Twisted tensor products of bialgebras and Frobenius algebras

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Pablo S Ocal and Amrei Oswald

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Twisted tensor product of algebras

We denote a unital, associative k-algebra A with multiplication $\nabla_A : A\otimes A \longrightarrow A$ and unit $\eta_A : k \longrightarrow A$ by (A, ∇_A, η_A)

Def: Given two unital, associative k-algebras (A, ∇_A, η_A) and (B, ∇_B, η_B), a *twisting map* is a bijective k-linear map $\tau : B\otimes A \longrightarrow A\otimes B$ so that the following diagrams commute

$$\begin{tikzcd} B\otimes k \arrow[r, "\sim"] \arrow[d, "1\otimes\eta_A"'] & k\otimes B \arrow[d, "\eta_A\otimes 1"] \\ B\otimes A \arrow[r, "\tau"] & A\otimes B \end{tikzcd} \qquad \begin{tikzcd} k\otimes A \arrow[r, "\sim"] \arrow[d, "\eta_B\otimes 1"'] & A\otimes k \arrow[d, "1\otimes\eta_B"] \\ B\otimes A \arrow[r, "\tau"] & A\otimes B \end{tikzcd}$$

$$\begin{tikzcd} B\otimes B\otimes A\otimes A \arrow[r, "\nabla_B\otimes\nabla_A"] \arrow[d, "1\otimes\tau\otimes 1"'] & B\otimes A \arrow[r, "\tau"] & A\otimes B \\ B\otimes A\otimes B\otimes A \arrow[r, "\tau\otimes\tau"] & A\otimes B\otimes A\otimes B \arrow[r, "1\otimes\tau\otimes 1"] & A\otimes A\otimes B\otimes B \arrow[u, "\nabla_A\otimes\nabla_B"'] \end{tikzcd}$$

Twisted tensor product of algebras

The diagram

$$\begin{tikzcd}
B\otimes B\otimes A\otimes A \arrow[r, "\nabla_B\otimes\nabla_A"] \arrow[d, "1\otimes\tau\otimes 1"'] & B\otimes A \arrow[r, "\tau"] & A\otimes B \\
B\otimes A\otimes B\otimes A \arrow[r, "\tau\otimes\tau"] & A\otimes B\otimes A\otimes B \arrow[r, "1\otimes\tau\otimes 1"] & A\otimes A\otimes B\otimes B \arrow[u, "\nabla_A\otimes\nabla_B"']
\end{tikzcd}$$

commutes if and only if the diagrams

$$\begin{tikzcd}
B\otimes B\otimes A \arrow[r, "1\otimes\tau"] \arrow[d, "\nabla_B\otimes 1"'] & B\otimes A\otimes B \arrow[r, "\tau\otimes 1"] & A\otimes B\otimes B \arrow[d, "1\otimes\nabla_B"] \\
B\otimes A \arrow[rr, "\tau"] & & A\otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
B\otimes A\otimes A \arrow[r, "\tau\otimes 1"] \arrow[d, "1\otimes\nabla_A"'] & A\otimes B\otimes A \arrow[r, "1\otimes\tau"] & A\otimes A\otimes B \arrow[d, "\nabla_A\otimes 1"] \\
B\otimes A \arrow[rr, "\tau"] & & A\otimes B
\end{tikzcd}$$

commute.

Twisted tensor product of algebras

Let (A,∇_A,η_A), (B,∇_B,η_B) be unital, associative k-algebras and $\tau: B\otimes A \longrightarrow A\otimes B$ a twisting map

Define

$\nabla_{A\otimes_\tau B}: A\otimes B\otimes A\otimes B \longrightarrow A\otimes B$ by

$$A\otimes B\otimes A\otimes B \xrightarrow{1\otimes\tau\otimes 1} A\otimes A\otimes B\otimes B \xrightarrow{\nabla_A\otimes\nabla_B} A\otimes B$$

$\eta_{A\otimes_\tau B}: k \longrightarrow A\otimes B$ by

$$k \xrightarrow{\sim} k\otimes k \xrightarrow{\eta_A\otimes\eta_B} A\otimes B$$

Then, the twisted tensor product algebra is the unital, associative k-algebra $(A\otimes_\tau B, \nabla_{A\otimes_\tau B}, \eta_{A\otimes_\tau B})$

where $A\otimes_\tau B = A\otimes B$ as a vector space

[Cap, Schichl, Vanžura, 1994]

Twisted tensor product of algebras

Examples:

