Universal quantum semigroupoids

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Online

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Big picture

Goal. [Lots of people here, throughout history, etc.]

Study symmetries in algebra.

Symmetries in algebra

- Fix a field $\Bbbk$.
- Want to study symmetries of

$$A = \bigoplus_{i \in \mathbb{N}} A_i = A_0 \oplus A_1 \oplus \cdots$$

an $\mathbb{N}$-graded, locally finite $\Bbbk$-algebra.

Favorite example.

The polynomial ring $A = \Bbbk[x_1, \ldots, x_n]$.

Also connected ($A_0 = \Bbbk$) and commutative.

How to study symmetries?

- Classically, symmetries are captured by group actions.

$C_2 \curvearrowright$ tiger

$D_{10} \curvearrowright$ pentagon

- $A = \left(\bigoplus_{i\in\mathbb{N}} A_i, \quad m : A \otimes_{\Bbbk} A \to A, \quad u : \Bbbk \to A\right)$ has structure.
- Should preserve algebra structure and grading.

Classical symmetry

- Suppose A is connected and commutative.
- Study group actions such that:

$$g \cdot (ab) = (g \cdot a)(g \cdot b) \quad \text{and}$$
$$g \cdot 1_A = 1_A,$$

[A is a G-module and m and u are G-module morphisms] and

$$g(A_i) \subseteq A_i \quad \text{for all } i$$

[each A_i is a G-module].

Example. [Extremely rich history]

Finite group $G \leq \mathrm{GL}_n(\Bbbk) \curvearrowright A = \Bbbk[x_1, \ldots, x_n]$.

Quantum rigidity

Example.

Let $q \in \Bbbk^\times$. Define the quantum plane

$$\Bbbk_q[x,y] := \frac{\Bbbk\langle x,y\rangle}{(yx - qxy)}.$$

- If g preserves grading, then $g(x) = \alpha x + \beta y$ and $g(y) = \gamma x + \delta y$.
- For g to extend to $\Bbbk_q[x,y]$, must send $yx - qxy$ to $\lambda(yx - qxy)$.

1. If $q = 1$, no restriction so
$$\mathrm{Aut}_{\mathrm{gr}}(\Bbbk[x,y]) = \mathrm{GL}_2(\Bbbk).$$
2. If $q = -1$, $\alpha = \delta = 0$ or $\beta = \gamma = 0$ so
$$\mathrm{Aut}_{\mathrm{gr}}(\Bbbk_{-1}[x,y]) = (\Bbbk^\times)^2 \rtimes \langle\sigma\rangle.$$
3. If $q \neq \pm 1$, $\beta = \gamma = 0$ so
$$\mathrm{Aut}_{\mathrm{gr}}(\Bbbk_q[x,y]) = (\Bbbk^\times)^2.$$

Bialgebras

- A bialgebra H over $\Bbbk$ is a $\Bbbk$-algebra (H, m, u) and a $\Bbbk$-coalgebra (H, Δ, ε) such that

  (a) m and u are coalgebra morphisms

  $$\Delta(ab) = \Delta(a)\Delta(b) \qquad \text{and} \qquad \Delta(1) = 1 \otimes 1$$

  (a′) Δ and ε are algebra morphisms

  $$\Delta(ab) = \Delta(a)\Delta(b) \qquad \text{and} \qquad \varepsilon(ab) = \varepsilon(a)\varepsilon(b).$$

- **Sweedler Notation.** $\Delta(a) = a_1 \otimes a_2$.
- A Hopf algebra is a bialgebra with a $\Bbbk$-linear antipode S

  $$S(a_1)a_2 = a_1 S(a_2) = \varepsilon(a).$$

Quantum symmetries

- Bialgebra (co)actions capture quantum symmetries.
- Let ${}_H\mathcal{M}$ be the category of left H-modules.
- A is called a left H-module algebra if $A \in {}_H\mathcal{M}$ such that

$$h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b)$$
$$h \cdot 1_A = \varepsilon(h)1_A.$$

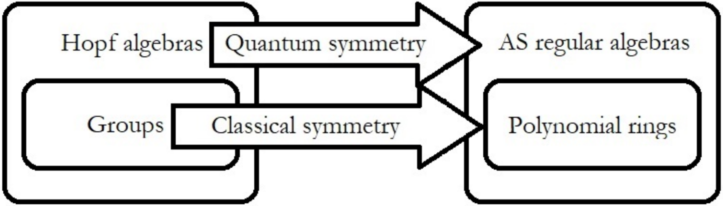
Theorem. [Etingof–Walton, 2013]

Let H be a semisimple Hopf algebra over an algebraically closed field of characteristic zero. If A is a commutative domain that is an H-module algebra, then the H-action factors through a group action.

