# On The Twists of Graded Poisson Algebras

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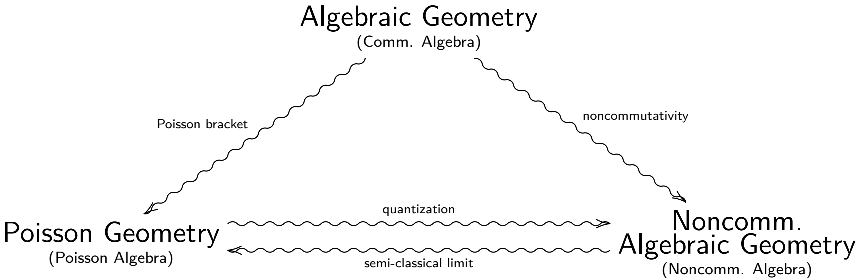

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- graded Poisson algebra $A = \mathbb{k}[x_1, \ldots, x_n]$: multiplication and bracket both graded.

# Examples of Poisson Algebras

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- $\{x, y\} = 2xy$.

# Poisson Structures on $\mathbb{k}[x, y, z]$

- Let $\Omega \in \mathbb{k}[x, y, z]$. One can define the following Poisson structure on $A = \mathbb{k}[x, y, z]$

$$\{f, g\} = \det\left(\frac{\partial(\Omega, f, g)}{\partial(x, y, z)}\right)$$

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- If $\Omega = xyz$, then

$$\begin{aligned} \{x, y\} &= xy; \\ \{y, z\} &= yz; \\ \{z, x\} &= zx. \end{aligned}$$

# Lecoutre-Sierra's Poisson algebra $A(n, a)$

**Definition (Lecoutre-Sierra, 19)**

Set $n \geq 1$ and $a \in \Bbbk$. Set $A(n, a) := \Bbbk[x_0, \ldots, x_n]$ with

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- $A(1, a) = \Bbbk[x_0, x_1]$ with $\{x_0, x_1\} = -ax_0^2$
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$$\begin{aligned}
\{x_0, x_1\} &= -ax_0^2, \\
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- $A(3, -\frac{5}{4})$ is Pym's exceptional Poisson algebra $E(3)$

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# Semi-Poisson Derivations

- Let $(A, \{,\})$ be a Poisson algebra. A derivation $\delta$ of $A$ is called a Poisson derivation if

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for any $a, b \in A$.

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$$\begin{aligned} p(\{-,-\}, \delta; a, b, c) :=& |a|a[\delta(\{b, c\}) - \{\delta(b), c\} - \{b, \delta(c)\}] \\ & - |b|b[\delta(\{a, c\}) - \{\delta(a), c\} - \{a, \delta(c)\}] \\ & + |c|c[\delta(\{a, b\}) - \{\delta(a), b\} - \{a, \delta(b)\}]. \end{aligned}$$

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$$\text{Poisson derivation} \Rightarrow \text{semi-Poisson derivation} \Rightarrow \text{derivation}$$

# Twisting of Graded Poisson Brackets

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The twist of $(A, \pi)$ is $(A^\delta, \pi_{new})$ such that

- $A = A^\delta$ as commutative algebras
- $\pi_{new} = \pi + E \wedge \delta$ or

$$\boxed{\{a, b\}_{new} = \{a, b\} + E(a)\delta(b) - \delta(a)E(b)}$$

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- $A(n, a)^{b\Delta} \cong A(n, a - b)$

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Let $A$ be a G-graded Poisson algebra where $G$ is cyclic.

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- A is rigid if $rgt(A) = 0$ and $-1$ rigid if $rgt(A) = -1$

# Modular Derivations and Unimodularity

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## Definition

The modular derivation $\mathfrak{m}$ of $A$ is

$$\boxed{\mathfrak{m}(a) := -\mathrm{Div}(H_a).}$$

$A$ is called unimodular if $\mathfrak{m} = 0$.

## Remark

$\mathfrak{m} \in Pder(A)$ *and* $\mathrm{Div}(\mathfrak{m}) = 0$

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$$\mathfrak{m}(x) = 0, \quad \mathfrak{m}(y) = 2x.$$

The derivation $\phi$ defined by $\phi(x) = -x, \phi(y) = y - x$ is a semi-Poisson derivation of $A$ under the bracket $\{x, y\} = x^2$.

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- $\delta \in Pder(A)$ of degree zero
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*Then*

$$\boxed{\mathbf{n} = \mathbf{m} + \left(\sum_{i=1}^{n} \deg(x_i)\right) \delta - \mathrm{Div}(\delta) E.}$$

# Unimodular Poisson Brackets

**Theorem (Wang-Zhang-T. 22)**

- $(A = \Bbbk[x_1, \ldots, x_n], \pi)$: *$\mathbb{Z}$-graded Poisson algebra*
- $\mathrm{Div}(E) = \deg(x_1) + \cdots + \deg(x_n) \neq 0$ *in* $\Bbbk$
- $\mathfrak{m}(-) = -\mathrm{Div}(H_-)$: *modular derivation of A.*

