Symmetries captured by weak Hopf algebra actions

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Cocommutative Hopf-like actions on algebras, **arXiv:2209.11903**

Big picture

Goal. [Lots of people here, throughout history, etc.]

Study symmetries in algebra.

How to study symmetries?

- Classically, symmetries are captured by group actions.

$C_2 \curvearrowright$ tiger

$D_{10} \curvearrowright$ pentagon

- V a vector space, G a group $G \curvearrowright V$ if

$$(gh) \cdot v = g \cdot (h \cdot v).$$

We call V a G-module.

Symmetries of algebras

- $\Bbbk$ algebraically closed field of characteristic zero.
- Want to study symmetries of $\Bbbk$-algebras A.

Favorite examples.

The polynomial ring $A = \Bbbk[x_1, \ldots, x_n]$.
Also connected ($A_0 = \Bbbk$) and commutative.

The skew polynomial ring $A = \Bbbk_{\mathbf{p}}[x_1, \ldots, x_n]$.
Connected but noncommutative.

- **Motivation:** What if A is not connected?

Group actions

- $A = (A, \quad m : A \otimes_{\Bbbk} A \to A, \quad u : \Bbbk \to A)$ has structure.
- Should preserve algebra structure.
- Study group actions $G \curvearrowright A$ such that:

$$g \cdot (ab) = (g \cdot a)(g \cdot b) \quad \text{and}$$
$$g \cdot 1_A = 1_A,$$

[A is a G-module and m and u are G-module morphisms].

- We say that A is a G-module algebra.

Example. [Extremely rich history]

Finite group $G \leq \mathrm{GL}_n(\Bbbk) \curvearrowright A = \Bbbk[x_1, \ldots, x_n]$.

Group actions

- Group actions on a vector space V are captured by general linear group $\mathrm{GL}(V)$.
- Group actions on an algebra A are captured by the automorphism group $\mathrm{Aut}_{\mathsf{Alg}}(A)$.

Facts.

- G acts on a vector space V if and only if there is a group homomorphism $G \to \mathrm{GL}(V)$.
- A is a G-module algebra if and only if there is a group homomorphism $G \to \mathrm{Aut}_{\mathsf{Alg}}(A) \leq \mathrm{GL}(A)$.

Lie algebra actions

- Also classical symmetries by Lie algebras.
- V a vector space, $\mathfrak{g}$ a Lie algebra, $\mathfrak{g} \curvearrowright V$ if

$$[p, q] \cdot v = p \cdot (q \cdot v) - q \cdot (p \cdot v).$$

We call V a $\mathfrak{g}$-module.

- A an algebra, we say A is a $\mathfrak{g}$-module algebra if $\mathfrak{g} \curvearrowright A$ and

$$p \cdot (ab) = (p \cdot a)b + a(p \cdot b) \quad \text{and}$$
$$p \cdot 1_A = 0,$$

[A is a $\mathfrak{g}$-module and m and u are $\mathfrak{g}$-module morphisms].

Lie algebra actions

- Lie algebra actions on V are captured by general linear Lie algebra $\mathfrak{gl}(V)$.
- Lie algebra actions on algebras A are captured by the Lie algebra of derivations $\mathrm{Der}(A)$.

Facts.

- $\mathfrak{g}$ acts on a vector space V if and only if there is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$.
- A is a $\mathfrak{g}$-module algebra if and only if there is a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{Der}(A) \leq \mathfrak{gl}(A)$.

Symmetry capturing objects

- A is a G-module algebra $\iff$ morphism $G \to \mathrm{Aut}_{\mathsf{Alg}}(A)$.
  View $\mathrm{Aut}_{\mathsf{Alg}}(A)$ as capturing symmetries of A in Grp.

- A is a $\mathfrak{g}$-module algebra $\iff$ morphism $\mathfrak{g} \to \mathrm{Der}(A)$.
  View $\mathrm{Der}(A)$ as capturing symmetries of A in Lie.