1) $G, H =$ finite groups where G acts on H via $\varphi: G \longrightarrow \mathrm{Aut}(H)$

Define $\tau: kG \otimes kH \longrightarrow kH \otimes kG$ by $g \otimes h \mapsto \varphi(g)(h) \otimes g$ for $g \in G, h \in H$

Then $kG \otimes_\tau kH \simeq k(H \rtimes_\varphi G)$

2) $k[x], k[y]$ = polynomial rings

Define $\tau: k[y] \otimes k[x] \longrightarrow k[x] \otimes k[y]$ by $y \otimes x \mapsto q\, x \otimes y$ for $q \in k^\times$

Then $\left.\begin{matrix} k[x] \otimes_\tau k[y] \simeq k_q[x,y] \simeq k\langle x,y\rangle / (qxy - yx) \end{matrix}\right\}$ the quantum plane

Let $q = (q_{ij}) \in M_n(k^\times)$ st $q_{ii} = 1$ and $q_{ij} q_{ji} = 1$

Then, we obtain $k_q[x_1, \ldots, x_n] = k\langle x_1, \ldots, x_n \rangle / (x_i x_j - q_{ij} x_j x_i)$

via iterated applications of the above

Twisted tensor product of algebras

Examples:

3) Twisting by a bicharacter

$F, G =$ abelian groups

$A =$ k-algebra graded by F, $B =$ k-algebra graded by G

$t: F \otimes_{\mathbb{Z}} G \longrightarrow k^{\times}$ is a group homomorphism

Define $\tau: B \otimes A \longrightarrow A \otimes B$ by linearly extending $b \otimes a \longmapsto t(|a| \otimes_{\mathbb{Z}} |b|)\, a \otimes b$

Then, τ is a twisting map and we denote the resulting algebra $A \otimes_{\tau} B$ by $A \otimes^{t} B$

Twisted tensor product of coalgebras

We denote a counital, coassociative k-coalgebra A with

comultiplication $\Delta_A : A \longrightarrow A \otimes A$

and counit $\varepsilon_A : A \longrightarrow k$ by $(A, \Delta_A, \varepsilon_A)$

Def: Given two coassociative coalgebras $(A, \Delta_A, \varepsilon_A), (B, \Delta_B, \varepsilon_B)$

and a twisting map $\tau : B \otimes A \longrightarrow A \otimes B$

define $\Delta_{A \otimes_\tau B} : A \otimes B \longrightarrow A \otimes B \otimes A \otimes B$ by

$$A \otimes B \xrightarrow{\Delta_A \otimes \Delta_B} A \otimes A \otimes B \otimes B \xrightarrow{1 \otimes \tau^{-1} \otimes 1} A \otimes B \otimes A \otimes B$$

and $\varepsilon_{A \otimes_\tau B} : A \otimes B \longrightarrow k$ by

$$A \otimes B \xrightarrow{\varepsilon_A \otimes \varepsilon_B} k \otimes k \xrightarrow{\sim} k$$

Twisted tensor product of coalgebras

Def: Given two coassociative coalgebras $(A,\Delta_A,\varepsilon_A)$, $(B,\Delta_B,\varepsilon_B)$ and a twisting map $\tau: B\otimes A \longrightarrow A\otimes B$

define $\Delta_{A\otimes_\tau B}: A\otimes B \longrightarrow A\otimes B\otimes A\otimes B$ by $A\otimes B \xrightarrow{\Delta_A\otimes\Delta_B} A\otimes A\otimes B\otimes B \xrightarrow{1\otimes\tau^{-1}\otimes 1} A\otimes B\otimes A\otimes B$

$\varepsilon_{A\otimes_\tau B}: A\otimes B \longrightarrow k$ by $A\otimes B \xrightarrow{\varepsilon_A\otimes\varepsilon_B} k\otimes k \xrightarrow{\sim} k$

