Noncommutative Auslander theorem for PI algebras

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Outline

Commutative Auslander theorem

Theorem 1.1 (Auslander theorem I)

Let R be a commutative normal domain, and G be a finite subgroup of $\mathrm{Aut}(R)$. If $R^G \to R$ is unramified in codimension one, then the natural ring map $\varphi : R\#G \to \mathrm{End}_{R^G}(R)$ defined by $\varphi(r\#\sigma)(x) = r\sigma(x)$ is an isomorphism.

Theorem 1.2 (Auslander theorem II)

Let $R = \Bbbk[x_1, \ldots, x_d]$ and G be a finite subgroup of $\mathrm{GL}_{\Bbbk}(d)$. Then $R^G \to R$ is unramified in codimension one iff G is small, that is, it contains no pseudo-reflections.

Commutative Auslander theorem

Corollary 1.3 (Auslander theorem III)

Let $R = \Bbbk[x_1, \ldots, x_d]$ and G be a finite subgroup of $\mathrm{SL}_{\Bbbk}(d)$. Then the natural map $\varphi : R\#G \to \mathrm{End}_{R^G}(R)$ is an isomorphism.

A complete proof at

G. Leuschke, R. Wiegand, Cohen-Macaulay representations, Mathematical Surveys and Monographs, 181., 2012

Artin-Schelter regular algebras

Let $\mathbb{k}$ be an algebraically closed field of characteristic zero.

Definition 1.4

A connected graded algebra R is called Artin-Schelter Gorenstein of dimension d, if

(1) *R has left and right injective dimension $d < +\infty$,*

(2) *There is an integer l such that*

$$\underline{\mathrm{Ext}}^i_R(\mathbb{k}, R) \cong \underline{\mathrm{Ext}}^i_{R^{op}}(\mathbb{k}, R) \cong \begin{cases} 0 & i \neq d \\ \mathbb{k}(l) & i = d \end{cases}.$$

The integer l is called AS index of R.

(3) *If further, $\mathrm{gldim}(R) = d$,*

then R is called Artin-Schelter regular of dimension d.

Throughout, assume that

(1) $\mathbb{k}$ is an algebraically closed field of characteristic zero,

(2) R is a Noetherian connected graded Artin-Schelter regular $\mathbb{k}$-domain with $\operatorname{GKdim} R = d \geq 2$,

(3) H is a semisimple Hopf algebra acting on R inner faithfully and homogeneously,

(4) Let t be an integral of H with $\varepsilon(t) = 1$.

(1) The grade of a finite R-module

$$j(M) := \min\{i \mid \operatorname{Ext}_R^i(M, R) \neq 0\}.$$

(2) R is called a Cohen-Macaulay algebra, if for every finite R-module M,

$$\operatorname{GKdim}(M) + j(M) = \operatorname{GKdim}(R).$$

Definition 1.5 (Bao-He-Zhang, 2019)

(1) *The pertinency of the H-action on R is defined to be the number*

$$p(R, H) := \mathrm{GKdim}(R) - \mathrm{GKdim}(\frac{R\#H}{(1\#t)}).$$

(2) *The H-action on R is homologically small, if the grade $j(\frac{R\#H}{(1\#t)}) \geq 2$.*

Noncommutative Auslander theorem I & II

Theorem 1.6 (Bao-He-Zhang, 2019)

If R is a Noetherian connected graded AS regular CM algebra, then the following are equivalent.

(1) *The natural map $\varphi : R\#H \to \mathrm{End}_{R^H}(R)$ is an isomorphism,*

(2) *The pertinency $\mathrm{p}(R, H) \geq 2$,*

(3) *The H-action on R is homologically small.*

Noncommutative Auslander theorem I: (1) $\Leftrightarrow$ (2).

Noncommutative Auslander theorem II: (2) $\Leftrightarrow$ (3).

Theorem 1.7 (Bao-He-Zhang, 2018)

(1) *Let* $\mathfrak{g}$ *be a finite dimensional Lie algebra, and* $G \leq \mathrm{Aut}_{Lie}(\mathfrak{g})$ *a finite small subgroup. Then* $U(\mathfrak{g})\#G \cong \mathrm{End}_{U(\mathfrak{g})^G}(U(\mathfrak{g}))$.

(2) *Let* $R = \Bbbk_{p_{ij}}[x_1, \ldots, x_d]$ *be the skew polynomial algebra, and assume that* p_{ij} *generic. Let* G *be a finite small subgroup of* $\mathrm{Aut}_{gr}(R)$. *Then* $R\#G \cong \mathrm{End}_{R^G}(R)$.