- Other categories?

Categories of interest

Let X be a nonempty set.

- X-Grpd, category of groupoids with object set X, X-preserving groupoid homomorphisms.
  If $|X| = 1$, $X\text{-Grpd} = \text{Grp}$.

- X-Lie, category of X-Lie algebroids $\mathfrak{G} = \bigoplus_{x \in X} \mathfrak{g}_X$, X-Lie algebroid homomorphisms.
  If $|X| = 1$, $X\text{-Lie} = \text{Lie}$.

- X-WHA, category of weak Hopf algebras with a complete set $\{e_x\}_{x \in X}$ of grouplike idempotents, X-preserving weak Hopf algebra homomorphisms.
  If $|X| = 1$, $X\text{-WHA} = \text{Hopf}$.

Weak Hopf algebras

- A weak bialgebra H over $\Bbbk$ is a $\Bbbk$-algebra (H, m, u) and a $\Bbbk$-coalgebra (H, Δ, ε) such that

  (1) $\Delta(ab) = \Delta(a)\Delta(b)$,

  (2) $(\Delta \otimes \mathrm{Id}) \circ \Delta = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1)$,

  (3) $\varepsilon(abc) = \varepsilon(ab_1)\varepsilon(b_2c) = \varepsilon(ab_2)\varepsilon(b_1c)$.

- Bialgebra if and only if $\Delta(1) = 1 \otimes 1$ if and only if $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$.
- A weak Hopf algebra is a weak bialgebra with antipode S:

$$S(a_1)a_2 = 1_1\varepsilon(a1_2), \qquad a_1S(a_2) = \varepsilon(1_1a)1_2, \qquad S(a_1)a_2S(a_3) = S(a).$$

Counital maps

- The maps appearing in the antipode axioms:

$$S(a_1)a_2 = 1_1\varepsilon(a1_2), \qquad a_1S(a_2) = \varepsilon(1_1a)1_2$$

  are important.

source counital map

$$\varepsilon_s : H \to H$$

$$\varepsilon_s(a) = 1_1\varepsilon(a1_2)$$

target counital map

$$\varepsilon_t : H \to H$$

$$\varepsilon_t(a) = \varepsilon(1_1a)1_2$$

source counital subalgebra

$$H_s := \varepsilon_s(H)$$

target counital subalgebra

$$H_t = \varepsilon_t(H)$$

- H_s and H_t are antiisomorphic separable, semisimple, finite-dimensional, coideal sub-$\mathbb{k}$-algebras.
- A weak bialgebra is a bialgebra if and only if $H_s = H_t = \mathbb{k}$.

Why weak Hopf algebras?

- Introduced by [Böhm–Nill–Szlachanyi 1999], motivated by physics: study symmetries in conformal field theory.
- Axioms are self-dual, so the dual of a finite-dimensional weak Hopf algebra is again a weak Hopf algebra.

Example.

If H, K are bialgebras, then $H \oplus K$ is an algebra as usual and a coalgebra under

$$\Delta(h,k) = (h_1, 0) \otimes (h_2, 0) + (0, k_1) \otimes (0, k_2)$$

$$\varepsilon(h,k) = \varepsilon_H(h) + \varepsilon_K(k)$$

But $\Delta(1,1) = (1,0) \otimes (1,0) + (0,1) \otimes (0,1) \neq (1,1) \otimes (1,1)$.
$(H \oplus K)_t = (H \oplus K)_s = \Bbbk \oplus \Bbbk$.
So $H \oplus K$ not a bialgebra, only a weak bialgebra.

Why weak Hopf algebras?

If G, H are groups, then $G \sqcup H$ is not a group, but a groupoid.

Example.

