Automorphisms of the quantum grassmannian

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Background

- **The quantum grassmannian**

  The quantum grassmannian is a noncommutative deformation of the homogeneous coordinate ring of the classical grassmannian. (Throughout, K is a field and $0 \neq q \in K$ is not a root of unity.)

- **The classical grassmannian**

  The $k \times n$ **grassmannian** $G(k, n)$ is the set of k-dimensional vector subspaces of an n-dimensional vector space over some fixed field.

- **Example**

  The 1×3 **grassmannian** $G(1, 3)$ is the set of lines through the origin in 3-space; that is, $\mathbb{P}^2$

$G(2,4)$

- $G(2,4)$ is the grassmannian of 2-spaces in 4-space
- Specify P by two linearly independent vectors $\mathbf{v_1}, \mathbf{v_2}$
- Display in a 2×4 matrix

$$\begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \end{pmatrix}$$

- Many such matrices give the same P (change of basis, left multiplication by $GL(2)$)
- Let $[ij]$ be the 2×2 minor using columns i and j
- The ratios $[12] : [13] : [14] : [23] : [24] : [34]$ specify P uniquely

$G(2,4)$

- The ratios [12] : [13] : [14] : [23] : [24] : [34] are the **Plücker coordinates** of P; they specify P uniquely.
- There is a **Plücker relation** $[12][34] - [13][24] + [14][23] = 0$
- We can choose a **normal form** for $P \in G(2,4)$ by reducing to echelon form: the generic case is

$$\begin{pmatrix} 1 & 0 & \star & \star \\ 0 & 1 & \star & \star \end{pmatrix} \quad \approx \quad k^4 \quad \approx \quad M_2(k)$$

- The **dehomogenisation** of $G(2,4)$ at [12] is the space of 2×2 matrices (aka the **big cell**)

Quantum matrices

- The **algebra of quantum** 2×2 **matrices**

$$\mathcal{O}_q(M_{22}) = K \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

  is generated by four indeterminates a, b, c, d subject to the following rules:

$$ab = qba, \qquad cd = qdc$$
$$ac = qca, \qquad bd = qdb$$
$$bc = cb, \qquad ad - da = (q - q^{-1})bc.$$

- The **quantum determinant** $ad - qbc$ is a central element

2×4 quantum matrices

- **The algebra of** 2×4 **quantum matrices.**

$$\mathcal{O}_q(M_{24}) := K \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \end{bmatrix},$$

where each 2×2 sub-matrix is a copy of $\mathcal{O}_q(M_{22})$.

The quantum grassmannian $\mathcal{O}_q(G_{24})$

- $\mathcal{O}_q(M_{24}) := K \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \end{bmatrix}$
- The quantum grassmannian $\mathcal{O}_q(G_{24})$ is the subalgebra of $\mathcal{O}_q(\mathcal{M}_{24})$ generated by the quantum determinants of the 2×2 quantum submatrices.
- Denote by $[ij]$ the quantum determinant of the 2×2 quantum submatrix that uses columns i, j. These are the **quantum Plücker coordinates**. (Eg. $[12] = Y_{11}Y_{22} - qY_{12}Y_{21}$)
- $\mathcal{O}_q(G_{24})$ is a domain that is an $\mathbb{N}$-graded algebra with each $[ij]$ having degree 1.

The quantum grassmannian $\mathcal{O}_q(G_{24})$

- $\mathcal{O}_q(G_{24})$ is generated by the six quantum Plücker coordinates

$$[12], \quad [13], \quad [14], \quad [23], \quad [24], \quad [34]$$

- Some quantum minors $q^{\bullet}$-commute, for example,

$$[14]\,[23] = [23]\,[14]\,, \; [12]\,[13] = q\,[13]\,[12]\,, \; [12]\,[34] = q^2\,[34]\,[12]$$

- However,

$$[13]\,[24] = [24]\,[13] + (q - q^{-1})\,[14]\,[23]$$

- There is a quantum Plücker relation

$$[12]\,[34] - q\,[13]\,[24] + q^2\,[14]\,[23] = 0.$$

Noncommutative dehomogenisation

- Let $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ be an $\mathbb{N}$-graded algebra and $x \in R_1$ be a nonzerodivisor that is normal (ie. $xR = Rx$)
- Then $S := R[x^{-1}]$ is $\mathbb{Z}$-graded
- Set $\mathrm{Dhom}(R, x) := S_0 \quad (= R_0 + R_1x^{-1} + R_2x^{-2} + \dots)$, the **noncommutative dehomogenisation of** R **at** x.
- For $r \in R$, write $xr = \sigma(r)x$, with σ an automorphism of R.
- $$R[x^{-1}] \cong \mathrm{Dhom}(R, x)[z, z^{-1}; \sigma]$$

Noncommutative dehomogenisation of $\mathcal{O}_q(G_{24})$ at [12]

