

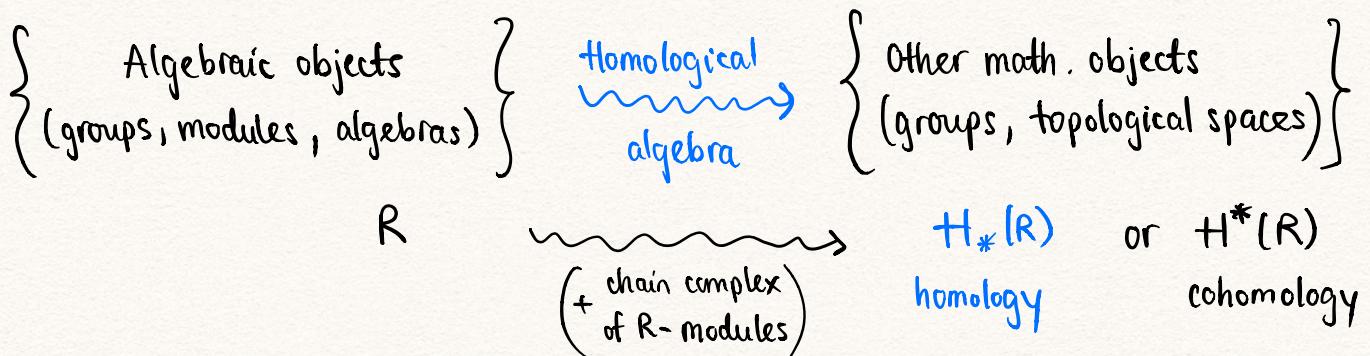
BUILDING RESOLUTIONS FROM KOSZUL COMPLEX

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Outline : I. Motivation / Background
II. Main Result, Construction
III. Applications

① Motivation / Background :



Idea : Use structure of $H_*(R)$ or $H^*(R)$
to gain insights about R .

- Let \mathbb{K} be a base field (e.g. $\mathbb{Q}, \mathbb{C}, \mathbb{Z}_p$)

- Local ring $(R, \mathfrak{m}, \mathbb{K})$

noetherian

\uparrow maximal ideal of R

$R/\mathfrak{m} = \mathbb{K}$, residue field

$$\dim_{R/\mathfrak{m}} (\mathfrak{m}/\mathfrak{m}^2) = \dim_{\mathbb{K}} (\mathfrak{m}/\mathfrak{m}^2)$$

$n = \text{embedding dim of } R$

$$c = \text{codepth}(R) = n - \underbrace{\text{depth}(R)}$$

$$\text{depth}_{\mathfrak{m}}(R) = \min \{ i : \underset{\mathbb{K}}{\text{Ext}}^i(R/\mathfrak{m}, R) \neq 0 \}$$

- Koszul complex K^R on a minimal set of generators of \mathfrak{m} :

$$(\partial_i^K \circ \partial_{i+1}^K = 0)$$

homological degree

$$\dots \rightarrow K_3 \xrightarrow{\partial_3^K} K_2 \xrightarrow{\partial_2^K} K_1 \xrightarrow{\partial_1^K} K_0 \rightarrow 0$$

3 2 1 0

\rightsquigarrow Obtain the Koszul homology $A := H_*(K^R) = \bigoplus_{i \geq 0} H_i(K^R)$

- + Differential graded (DG)-algebra induces graded-commutative algebra structure of Koszul complex K^R

$$\partial(a \wedge b) = \partial(a) \wedge b + (-1)^{\deg(a)} a \wedge \partial(b)$$

structure of $A = H_*(K^R)$

$$x \cdot y = (-1)^{\deg(x) \cdot \deg(y)} y \cdot x$$

- \mathbb{K} is an R -module \rightsquigarrow define $\text{Tor}_i^R(\mathbb{K}, \mathbb{K}) = i^{\text{th}}$ homology of \mathbb{K} over R (using free resolution of \mathbb{K})

- Koszul complex K^R of \mathfrak{m} over R $\rightsquigarrow A = H_*(K^R) \rightsquigarrow a_i := \text{rank}_{\mathbb{K}} A_i$

- Free resolution of \mathbb{K} over R $\rightsquigarrow \text{Tor}_*^R(\mathbb{K}, \mathbb{K}) \rightsquigarrow \beta_i := \text{rank}_{\mathbb{K}} \text{Tor}_i^R(\mathbb{K}, \mathbb{K})$
Betti numbers

[Serre]: There is an inequality between series (3)

Poincaré series of lk over R $P_{\text{lk}}^R(t) := \sum_{i=0}^{\infty} \beta_i t^i \leq \frac{(1+t)^n}{1 - \sum_{i=1}^c a_i t^{i+1}}$

[Grodz'1962]: Equality holds $\Leftrightarrow \begin{cases} A := H_*(K^R) \text{ has trivial multiplications} \\ \text{AND trivial Massey operations} \end{cases}$
↓
 R is Grod



- Construct a minimal free resolution of lk over R that involves algebraic invariants of $A := H_*(K^R) = A_0 \oplus \dots \oplus A_c$
- Measure how far the ring R is from being Grod.
- Gain more understanding of the Massey products.