$\mathcal{G}$ is a groupoid. $\Bbbk\mathcal{G}$ the groupoid algebra is a weak Hopf algebra.

$$\text{For } g \in \mathcal{G}: \qquad \Delta(g) = g \otimes g, \qquad \varepsilon(g) = 1, \qquad S(g) = g^{-1}.$$

$$\mathcal{G} = \; 1 \underset{\alpha^{-1}}{\overset{\alpha}{\rightleftarrows}} 2$$

Then $1 = e_1 + e_2$ but $\Delta(1) = e_1 \otimes e_1 + e_2 \otimes e_2 \neq 1 \otimes 1$.
$(\Bbbk\mathcal{G})_t = (\Bbbk\mathcal{G})_s = \Bbbk e_1 \oplus \Bbbk e_2$.

Why weak Hopf algebras?

Theorem. [Hayashi 1999, Szlachanyi 2001]

Every fusion category is equivalent to ${}_H\mathcal{M}_{\text{fd}}$ for some weak Hopf algebra H.

- If $(H, m, u, \Delta, \varepsilon)$ is an algebra and coalgebra such that $\Delta(ab) = \Delta(a)\Delta(b)$, then

  Δ axiom $\Rightarrow {}^H\mathcal{M}$ and $\mathcal{M}^H$ are monoidal,

  ε axiom $\Rightarrow {}_H\mathcal{M}$ and $\mathcal{M}_H$ are monoidal.

- But not $\otimes_{\Bbbk}$! [Nill 1998], [Böhm–Caenepeel–Janssen 2011], [Walton–Wicks–W 2022]

Symmetries in categories

Definition.

A a $\Bbbk$-algebra and $\mathcal{C}$ a category like X-Grpd, X-Lie, X-WHA. We denote by $\mathrm{Sym}_{\mathcal{C}}(A)$ an object in $\mathcal{C}$ (if it exists) such that:

1. A is a $\mathrm{Sym}_{\mathcal{C}}(A)$-module algebra; we write $f \triangleright a$ for the action.
2. For each H in $\mathcal{C}$, there is a bijection $\Psi_H : \mathrm{Act}(H, A) \to \mathrm{Hom}_{\mathcal{C}}(H, \mathrm{Sym}_{\mathcal{C}}(A))$ such that for any action $\cdot$ of H on A, if we denote $\Psi_H(\cdot) := \Psi_{(H,\cdot)}$, then

$$h \cdot a = \Psi_{(H,\cdot)}(h) \triangleright a.$$

Symmetries in categories

$\mathrm{Sym}_{\mathcal{C}}(A)$ may not exist.

Example.

- $\mathcal{C} = \mathsf{AbGrp}$, $A = \mathbb{k}[x]$.
- $G = \langle g \rangle$ of order 2.
- $G \curvearrowright A$ via $\cdot$ defined by $g \cdot x = -x$ and $*$ defined by $g * x = -x + 1$.
- $\mathrm{Sym}_{\mathcal{C}}(A)$ existed:

  $$\Psi_{(G,\cdot)} : G \to \mathrm{Sym}_{\mathsf{AbGrp}}(A) \quad \text{and} \quad \Psi_{(G,*)} : G \to \mathrm{Sym}_{\mathsf{AbGrp}}(A)$$

  so that $\Psi_{(G,\cdot)}(g) \triangleright x = -x$ and $\Psi_{(G,*)}(g) \triangleright x = -x + 1$.
- But these elements would not commute in $\mathrm{Sym}_{\mathsf{AbGrp}}(A)$.

Modules over groupoids

Definition.

A vector space V is X-decomposable if there exists a family $\{V_x\}_{x\in X}$ of subspaces of V such that $V = \bigoplus_{x\in X} V_x$.

Definition.

$\mathcal{G}$ a groupoid with object set X. An X-decomposable vector space $V = \bigoplus_{x\in X} V_x$ is a left $\mathcal{G}$-module if it is equipped with, for each $x, y \in X$, a linear map $\mathrm{Hom}_{\mathcal{G}}(x, y) \times V_x \to V_y$, denoted $(g, v) \mapsto g \cdot v$, such that

- $(gh) \cdot v = g \cdot (h \cdot v)$, for all $g, h \in \mathcal{G}_1$ with $t(h) = s(g)$ and all $v \in V_{s(h)}$, and
- $e_x \cdot v = v$, for all $x \in X$ and $v \in V_x$.

