Categories of hypergroups and hyperstructures

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Hypergroups and hyperstructures

History of hypergroups

Idea: A hypergroup is “just like a group,” but whose binary operation $G \times G \to G$ is replaced by a *multivalued* operation $G \times G \to \mathcal{P}(G)$.

Defined by Frédéric Marty in 1934:

- Mentions that the idea arose from groups of transformations, not much else known about the origin
- Was only able to publish three works on the idea before he died at age 29 in World War II

We will work with the following, close to Marty's original notion.

Def: A hypergroup is a set G with a hyperoperation:

$$\star\colon G \times G \to \mathcal{P}(G) \setminus \{\varnothing\},$$

that satisfies the following properties

- **associative:** $x \star (y \star z) = (x \star y) \star z$ as *subsets*;
- **identity:** exists $e \in G$ such that $x \star e = \{x\} = e \star x$;
- **inverses:** $e \in x^{-1} \star x$ and $e \in x \star x^{-1}$;
- **reversible:** $x \in y \star z \implies y \in x \star z^{-1}$ and $z \in y^{-1} \star x$.

First remarks on hypergroups

Why these particular axioms?

Weak units: If we only demand $x \in e \star x \cap x \star e$, then the unit e need not be unique.

Example: Any set X with $x \star y = X$ for all $x, y \in X$.

Reversibility: implies uniqueness of inverses, since

$$e \in x' \star x \implies x' \in e \star x^{-1} = \{x^{-1}\}$$

(Otherwise, no reason that inverses need be unique.)

Examples of hypergroups

Example 1: Given a non-normal subgroup K of a group H, the double cosets $G = H//K$ form a hypergroup under

$$KxK \star KyK = \{KwK \mid KwK \subseteq KxKyK\}.$$

Example 2: For $A \in \mathbf{Ab}$ and $G \subseteq \mathrm{Aut}(A)$, the quotient A/G forms a hypergroup under

$$Gx \boxplus Gy := \{Gz \mid z \in Gx + Gy\} = Gy \boxplus Gx.$$

(A commutative reversible hypergroup is said to be canonical.)

Some canonical hypergroups

Ex (Roth 1974): The set $\{\chi_i\}$ of complex irreducible characters of a finite group G is a canonical hypergroup under:

$$\chi \star \psi := \{\chi_i \mid \chi \cdot \psi = \sum c_i \chi_i,\ c_i \neq 0\}.$$

Ex (Prenowitz 1943): If G is a projective geometry with at least 3 points in each line, then $G \sqcup \{0\}$ is a hypergroup by defining hyperaddition of $x, y \in G$ as

$$x + y = \begin{cases} \ell(x,y) \setminus \{x,y\}, & x \neq y \\ \{x, 0\}, & x = y. \end{cases}$$

(Here $-x = x$ and reversibility follows from projective axioms…)

Morphisms of hypergroups

Def: A morphism of hypergroups is a function $\phi\colon G \to H$ satisfying

$$\phi(x \star y) \subseteq \phi(x) \star \phi(y).$$

The morphism is strict if $\phi(x \star y) = \phi(x) \star \phi(y)$.

Ex: For many of our hypergroups that are obtained as quotients, the quotient map is a (not necessarily strict) morphism.

Hyperrings

Def (Krasner 1957): A hyperring R is both a multiplicative monoid (in the ordinary sense) and an additive canonical hypergroup, satisfying distributivity of the form

$$x(y+z) = xy + xz \quad \text{and} \quad (y+z)x = yx + zx$$

in $\mathcal{P}(R)$.

Morphisms: $\phi(xy) = \phi(x)\phi(y)$ and $\phi(x+y) \subseteq \phi(x) + \phi(y)$.

Ex (Krasner 1983): If R is a ring and $G \leq U(R)$ normalizes R, then R/G becomes a hypergroup under $\bar{x} + \bar{y} = \{\bar{z} \mid zG \subseteq xG + yG\}$.

A hyperfield is a commutative hyperring whose nonzero elements form a multiplicative group.

Ex: The Krasner hyperfield $\mathsf{K} = \{0, 1\}$ has usual multiplication, hyperaddition given by $1 + 1 = \{0, 1\}$. (Note $\mathsf{K} \cong F/F^\times$ for any field of $|F| > 2$.) It is the *terminal* hyperfield.

Ex: The hyperfield of signs $\mathbb{S} = \{1, 0, -1\} \cong \mathbb{R}/\mathbb{R}_{>0}$ with usual multiplication and

$$1 + 1 = 1, \quad -1 - 1 = -1, \quad 1 - 1 = \{1, 0, -1\}.$$

Representing Spec

One reason we might study hyperstructures even if we are primarily interested in “ordinary” algebraic structures: functors that were not representable may “become” representable!