Q: Geometric interpretations?

II Result + Construction:

$$a_i := \text{rank}_{\text{lk}}(A_i) \quad \forall i \geq 0$$

$$g_{ij} := \text{rank}_{\text{lk}}(A_i \cdot A_j) \quad \text{multiplication}$$

Study in detail the multiplicative structure of A in low degrees.

- Degree 1: A_1 is generated by $\{[z_i^1]\}_{1 \leq i \leq a_1}$

- Degree 2: Elements in $A_2 \xrightarrow{\{[z_i^1 \wedge z_j^1]\}_{1 \leq i, j \leq a_1}} A_1 \cdot A_1$

$$A_2 = (A_1 \cdot A_1) \oplus \overline{A}_2 \xrightarrow{\{[z_e^2]\}_{1 \leq e \leq a_2 - q_{11}}} \overline{A}_2$$

- Degree 3: $A_3 = (A_1 \cdot A_2) \oplus \overline{A}_3$

$$= (A_1 \cdot A_1 \cdot A_1 + A_1 \cdot \overline{A}_2) \oplus \overline{A}_3$$

$$\{[z_i^1 \wedge z_j^1 \wedge z_k^1]\} \uparrow \quad \{[z_i^1 \wedge z_e^2]\} \uparrow \quad \{[z_t^3]\}_{1 \leq t \leq a_3 - q_{12}}$$

$$\Psi: A_1 \otimes A_1 \otimes A_1 \rightarrow (A_1 \cdot A_1) \otimes A_1 \oplus A_1 \otimes (A_1 \cdot A_1)$$

$$[x] \otimes [y] \otimes [z] \mapsto ([x] \wedge [y] \otimes [z], [x] \otimes [y] \wedge [z])$$

b := \text{rank}_{IK}(\text{Coker } \Psi)

- Degree 4: $A_4 = (\underbrace{A_1 \cdot A_3 + A_2 \cdot A_2 + \langle A_1, A_1, A_1 \rangle}_{\text{rank}_{IK} =: a}) \oplus \overline{A}_4$

Massey product

does not come from any product

(triple) Massey product:

Let $[x], [y], [z] \in A_1$

s.t. $\begin{cases} [x] \cdot [y] = 0 \text{ in } A_2 \Rightarrow \partial_3^k(s) = \bar{x} \cdot y & \text{some } s \in K_3 \\ [y] \cdot [z] = 0 \Rightarrow \partial_3^k(t) = \bar{y} \cdot z & t \in K_3 \end{cases}$

[N-Veliche] Describe the elements of this set explicitly.

Notation: $\bar{u} = (-1)^{\deg(u) \pm 1} \cdot u$

$\langle [x], [y], [z] \rangle = \{[\bar{s}z + \bar{x} \cdot t]\} \in A_4$

If of degree $i \quad j \quad k$

\hookrightarrow degree $i+j+k+1$ homology

\hookrightarrow degree $i+j+k-1$ cohomology

Construction: (truncated) minimal free resolution of \mathbb{K} over R by using K^R
 [N-Veliche]

$$F_0 : \dots \rightarrow F_5 \xrightarrow{\partial_5^F} F_4 \xrightarrow{\partial_4^F} F_3 \xrightarrow{\partial_3^F} F_2 \xrightarrow{\partial_2^F} F_1 \xrightarrow{\partial_1^F} F_0$$

$$F_0 := K_0$$

$$F_1 := K_1$$

$$F_2 := K_2 \oplus K_0^{a_1} = \underbrace{K_0 \oplus \dots \oplus K_0}_{a_1 \text{ times}}$$

$$F_3 := K_3 \oplus K_1^{a_1} \oplus K_0^{a_2 - q_{11}}$$

$$F_4 := K_4 \oplus K_2^{a_1} \oplus K_1^{a_2 - q_{11}} \oplus K_0^{a_3 - q_{12}} \oplus K_0^{a_1^2 - q_{11}}$$

$$F_5 := K_5 \oplus K_3^{a_1} \oplus K_2^{a_2 - q_{11}} \oplus K_1^{a_3 - q_{12}} \oplus K_1^{a_1^2 - q_{11}} \oplus K_0^{a_4 - a}$$

$$\oplus K_0^{a_1 a_2 - a_1 q_{11} - q_{12} + b} \oplus K_0^{a_1 a_2 - a_1 q_{11}}$$

[Fibonacci sequence?]

