Homological Regularities
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Theorem (Stanley)

Let A be a commutative finitely generated connected graded Cohen–Macaulay domain. Then A is Gorenstein if and only if its Hilbert series $h_A(t)$ satisfies the equation

$$h_A(t^{-1}) = \pm t^{\ell} h_A(t)$$

for some integer ℓ.

There are noncommutative versions of this theorem due to Jørgensen-Zhang.

Homological Regularities:

Tor-regularity

Castelnuovo-Mumford regularity

Artin-Schelter regularity

Concavity of homological invariants

Weighted regularities

Let $\mathbb{k}$ be a field, A be a connected $\mathbb{N}$-graded Noetherian $\mathbb{k}$-algebra with a balanced dualizing complex, and X be a complex of graded left A-modules.

Gradings on X:

- Internal degree
- Homological degree

Properties of A reflected in the relationship between these degrees.

A is Koszul $\Leftrightarrow \mathbb{k}$ has a minimal free graded resolution of the form

$$\cdots \to A(-i)^{\beta_i} \to A(-i+1)^{\beta_{i-1}} \to \cdots \to A(-1)^{\beta_1} \to A \to \mathbb{k} \to 0,$$

A is Koszul $\Leftrightarrow$ $\mathrm{Tor}_i^A(\mathbb{k}, \mathbb{k})_j = 0$ for all $j \neq i$.

Homological degree i = Internal degree j

Measure of growth of degrees of generators in a resolution

Definition [Jørgensen, Dong-Wu]

The *Tor-regularity* of a nonzero complex X of graded left A-modules

$$\mathrm{Torreg}({}_A X) = \sup_{i,j \in \mathbb{Z}} \{ j - i \mid \mathrm{Tor}_i^A(\Bbbk, X)_j \neq 0 \}.$$

$\mathrm{Torreg}({}_A \Bbbk) \geq 0$ and $= 0 \iff A$ is Koszul.

$\mathrm{Torreg}({}_A X)$ measures the growth of the degrees of the generators of the free modules in a minimal free resolution of X.

A commutative, generated in degree one,
$\mathrm{Torreg}({}_A\Bbbk) = 0$ or ∞ (Avramov and Peeva)

In this case $\mathrm{Torreg}({}_A\Bbbk)$ tells only whether A is Koszul.

A noncommutative,
$\mathrm{Torreg}({}_A\Bbbk)$ can be any value in $\mathbb{N} \cup \{+\infty\}$.

Torreg(k) for AS regular algebra

Example: Let T be an AS-regular ring of dimension 3 that is not Koszul

$$\deg(\mathsf{Tor}_i^T(\Bbbk,\Bbbk)) = \begin{cases} 0, & i=0, \\ 1, & i=1, \\ 3, & i=2, \\ 4, & i=3, \\ -\infty & i>3. \end{cases}$$

$$\mathrm{Torreg}({}_T\Bbbk) = \max\{0, 1-1, 3-2, 4-3, -\infty\} = 1.$$

In general, T AS regular of type (d, ℓ) then

$$\mathrm{Torreg}({}_T\Bbbk) = \ell - d.$$

Torreg(k) can be any value in $\mathbb{N} \cup \{+\infty\}$

$$\mathrm{Torreg}({}_T\Bbbk) = 1$$

Let A be any finitely generated commutative Koszul algebra (but not regular) and let $B = A \otimes T^{\otimes n}$.

Then $\mathrm{Torreg}({}_B\Bbbk) = n$

(B is neither AS regular nor Koszul)

The CM regularity of X

The *ith local cohomology* of a complex X of graded left A-module is

$$H^i_{\mathfrak{m}}(X) = \lim_{n\to\infty} \operatorname{Ext}^i_A(A/\mathfrak{m}^n, X).$$

Definition (Jørgensen, Dong-Wu):
For A a noetherian connected graded algebra the *Castelnuovo–Mumford regularity* (or *CM regularity)* of X, a complex of graded left A-modules:

$$\mathrm{CMreg}(X) = \sup_{i,j\in\mathbb{Z}}\{j+i \mid H^i_{\mathfrak{m}}(X)_j \neq 0\}.$$

CM regularity of an AS regular algebra

Let T be AS regular of type (d, ℓ)

$\text{Ext}^i_T(\Bbbk, T) = 0$ for $i \neq d$
$\text{Ext}^d_T(\Bbbk, T) = \Bbbk(\ell)$

$H^i_{\mathfrak{m}}(T) = 0$ for $i \neq d$
$H^d_{\mathfrak{m}}(T) = T^*(-\ell)$

$$\mathrm{CMreg}(T) = d - \ell = \mathrm{gldim}(T) + \deg h_T(t)$$
$$\mathrm{CMreg}(\Bbbk[x_1, \ldots, x_d]) = d + \deg 1/(1-t)^d$$
$$= d - d = 0.$$

T non-Koszul AS regular of type $(3, 4)$
$$\mathrm{CMreg}(T) = d - \ell = 3 - 4 = 3 + \deg 1/((1-t)^2(1-t^2)) = -1$$

Generalization of a result of Symonds on bounds of degrees of generators:

$\beta(R)$ is the maximal degree of a minimal set of generators of R

A graded algebra map $\phi : T \to F$ is called *finite* if ${}_T F$ and F_T are finitely generated.

