Hopf Actions of some quantum doubles on Artin-Schelter regular algebras

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March 18th, 2023

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Plan of the talk:

1. Hopf actions
2. Quantum double $D(G)$ of a group G
3. $D(G)$-modules
4. A detailed example: $G = Q_8$

Throughout, let H be a finite dimensional Hopf algebra over a field k.

Motivating Question: Given a Hopf algebra H, determine 'nice' algebras A on which H acts homogeneously and inner-faithfully, and in turn study properties of the invariant subalgebra A^H.

In this talk, we will focus on when H is semisimple, and search for quadratic Artin-Schelter regular algebras A.

Hopf Actions

When we say that a Hopf algebra H acts on an algebra A, we mean that A is an H-comodule algebra.

That is, the action of H on A is associative, and for any elements $h \in H$ and $a, b \in A$, one has

$$(ab).h = \sum (a.h_{(1)})(b.h_{(2)})$$

In this context, the invariant subalgebra is defined as:

$$A^H = \{f \in A \mid f.h = \epsilon(h)f\}$$

We also often assume that A is generated in degree one, so that A_1 is an H-module, and this H-module structure determines the action of H on A.

Inner-faithful actions

An inner-faithful H-module is an H-module V that is not annihilated by a nontrivial Hopf ideal.

Theorem (Rieffel)

Let H be a semisimple Hopf algebra. Then V is an inner-faithful H-module if and only if every simple H-module appears as a direct summand of some tensor power of V.

Examples

1. kG for G a finite group, char$k \nmid |G|$. A^G is the usual ring of invariants, and inner-faithful actions are just the faithful actions.

2. $H = (kG)^*$ for G a group. A homogeneous action of H on A is a G-grading on A that is compatible with the $\mathbb{N}$-grading. This action is inner-faithful if the group grades of the G-homogeneous elements in A_1 generate G.

The Quantum Double $D(G)$

Let G be a group and k a field. The quantum double $D(G)$ of G is the algebra

$$D(G) = (kG)^* \# kG$$

where we let G act on $(kG)^*$ via $(f.g)(x) = f(g^{-1}xg)$ for $f \in (kG)^*$ and $g \in G$.

Alternatively, we may write (suppressing the $\#$ or $\otimes$ sign):

$$(\phi_a g)(\phi_b h) = \phi_a \phi_{g^{-1}bg} gh$$

which is nonzero if and only if $a = g^{-1}bg$.

Invariants under $D(G)$

Note that if $D(G)$ acts on A, then $(kG)^*$ acts on A, kG acts on A, and these actions are compatible (explained in detail on next slide).

Lemma

Let $D(G)$ act on an algebra A, so that A is G-graded and carries a compatible G-action. Then:

1. *A_e (the identity component of A in the G-grading) also carries a G-action.*
2. *While A^G need not be G-graded, the identity component of a G-invariant element is again G-invariant.*
3. $A^H = (A_e)^G = A_e \cap A^G$.

G-equivariant vector bundles

Let G act on itself on the right by conjugation. That is, $a.g = a^g = g^{-1}ag$.

A G-equivariant vector bundle on G is a collection of vector spaces $\{V_a\}_{a\in G}$ together with a representation of G on $V = \bigoplus_{a\in G} V_a$ such that for each $v \in V_a$ and $g \in G$, one has $v.g \in V_{a^g}$.

The collection of all such objects and morphisms between them forms a category $\text{vect}_G(G)$.

Theorem (Witherspoon)

There is an equivalence of categories between $\text{vect}_G(G)$ *and* mod $D(G)$*.*

In this way, to search for $D(G)$-modules, we instead may search for G-graded vector spaces that carry an action of G that is compatible with the grading using the conjugation action.

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Simple $D(G)$-modules

Let $\{a_1, \ldots, a_r\}$ be a set of representatives of the conjugacy classes of G. Let G_i denote the centralizer $C_G(a_i)$.

The simples of $D(G)$ are in bijection with the set

$$\{(a_i, V) \mid V \text{ is a simple } kG_i \text{ module}\}.$$

The simple $D(G)$-module corresponding to (a_i, V) is the induced right kG-module

$$V \otimes_{kG_i} kG$$

where the group grade of $v \otimes g$ is a_i^g (so that this module is concentrated in degrees given by the conjugacy class of a_i).