GW

General linear groupoid

Definition.

Let $V = \bigoplus_{x \in X} V_x$ be an X-decomposable vector space. We define the X-general linear automorphism groupoid $\mathrm{GL}_X(V)$:

- the object set is X,
- for any $x, y \in X$, $\mathrm{Hom}_{\mathrm{GL}_X(V)}(x, y)$ is the space of vector space isomorphisms between V_x and V_y.

If $X = \{1, \ldots, n\}$ and V_i has dimension d_i, then we also denote $\mathrm{GL}_X(V)$ by $\mathrm{GL}_{(d_1,\ldots,d_n)}(\Bbbk)$ for $d_1 \leq d_2 \leq \cdots \leq d_n$.

This generalizes the classical notation $GL(V) = \mathrm{GL}_d(\Bbbk)$ when V has dimension d.

General linear groupoid

- A vector space V is a $\mathcal{G}$-module if and only if there is an X-groupoid morphism $\mathcal{G} \to \mathrm{GL}_X(V)$.

Example.

If $X = \{x, y\}$, then $\Bbbk^4$ is X-decomposable by taking $(\Bbbk^4)_x := (\Bbbk, \Bbbk, 0, 0)$ and $(\Bbbk^4)_y := (0, 0, \Bbbk, \Bbbk)$. Moreover, we have

$$\mathrm{GL}_X(\Bbbk^4) = \mathrm{GL}_{(2,2)}(\Bbbk) = \circlearrowleft\ \Bbbk^2 \ \underset{\dashleftarrow}{\overset{\dashrightarrow}{}}\ \Bbbk^2\ \circlearrowright ,$$

where the dashed arrows can be identified with $\mathrm{GL}_2(\Bbbk)$.

Module algebras over groupoids

Definition.

Let A be a $\Bbbk$-algebra. We say that A is an X-decomposable algebra if there exists a family $\{A_x\}_{x\in X}$ of $\Bbbk$-algebras (some of which may be 0) such that $A = \bigoplus_{x\in X} A_x$ as algebras.

Lemma.

$A = \bigoplus_{x\in X} A_x$ is a $\mathcal{G}$-module algebra if and only if A is a $\mathcal{G}$-module such that

$$g \cdot (ab) = (g\cdot a)(g\cdot b),$$
$$g \cdot 1_{s(g)} = 1_{t(g)},$$

for all $g \in \mathcal{G}_1$ and $a, b \in A_{s(g)}$.

Module algebras over groupoids

Example.

Let $A = \Bbbk[x] \oplus \Bbbk[x]$. $\mathcal{G} = \; 1 \underset{\alpha^{-1}}{\overset{\alpha}{\rightleftarrows}} 2$.

Let $\sigma \in \mathrm{Aut}_{\mathsf{Alg}}(\Bbbk[x])$. For $(f, g) \in A$, let

$$\begin{aligned} \alpha \cdot (f, g) &= (0, \sigma(f)) \\ \alpha^{-1} \cdot (f, g) &= (\sigma^{-1}(g), f) \\ e_1 \cdot (f, g) &= (f, 0) \\ e_2 \cdot (f, g) &= (0, g). \end{aligned}$$

Then A is a $\mathcal{G}$-module algebra.

Symmetries by groupoids

Definition.

Let $A = \bigoplus_{x \in X} A_x$ be an X-decomposable algebra. We define $\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$, the X-algebra automorphism groupoid of A:

- the object set is X,
- for any $x, y \in X$, $\mathrm{Hom}_{\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)}(x, y)$ is the space of algebra isomorphisms between the unital algebras A_x and A_y.

Theorem.