Hopf algebras as quantum symmetries

Theorem. [Etingof–Walton]

Semisimple Hopf actions on commutative domains are captured by group actions.

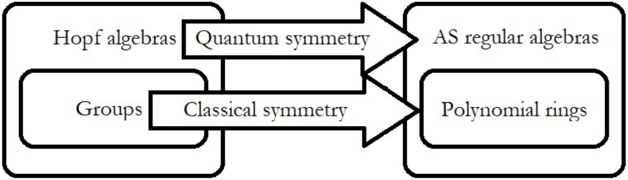


Hopf algebras	"="	Quantum groups
Bialgebras	"="	Quantum semigroups

[Extremely rich theory.]

Hopf algebras as quantum symmetries

In this picture:



$$\text{noncommutative connected}$$
$$\rightsquigarrow$$
$$\text{commutative connected}$$

Motivating question.

How do we enlarge this picture?

(Quotients of) path algebras of quivers: so $A_0 \neq \Bbbk$.

What is symmetry?

- Group: $g.(ab) = (g.a)(g.b)$ and $.1 = 1$.
- Hopf algebra: $h.(ab) = (h_1.a)(h_2.b)$ and $h.1 = \varepsilon(h)1$.
- $(\mathsf{C}, \otimes, \mathbb{1})$ a monoidal category. An algebra object in C is (A, m, u) where A is an object, m and u are morphisms satisfying:

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[r, "\alpha_{A,A,A}"] \arrow[d, "m \otimes \mathrm{Id}_A"'] & A \otimes (A \otimes A) \arrow[r, "\mathrm{Id} \otimes m"] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[rr, "m"'] & & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \otimes A \arrow[r, "u \otimes \mathrm{Id}"] \arrow[dr, "l_A"'] & A \otimes A \arrow[d, "m"] & A \otimes \mathbb{1} \arrow[l, "\mathrm{Id} \otimes u"'] \arrow[dl, "r"] \\
& A &
\end{tikzcd}$$

- Group: (A, m, u) an algebra object in monoidal category ${}_{\Bbbk G}\mathcal{M}$.
- Hopf: (A, m, u) an algebra object in monoidal category ${}_{H}\mathcal{M}$.

A more general framework?

Look for H with ${}_H\mathcal{M}$ or $\mathcal{M}^H$ monoidal.

Many generalizations of Hopf algebras:

1. Weak Hopf algebras
2. Quasi-bialgebras with antipode
3. Hopf algebroids
4. Hopf bimonoids in duoidal categories
5. Hopfish algebras

Weak Hopf algebras

- A weak bialgebra H over $\Bbbk$ is a $\Bbbk$-algebra (H, m, u) and a $\Bbbk$-coalgebra (H, Δ, ε) such that

  (1) $\Delta(ab) = \Delta(a)\Delta(b)$,

  (2) $(\Delta \otimes \mathrm{Id}) \circ \Delta = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1)$,

  (3) $\varepsilon(abc) = \varepsilon(ab_1)\varepsilon(b_2c) = \varepsilon(ab_2)\varepsilon(b_1c)$.

- Bialgebra if and only if $\Delta(1) = 1 \otimes 1$ if and only if $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$.

