} \quad P = \begin{cases} P_+ & \text{for } x_2 \geq 1 \\ P_- & \text{for } x_2 \leq -1 \end{cases}.$$



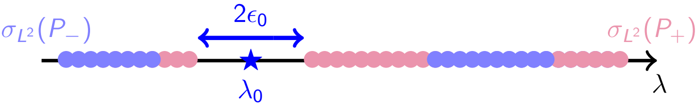
The bulk, $|x_2| \geq 1$, insulates at energy λ_0. **The interface may still support currents.**

Conduction at the interface: the edge index



Fix $f(x_1) \in C^\infty(\mathbb{R})$, $g(\lambda) \in C^\infty(\mathbb{R})$ with:

$$f(x_1) = \begin{cases} 1 & \text{for} \quad x_1 \geq 1 \\ 0 & \text{for} \quad x_1 \leq -1 \end{cases}, \qquad g(\lambda) = \begin{cases} 1 & \text{for} \quad \lambda \leq \lambda_0 - \epsilon_0 \\ 0 & \text{for} \quad \lambda \geq \lambda_0 + \epsilon_0 \end{cases}$$



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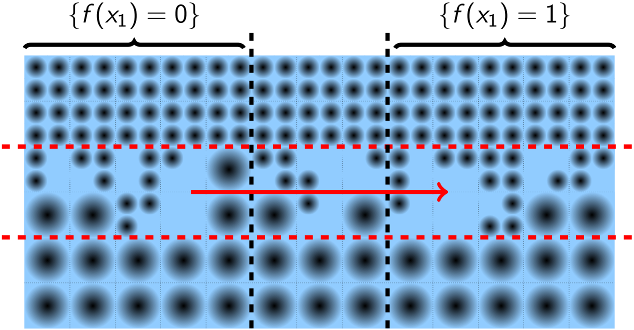

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[Kellendonk–Richter–Schulz-Baldes '02] : $\quad \mathcal{I}(P) \stackrel{\text{def}}{=} \mathrm{Tr}_{L^2}\Big(i\big[P, f(x_1)\big]g'(P)\Big)$

Physically: At the quantum level:

- $i[P, f(x_1)] \sim \partial_t e^{itP} f(x_1) e^{-itP}$ is the particle number moving left to right, per unit time: the current.

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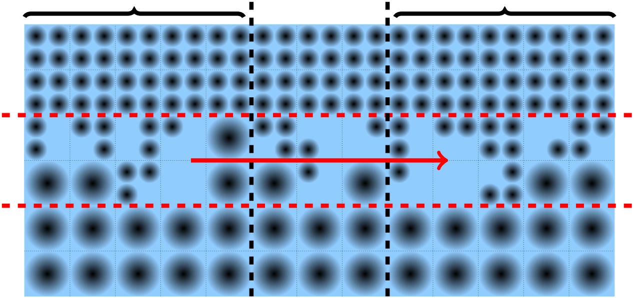

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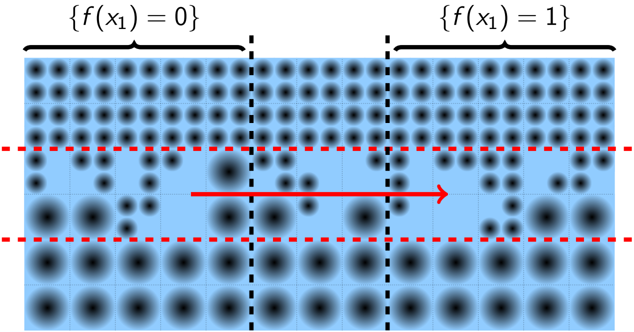
Physically: At the quantum level:

- $g'(P)$ is the density of states within the bulk spectral gap.

Conduction at the interface: the edge index



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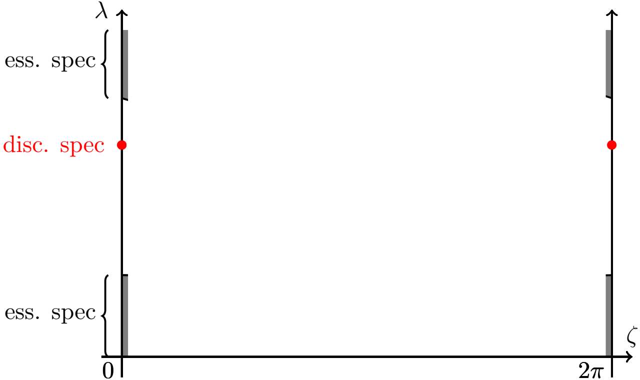
At the quantum level:
$\mathcal{I}(P)$ is the density of current, per unit energy (near λ_0), moving left to right: **the quantum conductivity of the interface** (Ohm's law).