$\mathcal{G}$ an X-groupoid, then A is a $\mathcal{G}$-module algebra if and only if there exists a morphism $\pi : \mathcal{G} \to \mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$.

Hence $\mathrm{Sym}_{X\text{-}\mathsf{Grpd}}(A) = \mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$.

Algebras that groupoids act on

- $A = \bigoplus_{x \in X} A_x$ seems like a strong hypothesis.

Theorem.

Let H be a weak Hopf algebra with $H_s = H_t$ and let A be an H-module algebra. Then $A = \bigoplus_{i=1}^{n} A_i$ is an X-decomposable algebra where $X = \{e_1, \ldots, e_n\}$ is a complete set of primitive idempotents of H.
The local identities of A are given by the family of orthogonal idempotents $\{e_i \cdot 1_A \mid 1 \leq i \leq n\}$.

- $\Bbbk\mathcal{G}$-mod $\cong \mathcal{G}$-mod.

"No weak quantum symmetries"

Corollary.

Suppose that $\mathcal{G}$ is a finite groupoid and A is a domain such that A is an inner-faithful $\Bbbk\mathcal{G}$-module algebra. Then $\mathcal{G}$ is a disjoint union of groups, and at most one of the groups is nontrivial.

Conjecture.

If H is a weak Hopf algebra and A is a domain that is an inner-faithful H-module algebra, then H is a Hopf algebra.

Back to Lie symmetries

Definition.

An X-Lie algebroid $\mathfrak{G}$ is a direct sum of vector spaces

$$\mathfrak{G} := \bigoplus_{x \in X} \mathfrak{g}_x$$

where each $\mathfrak{g}_x$. We regard $\mathfrak{G}$ as having a partially defined bracket $[-,-]$.
The X-universal enveloping algebra of $\mathfrak{G}$ is $U_X(\mathfrak{G}) = \bigoplus U(\mathfrak{g}_x)$.

Theorem. [Nikshych thesis]

Any cocommutative weak Hopf algebra is isomorphic to $U_X(\mathfrak{G})\#\Bbbk\mathcal{G}$ for an X-groupoid $\mathcal{G}$ and an X-Lie algebroid $\mathfrak{G}$.

- Analogue of Gabriel–Kostant–Milnor–Moore.

Modules over Lie algebroids

Definition.

$V = \bigoplus_{x \in X} V_x$ is a $\mathfrak{G}$-module if each V_x is a $\mathfrak{g}_x$-module.

Definition.

The X-general linear algebroid $\mathfrak{GL}_X(V) = \bigoplus_{x \in X} \mathfrak{gl}(V_x)$, viewed as a Lie algebroid.

Lemma.

V is a $\mathfrak{G}$-module if and only if there is an X-Lie algebroid homomorphism $\mathfrak{G} \to \mathfrak{GL})_X(V)$.

Modules algebras of Lie algebroids

Lemma.

$A = \bigoplus_{x \in X} A_x$ is a $\mathfrak{G}$-module algebra if and only if A is a $\mathfrak{G}$-module such that

$$p \cdot (ab) = a(p \cdot b) + (p \cdot a)b,$$
$$p \cdot 1_x = 0,$$

for all $p \in \mathfrak{g}_x$ and $a, b \in A_x$.

Symmetries by Lie algebroids

Definition.

$A = \bigoplus_{x \in X} A_x$. The X-Lie algebroid of derivations is $\mathrm{Der}_X(A) = \bigoplus_{x \in X} \mathrm{Der}(A_x)$.

Theorem.

If $\mathfrak{G}$ is an X-Lie algebroid, then A is a $\mathfrak{G}$-module algebra if and only if there exists an X-Lie algebroid homomorphism $\tau : \mathfrak{G} \to \mathrm{Der}_X(A)$.

Hence $\mathrm{Sym}_{\mathsf{Lie}}(A) = \mathrm{Der}_X(A)$.