- A weak Hopf algebra is a weak bialgebra with antipode S:

$$S(a_1)a_2 = 1_1\varepsilon(a1_2), \qquad a_1S(a_2) = \varepsilon(1_1a)1_2, \qquad S(a_1)a_2S(a_3) = S(a).$$

Counital maps

- The maps appearing in the antipode axioms:

$$S(a_1)a_2 = 1_1\varepsilon(a1_2), \qquad a_1S(a_2) = \varepsilon(1_1a)1_2$$

are important.

source counital map

$$\varepsilon_s : H \to H$$
$$\varepsilon_s(a) = 1_1\varepsilon(a1_2)$$

target counital map

$$\varepsilon_t : H \to H$$
$$\varepsilon_t(a) = \varepsilon(1_1a)1_2$$

source counital subalgebra

$$H_s := \varepsilon_s(H)$$

target counital subalgebra

$$H_t = \varepsilon_t(H)$$

- H_s and H_t are antiisomorphic separable, semisimple, finite-dimensional, codieal sub-$\Bbbk$-algebras.
- A weak bialgebra is a bialgebra if and only if $H_s = H_t = \Bbbk$.

Why weak Hopf algebras?

- Introduced by [Böhm–Nill–Szlachanyi 1999], motivated by physics: study symmetries in conformal field theory.
- Axioms are self-dual, so the dual of a finite-dimensional weak Hopf algebra is again a weak Hopf algebra.

Example.

If H, K are bialgebras, then $H \oplus K$ is an algebra as usual and a coalgebra under

$$\Delta(h,k) = (h_1, 0) \otimes (h_2, 0) + (0, k_1) \otimes (0, k_2)$$

$$\varepsilon(h,k) = \varepsilon_H(h) + \varepsilon_K(k)$$

But $\Delta(1,1) = (1,0) \otimes (1,0) + (0,1) \otimes (0,1) \neq (1,1) \otimes (1,1)$.
$(H \oplus K)_t = (H \oplus K)_s = \Bbbk \oplus \Bbbk$.
So $H \oplus K$ not a bialgebra, only a weak bialgebra.

Why weak Hopf algebras?

If G, H are groups, then $G \sqcup H$ is not a group, but a groupoid.

Example.

$\mathcal{G}$ is a groupoid. $\Bbbk\mathcal{G}$ the groupoid algebra is a weak Hopf algebra.

$$\text{For } g \in \mathcal{G}: \qquad \Delta(g) = g \otimes g, \qquad \varepsilon(g) = 1, \qquad S(g) = g^{-1}.$$

$$\mathcal{G} = \begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\alpha^{-1}"] \end{tikzcd}$$

Then $1 = e_1 + e_2$ but $\Delta(1) = e_1 \otimes e_1 + e_2 \otimes e_2 \neq 1 \otimes 1$.
$(\Bbbk\mathcal{G})_t = (\Bbbk\mathcal{G})_s = \Bbbk e_1 \oplus \Bbbk e_2$.

Weak Hopf algebras “=” Quantum groupoids

Weak bialgebras “=” Quantum semigroupoids

Why weak Hopf algebras?

Theorem. [Hayashi 1999, Szlachanyi 2001]

Every fusion category is equivalent to ${}_H\mathcal{M}_{\text{fd}}$ for some weak Hopf algebra H.

- If $(H, m, u, \Delta, \varepsilon)$ is an algebra and coalgebra such that $\Delta(ab) = \Delta(a)\Delta(b)$, then

  Δ axiom $\Rightarrow$ ${}^H\mathcal{M}$ and $\mathcal{M}^H$ are monoidal,

  ε axiom $\Rightarrow$ ${}_H\mathcal{M}$ and $\mathcal{M}_H$ are monoidal.

- But not $\otimes_{\Bbbk}$! [Nill 1998], [Böhm–Caenepeel–Janssen 2011], [Walton–Wicks–W arXiv: 1911.12847]

An important example

Example. [Hayashi]

Q a finite quiver, e.g.:

$$Q = \; 1 \xrightarrow{p} 2 \xrightarrow{q} 3 \; .$$

Define a graded weak bialgebra $\mathfrak{H}(Q)$:

- $\Bbbk$-basis: $\{x_{a,b} \mid a, b \in Q_\ell, \ell \in \mathbb{N}\}$

$$= \{x_{i,j} \mid 1 \leq i, j \leq 3\} \cup \{x_{p,p}, x_{p,q}, x_{q,p}, x_{q,q}\} \cup \{x_{pq,pq}\}.$$

- Multiplication: $x_{a,b}x_{c,d} = \begin{cases} x_{ac,bd}, & \text{if } ac \text{ and } bd \text{ are paths} \\ 0, & \text{otherwise.} \end{cases}$
- Unit: $1_{\mathfrak{H}(Q)} = \sum_{i,j \in Q_0} x_{i,j}$.
- As an algebra, $\mathfrak{H}(Q) \cong \Bbbk\widehat{Q}$ [Calderon–Walton, 2021].