Note that a basis of $V \otimes_{kG_i} kG$ may be taken to be the elementary tensors between a basis of V and elements of a right transversal of G_i in G, so that

$$\dim_k(V \otimes_{kG_i} kG) = (\dim V)[G : G_i].$$

An Example

Consider $Q_8 = \langle r, s \mid r^4 = 1, s^2 = r^2, srs^{-1} = r^{-1} \rangle$.

Conjugacy class	Centralizer	Representations	Transversal
$\{e\}$	Q_8	$\psi_0, \psi_1, \psi_2, \psi_3, \chi$	$\{1\}$
$\{r^2\}$	Q_8	$\psi_0, \psi_1, \psi_2, \psi_3, \chi$	$\{1\}$
$\{r, r^3\}$	$\langle r \rangle$	$\alpha_0, \alpha_1, \alpha_2, \alpha_3$	$\{1, s\}$
$\{s, sr^2\}$	$\langle s \rangle$	$\beta_0, \beta_1, \beta_2, \beta_3$	$\{1, r\}$
$\{sr, sr^3\}$	$\langle sr \rangle$	$\gamma_0, \gamma_1, \gamma_2, \gamma_3$	$\{1, s\}$

Here, α_ℓ, β_ℓ and γ_ℓ send the generator of their corresponding centralizer to i^ℓ.

Also, ψ_0 is the trivial module of kG, ψ_1, ψ_2, ψ_3 are the one-dimensional reps corresponding to the index two subgroups $\langle r \rangle$, $\langle s \rangle$ and $\langle rs \rangle$ respectively, and χ is the unique two-dimensional simple representation of G.

We number the corresponding simple modules of $D(G)$ from the above table

$$V_0, V_1, \ldots, V_{21}.$$

Inner Faithful Q_8-modules

Theorem (Kirkman-Oke-M)

Let V be a $D(Q_8)$-module. Then:

1. *If V has only one or two distinct simple summands up to isomorphism, then V is not inner-faithful.*
2. *Suppose $V = W_1 \oplus W_2 \oplus W_3$ with each W_ℓ simple. Then V is an inner-faithful $D(G)$-module if and only if W_1 and W_2 correspond to distinct non-singleton conjugacy classes and W_3 is:*
   a. V_4 *or* V_9, *or*
   b. *One of two simples arising from each of the other three non-singleton conjugacy classes (six total choices) depending on your choices of W_1 and W_2.*

In total, there are 224 non-isomorphic inner-faithful $D(Q_8)$-modules with three distinct simple summands, and all of them dimension six.

Example: $V = V_{17} \oplus V_{20} \oplus V_{21}$

To determine a quadratic Artin-Schelter regular algebra A on which $D(Q_8)$ acts, we must choose a 15-dimensional $D(Q_8)$-submodule W of $V \otimes V$. So we must decompose $V \otimes V$ as a direct sum of simple $D(Q_8)$-modules.

Using either Witherspoon's character theory for almost cocommutative algebras or the S-matrix of this group due to Coste-Gannon-Ruelle, one has:

$$
\begin{aligned}
V_{17} \otimes V_{17} &= V_0 \oplus V_2 \oplus V_6 \oplus V_8 \\
V_{20} \otimes V_{20} &= V_0 \oplus V_3 \oplus V_5 \oplus V_8 \\
V_{21} \otimes V_{21} &= V_0 \oplus V_3 \oplus V_6 \oplus V_7 \\
V_{17} \otimes V_{20} \cong V_{20} \otimes V_{17} &= V_{11} \oplus V_{13} \\
V_{17} \otimes V_{21} \cong V_{21} \otimes V_{17} &= V_{10} \oplus V_{12} \\
V_{20} \otimes V_{21} \cong V_{20} \otimes V_{21} &= V_4 \oplus V_9
\end{aligned}
$$

However, we need the actual decompositions that realize these isomorphisms.

