On The Endomorphisms of Some Noncommutative Algebras

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Automorphisms of Polynomial Algebras

Let $\mathbb{k}$ be a field. The $\mathbb{k}$-algebra automorphisms of $\mathbb{k}[x_1, \cdots, x_n]$ have been extensively researched in the literature.

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- (H.W.E. Jung 1942 & W. Van der Kulk 1953) The structure of $\mathrm{Aut}_{\mathbb{k}}\mathbb{k}[x, y]$ is also well understood.
- (M. Nagata 1972) It was conjectured by Nagata that $\mathbb{k}[x, y, z]$ has a wild automorphism

$$(x, y, z) \mapsto (x - 2(xz + y^2)y - (xz + y^2)^2 z, y + (xz + y^2)z, z).$$

This conjecture was proved by U.U. Umirbaev and I.P. Shestakov in 2004.

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- The Jacobian Conjecture remains open (except for the case $n = 1$ which is trivial).

Weyl Algebras and The Dixmier Conjecture

Let $A_1(\mathbb{C})$ be the $\mathbb{C}-$ algebra generated by x, y subject to the relation $xy - yx = 1$.

- (J. Dixmier 1968 & L. Makar-Limanov 1983) The group $\mathrm{Aut}_{\mathbb{C}} A_1(\mathbb{C})$ was classified by Dixmier. Makar-Limanov determined $\mathrm{Aut}_{\Bbbk}(A_1(\Bbbk))$ for $\Bbbk$ of positive characteristic.

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- (Y. Tsuchimoto 2003, A. Belov-Kanel & M. Kontsevich 2005) The Dixmier Conjecture is **stably equivalent to** the Jacobian Conjecture.

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- Determine the automorphism group for a noncommutative algebra.
- Identify noncommutative algebras whose endomorphisms are all automorphisms.
- Establish a criterion for an algebra endomorphism to be an automorphism.

Classifying the Automorphism Group

The automorphism groups have been classified for the following algebras.

- (J. Alev & M. Chamarie 1992) The tensor products of quantum plane algebras $\mathbb{k}_q[x, y]$, the 2×2 quantum matrix algebra $R_q[M_2]$, the quantum space $\mathbb{k}_q[x_1, \cdots, x_n]$, and the quantized enveloping algebra $U_q(sl_2)$.

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- (J. Alev & F. Dumas 1993) The quantum Heisenberg algebra $U_q^+(sl_3)$, the quantum Weyl algebra $A_1^q(\mathbb{k})$ of rank 1, the Weyl-Hayashi algebra, and other related algebras.

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- (L. Rigal 1996) The quantized Weyl algebras of higher ranks.
- (J. Gómez-Torrecillas & L. EL Kaoutit 2002) The coordinate ring of quantum Symplectic space.

Andruskiewitsch-Dumas Conjecture and Launois-Lenagan Conjecture

- In 2003, N. Andruskiewitsch and F. Dumas conjectured that

$$\mathrm{Aut}_{\mathbb{k}}(U_q^+(\mathfrak{g})) \cong (\mathbb{k}^\times)^n \rtimes \mathrm{Aut}(\Gamma).$$

  This was verified for $U_q^+(so_5)$ by S. Launois in 2004 and for $U_q^+(sl_4)$ by S. Launois and S. Lopes in 2006.

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- In 2007, S. Launois and T.H. Lenagan conjectured that

$$\mathrm{Aut}_{\Bbbk}(R_q[M_n]) \cong (\Bbbk^\times)^{2n-1} \rtimes \mathbb{Z}_2.$$

They proved their conjecture in the case $n = 3$ in 2011.

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- (S. Ceken, J.H. Palmieri, Y.-H. Wang & J.J. Zhang 2015) In the root of unity case, the automorphism group is determined for many quantum algebras using the idea of discriminants. There have been many works following this direction.

Algebras with Only Bijective Endomorphisms

Let $A = \Bbbk_Q[x_1^{\pm 1}, \cdots, x_m^{\pm 1}]$ with $n \geq 2$ be the quantum Laurent polynomial algebra defined by $Q = (q_{ij})_{1 \leq i,j \leq m}$.

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- (L. Richard 2002) Otherwise, any $\Bbbk$-algebra endomorphism φ of A is an automorphism if and only if φ preserves the center of A and the restriction of φ to the center is an automorphism.
- One may refer the property with "all algebra endomorphisms being bijective" to as the **Dixmier Property or Condition**.