$(A\otimes_\tau B, \Delta_{A\otimes_\tau B}, \varepsilon_{A\otimes_\tau B})$ is a counital coassociative k-coalgebra if and only if the following diagrams commute

$$\begin{tikzcd} A\otimes B \arrow[r, "\tau^{-1}"] \arrow[d, "\varepsilon_A\otimes 1"'] & B\otimes A \arrow[d, "1\otimes\varepsilon_A"] \\ k\otimes B \arrow[r, "\sim"] & B\otimes k \end{tikzcd} \qquad \begin{tikzcd} A\otimes B \arrow[r, "\tau^{-1}"] \arrow[d, "1\otimes\varepsilon_B"'] & B\otimes A \arrow[d, "\varepsilon_B\otimes 1"] \\ A\otimes k \arrow[r, "\sim"] & k\otimes A \end{tikzcd}$$

$$\left.\begin{tikzcd} A\otimes B \arrow[r, "\tau^{-1}"] \arrow[d, "\Delta_A\otimes\Delta_B"'] & B\otimes A \arrow[r, "\Delta_B\otimes\Delta_A"] & B\otimes B\otimes A\otimes A \\ A\otimes A\otimes B\otimes B \arrow[r, "1\otimes\tau^{-1}\otimes 1"] & A\otimes B\otimes A\otimes B \arrow[r, "\tau^{-1}\otimes\tau^{-1}"] & B\otimes A\otimes B\otimes A \arrow[u, "1\otimes\tau^{-1}\otimes 1"'] \end{tikzcd}\right\} \text{Diagram } \star$$

Twisted tensor product of coalgebras

The diagram

$$\begin{tikzcd}
A\otimes B \arrow[r, "\tau^{-1}"] \arrow[d, "\Delta_A\otimes\Delta_B"'] & B\otimes A \arrow[rr, "\Delta_B\otimes\Delta_A"] & & B\otimes B\otimes A\otimes A \\
A\otimes A\otimes B\otimes B \arrow[r, "1\otimes\tau^{-1}\otimes 1"] & A\otimes B\otimes A\otimes B \arrow[rr, "\tau^{-1}\otimes\tau^{-1}"] & & B\otimes A\otimes B\otimes A \arrow[u, "1\otimes\tau^{-1}\otimes 1"']
\end{tikzcd}$$

Diagram ☆

commutes if and only if the diagrams

$$\begin{tikzcd}
B\otimes A \arrow[rr, "\tau"] \arrow[d, "1\otimes\Delta_A"'] & & A\otimes B \arrow[d, "\Delta_A\otimes 1"] \\
B\otimes A\otimes A \arrow[r, "\tau\otimes 1"] & A\otimes B\otimes A \arrow[r, "1\otimes\tau"] & A\otimes A\otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
B\otimes A \arrow[rr, "\tau"] \arrow[d, "\Delta_B\otimes 1"'] & & A\otimes B \arrow[d, "1\otimes\Delta_B"] \\
B\otimes B\otimes A \arrow[r, "1\otimes\tau"] & B\otimes A\otimes B \arrow[r, "\tau\otimes 1"] & A\otimes B\otimes B
\end{tikzcd}$$

commute.

Bialgebra structures on twisted tensor products

Let $(A,\nabla_A,\eta_A,\Delta_A,\varepsilon_A),(B,\nabla_B,\eta_B,\Delta_B,\varepsilon_B)$ be bialgebras and $\tau: B\otimes A \longrightarrow A\otimes B$ a twisting map

$$\begin{tikzcd}
A\otimes B \arrow[rrr, "\tau^{-1}"] \arrow[dd, "1\otimes\varepsilon_B"'] & & & B\otimes A \arrow[dd, "\varepsilon_B\otimes 1"] \\
 & A\otimes k \arrow[ul, "1\otimes\eta_B"'] \arrow[r, "\sim"] \arrow[dl, "1\otimes 1"] \arrow[u, phantom, "\circlearrowright \text{ twisting map}"] \arrow[l, phantom, "\circlearrowright \text{ B is a bialg}"] \arrow[d, phantom, "\circlearrowright"] & k\otimes A \arrow[ur, "\eta_B\otimes 1"] \arrow[dr, "1\otimes 1"'] \arrow[r, phantom, "\circlearrowright \text{ B is a bialg}"] & \\
A\otimes k \arrow[rrr, "\sim"] & & & k\otimes A
\end{tikzcd}$$

$\Longrightarrow$ compatibility btwn τ and the counits is guaranteed

We just need Diagram ☆ to commute to get a coalg structure

$$\left.\begin{tikzcd}
A\otimes B \arrow[r, "\tau^{-1}"] \arrow[d, "\Delta_A\otimes\Delta_B"'] & B\otimes A \arrow[rr, "\Delta_B\otimes\Delta_A"] & & B\otimes B\otimes A\otimes A \\
A\otimes A\otimes B\otimes B \arrow[r, "1\otimes\tau^{-1}\otimes 1"] & A\otimes B\otimes A\otimes B \arrow[rr, "\tau^{-1}\otimes\tau^{-1}"] & & B\otimes A\otimes B\otimes A \arrow[u, "1\otimes\tau^{-1}\otimes 1"']
\end{tikzcd}\right\}\ \text{Diagram ☆}$$