- Recall from earlier that in the classical grassmannian, the dehomogenisation of $G(2,4)$ at the Plücker coordinate [12] is isomorphic to 2×2 matrices.
- In $\mathcal{O}_q(G_{24})$ the quantum Plücker coordinate $u := [12]$ $q^{\bullet}$-commutes with each $[ij]$ and so is normal. Consequently, the Ore localisation at the powers of u exists and

- Theorem

$$\mathrm{Dhom}(\mathcal{O}_q(G_{24}), u) \cong \mathcal{O}_q(M_{22})$$

The dehomogenisation equality

- The dehomogenisation equality

$$\mathcal{O}_q(G_{24})(u^{-1}) = \mathcal{O}_q(M_{22})[y, y^{-1}; \sigma]$$

  can be used either to get properties of quantum matrices from the quantum grassmannian or, vice versa, to get properties of the quantum grassmannian from quantum matrices.
- Today, we use the known automorphism group of quantum matrices to calculate the automorphism group of the quantum grassmannian.

Obvious automorphisms of quantum matrices

- Recall $\mathcal{O}_q(M_{22}) = K\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with

$$ab = qba, \quad cd = qdc, \quad ac = qca, \quad bd = qdb,$$

$$bc = cb, \quad ad - da = (q - q^{-1})bc.$$

- The torus $\mathcal{H} := (K^*)^4$ acts on $\mathcal{O}_q(M_{22})$ so that $h := (\alpha_1, \alpha_2; \beta_1, \beta_2)$ multiplies row i by α_i and column j by β_j
- Transposition (flip over the diagonal) gives an automorphism of $\mathcal{O}_q(M_{22})$ because b and c satisfy the same commutation rules.

The automorphism group of quantum matrices

- Theorem

> The automorphism group of $\mathcal{O}_q(M_{mn})$ is $\mathcal{H} := (K^*)^{(m+n)}$ when $m \neq n$, and $(K^*)^{2n} \rtimes \langle \tau \rangle$ when $m = n$

- History:

  Alev and Chamarie did the 2×2 case (1992).

  Launois and Lenagan did the nonsquare case and the 3×3 case (2007, 2013).

  Yakimov did the $n \times n$ case in general (2013).

Obvious automorphisms of the quantum grassmannian

- Recall that $\mathcal{O}_q(G_{24})$ is the subalgebra of $\mathcal{O}_q(M_{24})$ generated by the 2×2 quantum minors $[ij]$.
- The torus $\mathcal{H} := (K^*)^4$ of column automorphisms of $\mathcal{O}_q(M_{24})$ acts on $\mathcal{O}_q(G_{24})$ by restriction so that

$$(h_1, h_2, h_3, h_4) \circ [ij] \;=\; h_i h_j [ij]$$

Strategy for the quantum grassmannian

- Given any automorphism ρ of $\mathcal{O}_q(G_{24})$ show that by adjusting ρ by elements of $\mathcal{H}$ we can assume that the quantum Plücker coordinates [12] and [34] are fixed by ρ.
- With this assumption, we may extend ρ to act on the left hand side of the dehomogenisation equality

$$\mathcal{O}_q(G_{24})(u^{-1}) = \mathcal{O}_q(M_{22})[y, y^{-1}; \sigma]$$

  and this transfers to an action on the right hand side.
- In this equality, y and u are essentially the same element, and so ρ fixes y and the quantum determinant $(= [34][12]^{-1})$.

Strategy for the quantum grassmannian

- Now ρ acts on
$$\mathcal{O}_q(M_{22})[y, y^{-1}; \sigma]$$
and fixes y.
- Show that ρ takes $\mathcal{O}_q(M_{22})$ to itself. Now we know how ρ acts on the right hand side of the dehomogenisation equality as we know the automorphism group of quantum matrices.
- Use the dehomogenisation equality
$$\mathcal{O}_q(G_{24})(u^{-1}) = \mathcal{O}_q(M_{22})[y, y^{-1}; \sigma]$$
to transfer this information back to $\mathcal{O}_q(G_{24})$.

The automorphism group of the quantum grassmannian

- Theorem

The automorphism group of $\mathcal{O}_q(G_{24})$ is $(K^*)^4 \rtimes \langle\tau\rangle$ where $h = (h_1, h_2, h_3, h_4)$ acts on $[ij]$ by multiplying by $h_i h_j$ and τ is the diagram automorphism which fixes $[12], [13], [24]$ and $[34]$ and interchanges $[14]$ and $[23]$.

- Theorem

The automorphism group of $\mathcal{O}_q(G_{kn})$ is $(K^*)^n$ when $2k \neq n$ and $(K^*)^n \rtimes \langle\tau\rangle$ when $2k = n$ (here, τ is the diagram automorphism).

The poset diagram of $\mathcal{O}_q(G_{36})$

$[a_1 a_2 a_3] \leq [b_1 b_2 b_3]$ if and only if $a_i \leq b_i$ for $i = 1, 2, 3$