Endomorphisms of Quantized Weyl Algebras

Let $R = \mathbb{C}[t, t^{-1}]$ and $A_n^{(t)}$ be the $R-$algebra generated by $x_1, y_1, \cdots, x_n, y_n$ subject to the relations: $x_i y_i - t y_i x_i = 1$, $x_i y_j = y_j x_i$, $x_i x_j = x_j x_i$, and $y_i y_j = y_j y_i$. Set $A_n^q = A_n^{(t)}/(t-q)$ for $q \neq 1 \in \mathbb{C}^\times$ and $z = \prod_{i=1}^n (x_i y_i - y_i x_i)$.

- (E. Backelin 2011) Each $R-$algebra endomorphism of $(A_n^{(t)})_z$ is an automorphism.

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- (E. Backelin 2011) When q is a root of unity, the localization $(A_n^q)_z$ is an Azumaya algebra over its center.

More Algebras with the Dixmier Property

For $0 \neq a(h) \in \mathbb{K}[h^{\pm 1}]$, let $A(a(h), q)$ be the $\Bbbk$-algebra generated by $x, y, h^{\pm 1}$ subject to the relations

$$xh = qhx, \quad yh = q^{-1}hy, \quad xy = a(qh), \quad yx = a(h).$$

- (S. Launois & A. Kitchin 2014) Assume that a is not a monomial. Each $\Bbbk$-algebra monomorphism of $A(a(h), q)$ is an automorphism. If $A(a(h), q)$ is simple, then each $\Bbbk$-algebra endomorphism of $A(a(h), q)$ is an algebra automorphism.

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- (S. Launois & A. Kitchin 2015) When q is not a root of unity, each $\Bbbk$-algebra endomorphism of $A(n, d, q)$ is an automorphism, where $A(n, d, q) = \otimes_{i=1}^{n} A(h^d - 1, q)$ for $d \in \mathbb{N}$.

More Algebras with the Dixmier Property

- (Tang 2015) The localized down-up algebra: $A_{r,s}(\mathbb{k})_{\mathcal{S}}$ with r, s being independent in $\mathbb{K}^{\times}$. Here $A_{r,s}(\mathbb{k})$ is a $\mathbb{k}$-algebra generated by u, d subject to: $u^2d - (r+s)udu + rsdu^2 = 0$, $d^2u - (r+s)dud + rsud^2 = 0$. We denote by $\mathcal{S}$ the set $\{[(ud - rdu)(ud - sdu)]^i \mid i \geq 0\}$.

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- (Tang 2017) The algebra: $\bigotimes_{i=1}^{n} A(\frac{h_i - 1}{q_i - 1}, q_i)$ with $q_1, \cdots, q_n$ being independent in $\Bbbk^{\times}$.

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- (Tang 2017) The algebra: $\mathbb{k}_p[s^{\pm 1}, t^{\pm 1}] \bigotimes A(\frac{h-1}{q-1}, q)$ with $p, q \in \mathbb{k}^{\times}$ being non-root of unity.

Some Criteria for Bijectivity

There have been some criteria in the literature for singling algebra automorphisms out of algebra endomorphisms:

- (V.A. Artamonov & R. Wisbauer 2001) Assume that $n \geq 3$ and q_{ij} are independent in $\mathbb{k}^{\times}$. Let φ be a $\mathbb{k}$-algebra endomorphism of $\mathbb{k}_Q[x_1^{\pm 1}, \cdots, x_r^{\pm 1}, x_{r+1}, \cdots, x_n]$. If there exist i, j, k such that $\varphi(x_i), \varphi(x_j), \varphi(x_k) \neq 0$, then φ is an algebra automorphism.

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- (N. Lauritzen & J.F. Thomsen 2017) Assume that $\mathrm{char}\, \mathbb{k} = 0$. Each birational $\mathbb{k}$-algebra endomorphism of $A_n(\mathbb{k})$ is an automorphism.