Dynamical interpretation under $\mathbb{Z}e_1$-invariance

If P is $\mathbb{Z}e_1$-invariant, then $\mathcal{I}(P)$ **is a spectral flow:** signed number of $\mathfrak{L}^2_\zeta$-eigenvalues of P crossing gap at λ_0, where

$$\mathfrak{L}^2_\zeta \overset{\text{def}}{=} \left\{ u \in L^2_{\text{loc}}, \quad u(x+e_1) = e^{i\zeta} u(x), \quad \int_{[0,1]\times\mathbb{R}} |u|^2 < \infty \right\}.$$

[Avila–Schulz-Baldes–Villegas-Blas '13]



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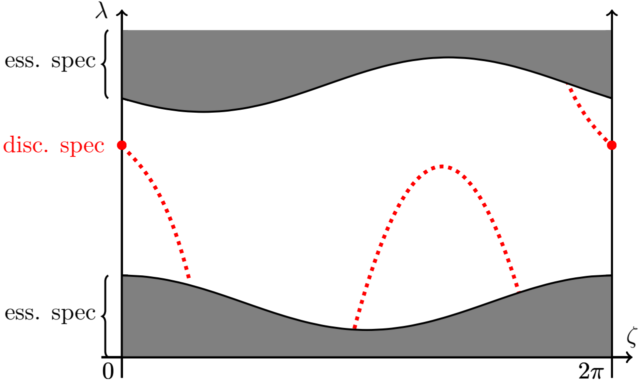

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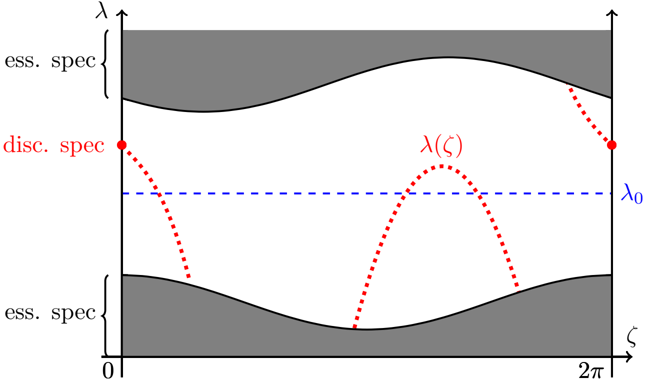

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[Avila–Schulz-Baldes–Villegas-Blas '13]

Let $\zeta \mapsto \lambda(\zeta)$ be an eigenvalue curve. The eigenstate ψ_ζ decays in x_2 **but not in** x_1**.** Take χ supported in $(-\delta, \delta)$ with $\chi' = O(\delta^{-1})$. Form

$$u_0(x) \ \overset{\text{def}}{=} \ \int_{\mathbb{R}} \chi(\zeta - \zeta_0) \cdot u_\zeta(x) d\zeta \quad : \quad \textbf{wavepacket in } L^2(\mathbb{R}^2).$$

Schrödinger evolution of u_0:

$$u(t,x) \ \overset{\text{def}}{=} \ e^{-itP} u_0(x) = \int_{\mathbb{R}} \chi(\zeta - \zeta_0) e^{-itP} u_\zeta(x) d\zeta = \dots$$

$$= \ \widehat{\chi}\big(\lambda'(\zeta_0)t - x_1\big) \cdot e^{-it\lambda(\zeta_0)} u_{\zeta_0}(x) \ + \ \mathcal{O}\big(\delta + \delta^2 t\big)$$

Wavepackets propagate balistically along $\mathbb{R}e_1$, at speed $\lambda'(\zeta_0)$.

Bulk-edge correspondence

Recall:

- λ_0 is insulating energy for $P_\pm$: $\lambda_0 \notin \sigma_{L^2}(P_+) \cup \sigma_{L^2}(P_-)$;
- $\mathcal{E}_\pm$ are eigenbundles of $P_\pm$ below λ_0 with Chern numbers $c_1(\mathcal{E}_\pm)$;
- $\mathcal{I}(P)$ is the conductivity of an interface between P_+ and P_-:

$$\mathcal{I}(P) = \mathrm{Tr}_{L^2}\left(i\left[P, f(x_1)\right] \cdot g'(P)\right).$$

Theorem: [D. '19] *With the above assumptions,*

$$\boxed{2\pi \cdot \mathcal{I}(P) = c_1(\mathcal{E}_+) - c_1(\mathcal{E}_-).} \tag{2}$$

Interpretation: If $c_1(\mathcal{E}_+) \neq c_1(\mathcal{E}_-)$ then $g'(P) \neq 0$, hence $\lambda_0 \in \sigma_{L^2}(P)$.
Interfaces between topologically distinct insulators are conductors.
Applications to engineering of very robust waveguides.

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Comment: (2) accounts for:

- quantization of $\mathcal{I}(P)$: $\boldsymbol{2\pi \cdot \mathcal{I}(P) \in \mathbb{Z}}$**.**
- Robustness of $\mathcal{I}(P)$: **it depends only on $\boldsymbol{P_+}$ and $\boldsymbol{P_-}$.**

These are standard facts that can be shown a priori **[Kellendonk–Schulz-Baldes '05, Combes–Germinet '05, Avila–Schulz-Baldes–Villegas-Blas '13, Bal '18]**

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History of (2): Functional Analysis:

- **[Hatsugai '93, Elbau–Graf '02, Graf–Porta '13]**: discrete Landau-type Hamiltonians.
- **[Elgart–Graf–Schenker '05, Taarabt '14]**: addition of disorder.