(K, Won, Zhang (2021))
Suppose there is a finite map $S \to A^H$,
where S is a noetherian AS regular algebra.
Let $\delta(A/S) = \mathrm{CMreg}(A) - \mathrm{CMreg}(S)$.
Then $\beta(A^H) \leq \max\{\beta(S), \delta(A/S)\}$

Relation between Tor and CM regularities

(Eisenbud-Goto) $T = \Bbbk[x_1, \ldots, x_n]$:

$$\mathrm{Torreg}(M) = \mathrm{CMreg}(M)$$

for all finitely generated graded T-module M.

Example: A non-Koszul AS-regular of dimension 3
$\mathrm{Torreg}(\Bbbk) = 1 \neq \mathrm{CMreg}(\Bbbk) = 0$.

(Jørgensen, Dong-Wu) A is a Noetherian connected graded algebra with a balanced dualizing complex. Then A is a Koszul AS regular algebra if and only if

$$\mathrm{Torreg}(M) = \mathrm{CMreg}(M)$$

for all finitely generated graded A-module M.

Is CMreg(M) finite?

If A is f.g. commutative then $\operatorname{CMreg}(M)$ is finite for every f.g. A-module M.

There exist noetherian Koszul algebras of gldim $A = 4$ with $\operatorname{CMreg}(A)$ infinite (Rogalski and Sierra).

Here χ-condition fails and A does not have a balanced dualizing complex.

Let T be an AS regular algebra of type (d, ℓ). Then

$$\mathrm{CMreg}(T) = d - \ell = -\,\mathrm{Torreg}(\Bbbk).$$

Definition:
Let A be a noetherian connected graded algebra. The *Artin–Schelter regularity* of A is defined to be

$$\mathrm{ASreg}(A) = \mathrm{Torreg}(\Bbbk) + \mathrm{CMreg}(A).$$

AS regularity measures how from from being AS regular A is

Extending a result of Jørgensen, Dong-Wu to the non-Koszul case:

Let A be a noetherian connected graded algebra with balanced dualizing complex. Then

$$A \text{ is AS regular} \iff \mathrm{ASreg}(A) = 0.$$

$\mathrm{ASreg}(A)$ can be any value in $\mathbb{N}$

Let $B = \Bbbk[x]/(x^2)$.

Then $\mathrm{CMreg}(B) = 1$ and $\mathrm{Torreg}({}_B\Bbbk) = 0$ so $\mathrm{ASreg}(B) = 1$.

Let $C = B^{\otimes m}$.

Then $\mathrm{CMreg}(C) = m$ and $\mathrm{Torreg}({}_C\Bbbk) = 0$ and $\mathrm{ASreg}(C) = m$.

Concavity

For A a locally finite $\mathbb{N}$-graded noetherian ring:
$\Phi(A) := \{T \mid \text{there is a finite map } \phi : T \to A\}$
where T denotes a connected graded noetherian AS regular algebra.

The *concavity* of A is defined to

$$c(A) := \inf_{T \in \Phi(A)} \{-\operatorname{CMreg}(T)\} \geq 0$$

If $A = T$ is a noetherian AS regular algebra, then $c(T) = 0$ if and only if T is Koszul.

Concavity of F measures how far F is from being “flat" = Koszul

Imagine $\operatorname{Spec} T$ as a noncommutative space associated to T, which is “flat" if T is Koszul.

If $T_1 \to T_2$ is a finite map between two noetherian AS regular algebras (or by analogy, if there is a finite map $\operatorname{Spec} T_2 \to \operatorname{Spec} T_1$), then $c(T_2) \leq c(T_1)$.

If T_1 is Koszul, then so is T_2.

Hence, in some sense, “concavity” measures how far away a noncommutative space is from being “flat”.

An Application

Let H be a semisimple Hopf algebra acting on a noetherian AS regular algebra T homogeneously. Let $R = T^H$ denote the invariant subring. Then

$$c(R) \geq \beta(R) - 1$$

where $\beta(R)$ is the maximal degree of a minimal set of generators of R.

Let $A = \Bbbk[x_1, \cdots, x_n][t]/(t^2 = f(x_1, \cdots, x_n))$ where $\deg x_i = 1$, $\deg t \geq 2$, and f an irreducible homogeneous polynomial in x_i of degree $(2 \deg t)$.

$$0 = c(A) < 1 \leq \deg t - 1 = \beta(A) - 1.$$

Therefore A cannot be isomorphic to T^H.

Theorem: A noetherian connected graded with balanced dualizing complex. Let X be a nonzero object in $\mathsf{D}^{\mathsf{b}}_{\mathsf{fg}}(A\,\mathrm{Gr})$ with finite projective dimension. Then

$$\mathrm{CMreg}(X) = \mathrm{Torreg}(X) + \mathrm{CMreg}(A).$$

Let $\phi : T \to A$ be a finite map with T AS regular.