An important example

Example. [Hayashi]

$$Q = \; 1 \xrightarrow{p} 2 \xrightarrow{q} 3$$

- Comultiplication: $\Delta(x_{a,b}) = \sum_{c \in Q_\ell} x_{a,c} \otimes x_{c,b}$

$$\Delta(x_{p,q}) = x_{p,p} \otimes x_{p,q} + x_{p,q} \otimes x_{q,q}.$$

- Counit: $\varepsilon(x_{a,b}) = \delta_{a,b}$.

An important example

Example. [Hayashi]

There are natural left and right coactions of $\mathfrak{H}(Q)$ on $\Bbbk Q$:

$$\lambda^{\mathfrak{H}(Q)} : \Bbbk Q \to \mathfrak{H}(Q) \otimes \Bbbk Q \qquad\qquad \rho^{\mathfrak{H}(Q)} : \Bbbk Q \to \Bbbk Q \otimes \mathfrak{H}(Q)$$

$$a \mapsto \sum_{b \in Q_\ell} x_{a,b} \otimes b \qquad\qquad a \mapsto \sum_{b \in Q_\ell} b \otimes x_{b,a} \; a \mapsto \sum_{b \in Q_\ell} x_{a,b} \otimes$$

Making $\Bbbk Q$ a left and right $\mathfrak{H}(Q)$-comodule algebra.

($\approx$ algebra object in ${}^{H}\mathcal{M}$ and $\mathcal{M}^{H}$, see [Walton–Wicks–W].)

Back to the beginning

- **Title:** Universal quantum semigroupoids.
- Quantum semigroupoid = weak bialgebra.
- Universal?

Main theorem. [Huang–Walton–Wicks–W]

The universal quantum semigroupoid of $\Bbbk Q$ is $\mathfrak{H}(Q)$.

Universal coactions

Suppose A is connected.

Definition. [Manin]

A left universal quantum semigroup (UQSG) of A is a bialgebra $O^{\text{left}}(A)$ that left coacts on A so that for every bialgebra H that left coacts on A, there exists a unique bialgebra map $\pi : O^{\text{left}}(A) \to H$ such that

$$\begin{tikzcd} A \arrow[r, "\lambda^O"] \arrow[dr, "\lambda^H"'] & O^{\text{left}}(A) \otimes A \arrow[d, "\pi \otimes \mathrm{Id}_A"] \\ & H \otimes A. \end{tikzcd}$$

Universal coactions

Example. [Manin]

Let $A = \Bbbk[y_1, y_2]$. Then

$$O^{\text{left}}(A) = \frac{\Bbbk\langle x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}\rangle}{R}$$

where

$$y_1 \mapsto x_{1,1} \otimes y_1 + x_{1,2} \otimes y_2$$

$$y_2 \mapsto x_{2,1} \otimes y_1 + x_{2,2} \otimes y_2.$$

Since $[y_1, y_2] = 0$,

$$R = \left([x_{1,1}, x_{2,1}], [x_{1,2}, x_{2,2}], [x_{1,1}x_{2,2}] - [x_{1,2}, x_{2,1}]\right)$$

Universal coactions

Definition. [Manin]

A right UQSG of A is a bialgebra $O^{\text{right}}(A)$ that right coacts on A so that for every bialgebra H that right coacts on A, there exists a unique bialgebra map $\pi : O^{\text{right}}(A) \to H$ such that

$$\begin{tikzcd} A \arrow[r, "\rho^O"] \arrow[dr, "\rho^H"'] & A \otimes O^{\text{right}}(A) \arrow[d, "\mathrm{Id}_A \otimes \pi"] \\ & A \otimes H. \end{tikzcd}$$

A transposed UQSG $O^{\text{trans}}(A)$ left and right coacts on A universally via λ^O and ρ^O such that

$$(\rho^O)^\top : A^* \to O \otimes A^*$$

is λ^O (after identifying a basis of A with the dual basis of A^*).