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History of (2): K-theory:

- **[Kellendonk–Richter–Schulz-Baldes '02, Kellendonk–Schulz-Baldes '04]**: disordered Landau-type Hamiltonians.
- **[Kubota '17, Bourne–Kellendonk–Rennie '17, Bourne–Rennie '18, Braverman '19]**: general K- and KK-theoretic approaches.

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History of (2): PDE side:

- **[Bal '18, '19, Faure '19]**: Quantitative forms of (2) for Dirac operators (non asymptotically periodic).
- **[D. '18, D.-Weinstein '19]**: Quantitative forms of (2) for weakly deformed graphene: indexes are ± 2.

Sketch of proof

$$2\pi \cdot \mathcal{I}(P) = c_1(\mathcal{E}_+) - c_1(\mathcal{E}_-).$$

1. **Deform** P to a semiclassical operator $\mathbb{P}_h$, with $\mathcal{I}(P) = \mathcal{I}(\mathbb{P}_h)$.
2. **Expand the semiclassical trace** in powers of h:

$$\mathcal{I}(\mathbb{P}_h) \;=\; \mathrm{Tr}\big(i[\mathbb{P}_h, f(x_1)] \cdot g'(\mathbb{P}_h)\big) \;\sim\; \sum_{j \geq 0} a_j \; h^{j-2}.$$

3. Use that $\mathcal{I}(\mathbb{P}_h)$ **is independent of** h to justify $a_2 = \mathcal{I}(P)$.
4. Use **symbolic calculus** to prove $a_2 = c_1(\mathcal{E}_+) - c_1(\mathcal{E}_-)$.

Inspiration for **1** comes from **[Fedosov '70]**: "semiclassical" proof of the index theorem. Technical aspects use **[Gérard–Martinez–Sjöstrand '91]**.

Techniques of **2** adapt arguments of **[Dimassi '93]**: spectral asymptotics for two-scale operators.

Use of a calculation of **[Elbau–Graf '02]** to prove that a_2 depends only on principal symbols in **4**.

Sketch of proof: A. Index invariance

Prove a priori that $\mathcal{I}(P)$ depends only on P_+ and P_-:

$$\mathcal{I}(P) = \mathcal{I}(P_+, P_-).$$

Standard fact; **[D.'19]** gives a pseudodifferential proof.

Then draw inspiration from **[Fedosov '70]**. Deform P to an operator P_h **that transitions slowly from P_+ to P_-:** if

$$\chi_+(x_2) = \begin{cases} 1 & \text{for} \quad x_2 \geq 1 \\ 0 & \text{for} \quad x_2 \leq -1 \end{cases}, \qquad \chi_- = 1 - \chi_+,$$

and $\quad P_h \overset{\text{def}}{=} \chi(hx_2)P_+ + \chi_-(hx_2)P_-,$

then $\boxed{\mathcal{I}(P_h) = \mathcal{I}(P_+, P_-) = \mathcal{I}(P).}$

P_h **decouples a slow scale from a periodic scale.** We write

$$P_h = \sum_\alpha c_\alpha(hx, x) \cdot D_x^\alpha, \qquad c_\alpha(x, y) \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^2).$$

B. Semiclassical deformation

Justify that $P_h = \sum_\alpha c_\alpha(hx, x) \cdot D_x^\alpha$ is a semiclassical operator:

$$U(x,y) \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^2), \quad u_h(x) \stackrel{\text{def}}{=} U(hx, x)$$
$$\Rightarrow \quad (D_x u_h)(x) = \big((hD_x + D_y)U\big)(hx, x),$$

$$\Rightarrow \quad (P_h u_h)(x) = (\mathbb{P}_h U)(hx, x), \qquad \boxed{\mathbb{P}_h \stackrel{\text{def}}{=} \sum_\alpha c_\alpha(x,y) \cdot (hD_x + D_y)^\alpha.}$$

Semiclassical operator **in** x **with** symbol acting on functions of $y \in \mathbb{T}^2$:

$$\boxed{\mathbb{P}(x,\xi) = \sum_\alpha c_\alpha(x,y) \cdot (\xi + D_y)^\alpha \ : \ L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2).}$$

P_h **on** $L^2(\mathbb{R}^2)$ **and** $\mathbb{P}_h$ **on** $L^2(\mathbb{R}^2 \times \mathbb{T}^2)$ **do not have the same spectra.**

[Gérard–Martinez–Sjöstrand '91]: construction of $\mathcal{H} \subset \mathcal{S}'(\mathbb{R}^2 \times \mathbb{T}^2)$ such that P_h **on** $L^2(\mathbb{R}^2)$ **and** $\mathbb{P}_h$ **on** $\mathcal{H}$ **are unitarily equivalent.**