Nullstellensatz: for commutative affine algebras A over $k = \overline{k}$,

$$\mathrm{Max}(A) \cong \mathrm{Hom}(A, k).$$

So the maximal spectrum is *representable* on the category of such algebras.

We are not so lucky for the Zariski spectrum in general...

While there is a sense in which each commutative ring has

$$\mathsf{Spec}(R) \cong \mathsf{Hom}(R, \text{“fields”}),$$

there is no single field (or ring) that represents this functor…
Or is there?

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Or is there?

Connes-Consani (2011): For any commutative (hyper)ring R,

$$\mathrm{Spec}(R) \cong \mathrm{Hom}(R, \mathrm{K})$$

Even more, if X is any scheme, $|X| \cong \mathrm{Hom}(\mathrm{Spec}(\mathrm{K}), X)$.

Jun (2021) has also showed how to recover the Zariski topology in this context.

Today's topic

Current work-in-progress with So Nakamura:

How to organize canonical hypergroups into a category with "good" properties?

Would like (co)completeness, plus correspondences similar to what we have with rings R and modules M:

$$\text{bilinear } R \times M \to M \iff R \otimes_{\mathbb{Z}} M \to M \iff R \to \mathrm{Hom}_{\mathbb{Z}}(M, M)$$

Eventual hope is to understand K-represenation theory for noncommutative rings...

Hyperstructures of interest

Note: We will now allow hyperoperations that possibly have empty products/sums: $M \times M \to \mathcal{P}(M)$

Def: A hypermagma $(M, \star)$ is a set with a hyperoperation. We also define:

- A unit $e \in M$ is an element satisfying $e \star x = x \star e = x$
- M is associative if $x \star (y \star z) = (x \star y) \star z$
- M is reversible if there is an operation $(-)^{-1}$ satisfying $x \in y \star z \implies y \in x \star z^{-1}$ and $z \in y^{-1} \star x$
- An inverse x' of x satisfies $e \in x \star x' \cap x' \star x$

A hypermonoid is an associative hypermagma with unit.

Categories of interest

HMag = category of hypermagmas with morphisms satisfying $\phi(x \star y) \subseteq \phi(x) \star \phi(y)$

uHMag = category of unital hypermagmas with morphisms as above that preserve unit.

Full subcategories of hypermonoids and (canonical) hypergroups:

$$\textbf{Can} \subseteq \textbf{HGrp} \subseteq \textbf{HMon} \subseteq \textbf{uHMag}$$

(Note: Morphisms automatically satisfy $f(x^{-1}) = f(x)^{-1}$ by uniqueness of inverses.)

Categories of hypermagmas

The category of hypermagmas

The forgetful functor **HMag** $\to$ **Set** has *both* a left adjoint and a right adjoint:

- Free functor $F\colon$ **Set** $\to$ **HMag** has $F(X) = X$ with $x \star y = \varnothing$
- Cofree functor $D\colon$ **Set** $\to$ **HMag** has $D(X) = X$ with $x \star y = X$.

Theorem

The forgetful functor **HMag** $\to$ **Set** *creates all limits and colimits. Thus* **HMag** *is complete and cocomplete*

Unital hypermagmas

For *unital* hypermagmas, the forgetful functor **uHMag** $\to$ **Set$_\bullet$** has a left adjoint, and it creates limits and coproducts:

- Products are $\prod M_i$ with identity $(e_i)_I$.
- Coproducts are wedge sums $\bigvee(M_i, e_i)$, $M_i \star M_j = \varnothing$ for $i \neq j$

Constructing coequalizers requires an extra step...

Unital hypermagmas

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Constructing coequalizers requires an extra step...

Lemma: For any hypermagma M and $E \subseteq M$, there exists $M_E \in \mathbf{uHMag}$ that represents the functor $\mathbf{uMag} \to \mathbf{Set}$,

$$F(N) = \{f \in \mathbf{HMag}(M, N) \mid E \subseteq f^{-1}(e_N)\}.$$

We call M_E the unitization of M at E.

Cocompleteness of uHMag

Theorem

The category **uHMag** *is complete and cocomplete.*

Proof: Since it has limits and coproducts, just need to construct coequalizers. Take $f, g \colon M \to N$ in **uHMag**.

Let $\pi_L \colon N \to L$ be their coequalizer in **HMag**.

Problem: The image of e_N in L may not be an identity.

Solution: Set $E = \{\pi_L(e_N)\} \subseteq L$, and form $\pi_E \colon L \to L_E$.