Universal coactions

Example. [Manin]

Let $A = \Bbbk[y_1, y_2]$. Then

$$O^{\text{right}}(A) = \frac{\Bbbk\langle x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}\rangle}{R'}$$

where

$$y_1 \mapsto y_1 \otimes x_{1,1} + y_2 \otimes x_{2,1}$$

$$y_2 \mapsto y_1 \otimes x_{1,2} + y_2 \otimes x_{2,2}.$$

Since $[y_1, y_2] = 0$,

$$R' = \left([x_{1,1}, x_{1,2}], [x_{2,1}, x_{2,2}], [x_{1,1}x_{2,2}] + [x_{1,2}, x_{2,1}]\right).$$

Universal coactions

$A = \Bbbk[y_1, y_2]$.

- $O^{\text{left}}(A)$ gave relations

$$R = \left([x_{1,1}, x_{2,1}], [x_{1,2}, x_{2,2}], [x_{1,1}x_{2,2}] - [x_{1,2}, x_{2,1}]\right).$$

- $O^{\text{right}}(A)$ gave relations

$$R' = \left([x_{1,1}, x_{1,2}], [x_{2,1}, x_{2,2}], [x_{1,1}x_{2,2}] + [x_{1,2}, x_{2,1}]\right).$$

- So $O^{\text{trans}}(A) \cong \Bbbk[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}]$.
- Coassociativity and counitality of coaction give

$$\begin{aligned}\Delta(x_{i,j}) &= x_{i,1} \otimes x_{1,j} + x_{i,2} \otimes x_{2,j} \\ \varepsilon(x_{i,j}) &= \delta_{i,j}.\end{aligned}$$

- So $O^{\text{trans}}(A) \cong \mathcal{O}(\mathrm{Mat}_2(\Bbbk))$ as a bialgebra.

Universal UQSGds

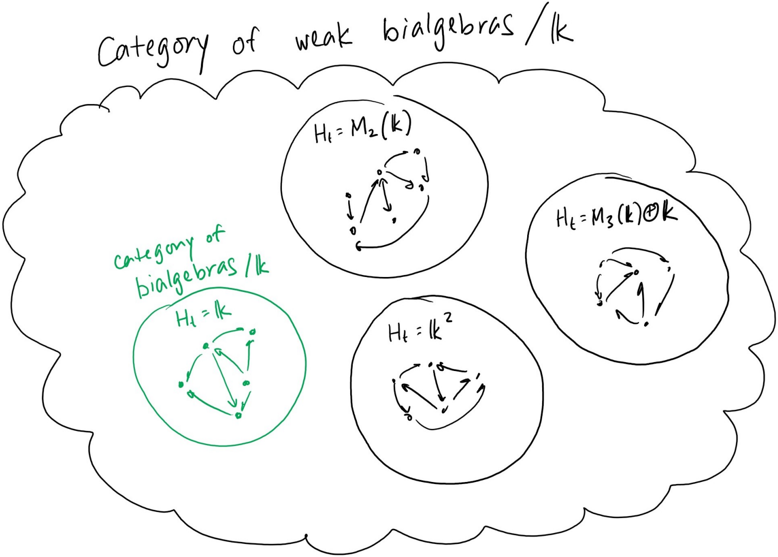
Suppose A is not necessarily connected.

Definition?

A left UQSGd of A is a weak bialgebra $\mathcal{O}^{\text{left}}(A)$ that left coacts on A so that for every weak bialgebra H that left coacts on A, there exists a unique weak bialgebra map $\pi : \mathcal{O}^{\text{left}}(A) \to H$ such that

$$\begin{tikzcd} A \arrow[r, "\lambda^{\mathcal{O}}"] \arrow[dr, "\lambda^H"'] & \mathcal{O}^{\text{left}}(A) \otimes A \arrow[d, "\pi \otimes \mathrm{Id}_A"] \\ & H \otimes A. \end{tikzcd}$$

Fact. If $\phi : H \to K$ is a nonzero wba morphism, then $H_t \cong K_t$.