Elements in $\mathcal{H}$ are (up to normalization) $L^2(\mathbb{R}^2)$ multiples of the Dirac mass on $\big\{(x,y) = (h\tilde{x}, \tilde{x}) \in \mathbb{R}^2 \times \mathbb{T}^2\big\}$ i.e.

$$\big\{(x,y) \in \mathbb{R}^2 \times \mathbb{T}^2 \ : \ x = h(y+m), \ m \in \mathbb{Z}^2\big\}.$$

B. Semiclassical deformation

Justify that $P_h = \sum_\alpha c_\alpha(hx, x) \cdot D_x^\alpha$ is a semiclassical operator:

$$U(x,y) \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^2), \quad u_h(x) \stackrel{\text{def}}{=} U(hx, x)$$
$$\Rightarrow \quad (D_x u_h)(x) = \big((hD_x + D_y)U\big)(hx, x),$$

$$\Rightarrow \quad (P_h u_h)(x) = (\mathbb{P}_h U)(hx, x),$$

$$\boxed{\mathbb{P}_h \stackrel{\text{def}}{=} \sum_\alpha c_\alpha(x,y) \cdot (hD_x + D_y)^\alpha.}$$

Semiclassical operator **in** x **with** symbol acting on functions of $y \in \mathbb{T}^2$:

$$\boxed{\mathbb{P}(x,\xi) = \sum_\alpha c_\alpha(x,y) \cdot (\xi + D_y)^\alpha \ : \ L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2).}$$

P_h **on** $L^2(\mathbb{R}^2)$ **and** $\mathbb{P}_h$ **on** $L^2(\mathbb{R}^2 \times \mathbb{T}^2)$ **do not have the same spectra.**

[Gérard–Martinez–Sjöstrand '91]: construction of $\mathcal{H} \subset \mathcal{S}'(\mathbb{R}^2 \times \mathbb{T}^2)$ such that P_h **on** $L^2(\mathbb{R}^2)$ **and** $\mathbb{P}_h$ **on** $\mathcal{H}$ **are unitarily equivalent.**

$$\boxed{\mathcal{I}(P) = \mathcal{I}(P_h) = \mathcal{I}(\mathbb{P}_h) = \mathrm{Tr}_{\mathcal{H}}\big(i\left[\mathbb{P}_h, f(x_1)\right] \cdot g'(\mathbb{P}_h)\big).}$$

C. Trace expansion

Recall $\mathcal{I}(P) = \mathcal{I}(\mathbb{P}_h)$: a semiclassical trace. Expect:

$$\boxed{\mathcal{I}(\mathbb{P}_h) = \mathrm{Tr}_{\mathcal{H}}\left(i\left[\mathbb{P}_h, f(x_1)\right] \cdot g'(\mathbb{P}_h)\right) \sim \sum_{j \geq 0} a_j \ h^{j-2}.} \tag{3}$$

But $\mathcal{I}(P)$ is h-independent: for all $j \neq 2$, $a_j = 0$; and $a_2 = \mathcal{I}(P)$.

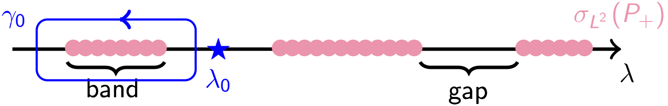
Using **symbolic calculus and ideas from [Dimassi '93, Elgart–Graf–Schenker '05]**, one can compute a_2:

$$\begin{aligned}\mathcal{I}(P) = a_2 &= \sum_{\pm} \pm \frac{i}{2\pi} \int_{(\mathbb{T}^2)^*} \mathrm{Tr}_{L^2(\mathbb{T}^2)}\Big(\Pi_{\pm,\xi}\left[\partial_{\xi_1}\Pi_{\pm,\xi}, \partial_{\xi_2}\Pi_{\pm,\xi}\right]\Big) d\xi \\ &= c_1(\mathcal{E}_+) - c_1(\mathcal{E}_-).\end{aligned}$$

See the appendix for the details...

Future perspectives

Reminder: Spectrum of P_+:



The eigenspace $1_{(-\infty,\lambda_0]}(P_+) \subset L^2(\mathbb{R}^2)$ identifies with a bundle $\mathcal{E}_+ \to (\mathbb{T}^2)^*$: the fibers are

$$\mathcal{E}_+(\xi) \simeq \text{Range}\big(\Pi_+(\xi)\big),$$

$$\Pi_+(\xi) \simeq \frac{1}{2i\pi}\oint_{\gamma_0} \big(z - \mathbb{P}(x,\xi)\big)^{-1} dz \quad \text{with} \quad x_2 \gg 1.$$

$$\mathbb{P}(x,\xi) = \sum_\alpha c_\alpha(x,y)(D_y+\xi)^\alpha \ : \ L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2).$$

Assume now that $\text{rk}(\mathcal{E}_+) = \text{rk}(\mathcal{E}_-) = n$**.**

Future perspectives

Write $\{\lambda_j(x,\xi)\} = \sigma_{L^2(\mathbb{T}^2)}\big(\mathbb{P}(x,\xi)\big)$. **If**

$$\forall (x,\xi), \qquad \lambda_n(x,\xi) < \lambda_{n+1}(x,\xi) \tag{4}$$

then the projector

$$\Pi(x,\xi) \overset{\text{def}}{=} \frac{1}{2\pi i} \oint \big(z - \mathbb{P}(x,\xi)\big)^{-1} dz$$

interpolates smoothly from $\Pi_-(\xi)$ to $\Pi_+(\xi)$ as x_2 runs from $-\infty$ to $+\infty$.