Examining a particular case

If we let

$$V_{17} = \text{span}_k\{x_1, x_2\}, V_{20} = \text{span}_k\{y_1, y_2\}, V_{21} = \text{span}_k\{z_1, z_2\},$$

then one may show

$$V_{17} \otimes V_{17} = \text{span}_k\{x_1x_2 - x_2x_1\} \oplus \text{span}_k\{x_1x_2 + x_2x_1\} \oplus \text{span}_k\{x_1^2 + x_2^2\} \oplus \text{span}_k\{x_1^2 - x_2^2\}$$

$$V_{20} \otimes V_{20} = \text{span}_k\{y_1y_2 + y_2y_1\} \oplus \text{span}_k\{y_1y_2 - y_2y_1\} \oplus \text{span}_k\{y_1^2 + y_2^2\} \oplus \text{span}_k\{y_1^2 - y_2^2\}$$

$$V_{21} \otimes V_{21} = \text{span}_k\{z_1z_2 - z_2z_1\} \oplus \text{span}_k\{z_1z_2 + z_2z_1\} \oplus \text{span}_k\{z_1^2 - z_2^2\} \oplus \text{span}_k\{z_1^2 + z_2^2\}$$

$$V_{17} \otimes V_{20} = \text{span}_k\{x_2y_1 - \mathbb{i}x_1y_2, \mathbb{i}x_2y_2 - x_1y_1\} \oplus \text{span}_k\{x_2y_1 + \mathbb{i}x_1y_2, \mathbb{i}x_2y_2 + x_1y_1\}$$

$$V_{20} \otimes V_{17} = \text{span}_k\{y_1x_1 + \mathbb{i}y_2x_2, -\mathbb{i}y_2x_1 - y_1x_2\} \oplus \text{span}_k\{y_1x_1 - \mathbb{i}y_2x_2, -\mathbb{i}y_2x_1 + y_1x_2\}$$

$$V_{17} \otimes V_{21} = \text{span}_k\{x_2z_1 + \mathbb{i}x_1z_2, \mathbb{i}x_2z_2 - x_1z_1\} \oplus \text{span}_k\{x_2z_1 - \mathbb{i}x_1z_2, \mathbb{i}x_2z_2 + x_1z_1\}$$

$$V_{21} \otimes V_{17} = \text{span}_k\{\mathbb{i}z_1x_1 + z_2x_2, z_2x_1 - \mathbb{i}z_1x_2\} \oplus \text{span}_k\{\mathbb{i}z_1x_1 - z_2x_2, z_2x_1 + \mathbb{i}z_1x_2\}$$

$$V_{20} \otimes V_{21} = \text{span}_k\{y_1z_2 + \mathbb{i}y_2z_1, \mathbb{i}y_2z_1 - y_1z_2\} \oplus \text{span}_k\{y_2z_2 + \mathbb{i}y_1z_1, \mathbb{i}y_1z_1 - y_2z_2\}$$

$$V_{21} \otimes V_{20} = \text{span}_k\{z_2y_1 + \mathbb{i}z_1y_2, \mathbb{i}z_1y_2 - z_2y_1\} \oplus \text{span}_k\{z_2y_2 + \mathbb{i}z_1y_1, \mathbb{i}z_1y_1 - z_2y_2\}$$

One way to choose relations from this list is to use the almost cocommutative property of $D(Q_8)$ to help create commutativity relations in A. For example, let W be given by the span of the following elements (where b, c, d, e are arbitrary parameters, and $\alpha, \beta, \gamma \in \{1, -1\}$):