From groupoids to groupoid algebras

- $\mathcal{G}$ an X-groupoid, $A = \bigoplus_{x \in X} A_x$.

Theorem.

A is a $\mathcal{G}$-module algebra if and only if there exists an X-groupoid morphism $\pi : \mathcal{G} \to \mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$.

Hence $\mathrm{Sym}_{X\text{-}\mathsf{Grpd}}(A) = \mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$.

Theorem.

A is a $\Bbbk\mathcal{G}$-module algebra if and only if there exists a weak Hopf morphism $\widetilde{\pi} : \Bbbk\mathcal{G} \to \Bbbk\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$.

- The groupoid algebra functor X-$\mathsf{Grpd} \to X$-WHA is left adjoint to the grouplike elements functor X-$\mathsf{WHA} \to X$-Grpd.

From Lie algebroids to universal enveloping algebras

- $\mathfrak{G}$ an X-Lie algebroid, $A = \bigoplus_{x \in X} A_x$.

Theorem.

A is a $\mathfrak{G}$-module algebra if and only if there exists an X-Lie algebroid morphism $\tau : \mathfrak{G} \to \mathrm{Der}_X(A)$.

Hence $\mathrm{Sym}_{X\text{-Lie}}(A) = \mathrm{Der}_X(A)$.

Theorem.

A is a $U_X(\mathfrak{G})$-module algebra if and only if there exists an weak Hopf morphism $\widetilde{\tau} : U_X(\mathfrak{G}) \to U_X(\mathrm{Der}_X(A))$.

Weak Hopf algebra symmetries

- By Nikshych's analogue to Gabriel–Kostant–Milnor–Moore, we have the following:

Theorem.

Let $H := U_X(\mathfrak{G})\#\Bbbk\mathcal{G}$ be a cocommutative weak Hopf algebra and $A = \bigoplus_{x\in X} A_x$ be a X-decomposable algebra. Then the following statements are equivalent:

1. A is an H-module algebra.
2. There exists an X-weak Hopf algebra homomorphism

$$\phi : H \longrightarrow U_X(\mathrm{Der}_X(A))\#\Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)).$$

Hence $\mathrm{Sym}_{X\text{-}\mathsf{CocomWHA}}(A) = U_X(\mathrm{Der}_X(A))\#\Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))$.

Future work

- Let $A = \bigoplus_{x \in X} A_x$.

Question.

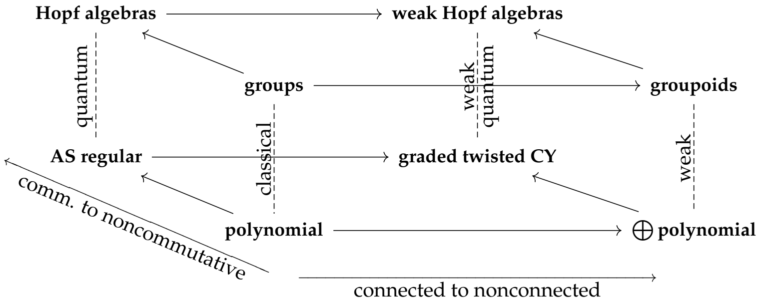
Take a finite subgroupoid $\mathcal{G}$ of $\mathrm{Sym}_{\mathsf{Grpd}}(A) = \mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$. Study $\mathcal{G}$ actions on A.

Chevalley–Shephard–Todd? Watanabe's Theorem? Auslander's Theorem? Homological determinants?

Question.

Take a finite-dimensional weak Hopf subalgebra H of $\mathrm{Sym}_{X\text{-}\mathsf{CocomWHA}}(A) = U_X(\mathrm{Der}_X(A))\#\Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))$. Study H actions on A.

Future work



Hopf algebras
weak Hopf algebras
groups
groupoids
quantum
weak quantum
classical
weak
AS regular
graded twisted CY
polynomial
⊕ polynomial
comm. to noncommutative
connected to nonconnected