This provides a continuous deformation from $\mathcal{E}_-(\xi)$ to $\mathcal{E}_+(\xi)$. **Hence if (4) holds, $c_1(\mathcal{E}_-) = c_1(\mathcal{E}_+)$.**

In other words:

$$\boxed{\mathcal{I}(P) \neq 0 \quad \Leftrightarrow \quad c_1(\mathcal{E}_-) \neq c_1(\mathcal{E}_+) \quad \Rightarrow \quad \exists (x,\xi), \ \ \lambda_n(x,\xi) = \lambda_{n+1}(x,\xi).}$$

Non-zero conductivity $\Rightarrow$ semiclassical eigenvalue crossing.

Future perspectives

Define a subset of **singularities in the Bloch variety:**

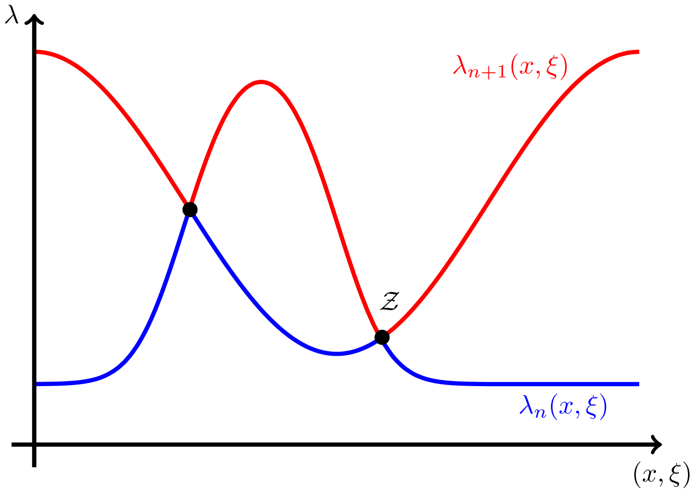
$$\boxed{\mathcal{Z} \overset{\text{def}}{=} \big\{(x,\xi) \ : \ \lambda_n(x,\xi) = \lambda_{n+1}(x,\xi)\big\}.}$$

If $\mathcal{I}(P) \neq 0$ then $\mathcal{Z} \neq \emptyset$. **Microlocally,** everything should happen near $\mathcal{Z}$.

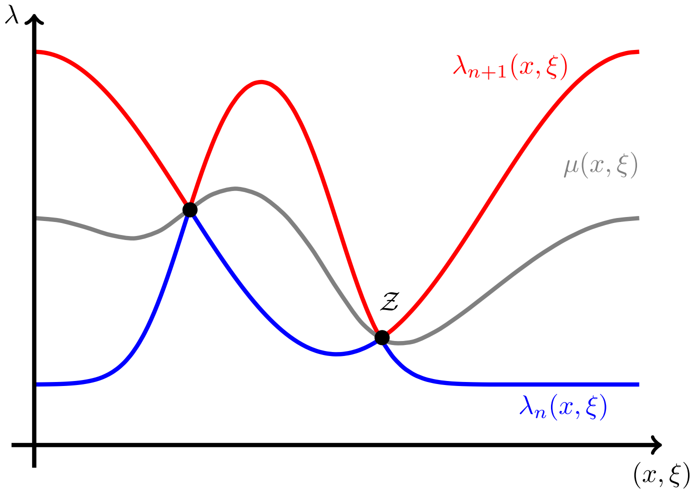
Fix $\varepsilon_0 > 0$ small enough and define:

$$G(x,\xi,\lambda) = \begin{cases} 1 & \text{if} \quad \lambda < \mu(x,\xi) - \varepsilon_0 \\ 0 & \text{if} \quad \lambda > \mu(x,\xi) + \varepsilon_0 \end{cases}, \quad \mu(x,\xi) \overset{\text{def}}{=} \frac{\lambda_n(x,\xi) + \lambda_{n+1}(x,\xi)}{2}.$$