Then we get the coequalizer in **uHMag** by composing

$$M \underset{g}{\overset{f}{\rightrightarrows}} N \xrightarrow{\pi_L} L \xrightarrow{\pi_E} L_E, \qquad \pi = \pi_E \circ \pi_L \colon N \twoheadrightarrow L_E$$

Epimorphisms and monomorphisms

The mono- and epimorphisms are the usual suspects:

Proposition: The a morphism in **HMag** (resp., **uHMag**) is a monomorphism if and only if it is injective, and it is an epimorphism if and only if it is surjective.

(For **HMag** this is directly from co/free constructions. Surjections in **uHMag** take just a little more work…)

In certain categories (like rings), epi/monomorphisms may not be so “nice” in practice. So it can be useful to consider more restricted classes of epi/mono’s…

Regular and normal epi's and mono's

In any category, one defines:

- regular epimorphism = coequalizer
- regular monomorphism = equalizer

If the category has a zero object, then one defines:

- normal epimorphism = cokernel = $\mathsf{Coeq}(f, 0)$
- normal monomorphism = kernel = $\mathsf{Eq}(f, 0)$

Why we're often blind to the distinction: in any abelian category, the various epi's and mono's are all equivalent!

Special morphisms and subobjects

Def: A morphism of hypermagmas $f : M \to N$ is:

- short if f is surjective and satisfies, for all $x, y \in N$:

$$x \star y = f(f^{-1}(x) \star f^{-1}(y)).$$

- coshort if f is injective and satisfies, for all $a, b \in M$:

$$f^{-1}(f(a) \star f(b)) \subseteq a \star b.$$

Note: Coshort morphisms correspond to *weak subhypermagmas*: $M \subseteq N$ with $x \star_M y = (x \star_N y) \cap M$.

Let's say a *strong subhypermagma* is $M \subseteq N$ such that $x, y \in M \implies x \star_N y \subseteq M$.

Regular and normal epi's and mono's

Theorem

In each of HMag *and* uHMag*, we have*

- *the regular epimorphisms are the short morphisms;*
- *the regular monomorphisms are the coshort morhpisms (i.e., weak subhypermagmas).*

In uHMag*, we have:*

- *the normal epimorphisms* $p\colon M \to N$ *correspond to the unitizations* $M \to M_E$*;*
- *the normal monomorphisms correspond to strong subhypermagmas.*

Closed monoidal structure

There is an obvious way to equip the hom sets with a hyperoperation: for $f, g \in \mathbf{HMag}(M, N)$,

$$f \star g := \{h \in \mathbf{HMag}(M, N) \mid h(x) \in f(x) \star g(x) \text{ for all } x \in M\}$$

This forms the enriched hom for a closed monoidal structure:

Set $M \boxdot N = M \times N$ with "minimal" hyperoperation satisfying

$$(x \star x') \boxdot y \subseteq x \boxdot y \star x' \boxdot y$$
$$x \boxdot (y \star y') \subseteq x \boxdot y \star x \boxdot y'$$

Unit: $1_\varnothing$ is $1 = \{*\}$ with empty product $\varnothing \subseteq \mathbf{Rel}(1 \times 1, 1)$

Closed monoidal structure

Universal property: A *bimorphism* is a function $B \colon M \times N \to L$ such that each $B(x, -) \colon N \to L$ and $B(-, y) \colon M \to L$ are morphisms.

Then $\mathsf{Bim}(M, N; -) \cong \mathsf{HMag}(M \boxdot N, -) \colon \ \mathsf{HMag} \to \mathbf{Set}$.

Theorem

Thm: *The structure* $(\mathsf{HMag}, \boxdot, 1_{\varnothing})$ *is a symmetric closed monoidal category, whose internal hom is* $\mathsf{HMag}(M, N)$ *with the above hyperoperation:*

$$\mathsf{HMag}(L \boxdot M, N) \cong \mathsf{HMag}(L, \mathsf{HMag}(M, N))$$

Closed monoidal structure: unital case

Similarly **uHMag** has a closed monoidal structure, "created" by the closed monoidal structure of $(\mathbf{Set}_\bullet, \wedge, 1)$.

The same hyperopration on $\mathbf{uHMag}(M, N)$ makes it a unital hypermamga, whose unit is the constant morphism e_N.