Theorem 1.8 (Gaddis-Kirkman-Moore-Won, 2019)

The natural map $\varphi : R\#G \to \mathrm{End}_{R^G}(R)$ *is an isomorphism for the following groups acting on the following algebras:*

(1) *any subgroup of* $\mathcal{S}_3$ *acting on Sklyanin algebra* $S(1,1,-1)$,

(2) $\langle(1\,2\,3)\rangle$ *acting on a generic Sklyanin algebra* $S(a,b,c)$.

(3) $\cdots$

Theorem 1.9

Let $R = \Bbbk\langle x, y\rangle/(x^2y - \alpha xyx - \beta yx^2, xy^2 - \alpha yxy - \beta y^2x)$ be the graded down-up algebra. Assume that $\beta \neq 0$.

(1) *(Bao-He-Zhang, 2018)*
If $\beta \neq -1$ or $(\alpha, \beta) = (2, -1)$, then $R\#G \cong \mathrm{End}_{R^G}(R)$ for any finite subgroup G of $\mathrm{Aut}_{gr}(R)$.

(2) *(Chen-Kirkman-Zhang, 2020)*
If $H = (\Bbbk G)^*$ and hdet is trivial, then $R\#H \cong \mathrm{End}_{R^H}(R)$.

Noncommutative cyclic isolated singularties

Let $A = \Bbbk_{-1}[x_1, \ldots, x_d]$ be the (-1)-skew polynomial ring, and $G := \langle (1\,2\,\cdots\,d) \rangle$.

Theorem 1.10 (Bao-He-Zhang, 2019)

If $d = 2^n$ for some $n \geq 1$, then $\mathrm{p}(R, G) = d$. Consequently, R^G is a graded isolated singularity.

Theorem 1.11 (Chan-Young-Zhang, 2020)

If d is divisible by 3 or 5, then $\mathrm{p}(R, G) < d$.

Corollary 1.12 (Chan-Young-Zhang' conjecture)

d is not divisible by 3 or 5 if and only if $\mathrm{p}(R, G) = d$.

Conjecture 1.13 (Chan-Kirkman-Walton-Zhang, 2018)

If the homological determinant is trivial, then the natural map $\varphi : R\#H \to \mathrm{End}_{R^H}(R)$ is an isomorphism.

Theorem 1.14 (Chan-Kirkman-Walton-Zhang, 2018)

If R is a Noetherian AS regular of dimension 2, then $\varphi : R\#H \to \mathrm{End}_{R^H}(R)$ is an isomorphism.

Two necessary conditions and main theorem

Lemma 2.1

If the map $\varphi : R\#H \to \mathrm{End}_{R^H}(R)$ *is bijective, then*

(a) φ *is injective,*

(b) $\widetilde{t} : R \to R^*(:= \mathrm{Hom}_{R^H}(R, R^H)),\ r \mapsto \widehat{t} \cdot r$ *is an isomorphism, where* $\widehat{t}(x) = t \rightharpoonup x$ *for all* $x \in R$.

Theorem 2.2 (Main theorem)

Suppose that R *is a Noetherian PI connected graded AS regular algebra with GK-dimension* $d \geq 2$. *If the condition (a) and (b) are satisfying, then the natural map*

$$\varphi : R\#H \longrightarrow \mathrm{End}_{R^H}(R)$$

is bijective.

Equivalent conditions for condition (a)

Lemma 2.3 (Bao-He-Zhang, 2019)

Suppose that R is a Noetherian connected graded AS regular CM algebra with GK-dimension $d \geq 2$. The following statements are equivalent:

1. $\varphi : R\#H \to \mathrm{End}_{R^H}(R)$ *is injective,*
2. $R\#H$ *is prime,*
3. $\mathrm{p}(R, H) \geq 1$,
4. $j(R\#H/(1\#t)) \geq 1$.

Let Q be the quotient division ring $Q(R)$ of R.

Theorem 2.4 (Cohen-Fischman-Montgomery, 1990)

The following are equivalent:

(1) $Q^H \subseteq Q$ *is* H^**-Galois,*

(2) $Q\#H$ *is simple.*

Theorem 2.5 (Cuadra-Etingof, 2017)

Suppose that $H = \Bbbk\, G$*. Then the extension* Q/Q^G *is* $(\Bbbk\, G)^*$*-Galois.*

Corollary 2.6

Suppose that R *is PI, and* $H = \Bbbk\, G$*. If* $\widetilde{t} : R \to R^*$ *is bijective, then so is* φ*.*

Homological determinant

Jørgensen-Zhang, 2000; Kirkman-Kuzmanovich-Zhang, 2009

To study noncommutative invariant theory of AS-regular algebras, the *homological determinant* hdet of graded algebra had defined by using local cohomology.

Definition 2.7 (Jørgensen-Zhang, Kirkman-Kuzmanovich-Zhang)

Let e *be a nonzero element in* $\mathrm{Ext}^d_R(\Bbbk, R)$. *Then the* *homological determinant* hdet *of the* H*-action on* R *is given by*

$$h \rightharpoonup e = \mathrm{hdet}(S^{-1}h)e$$

for any $h \in H$.