Bialgebra structures on twisted tensor products

Examples:

1) G, H = finite groups where G acts on H via $\varphi: G \longrightarrow \mathrm{Aut}(H)$

Define $\tau: kG \otimes kH \longrightarrow kH \otimes kG$ by $g \otimes h \mapsto \varphi(g)(h) \otimes g$ for $g \in G, h \in H$

Then $kG \otimes_\tau kH \simeq k(H \rtimes_\varphi G)$

kG is a bialgebra with $\Delta_G: g \mapsto g \otimes g$ and $\varepsilon_G: g \mapsto 1$ for $g \in G$

Then, diagram ☆ does not commute $\Rightarrow$ $kG \otimes_\tau kH \simeq k(H \rtimes_\varphi G)$ as algebras,

but $kG \otimes_\tau kH$ with $\Delta_{kG \otimes_\tau kH}$ is not a coalgebra, let alone a bialgebra

Note, $kG \otimes_\tau kH \simeq k(H \rtimes_\varphi G)$ is a group algebra, so it has a bialgebra structure

Bialgebra structures on twisted tensor products

Examples:

2) $k[x], k[y]$ = polynomial rings. Define $\tau: k[y]\otimes k[x] \longrightarrow k[x]\otimes k[y]$ by $y\otimes x \mapsto q\, x\otimes y$ for $q \neq 0 \in k$

Then $k[x]\otimes_\tau k[y] \simeq k_q[x,y] \simeq k[x,y]/\langle qxy-yx\rangle$ ie the quantum plane

$k[x]$ is a bialgebra with $\Delta_x : x \mapsto 1\otimes x + x\otimes 1$ and $\varepsilon_x : x \mapsto 0$

Then, diagram ☆ commutes, and $(k[x]\otimes_\tau k[y], \Delta_{k[x]\otimes_\tau k[y]}, \varepsilon_{k[x]\otimes_\tau k[y]})$ is a coalgebra,

but $(k[x]\otimes_\tau k[y], \nabla_{k[x]\otimes_\tau k[y]}, \eta_{k[x]\otimes_\tau k[y]}, \Delta_{k[x]\otimes_\tau k[y]}, \varepsilon_{k[x]\otimes_\tau k[y]})$ is not a bialgebra

Similarly, for $q = (q_{ij}) \in M_n(k^\times)$ st $q_{ii} = 1$ and $q_{ij}q_{ji} = 1$,

$k_q[x_1,\dots,x_n] = k\langle x_1,\dots,x_n\rangle/(x_ix_j - q_{ij}x_jx_i)$ is a coalgebra but not a bialgebra

Bialgebra structures on twisted tensor products

Theorem: [Ocal, O.] Let $(A, \nabla_A, \eta_A, \Delta_A, \varepsilon_A), (B, \nabla_B, \eta_B, \Delta_B, \varepsilon_B)$ be bialgebras and

$\tau : B \otimes A \longrightarrow A \otimes B$ a twisting map. Then, $(A \otimes_\tau B, \nabla_{A\otimes_\tau B}, \eta_{A\otimes_\tau B}, \Delta_{A\otimes_\tau B}, \varepsilon_{A\otimes_\tau B})$

is a bialgebra if only if τ is trivial.

proof:

$$
\begin{tikzcd}
B\otimes A\otimes B\otimes A & B\otimes B\otimes A\otimes A \arrow[l, "1\otimes\tau\otimes 1"'] \arrow[r, "(1\otimes\tau\otimes 1)(\tau\otimes\tau)(1\otimes\tau\otimes 1)"] \arrow[dr, phantom, "\circlearrowright\ \bigstar"] & A\otimes A\otimes B\otimes B \arrow[r] & A\otimes B\otimes A\otimes B \\
{} \arrow[ur, phantom, "\circlearrowright\ \text{bialg}"] & B\otimes A \arrow[u, "\Delta_B\otimes\Delta_A"] \arrow[r, "\tau"] \arrow[dr, phantom, "?"] & A\otimes B \arrow[u, "\Delta_A\otimes\Delta_B"'] & {} \arrow[ul, phantom, "\circlearrowright\ \text{bialg}"] \\
B\otimes B\otimes A\otimes A \arrow[uu, "\sigma_{23}"] \arrow[d, "\varepsilon_B\otimes 1\otimes 1\otimes\varepsilon_A"'] \arrow[dr, phantom, "\circlearrowright\ \text{coalg}"] & B\otimes A \arrow[l, "\Delta_B\otimes\Delta_A"'] \arrow[u, "="'] \arrow[r, "\sigma_{12}"'] \arrow[d, "="] \arrow[dr, phantom, "\circlearrowright"] & A\otimes B \arrow[u, "="] \arrow[r, "\Delta_A\otimes\Delta_B"] \arrow[d, "="'] \arrow[dr, phantom, "\circlearrowright\ \text{coalg}"] & A\otimes A\otimes B\otimes B \arrow[uu, "\sigma_{23}"'] \arrow[d, "1\otimes\varepsilon_A\otimes\varepsilon_B\otimes 1"] \\
k\otimes B\otimes A\otimes k \arrow[r, "\sim"'] & B\otimes A \arrow[r, "\sigma_{12}"'] & A\otimes B \arrow[r, "\sim"'] & A\otimes k\otimes k\otimes B
\end{tikzcd}
$$

and the outside commutes b/c "counit commutes w/ τ and σ_{23}"

Bialgebra structures on twisted tensor products

Theorem: [Ocal, O.] Let $(A, \nabla_A, \eta_A, \Delta_A, \varepsilon_A)$, $(B, \nabla_B, \eta_B, \Delta_B, \varepsilon_B)$ be bialgebras and

$\tau : B \otimes A \longrightarrow A \otimes B$ a twisting map. Then, $(A \otimes_\tau B, \nabla_{A \otimes_\tau B}, \eta_{A \otimes_\tau B}, \Delta_{A \otimes_\tau B}, \varepsilon_{A \otimes_\tau B})$

is a bialgebra if only if τ is trivial.

Corollary: Let A and B be Hopf algebras and $\tau : B \otimes A \longrightarrow A \otimes B$ a twisting map.

Then, if $A \otimes_\tau B$ is a Hopf algebra, it is the tensor product Hopf algebra $A \otimes B$

Twisted tensor products of Frobenius algebras

Def: A tuple $(A, \nabla_A, \eta_A, \Delta_A, \varepsilon_A)$ is a Frobenius algebra if (A, ∇_A, η_A) is an algebra, $(A, \Delta_A, \varepsilon_A)$ is a coalgebra and the following diagrams commute

$$\begin{tikzcd} A\otimes A \arrow[r, "\nabla_A"] \arrow[d, "\Delta_A\otimes 1"'] & A \arrow[d, "\Delta_A"] \\ A\otimes A\otimes A \arrow[r, "1\otimes\nabla_A"] & A\otimes A \end{tikzcd} \qquad \begin{tikzcd} A\otimes A \arrow[r, "\nabla_A"] \arrow[d, "1\otimes\Delta_A"'] & A \arrow[d, "\Delta_A"] \\ A\otimes A\otimes A \arrow[r, "\nabla_A\otimes 1"] & A\otimes A \end{tikzcd}$$

Examples:

1) For a finite group G, kG is a Frobenius algebra with

$$\tilde{\Delta}_G : g \mapsto \sum_{r\in G} gr \otimes r^{-1} \quad \text{and} \quad \tilde{\varepsilon}_G : g \mapsto \delta_{g,1} \text{ for } g\in G$$

2) $k[x]/\langle x^n\rangle$ is Frobenius with

$$\tilde{\Delta}_x : p(x) \mapsto \sum_{i=0}^{n-1} x^i p(x) \otimes x^{n-1-i} \quad \text{and} \quad \tilde{\varepsilon}_x : x^i \mapsto \delta_{i,n-1}$$

Twisted tensor products of Frobenius algebras

Example:

$G, H =$ finite groups where G acts on H via $\varphi: G \longrightarrow \mathrm{Aut}(H)$

Define $\tau: kG \otimes kH \longrightarrow kH \otimes kG$ by $g \otimes h \mapsto \varphi(g)(h) \otimes g$ for $g \in G, h \in H$

Then $kG \otimes_\tau kH \simeq k(H \rtimes_\varphi G)$

kG is a Frobenius algebra with $\tilde{\Delta}_G: g \mapsto \sum_{r \in G} gr \otimes r^{-1}$ and $\tilde{\varepsilon}_G: g \mapsto \delta_{g,1}$ for $g \in G$

kH is a Frobenius algebra w/ $\tilde{\Delta}_H: h \mapsto \sum_{r \in H} hr \otimes r^{-1}$ and $\tilde{\varepsilon}_H: h \mapsto \delta_{h,1}$ for $h \in H$