*Then* $\left(A^{-\frac{1}{\mathrm{Div}(E)}\mathfrak{m}}, \pi_{unim}\right)$ *is unimodular and*

$$\boxed{\pi = \pi_{unim} + \frac{1}{\mathrm{Div}(E)}(E \wedge \mathfrak{m}).}$$

# More on Unimodularity

**Theorem (Wang-Zhang-T. 22)**

*Let $A$ be a $\mathbb{Z}$-graded Poisson algebra. Then*

- $A$ *is unimodular* $\Rightarrow Gspd(A) = Gpd(A)$

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- $rgt(A) = 0 \Rightarrow A$ *is unimodular*
- $rgt(A) = -1 \Rightarrow \dim_{\Bbbk} Gspd(A) = \dim_{\Bbbk} Gpd(A) = 2$

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**Proposition (Wang-Zhang-T. 22)**

| $\Omega$ | $0$ | $x^3$ | $x^2y$ | $xyz$ | $xy(x+y)$ | $xyz + x^3$ | $xy^2 + x^2z$ | *irred.* |
|---|---|---|---|---|---|---|---|---|
| $rgt(A)$ | $-8$ | $-5$ | $-3$ | $-2$ | $-2$ | $-1$ | $-1$ | $0$ |

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- $A(n, a)$ is unimodular $\Leftrightarrow a = \frac{(n+2)(1-n)}{2(n+1)} =: a_0$ $(n = 3, a_0 = -\frac{5}{4})$
- $Gspd(A(n, a_0)) = Gpd(A(n, a_0)) = \mathrm{span}_{\mathbb{k}}(E, \Delta)$

# Revisit to L-S Poisson algebra $A(n, a)$

**Definition (Lecoutre-Sierra, 19)**

Set $n \geq 1$ and $a \in \Bbbk$. Set $A(n, a) := \Bbbk[x_0, \dots, x_n]$ with

$$\boxed{\{x_i, x_j\} := (a + j)x_{i-1}x_j - (a + i)x_{j-1}x_i.}$$

- $\Delta := x_0 \frac{\partial}{\partial x_1} + \cdots + x_{n-1} \frac{\partial}{\partial x_n}$: downward Poisson derivation
- modular derivation $\mathfrak{m} = ((n+1)a + \binom{n+1}{2} - 1)\Delta$
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# $H$-ozoness and $PH^1$-Minimality

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- A Poisson derivation $\phi$ of $A$ is called ozone if $\phi(Z) = 0$
- $A$ is $H$-ozone if every ozone Poisson derivation is Hamiltonian
- Let $A$ be a nontrivial connected graded Poisson algebra with its Poisson center $Z$ being a domain. $A$ is said to be $PH^1$-minimal if $PH^1(A) \cong ZE$

# $H$-ozoness

**Theorem (Wang-Zhang-T. 22)**

*Let $A = \Bbbk[x_1, \cdots, x_n]$ be a connected graded Poisson algebra with its Poisson center $Z \neq \Bbbk$. Then*

- $$\begin{array}{ccc} A \text{ is } PH^1\text{-minimal} & \Longrightarrow & A \text{ is } H\text{-ozone} \\ \Downarrow & & \Downarrow \\ rgt(A) = 0 & \Longrightarrow & A \text{ is unimodular} \end{array}$$

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- *If $Z = \Bbbk[z]$ for some $\deg(z) > 0$, then $A$ is $H$-ozone implies that $rgt(A) = 0$.*

**Theorem (Wang-Zhang-T. 22)**

*Let $A = \Bbbk[x, y, z]$ be a connected graded Poisson algebra with Poisson center $Z$ and $\deg(x, y, z) = 1$. TFAE.*

(1) *$A$ is $PH^1$-minimal.*

(2) $rgt(A) = 0$.

(3) *Any graded twist of $A$ is isomorphic to $A$.*

(4) $h_{Pder(A)}(t) = \frac{1}{(1-t)^3}$.

(5) $h_{PH^1(A)}(t) = \frac{1}{1-t^3}$.

(6) $h_{PH^1(A)}(t) = h_Z(t)$.

(7) $h_{PH^3(A)}(t) - h_{PH^2(A)}(t) = t^{-3}$.

(8) *$A$ is unimodular with irreducible $\Omega$.*

**Corollary (Wang-Zhang-T. 22)**

*$A = \mathbb{k}[x, y, z]$ unimodular quadratic Poisson algebra with irreducible potential $\Omega$. Then*

# Poisson cohomology of $\mathbb{k}[x, y, z]$

**Corollary (Wang-Zhang-T. 22)**

*$A = \mathbb{k}[x, y, z]$ unimodular quadratic Poisson algebra with irreducible potential $\Omega$. Then*

(1) $h_{PH^0(A)}(t) = \frac{1}{1-t^3}$

(2) $h_{PH^1(A)}(t) = \frac{1}{1-t^3}$

(3) $h_{PH^2(A)}(t) = \frac{1}{t^3}\left(\frac{(1+t)^3}{1-t^3} - 1\right)$

(4) $h_{PH^3(A)}(t) = \frac{(1+t)^3}{t^3(1-t^3)}$

# Thank You!