For $E := M \boxdot e_N \cup e_M \boxdot N = M \vee N \subseteq M \boxdot N$, set

$$M \boxslash N = (M \boxdot N)_E$$

This has underlying set $(M, e_M) \wedge (N, e_N) = M \times N/(M \vee N)$ given by the wedge product. It has the effect of setting

$$e_M \boxslash n = m \boxslash e_N = e_M \boxslash e_N.$$

As before, this represents bimorphisms in the category **uHMag**:

$$\mathrm{Bim}_{\mathbf{uHMag}}(M, N; -) \cong \mathbf{uHMag}(M \boxtriangle N, -)$$

Unit is the terminal hypermagma 1 (i.e., the trivial monoid)

Theorem

The symmetric monoidal category $(\mathbf{uHMag}, \boxtriangle, 1)$ *is closed, with internal hom being the natural structure on hom sets:*

$$\mathbf{uHMag}(M \boxtriangle N, L) \cong \mathbf{uHMag}(M, \mathbf{uHMag}(N, L)).$$

Categories of hypergroups and generalizations

A difficult lesson: While hypergroups form attractive objects, they do not form such a nice category!

Both **HGrp** and **Can** are closed under products in **uHMag**.

But we are not so lucky when it comes to coproducts.

Theorem

The coproduct $\mathbb{F}_2 \coprod \mathbb{F}_2$ *does not exist in* **HGrp** *or* **Can**.

Argument uses the fact that $\mathsf{Hom}(-, \mathsf{K})$ deterines all possible kernels of morphisms out of an object...

Equalizers are also problematic...

Let $H = \mathbb{F}_9/\mathbb{F}_3^\times$, a finite projective plane with zero.

Let $F\colon H \to H$ be the morphism induced by the Frobenius map $x \mapsto x^3$ on $\mathbb{F}_9$.

Theorem

There is no equalizer of the morphisms

$$H \underset{F}{\overset{\mathrm{id}_H}{\rightrightarrows}} H$$

in the categories HGrp *or* Can.

Internal hom

What about internal homs? We know the natural structure on hom sets gives a unital hypermagma: for $f, g \in \mathsf{Can}(G, H)$ we set

$$f + g = \{h \in \mathsf{Can}(G, H) \mid h(x) \in f(x) + g(x) \text{ for all } x \in G\}.$$

It also inherits reversibility if we assume H is reversible, so it only needs associativity.

As discussed before, associativity would then imply $f + g = \varnothing$ for all f, g...

Failure of internal homs

Define a comutative ring by generators and relations:

$$R = \mathbb{Z}[x, y, z \mid 2x = y(z+1) = 0,\ z^2 = 1].$$

Then $G = \{1, z\} \subseteq R$ is a multiplicative group, and we may form the quotient hyperring R/G.

Theorem

Let $f, g \in \mathsf{Can}(\mathbb{F}_2, R/G)$ be given by $f(1) = [x]$ and $g(1) = [y]$. Then

$$f + g = \varnothing,$$

so the commutative unital reversible hypermagma $\mathsf{Can}(\mathbb{F}_2, R/G)$ is not a hypergroup.

It is also not possible to form a “tensor product” over **Can** that represents the “bilinear” mappings.

Theorem

Let $V = \mathbb{F}_2 \times \mathbb{F}_2$ denote the Klein four-group. Then the functor of bimorphisms

$$\mathrm{Bim}_{\mathsf{Can}}(V, V; -)\colon\ \mathsf{Can} \to \mathsf{Set}$$

is not representable.

A convenient category of canonical hypergroups

Def: We will say that a unital, reversible hypermagma is a mosaic.

So we have full subcategories **HGrp** $\subseteq$ **Msc** $\subseteq$ **uHMag** and **Can** $\subseteq$ **cMsc**.

Happily, reversibility behaves well under many operations:

Theorem

The categories **Msc** *and* **cMsc** *are closed under limits and colimits in* **uHMag**. *Thus they are both complete and cocomplete.*

Free mosaics are commutative

There are also free mosaics:

Theorem

The forgetful functor $U\colon \mathsf{Msc} \to \mathsf{Set}$ *has a left adjoint.*

Construction: $F(X) = X \sqcup -X \sqcup \{0\}$ with addition of nonzero elements given by

$$a + b = \begin{cases} 0 & \text{if } a = -b, \\ \varnothing & \text{otherwise} \end{cases}$$

Tensor products of commutative mosaics

Is the absence of associativity enough for us to form “tensor products?”

Tensor products of commutative mosaics

Is the absence of associativity enough for us to form "tensor products?" Yes!