Then, diagram ☆ commutes $\Rightarrow (kG \otimes_\tau kH, \tilde{\Delta}_{kG \otimes_\tau kH}, \tilde{\varepsilon}_{kG \otimes_\tau kH})$ is a coalgebra

$(kG \otimes_\tau kH, \nabla_{kG \otimes_\tau kH}, \eta_{kG \otimes_\tau kH}, \tilde{\Delta}_{kG \otimes_\tau kH}, \tilde{\varepsilon}_{kG \otimes_\tau kH})$ is a Frobenius algebra

and $\tilde{\Delta}_{kG \otimes_\tau kH}$ recovers exactly the Frobenius algebra structure on $k(H \rtimes_\varphi G)$

Twisted tensor products of Frobenius algebras

Theorem: [Ocal, O.] Let $(A,\nabla_A,\eta_A,\Delta_A,\varepsilon_A),(B,\nabla_B,\eta_B,\Delta_B,\varepsilon_B)$ be Frobenius algebras and $\tau: B\otimes A \longrightarrow A\otimes B$ a twisting map so that diagram ☆ commutes. Then,

$(A\otimes_\tau B,\nabla_{A\otimes_\tau B},\eta_{A\otimes_\tau B},\Delta_{A\otimes_\tau B},\varepsilon_{A\otimes_\tau B})$ is a Frobenius algebra.

Twisted tensor products of Frobenius algebras

Def: Let (A, ∇_A, η_A) be an algebra. A pairing is a k-linear map $\beta: A\otimes A \longrightarrow k$.

A copairing is a k-linear map $\alpha: k \longrightarrow A\otimes A$.

A pairing β is non-degenerate if there is a copairing α so that the following diagram commutes

$$\begin{tikzcd}
k\otimes A \arrow[r, "\alpha\otimes 1"] \arrow[d, "\sim"'] & A\otimes A\otimes A \arrow[r, "1\otimes\beta"] & A\otimes k \arrow[d, "\sim"] \\
A \arrow[rr, "1"] & & A
\end{tikzcd}$$

A pairing β is associative if the following diagram commutes

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "1\otimes\nabla_A"] \arrow[d, "\nabla_A\otimes 1"'] & A\otimes A \arrow[d, "\beta"] \\
A\otimes A \arrow[r, "\beta"] & k
\end{tikzcd}$$

An algebra (A, ∇_A, η_A) is Frobenius if and only if there is a nondegenerate, associative pairing $\beta: A\otimes A \longrightarrow k$.

Twisted tensor products of Frobenius algebras

Given a Frobenius algebra $(A, \nabla_A, \eta_A, \Delta_A, \varepsilon_A)$, we can define

$$\beta : A\otimes A \xrightarrow{\nabla_A} A \xrightarrow{\varepsilon_A} k \quad \text{and} \quad \alpha : k \xrightarrow{\eta_A} A \xrightarrow{\Delta_A} A\otimes A$$

Then β is an associative, non-degenerate pairing with copairing α.

Given an algebra (A, ∇_A, η_A) with associative, non-degenerate pairing β and copairing α, we have that $(A, \nabla_A, \eta_A, \Delta_A, \varepsilon_A)$ is a Frobenius algebra where Δ_A and ε_A are defined by

$$\begin{tikzcd}
A\otimes k \arrow[r, "1\otimes\alpha"] & A\otimes A\otimes A \arrow[d, "\nabla_A\otimes 1"] \\
A \arrow[u, "\sim"] \arrow[r, "\Delta_A"] \arrow[d, "\sim"'] & A\otimes A \\
k\otimes A \arrow[r, "\alpha\otimes 1"] & A\otimes A\otimes A \arrow[u, "1\otimes\nabla_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes k \arrow[r, "1\otimes\eta_A"] & A\otimes A \arrow[d, "\beta"] \\
A \arrow[u, "\sim"] \arrow[r, "\varepsilon_A"] \arrow[d, "\sim"'] & k \\
k\otimes A \arrow[r, "\eta_A\otimes 1"] & A\otimes A \arrow[u, "\beta"']
\end{tikzcd}$$