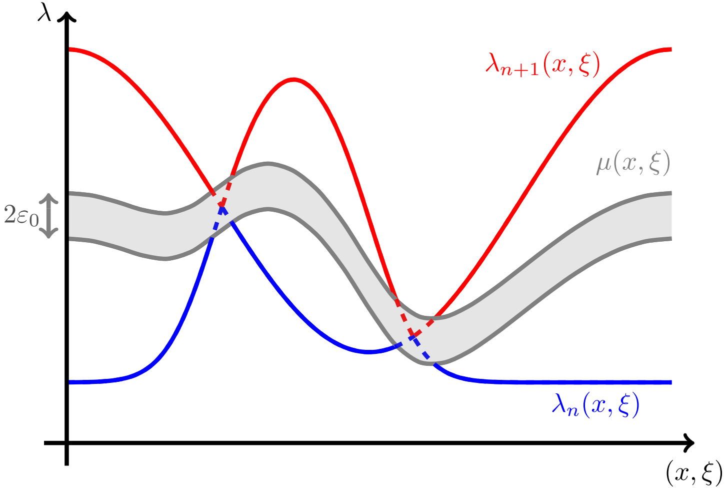
Future perspectives



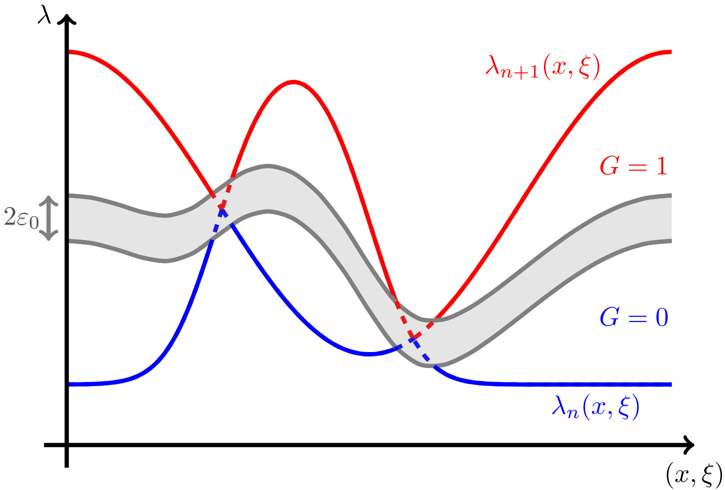
Future perspectives



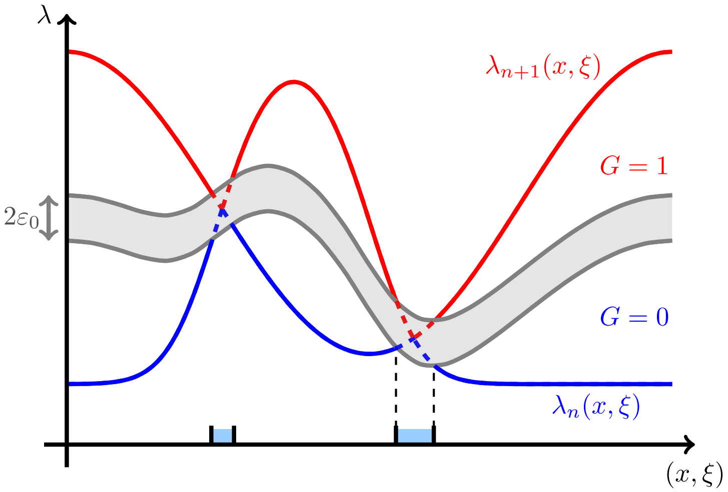
Future perspectives



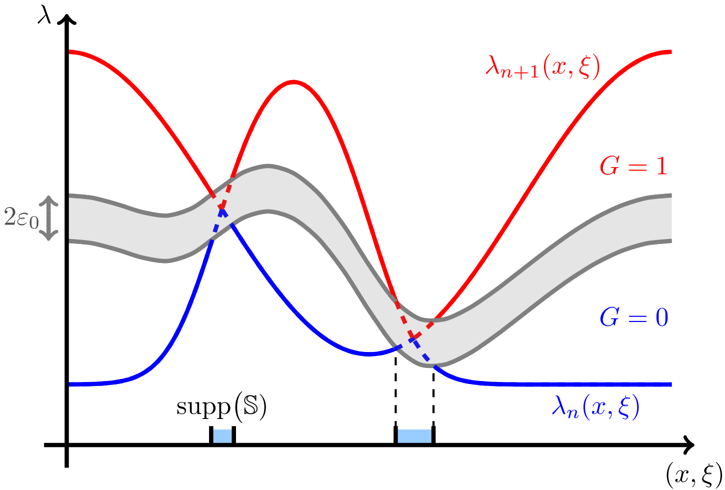
Future perspectives



Future perspectives



Future perspectives



$$\mathbb{W}(x,\xi) \stackrel{\mathsf{def}}{=} \partial_\lambda G\big(x,\xi,\mathbb{P}(x,\xi)\big).$$

Future perspectives

$$\boxed{\mathcal{Z} = \{(x,\xi)\ :\ \lambda_n(x,\xi) = \lambda_{n+1}(x,\xi)\}.}$$

$$G(x,\xi,\lambda) = \begin{cases} 1 & \text{if} \quad \lambda < \mu(x,\xi) - \varepsilon_0 \\ 0 & \text{if} \quad \lambda > \mu(x,\xi) + \varepsilon_0 \end{cases}, \quad \mu(x,\xi) \stackrel{\text{def}}{=} \frac{\lambda_n(x,\xi) + \lambda_{n+1}(x,\xi)}{2}.$$

$\mathbb{W}(x,\xi) = \partial_\lambda G\big(x,\xi,\mathbb{P}(x,\xi)\big)$: Operator valued-symbol.