$\mathrm{CMreg}(A) = \mathrm{CMreg}({}_T A) = \mathrm{Torreg}({}_T A) + \mathrm{CMreg}(T)$

$-\mathrm{CMreg}(T) = \mathrm{Torreg}({}_T A) - \mathrm{CMreg}(A) \geq -\mathrm{CMreg}(A)$

$c(A) \geq -\mathrm{CMreg}(A)$

$c(A) + \mathrm{CMreg}(A) \geq 0$

If A is regular then $c(A) \leq -\mathrm{CMreg}(A)$
and $c(A) + \mathrm{CMreg}(A) = 0$. Converse is true.

$$\mathrm{CMreg}(X) = \mathrm{Torreg}(X) + \mathrm{CMreg}(A).$$

Let $\phi : T \to A$ finite map T AS regular and Koszul.
$0 \leq c(A) \leq -\,\mathrm{CMreg}(T) = 0$ so $c(A) = 0$

$$\mathrm{CMreg}(A) = \mathrm{CMreg}({}_T A) = \mathrm{Torreg}({}_T A) + \mathrm{CMreg}(T)$$

$$\mathrm{CMreg}(A) = \mathrm{Torreg}({}_T A) \geq 0$$

A is AS regular $\Leftrightarrow \mathrm{CMreg}(A) = 0 \Leftrightarrow \mathrm{Torreg}({}_T A) = 0$

A is AS regular if and only if $c(A) + \mathrm{CMreg}(A) = 0$

If there exists a finite map $T \to A$ with T noetherian Koszul AS regular, then

- $\mathrm{CMreg}(A) \geq 0$ and $\mathrm{CMreg}(A) = 0$ if and only if A is AS regular (and Koszul)
- If A is graded s-Cohen-Macaulay then A is AS regular if and only if $\deg(h_A(x)) = -s$
- A is AS regular if and only if $\mathrm{Torreg}({}_T A) = 0$

Numerical invariants indicative of AS regularity:

- $\mathrm{ASreg}(A)$
- $c(A) + \mathrm{CMreg}(A)$
- $\deg(h_A(x))$
- $\mathrm{CMreg}(A)$
- $\mathrm{Torreg}({}_T A)$

(K, Won, Zhang: arXiv 2204.06679)
For a cochain complex X and $\xi_0, \xi_1 \in \mathbb{R}$, $\xi = (\xi_0, \xi_1)$, the ξ-*weighted degree* of X is

$$\deg_\xi(X) = \sup_{m,n\in\mathbb{Z}} \{\xi_0 m + \xi_1 n \mid H^n(X)_m \neq 0\}.$$

Example:

$$\deg_{(1,0)}(X) = \deg(X)$$

$$\deg_{(0,1)}(X) = \sup(X)$$

$$\mathrm{Torreg}_\xi(X) = \deg_\xi(\Bbbk \otimes_A^L X)$$

When $\xi_0 \neq 0$

$$\mathrm{Torreg}_\xi(X) = \sup_{i\in\mathbb{Z}}\{\xi_0 \deg(\mathsf{Tor}_i^A(\Bbbk, X)) - \xi_1 i\}$$

$$\mathrm{CMreg}_\xi(X) = \deg_\xi(\mathsf{R}\Gamma_{\mathfrak{m}}(X)).$$

When $\xi_0 \neq 0$

$$\mathrm{CMreg}_\xi(X) = \sup_{i\in\mathbb{Z}}\{\xi_0 \deg(H_{\mathfrak{m}}^i(X)) + \xi_1 i\}.$$

$$\mathrm{CMreg}_{(0,-1)}(X) = -\mathrm{depth}(X).$$

Theorem:
If there is a finite map $\phi : T \to A$, then there is $c \in \mathbb{R}$ such that for all $\xi_1 \geq c$ and $\xi = (1, \xi_1)$ $\mathrm{Torreg}_\xi(X) < \infty$ for all $X \in \mathsf{D}^{\mathsf{b}}_{\mathsf{fg}}(A\,\mathrm{Gr})$,

e.g. if A is a noetherian commutative graded algebra there exists $\xi = (1, \xi_1)$ with

$$\mathrm{Torreg}_\xi(\Bbbk) < \infty.$$

Theorem: If

- A generated in degree 1
- there is a finite map $T \to A$ where T is a noetherian connected graded algebra of finite global dimension.

Then $A^{(d)}$ is Koszul for $d \gg 0$,

recovering a result of Mumford for commutative algebras.

For commutative algebras:

$$\text{regular} \Longrightarrow \text{hypersurface} \Longrightarrow \text{complete intersection}$$
$$\Longrightarrow \text{Gorenstein} \Longrightarrow \text{Cohen-Macaulay.}$$

Problem: Find computable invariants for other homological properties.

THANKS!