Twisted tensor products of Frobenius algebras

Proposition: [Ocal, O.] Let $(A,\nabla_A,\eta_A,\Delta_A,\varepsilon_A),(B,\nabla_B,\eta_B,\Delta_B,\varepsilon_B)$ be Frobenius algebras with pairings β_A,β_B and copairings α_A,α_B resp. and $\tau: B\otimes A \longrightarrow A\otimes B$ a twisting map so that diagram ☆ commutes. Then,

$$\beta_{A\otimes_\tau B}: A\otimes B\otimes A\otimes B \xrightarrow{1\otimes\tau\otimes 1} A\otimes A\otimes B\otimes B \xrightarrow{\beta_A\otimes\beta_B} k\otimes k \xrightarrow{\sim} k$$

$$\alpha_{A\otimes_\tau B}: k \xrightarrow{\sim} k\otimes k \xrightarrow{\alpha_A\otimes\alpha_B} A\otimes A\otimes B\otimes B \xrightarrow{1\otimes\tau^{-1}\otimes 1} A\otimes B\otimes A\otimes B$$

are an associative, non-degenerate pairing and copairing of $(A\otimes_\tau B,\nabla_{A\otimes_\tau B},\eta_{A\otimes_\tau B},\Delta_{A\otimes_\tau B},\varepsilon_{A\otimes_\tau B})$

proof:

pairing on $A\otimes_\tau B$ induced by $\varepsilon_{A\otimes_\tau B}$

$$\begin{tikzcd}
A\otimes B\otimes A\otimes B \arrow[r, "1\otimes\tau\otimes 1"] \arrow[ddd, "="'] & A\otimes A\otimes B\otimes B \arrow[rrr, "\nabla_A\otimes\nabla_B"] \arrow[ddd, "="] \arrow[dr, "\sim"] & & & A\otimes B \arrow[dl, "\sim"'] \arrow[ddd, "\varepsilon_A\otimes\varepsilon_B"] \\
& & k\otimes A\otimes A\otimes B\otimes B\otimes k \arrow[r, "1\otimes\nabla_A\otimes\nabla_B\otimes 1"] \arrow[d, "\eta_A\otimes 1\otimes 1\otimes 1\otimes 1\otimes\eta_B"'] & k\otimes A\otimes B\otimes k \arrow[d, "\eta_A\otimes 1\otimes 1\otimes\eta_B"'] & \\
& & A\otimes A\otimes A\otimes B\otimes B\otimes B \arrow[r, "1\otimes\nabla_A\otimes\nabla_B\otimes 1"'] \arrow[dl, "\nabla_A\otimes 1\otimes 1\otimes\nabla_B"] & A\otimes A\otimes B\otimes B \arrow[dr, "\beta_A\otimes\beta_B"] & \\
A\otimes B\otimes A\otimes B \arrow[r, "1\otimes\tau\otimes 1"] & A\otimes A\otimes B\otimes B \arrow[rrr, "\beta_A\otimes\beta_B"] & & & k\otimes k
\end{tikzcd}$$

A,B are unital algs

counits induced by pairings

assoc of pairings

$\beta_{A\otimes_\tau B}$

Twisted tensor products of Frobenius algebras

F, G = abelian groups, A = k-algebra graded by F, B = k-algebra graded by G

$t: F \otimes_{\mathbb{Z}} G \longrightarrow k^{\times}$ is a group homomorphism

Define $\tau: B \otimes A \longrightarrow A \otimes B$ by linearly extending $b \otimes a \mapsto t(|a| \otimes_{\mathbb{Z}} |b|)\, a \otimes b$

Then, τ is a twisting map and we denote the resulting algebra $A \otimes_{\tau} B$ by $A \otimes^{t} B$

Theorem: [Ocal, O.] Let A and B be Frobenius k-algebras graded by F and G resp. where the comultiplication maps Δ_A, Δ_B are graded maps of degree d_A and d_B resp.

Then, $A \otimes^{t} B$ is a Frobenius algebra if and only if

$$t(|a| \otimes_{\mathbb{Z}} |b|) = t((|a| + d_A) \otimes_{\mathbb{Z}} |b|) = t(|a| \otimes_{\mathbb{Z}} (|b| + d_B))$$

Corollary: If $d_A = 0_F$ and $d_B = 0_G$, $A \otimes^{t} B$ is a Frobenius algebra.