Homological determinant and condition (b)

Lemma 2.8

If the homological determinant is trivial, i.e., $\mathrm{hdet} = \varepsilon_H$, *then*

(1) *(Kirkman-Kuzmanovich-Zhang, 2009)*
R^H *is AS Gorenstein with a balanced dualizing complex* ${}^{\mu}(R^H)(-l)[d]$,

(2) *(Yekutieli-Zhang, 1999)*
$R \cong R^* = \mathrm{Hom}_{R^H}(R, R^H)$ *as graded* R^H-R*-bimodules.*

Lemma 2.9

Assume that $\varphi : R\#H \to \mathrm{End}_{R^H}(R)$ *is injective. If* hdet *is trivial, then* $\widetilde{t} : R \to R^*$ *is bijective.*

A commutative diagram

Lemma 3.1

There is a commutative diagram of R-R-bimodule morphisms

$$\begin{tikzcd} R \otimes_{R^H} R \arrow[r, "\beta"] \arrow[d, "\mathrm{id}\otimes\widetilde{t}"'] & R\#H \arrow[d, "\varphi"] \\ R \otimes_{R^H} R^* \arrow[r, "\gamma"] & \mathrm{End}_{R^H}(R) \end{tikzcd}$$

where $\beta(x\otimes y) = \sum_{(t)} x(t_1 \rightharpoonup y)\#t_2$, $\gamma(x\otimes f)(r) = xf(r)$.
If φ *is injective and* $\widetilde{t}$ *is bijective, then there is an exact sequence*

$$0 \longrightarrow \mathrm{Coker}\,\beta = \frac{R\#H}{(1\#t)} \longrightarrow \mathrm{Coker}\,\gamma \longrightarrow \mathrm{Coker}\,\varphi \longrightarrow 0.$$

PI AS regular algebras

Now, suppose that R is a Noetherian PI AS regular algebra.

Theorem 3.2 (Stafford-Zhang, 1994)

(1) *R is Auslander regular and Cohen-Macaulay.*

(2) *R is a maximal order.*

(3) *R is finite as a module over its center $Z(R)$, which is a Noetherian normal domain.*

Lemma 3.3

(1) *R^H is a finite module over its center $Z(R^H)$ which is affine.*

(2) *$R^H_{\mathfrak{p}}$ is hereditary, for any $\mathfrak{p} \in \operatorname{Spec}(Z(R^H))$ with $\operatorname{ht}(\mathfrak{p}) = 1$.*

GK-dimension of $\operatorname{Coker}\gamma$

Lemma 3.4

For any $\mathfrak{p} \in \operatorname{Spec}(Z(R^H))$ *with* $\operatorname{ht}(\mathfrak{p}) \leq 1$, *the map*

$$\gamma_{\mathfrak{p}} : R_{\mathfrak{p}} \otimes_{R^H_{\mathfrak{p}}} R^*_{\mathfrak{p}} \longrightarrow \operatorname{End}_{R^H_{\mathfrak{p}}}(R_{\mathfrak{p}})$$

is an isomorphism.

Lemma 3.5

$\operatorname{GKdim}(\operatorname{Coker}\gamma)_R = \operatorname{Kdim}(\operatorname{Coker}\gamma)_{Z(R^H)} \leq d-2.$

Noncommutative Auslander theorem III for PI algebras

Theorem 3.6

Let R be a Noetherian PI connected graded AS regular algebra of GK-dimension $d \geq 2$, and H be a semisimple Hopf algebra acting on R inner faithfully and homogeneously. Suppose that the homological determinant is trivial. If the natural map

$$\varphi : R\#H \longrightarrow \mathrm{End}_{R^H}(R)$$

is injective, then it is also surjective.

J.-W. He and Y.-H. Zhang, 2019

The *H-radical* of R is defined by

$$\mathfrak{r}(R,H) := \beta(R \otimes_{R^H} R) \cap R.$$

Then $\operatorname{GKdim}(\frac{R}{\mathfrak{r}(R,H)}) = \operatorname{GKdim}(\frac{R\#H}{(1\#t)})$.

Combine with some results in

J.T. Stafford, Auslander-regular algebras and maximal orders, J. London Math. Soc., 1994

Proposition 4.1

Suppose that R is connected graded Auslander regular and Cohen-Macaulay with GK-dimension $d \geq 2$. If

(1) *φ is injective and $\widetilde{t}$ is bijective,*

(2) $\operatorname{GKdim}(\frac{(1\#t)}{\mathfrak{r}(R,H)(R\#H)}) \leq d-2$,

then φ is bijective.

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Thank You!