For $M, N \in \mathbf{cMsc}$, define morphisms $i_{--} \colon M \mathbin{\boxed{\wedge}} N \to M \mathbin{\boxed{\wedge}} N$ by $i_{--}(x \mathbin{\boxed{\wedge}} y) = (-x) \mathbin{\boxed{\wedge}} (-y)$. Then for $i_{++} = \mathrm{id}_{M \mathbin{\boxed{\wedge}} N}$, we take the coequalizer

$$M \mathbin{\boxed{\wedge}} N \underset{i_{--}}{\overset{i_{++}}{\rightrightarrows}} M \mathbin{\boxed{\wedge}} N \twoheadrightarrow M \boxtimes N$$

This satisfies expected relations like

$$0 \boxtimes n = 0 \boxtimes 0 = m \boxtimes 0,$$
$$(-m) \boxtimes n = m \boxtimes (-n) =: -m \boxtimes n,$$

and it forms an object of $\mathbf{cMsc}$ (reversible).

Closed monoidal structure

As we expect by this point, this represents bimorphisms:

$$\mathrm{Bim}_{\mathsf{cMsc}}(M, N; L) \cong \mathsf{cMsc}(M \boxtimes N, L).$$

Unit is the free object on one element $F = \{1, 0, -1\}$

Theorem

The symmetric monoidal category $(\mathsf{cMsc}, \boxtimes, F)$ *is closed, with internal hom being the natural one:*

$$\mathsf{cMsc}(M \boxtimes N, L) \cong \mathsf{cMsc}(M, \mathsf{cMsc}(N, L)).$$

$$
\begin{array}{ccccc}
\mathsf{cRev} & \subseteq & \mathsf{Rev} & \subseteq & \mathsf{HMag} \\
\rotatebox{90}{$\hookrightarrow$} & & \rotatebox{90}{$\hookrightarrow$} & & \rotatebox{90}{$\hookrightarrow$} \\
\mathsf{cMsc} & \subseteq & \mathsf{Msc} & \subseteq & \mathsf{uHMag} \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
\mathsf{Can} & \subseteq & \mathsf{HGrp} & \subseteq & \mathsf{HMon} \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
\mathsf{Ab} & \subseteq & \mathsf{Grp} & \subseteq & \mathsf{Mon}
\end{array}
$$

Review of morphisms

normal monomorphism	$\Longrightarrow$	regular monomorphism	$\Longrightarrow$	mono.
$\Updownarrow$		$\Updownarrow$		$\Updownarrow$
strict submosaic	$\Longrightarrow$	weak submosaic	$\Longrightarrow$	injective morphism

normal epimorphism	$\Longrightarrow$	regular epimorphism	$\Longrightarrow$	epimorphism
$\Updownarrow$		$\Updownarrow$		$\Updownarrow$
unitization	$\Longrightarrow$	short morphism	$\Longrightarrow$	surjective morphism

Combinatorial examples of mosaics

A matroid is a set M with a closure operator $C\colon \mathcal{P}(M) \to \mathcal{P}(M)$ satisfying the *exchange axiom*:

$$x \in C(S \cup y) \setminus C(S) \implies y \in C(S \cup x).$$

A (strong) morphism of matroids $f\colon M \to N$ must satisfy $f(C_M(S)) \subseteq C_N(f(S))$ (equiv., preimage of closed sets is closed).

A pointed matroid $(M, 0)$ has a distinguished “loop” $0 \in C(\varnothing)$.

We have categories $\mathbf{Mat}$ and $\mathbf{Mat}_\bullet$ of (pointed) matroids.

Example: M a vector space, $C(S) = \mathsf{Span}(S)$, $0 \in C(\varnothing)$

Combinatorial examples of mosaics

A pointed matroid is simple if $C(\varnothing) = \{0\}$ and $C(x) = \{x, 0\}$.

Ex: $M = V/k^* = \mathbb{P}(V) \sqcup \{0\}$, closed subsets = linear subspaces

Inspired by similar construction for projective geometries:

Theorem: There is a functor $\mathbf{sMat_\bullet} \to \mathbf{cMsc}$ that turns a simple pointed matroid $(M, 0)$ into a mosaic with identity 0 by setting, for $x, y \neq 0$:

$$x + y = \begin{cases} C(x,y) \setminus \{x, y, 0\}, & x \neq y \\ \{x, 0\}, & x = y. \end{cases}$$

Conjecture: This restricts to a fully faithful embedding on pointed projective geometries $\mathbf{Proj_\bullet} \subseteq \mathbf{sMat_\bullet}$ (no assumptions about number of points on a line).

What's next?

Exactness: What exactness conditions does **Msc** satisfy? How close does **cMsc** come to being "non-additive abelian"?

Generalized hyperrings and modules: Do "rings" with additive mosaic structure also enjoy good categorical properties? What about their module categories?

Extraordinary representations of rings: What unusual modules can be constructed in this setting over an ordinary ring? Can they provide new tools to study the structure of rings?

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Thank you!