$$
\begin{aligned}
V_0 &: \quad x_1x_2 - \alpha x_2x_1 \\
V_3 &: \quad y_1y_2 - \beta y_2y_1 \\
V_0 &: \quad z_1z_2 - \gamma z_2z_1 \\
V_{11} &: \quad (x_2y_1 - \mathbb{i}x_1y_2) - b(y_1x_1 + \mathbb{i}y_2x_2), (\mathbb{i}x_2y_2 - x_1y_1) - b(-\mathbb{i}y_2x_1 - y_1x_2), \\
V_{13} &: \quad (x_2y_1 + \mathbb{i}x_1y_2) - b(y_1x_1 - \mathbb{i}y_2x_2), (\mathbb{i}x_2y_2 + x_1y_1) - b(-\mathbb{i}y_2x_1 + y_1x_2), \\
V_{10} &: \quad (x_2z_1 + \mathbb{i}x_1z_2) - c(\mathbb{i}z_1x_1 + z_2x_2), (\mathbb{i}x_2z_2 - x_1z_1) - c(z_2x_1 - \mathbb{i}z_1x_2), \\
V_{12} &: \quad (x_2z_1 - \mathbb{i}x_1z_2) - c(\mathbb{i}z_1x_1 - z_2x_2), (\mathbb{i}x_2z_2 + x_1z_1) - c(z_2x_1 + \mathbb{i}z_1x_2), \\
V_4 &: \quad (y_1z_2 + \mathbb{i}y_2z_1) - d(z_2y_1 + \mathbb{i}z_1y_2), (\mathbb{i}y_2z_1 - y_1z_2) - d(\mathbb{i}z_1y_2 - z_2y_1), \\
V_9 &: \quad (y_2z_2 + \mathbb{i}y_1z_1) - e(z_2y_2 + \mathbb{i}z_1y_1), (\mathbb{i}y_1z_1 - y_2z_2) - e(\mathbb{i}z_1y_1 - z_2y_2)
\end{aligned}
$$

We can clean these relations up a bit by adding and subtracting them to get binomial relations. However, it is no longer obvious that what we have afterwards is an $D(Q_8)$-module!

Cleaning up the relations in this way gives:

$$
\begin{array}{ll}
\multicolumn{2}{l}{x_1x_2 - \alpha x_2x_1 \ , \ y_1y_2 - \beta y_2y_1 \ , \ z_1z_2 - \gamma z_2z_1} \\
x_2y_1 - by_1x_1, & x_1y_2 + by_2x_2 \\
x_2y_2 + by_2x_1, & x_1y_1 - by_1x_2 \\
x_2z_1 - ciz_1x_1, & ix_1z_2 - cz_2x_2 \\
ix_2z_2 - cz_2x_1, & x_1z_1 - ciz_1x_2 \\
y_2z_1 - dz_1y_2, & y_1z_2 - dz_2y_1 \\
y_1z_1 - ez_1y_1, & y_2z_2 - ez_2y_2
\end{array}
$$

These relations define a (trimmed) double Ore extension (in the sense of Zhang-Zhang) of the skew polynomial ring $k_{p_{ij}}[y_1, y_2, z_1, z_2]$ by the variables x_1, x_2. In particular, this algebra (which we call A) is Artin-Schelter regular and Koszul.

Therefore this algebra is a derivation-quotient algebra defined by some twisted superpotential [Dubois-Violette].

The η-twisted superpotential of this algebra has the form:

$x_1x_2y_1y_2z_1z_2 + \beta\gamma x_1x_2y_2y_1z_2z_1 - \alpha\beta\gamma x_2x_1y_2y_1z_2z_1 +$ 717 other terms

with η (the Nakayama automorphism) given by the matrix

$$\begin{pmatrix} \alpha I & 0 & 0 \\ 0 & \alpha\beta I & 0 \\ 0 & 0 & -\alpha\gamma I \end{pmatrix}$$

where I is the 2×2 identity matrix.

To see that ω_A has this form is to note that in $A^!$ one has:

$$\begin{aligned} x_1x_2y_2y_1z_2z_1 &= (-\beta)(-\gamma)x_1x_2y_1y_2z_1z_2 \\ x_2x_1y_2y_1z_2z_1 &= (-\alpha)(-\beta)(-\gamma)x_1x_2y_1y_2z_1z_2 \end{aligned}$$

In particular, A is Calabi-Yau if and only if $(\alpha, \beta, \gamma) = (1, 1, -1)$.

The *homological determinant* of the action of H on A is a k-algebra homomorphism

$$\mathrm{hdet}_A : H \to k$$

defined by Jorgensen-Zhang in the case $H = kG$ and Kirkman-Kuzmanovich-Zhang when H is f.d. and semisimple.

This map is the character of a one-dimensional representation. When it is the character of the trivial representation (i.e. when $\mathrm{hdet}_A = \epsilon_H$), we say the homological determinant is *trivial*.