Benkart's Multi-paramter Weyl Algebras

- (G. Benkart 2013) For any $r, s \in \mathbb{k}^{\times}$, let $A_{r,s}(\mathbb{k})$ be the $\mathbb{k}$-algebra generated by $\rho^{\pm 1}, \sigma^{\pm 1}$ and x, y subject to the following relations: $\rho\sigma = \sigma\rho, \rho x = rx\rho, \rho y = r^{-1}\rho y, \sigma x = sx\sigma, \sigma y = s^{-1}y\sigma, yx = \frac{r^2\rho^2 - s^2\sigma^2}{r^2 - s^2}, xy = \frac{\rho^2 - \sigma^2}{r^2 - s^2}$. Set $A_{\overline{r},\overline{s}}(n) = \bigotimes_{i=1}^{n} A_{r_i,s_i}(\mathbb{k})$.

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- For any $r, s \in \mathbb{k}^{\times}$ and $b(\rho, \sigma) \in \mathbb{k}[\sigma^{\pm 1}, \rho^{\pm 1}]$, let $B(b, r, s)$ be the $\mathbb{k}$-algebra generated by $\rho^{\pm 1}, \sigma^{\pm 1}$ and x, y subject to the following relations $\rho\sigma = \sigma\rho$ and

$$\rho x = rx\rho, \quad \rho y = r^{-1}y\rho;$$
$$\sigma x = sx\sigma, \quad \sigma y = s^{-1}y\sigma;$$
$$xy = b(r\rho, s\sigma), \quad yx = b(\rho, \sigma).$$

Endomorphisms for Benkart's Algebras

Benkart's algebras have the Dixmier Property.

- If $b(\rho, \sigma)$ is a monomial and $r^i s^j = 1$ implies that $i = j = 0$, then each endomorphism of $B(b, r, s)$ is an automorphism.

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- If $b(\rho, \sigma) \in \mathbb{k}[\rho^{\pm 1}, \sigma^{\pm 1}]$ is not a monomial with no momials $m_1, m_2 \in \mathbb{k}[\rho^{\pm 1}, \sigma^{\pm 1}]$ such that $b(m_1, m_2) = b(rm_1, sm_2) = 0$ and $r^i s^j = 1$ implies that $i = j = 0$, then each endomorphism of $B(b, r, s)$ is an automorphism.

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- **An Example:** One can set $b(\rho, \sigma) = \frac{\rho^d - \sigma^d}{r - s}$ where $d \in \mathbb{Z}$ and $r^i s^j = 1$ implies that $i = j = 0$.

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- **An Example:** One can set $b(\rho, \sigma) = \frac{\rho^d - \sigma^d}{r - s}$ where $d \in \mathbb{Z}$ and $r^i s^j = 1$ implies that $i = j = 0$.
- **A Counter-Example:** Set $b(\rho, \sigma) = (\rho - \sigma)(s\rho - t\sigma)$. Then $B(b, r, s)$ has non-injective endomorphisms.

A Tensor Product Phenomenon

The following algebras have the Dixmier Property.

- The algebra: $\bigotimes_{i=1}^{n} \Bbbk[h^{\pm 1}](a_i(h), q_i)$ where q_i is a not a root of unity and a_i is a non-monomial with no $c_i \in \Bbbk^{\times}$ such that $a_i(c_i) = a_i(q_i c_i) = 0$ for $i = 1, \cdots, n$.

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- The algebra: $[\bigotimes_{i=1}^{n} A(a_i(h), q_i)] \bigotimes \mathbb{k}_Q[t_1^{\pm 1}, \cdots, t_m^{\pm 1}]$ where q_i is not a root of unity and $a_i(h)$ is not a monomial with no $c_i \in \mathbb{k}^{\times}$ such that $a_i(c_i) = a_i(q_i c_i) = 0$ for $i = 1, \cdots, n$ and $\mathbb{k}_Q[t_1^{\pm 1}, \cdots, t_m^{\pm 1}]$ is a simple algebra.

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- The algebra: $\bigotimes_{i=1}^{n} B(b_i(\rho, \sigma), r_i, s_i)$ where $r_1, s_1, \cdots, r_n, s_n$ are independent in $\mathbb{k}^{\times}$ and b_i is not a monomial with no monomials $m_1^i, m_2^i \in \mathbb{k}[\rho^{\pm 1}, \sigma^{\pm 1}]$ such that $b_i(m_1^i, m_2^i) = 0$ and $b_i(r_i m_1^i, s_i m_2^i) = 0$ for $i = 1, \cdots, n$.

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- Research on the root of unity case.

THANK YOU!