Quantize: $\mathbb{W}_h = \mathbb{W}(x, hD_x)$. It acts on $\mathcal{H}$ and **microlocalizes near $\mathcal{Z}$.**

Conjecture:

$$\boxed{\mathcal{I}(P) = \mathrm{Tr}_{\mathcal{H}}\Big(i[\mathbb{P}_h, f(x_1)]\mathbb{W}_h\Big) + \mathcal{O}(h^\infty).}$$

Comments:

- $\mathbb{W}(x,\xi) = 0$ if $\mathcal{Z} = \emptyset$ (and ε_0 is small enough).
  Thus conjecture is true if $\mathcal{Z} = \emptyset$:

  $$\mathcal{Z} = \emptyset \quad \Rightarrow \quad c_1(\mathcal{E}_+) = c_1(\mathcal{E}_-) \quad \Rightarrow \quad \mathcal{I}(P) = 0; \qquad (1)$$

  and $\mathbb{W}_h = 0$.

Future perspectives

$$\boxed{\mathcal{Z} = \big\{(x,\xi) \ : \ \lambda_n(x,\xi) = \lambda_{n+1}(x,\xi)\big\}.}$$

$$G(x,\xi,\lambda) = \begin{cases} 1 & \text{if} \quad \lambda < \mu(x,\xi) - \varepsilon_0 \\ 0 & \text{if} \quad \lambda > \mu(x,\xi) + \varepsilon_0 \end{cases}, \quad \mu(x,\xi) \overset{\text{def}}{=} \frac{\lambda_n(x,\xi) + \lambda_{n+1}(x,\xi)}{2}.$$

$\mathbb{W}(x,\xi) = \partial_\lambda G\big(x,\xi,\mathbb{P}(x,\xi)\big)$: Operator valued-symbol.

Quantize: $\mathbb{W}_h = \mathbb{W}(x, hD_x)$. It acts on $\mathcal{H}$ and **microlocalizes near $\mathcal{Z}$.**

Conjecture:

$$\boxed{\mathcal{I}(P) = \mathrm{Tr}_{\mathcal{H}}\Big(i[\mathbb{P}_h, f(x_1)]\mathbb{W}_h\Big) + \mathcal{O}(h^\infty).}$$

Comments:

- **[D. '18, D.-Weinstein'19]** supports the conjecture in the context of Dirac points (simplest eigenvalue crossings).

Future perspectives

$$\boxed{\mathcal{Z} = \big\{(x,\xi)\ :\ \lambda_n(x,\xi) = \lambda_{n+1}(x,\xi)\big\}.}$$

$$G(x,\xi,\lambda) = \begin{cases} 1 & \text{if} \quad \lambda < \mu(x,\xi) - \varepsilon_0 \\ 0 & \text{if} \quad \lambda > \mu(x,\xi) + \varepsilon_0 \end{cases}, \quad \mu(x,\xi) \overset{\text{def}}{=} \frac{\lambda_n(x,\xi) + \lambda_{n+1}(x,\xi)}{2}.$$

$\mathbb{W}(x,\xi) = \partial_\lambda G\big(x,\xi,\mathbb{P}(x,\xi)\big)$: Operator valued-symbol.

Quantize: $\mathbb{W}_h = \mathbb{W}(x, hD_x)$. It acts on $\mathcal{H}$ and **microlocalizes near $\mathcal{Z}$.**

Conjecture:

$$\boxed{\mathcal{I}(P) = \mathrm{Tr}_{\mathcal{H}}\Big(i[\mathbb{P}_h, f(x_1)]\mathbb{W}_h\Big) + \mathcal{O}(h^\infty).}$$

Comments:

- **Dynamical analog:** a normal form for $\mathbb{P}_h$ should govern the transport along the edge.
- Dynamical work without crossings (no topology!): **[Buslaev '87, Dimassi–Guillot–Ralston '02, Panati–Spohn–Teufel '02, ...]**

Future perspectives

- Conjecture!
- Semiclassical propagation of edge states for Dirac point crossings **in the presence of gaps of width 1?** See [Fefferman–Lee-Thorp–Weinstein '16, D.'18, D.–Weinstein '19] for small gaps (homogenization scaling).
- Semiclassical propagation along bended edges?
- Lieb lattice-type crossings: explain nonlinear phenomena [Marzuola–Rechtsman–Osting–Brandes '19]?

Thank you for your attention!

Appendix... C. Trace expansion

Recall $\mathcal{I}(P) = \mathcal{I}(\mathbb{P}_h)$: a semiclassical trace. Expect:

$$\boxed{\mathcal{I}(\mathbb{P}_h) = \mathrm{Tr}_{\mathcal{H}}\left(i\left[\mathbb{P}_h, f(x_1)\right] \cdot g'(\mathbb{P}_h)\right) \sim \sum_{j\geq 0} a_j \; h^{j-2}.} \tag{3}$$

But $\mathcal{I}(P)$ is h-independent: for all $j \neq 2$, $a_j = 0$; and $a_2 = \mathcal{I}(P)$.