Universal UQSGds

- Would like a condition on H_s that generalizes the bialgebra case.

Definition

A left UQSGd of A is a weak bialgebra $\mathcal{O} = \mathcal{O}^{\text{left}}(A)$ that left coacts on A, with $A_0 \cong \mathcal{O}_t$ in ${}^{\mathcal{O}}\mathcal{M}$, so that for every weak bialgebra H that left coacts on A, with $A_0 \cong H_t$, there exists a unique weak bialgebra map $\pi : \mathcal{O}^{\text{left}}(A) \to H$ such that the diagram commutes.

- Call this a base preserving coaction.
- $\mathcal{O}_t = \mathbb{1}$ in ${}^{\mathcal{O}}\mathcal{M}$.
- Analogous definition for $\mathcal{O}^{\text{right}}(A)$, with H_s instead of H_t.

Universal UQSGds

- A transposed UQSGd $\mathcal{O}^{\text{trans}}(A)$ should coact on A on left and right universally + transpose compatibility conditions.
- $\mathcal{O}_t \cong A_0 \cong \mathcal{O}_s$.
- Recall $\mathcal{O}_t$ antiisomorphic to $\mathcal{O}_s$.
- So in our work, we assume A_0 is commutative.

Main theorem

Theorem. [Huang–Walton–Wicks–W]

Let $A = \Bbbk Q$. Then
$$\mathcal{O}^*(A) \cong \mathfrak{H}(Q)$$
where $* =$ left, right, or trans.

- Surprising to us, but speaks to freeness of $\Bbbk Q$.
- $\Bbbk Q$ is the tensor algebra $T_{\Bbbk Q_0}(\Bbbk Q_1)$.

Theorem. [Huang–Walton–Wicks–W]

Let I be a graded ideal of $\Bbbk Q$. If $\mathcal{O}^*(\Bbbk Q/I)$ exists, then
$$\mathcal{O}^*(\Bbbk Q/I) \cong \mathfrak{H}(Q)/\mathcal{I}$$
for some biideal $\mathcal{I}$ of $\mathfrak{H}(Q)$.

Quadratic quiver algebras

Theorem. [Huang–Walton–Wicks–W]

If I is a graded ideal of $\Bbbk Q$ generated in degree 2, then

$$\mathcal{O}^{\text{left}}(\Bbbk Q/I) \cong \mathcal{O}^{\text{right}}((\Bbbk Q/I)^!)^{\text{op}}$$

$$\mathcal{O}^{\text{trans}}(\Bbbk Q/I) \cong \mathcal{O}^{\text{trans}}((\Bbbk Q/I)^!)^{\text{op}}$$

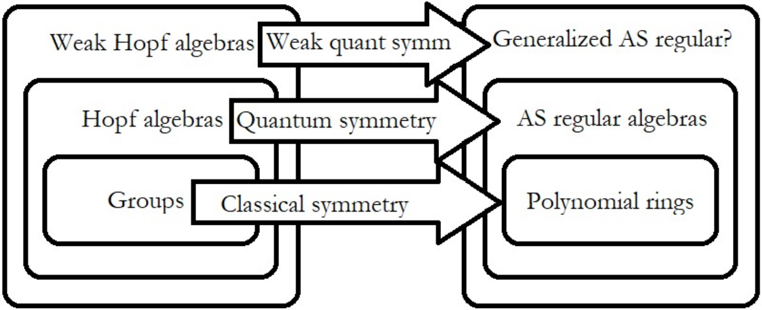
where $(\Bbbk Q/I)^!$ is the quadratic dual $\Bbbk Q^{\text{op}}/I^{\perp}_{\text{op}}$.

Example.

If I is the ideal of $\Bbbk Q$ generated by all paths of length 2, then $\Bbbk Q/I = \Bbbk Q^!$. Hence

$$\mathcal{O}^{\text{left}}(\Bbbk Q/I) \cong \mathfrak{H}(Q).$$

Thank you!



Weak Hopf algebras
Weak quant symm
Generalized AS regular?
Hopf algebras
Quantum symmetry
AS regular algebras
Groups
Classical symmetry
Polynomial rings