Twisted tensor products of Frobenius algebras

$k[x]/\langle x^n\rangle$ is Frobenius with $\tilde{\Delta}_x : p(x) \mapsto \sum_{i=0}^{n-1} x^i p(x) \otimes x^{n-1-i}$ and $\tilde{\varepsilon}_x : x^i \mapsto \delta_{i,n-1}$

Example: Let $q = (q_{ij}) \in M_n(k^\times)$ st $q_{ii} = 1$ and $q_{ij}q_{ji} = 1$ and $m = (m_1, \ldots, m_n)$

The complete quantum intersections $\Lambda_{q,m} = k_q[x_1, \ldots, x_n]/\langle x_1^{m_1}, \ldots, x_n^{m_n}\rangle$ can be understood as an iterated twisted tensor product where the twists come from the bicharacters $t_i^j(r \otimes_{\mathbb{Z}} s) = q_{ij}^{rs}$ using the $\mathbb{Z}$-grading on $k[x_i]/\langle x_i^{m_i}\rangle$

The comultiplication $\tilde{\Delta}_{x_i}$ is graded of degree $m_i - 1$

Corollary: If q_{ij} is a root of unity whose order divides $\gcd(m_i - 1, m_j - 1)$ for each i, j, then $\Lambda_{q,m}$ is a Frobenius algebra

Further, we can compute the pairing

$$\mathcal{B}_{\Lambda_{q,m}}(x_1^{a_1} \otimes \ldots \otimes x_n^{a_n} \otimes x_1^{b_1} \otimes \ldots \otimes x_n^{b_n}) = \prod_{j=2}^{n} \prod_{i=1}^{j-1} q_{ij}^{a_i b_j} \prod_{\ell=1}^{n} \delta_{a_\ell + b_\ell, m_\ell - 1}$$ which is symmetric

This partially recovers results from [Bergh, 2009]

Twisted tensor products of Frobenius algebras

Def: An algebra (A, ∇_A, η_A) is *separable* if ∇_A has a right inverse as an A-bimodule morphism

A Frobenius algebra $(A, \nabla_A, \eta_A, \Delta_A, \varepsilon_A)$ is *special* if

$$\begin{tikzcd} A \arrow[r, "\Delta_A"] \arrow[dr, "\sim"'] & A \otimes A \arrow[d, "\nabla_A"] \\ & A \end{tikzcd}$$

commutes ie (A, ∇_A, η_A) is separable w/ right inverse Δ_A

Twisted tensor products of Frobenius algebras

Theorem: [Ocal, O.] Let $(A,\nabla_A,\eta_A),(B,\nabla_B,\eta_B)$ be separable algebras w/ inverses Γ_A and Γ_B resp and $\tau: B\otimes A \longrightarrow A\otimes B$ a twisting map $(A\otimes_\tau B,\nabla_{A\otimes_\tau B},\eta_{A\otimes_\tau B})$ is a separable algebra with inverse

$$\Gamma_{A\otimes_\tau B}: A\otimes B \xrightarrow{\Gamma_A\otimes\Gamma_B} A\otimes A\otimes B\otimes B \xrightarrow{1\otimes\tau^{-1}\otimes 1} A\otimes B\otimes A\otimes B$$

if and only if the following diagrams commute

$$\begin{tikzcd}
B\otimes A \arrow[rr, "\tau"] \arrow[d, "1\otimes\Gamma_A"'] & & A\otimes B \arrow[d, "\Gamma_A\otimes 1"] \\
B\otimes A\otimes A \arrow[r, "\tau\otimes 1"] & A\otimes B\otimes A \arrow[r, "1\otimes\tau"] & A\otimes A\otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
B\otimes A \arrow[rr, "\tau"] \arrow[d, "\Gamma_B\otimes 1"'] & & A\otimes B \arrow[d, "1\otimes\Gamma_B"] \\
B\otimes B\otimes A \arrow[r, "1\otimes\tau"] & B\otimes A\otimes B \arrow[r, "\tau\otimes 1"] & A\otimes B\otimes B
\end{tikzcd}$$

Corollary: Let $(A,\nabla_A,\eta_A,\Delta_A,\varepsilon_A),(B,\nabla_B,\eta_B,\Delta_B,\varepsilon_B)$ be special Frobenius algebras and $\tau: B\otimes A\longrightarrow A\otimes B$ a twisting map so that diagram ✫ commutes. Then, $(A\otimes_\tau B,\nabla_{A\otimes_\tau B},\eta_{A\otimes_\tau B},\Delta_{A\otimes_\tau B},\varepsilon_{A\otimes_\tau B})$ is a special Frobenius algebra.

Thank you!

Questions?