Theorem [Dimassi '93]. Let $\mathbb{Q}(x,\xi) : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$, $I \subset \mathbb{R}$ s.t.:

$$\exists M > 0 \;\text{ such that }\; \forall |x| \geq M, \;\; \sigma_{L^2(\mathbb{T}^2)}\big(\mathbb{Q}(x,\xi)\big) \cap I = \emptyset.$$

Then for all $\psi(\lambda) \in C_0^\infty(I)$,

$$\mathrm{Tr}_{\mathcal{H}}\big(\psi(\mathbb{Q}_h)\big) \sim \sum_{j\geq 0} b_j \; h^{j-2}, \quad b_0 \overset{\text{def}}{=} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2 \times (\mathbb{T}^2)^*} \mathrm{Tr}_{L^2(\mathbb{T}^2)}\Big(\psi\big(\mathbb{Q}(x,\xi)\big)\Big) \, dx d\xi.$$

Need to go beyond **[Dimassi '93]**: we want a_2 in (3).

Quite different mechanism between [Dimassi '93] and [D. '19]:

- **[D.' 19]** shows that a_2 depends only on **eigenprojectors**. And $a_2 = O(1)$. For the spectral flow: lots of cancellations.

D. Localization

Calculation:

$$\mathcal{I}(\mathbb{P}_h) = \mathrm{Tr}_{\mathcal{H}}\big(i\big[\mathbb{P}_h, f(x_1)\big] \cdot g'(\mathbb{P}_h)\big)$$
$$= \mathrm{Tr}_{\mathcal{H}}\big(i\big[g(\mathbb{P}_h), f(x_1)\big]\big) = 0.$$

Wrong! The trace-class property fails.

Need: (frequency and) spatial localization.
If $|x_2| \geq 2$ then $\mathbb{P}(x,\xi) = \mathbb{P}_\pm(\xi)$ and

$$g'\big(\mathbb{P}(x,\xi)\big) = g'\big(\mathbb{P}_\pm(\xi)\big) = 0 \quad \text{(because } P_\pm \text{ are insulators).}$$

For $|x_1| \geq 2$, $f'(x_1) = 0$. Thus if $\phi(x) = 1$ on $[-2,2]^2$,

$$\mathcal{I}(\mathbb{P}_h) = \mathrm{Tr}_{\mathcal{H}}\Big(i\big[\mathbb{P}_h, f(x_1)\big] \cdot g'(\mathbb{P}_h) \cdot \phi(x)\Big) + \mathcal{O}(h^\infty).$$

E. Double commutator

Goal: adapt an idea of **[Elgart–Graf–Schenker '05]** to write

$$\mathcal{I}(\mathbb{P}_h) = \mathrm{Tr}_{\mathcal{H}}\Big(i\big[\mathbb{P}_h, f(x_1)\big]\cdot g'(\mathbb{P}_h)\cdot\phi(x)\Big) + \mathcal{O}(h^\infty)$$

as a double commutator. Modulo $\mathcal{O}(h^\infty)$:

$$\mathcal{I}(\mathbb{P}_h) = \mathrm{Tr}_{\mathcal{H}}\int \frac{\partial^2\tilde{g}(z,\overline{z})}{\partial z\partial\overline{z}} i\big[\mathbb{P}_h, f(x_1)\big](\mathbb{P}_h - z)^{-1}\phi(x)\frac{dm(z)}{\pi} = \dots$$

$$= \int \frac{\partial\tilde{g}(z,\overline{z})}{\partial\overline{z}}\cdot\underbrace{\mathrm{Tr}_{\mathcal{H}}\Big(i\big[(\mathbb{P}_h - z)^{-1}\phi(x), f(x_1)\big]\Big)}_{=0}\frac{dm(z)}{\pi}$$

$$+\,\mathrm{Tr}_{\mathcal{H}}\int \frac{\partial\tilde{g}(z,\overline{z})}{\partial\overline{z}}\mathrm{Tr}_{\mathcal{H}}\Big(\underbrace{i\big[\mathbb{P}_h, f(x_1)\big]}_{commutator}\cdot(\mathbb{P}_h - z)^{-1}\cdot\underbrace{\big[(\mathbb{P}_h - z)^{-1},\phi(x)\big]}_{commutator}\Big)\frac{dm(z)}{\pi}.$$

Rest of the proof: leading-order symbolic calculus – in a Grushin framework **[Gérard–Martinez–Sjöstrand '91, Dimassi '93]**. Eventually:

$$\mathcal{I}(P) = a_2 = \sum_{\pm}\pm\frac{i}{2\pi}\int_{(\mathbb{T}^2)^*}\mathrm{Tr}_{L^2(\mathbb{T}^2)}\Big(\Pi_{\pm,\xi}\big[\partial_{\xi_1}\Pi_{\pm,\xi},\partial_{\xi_2}\Pi_{\pm,\xi}\big]\Big)d\xi$$

$$= c_1(\mathcal{E}_+) - c_1(\mathcal{E}_-).$$