Theorem (KKZ)

When hdet_A *is trivial, then* A^H *is AS-Gorenstein.*

However, hdet_A can be a bit of a challenge (at least for me!) to compute as it is defined in terms of induced actions on local cohomology.

Theorem (Smith-Mori,Crawford)

If H is a semisimple Hopf algebra acting on a derivation-quotient algebra A defined by a twisted superpotential ω_A, then for all $h \in H$, one has

$$\omega_A.h = \omega_A \, \mathrm{hdet}_A(h).$$

When $H = D(G)$, for hdet to be a one-dimensional representation, we must have

$$\mathrm{hdet} = V \otimes_{kG_i} kG$$

where:

- $G_i = C_G(z) = G$ for some $z \in Z(G)$, and
- V a one-dimensional representation of G.

So the superpotential must be homogeneous with group grade in the center of G, and its span must be a one-dimensional rep of G.

In our example, we have:

We have that the group grades of the variables are given by:

	x_1	x_2	y_1	y_2	z_1	z_2
Grade:	s	sr^2	sr	sr^3	sr	sr^3

and the actions of r and s are given by the following matrices (acting on *rows* since we are acting on the *right*):

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathbb{i} \\ 0 & 0 & 0 & 0 & -\mathbb{i} & 0 \end{pmatrix} \qquad \begin{pmatrix} -\mathbb{i} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbb{i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

Using our superpotential

$$\omega_A = x_1x_2y_1y_2z_1z_2 + \beta\gamma x_1x_2y_2y_1z_2z_1 - \alpha\beta\gamma x_2x_1y_2y_1z_2z_1 + \cdots$$

we notice that ω_A is in group grade e_G, and one can check that:

$$\mathrm{hdet}_A(r) = -\alpha\beta\gamma \qquad \mathrm{hdet}_A(s) = -\beta\gamma.$$

Therefore this action has trivial homological determinant if and only if $\alpha = 1$ and $\beta = -\gamma$.

To see this, note that:

$$\begin{aligned}(-\alpha\beta\gamma x_2x_1y_2y_1z_2z_1).r &= -\alpha\beta\gamma(-x_1)(x_2)(-y_3)(-y_4)(-\mathbb{i}z_1)(-\mathbb{i}z_2) \\ &= -\alpha\beta\gamma x_1x_2y_1y_2z_1z_2.\end{aligned}$$

and similarly for s, acting on $x_1x_2y_2y_1z_2z_1$.

This led us to first consider the case

$$(\alpha, \beta, \gamma, b, c, d, e) = (1, 1, -1, 1, 1, 1, 1),$$

as this case is Calabi-Yau and has trivial homological determinant with simpler relations involving b, c, d, e.

Work in progress: The ring of invariants A^H is generated by the following central(!) elements:

Degree	Generators
2	y_1y_2
	z_1z_2
4	$x_1^2x_2^2$
	$x_1^4 + x_2^4$
	$y_1^4 + y_2^4$
	$z_1^4 + z_2^4$
	$(x_1^2 - x_2^2)(y_1^2 - y_2^2)$
	$(x_1^2 + x_2^2)(z_1^2 - z_2^2)$

Degree	Generators
6	$(x_1x_2)(x_1^2 - x_2^2)(x_1^2 + x_2^2)$
	$(x_1x_2)(x_1^2 + x_2^2)(y_1^2 - y_2^2)$
	$(x_1x_2)(x_1^2 - x_2^2)(z_1^2 - z_2^2)$
	$(x_1x_2)(y_1^2 + y_2^2)(z_1^2 + z_2^2)$
	$(x_1x_2)(y_1^2 - y_2^2)(z_1^2 - z_2^2)$
8	$(x_1^4 - x_2^4)(y_1^2z_2^2 + y_2^2z_1^2)$
	$(x_1^2 + x_2^2)(y_2^4z_1^2 - y_1^4z_2^2)$
	$(x_1^2 - x_2^2)(y_1z_2^2 - y_2z_1^2)(y_1z_2^2 + y_2z_1^2)$
	$y_2^4z_1^4 + y_1^4z_2^4$

We are working on proving that they generate, but there are many cases to consider!

Thanks for listening!