• Describe ∂_i^F explicitly, $1 \leq i \leq 5$

• Show exactness $\forall 1 \leq i \leq 5$

III Applications:

① Measure Golodness:

$$P_{\mathbb{K}}^R(t) := \sum_{i=0}^{\infty} \beta_i t^i \leq \frac{(1+t)^n}{1 - \sum_{i=1}^c a_i t^{i+1}} = \sum_{i=0}^{\infty} b_i t^i$$

Define sequence $\hat{\beta}_i := b_i - \beta_i$, $\forall i \geq 0$

- $\hat{\beta}_i \geq 0, \forall i$
 - R is Golod $\Leftrightarrow \hat{\beta}_i = 0, \forall i$
- $\left. \right\}$ sequence $\{\hat{\beta}_i\}_{i \geq 0}$ measures how far the ring R from being Golod

→ Question: $q_{ij} = 0, \forall i, j \geq 1 \stackrel{??}{\implies} \phi_i = 0, \forall i \geq 0?$ (6)

↪ TRUE if codepth $R \leq 3$ [Nagata '1962 - codepth 1, hypersurface]

[Scheja '1964 - codepth 2 ↗ Golod (is NOT c.i.)
complete intersection]

[Avramov '2012 - codepth 3, classification]

↪ FALSE if codepth $R \geq 4$ [Kath  n '2017, Roos '2017, counterexamples]

ex: $k[x, y, z, u, w]/(xy^2, zu^2, w^3, xyzu, u^2w^2, y^2w^2, xzw, y^2u^2w)$

$k[x, y, z, w]/(w^3, x^2w, xz^2 + yz^2, xy^2, x^2y + y^2w, y^2z + z^2w)$

→ Both have $\phi = (0, 0, 0, 0, 0, 1, \dots)$

deg 0 1 2 3 4 5 ↘

What does ϕ_5 mean here?

• Corollary 1: Describe the Betti #'s β_i in terms of n, a_i, q_{ij}, b, a
[N-Veliche]

$\forall 0 \leq i \leq 5$

• Corollary 2: Express ϕ_i explicitly $\underline{\hspace{10cm}} \underline{\hspace{10cm}}$
[N-Veliche]

$$\phi_0 = \phi_1 = \phi_2 = 0, \quad \phi_3 = q_{11}, \quad \phi_4 = (n+1)q_{11} + q_{12},$$

$$\phi_5 = \left[\binom{n+1}{2} + 2a_1 \right] q_{11} + (n+1)q_{12} - b + a, \quad \left(\begin{array}{l} \text{if } q_{11} = q_{12} = q_{13} = q_{22} = 0, \text{ then} \\ \phi_0 = \dots = \phi_4 = 0 \text{ and } \phi_5 = |A_1, A_1, A_1| \end{array} \right)$$

$\phi_{\geq 6} = ??$ [Conj: involve higher order Massey operations]

$$(\text{rational}) \quad P_{IK}^R(t) := \sum_{i=1}^{\infty} \beta_i t^i$$

- ② Describe the Poincaré series for special cases of R
- ↳ describe deviations $\varepsilon_i, 1 \leq i \leq 5$

codepth ≤ 3

(7)

codepth 4 + Gorenstein
"Yoshino" algebras

- ③ Illustrate the construction through an example [Avramov]

$$R = \mathbb{Q}[x, y, z, w] / (x^3, y^3, z^3 - xy^2, x^2z^2, xyz^2, y^2w, w^2)$$

- $n=c=4$
- $A = H_*(K^R)$ has nontrivial multiplications & Massey operations
- Compute all $a_{ij}, q_{ij}, a, b, f_i$

- ④ Hypersurfaces: $R = \mathbb{Q}/(f)$, \mathbb{Q} regular local ring of edim = n

[N-Veliche]

$$A = H_*(K^R) = \mathbb{k} \oplus A_1 \quad \text{and} \quad A_1 = \langle [z^1] \rangle \cong \mathbb{k}$$

Define $\xi : K \rightarrow \sum' K$ given by $\xi_i : K_i \rightarrow K_{i+1}$ via $\xi_i(\pi^i) = z^i \wedge \pi^i$

Let $F_i := K_i \oplus K_{i-2} \oplus K_{i-4} \oplus \dots \quad \forall i \geq 0$

$$\partial_i^F := \begin{pmatrix} \partial_i^K & \xi_{i-2} & 0 & 0 & \dots \\ & \partial_{i-2}^K & \xi_{i-4} & 0 & \dots \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

$$\begin{array}{ccc} F_i & & F_{i-1} \\ = & & = \\ K_i & \xrightarrow{\partial_i^K} & K_{i-1} \\ \oplus & \nearrow \xi_{i-2} & \oplus \\ K_{i-2} & \xrightarrow{\partial_{i-2}^K} & K_{i-3} \\ \oplus & \nearrow \xi_{i-4} & \oplus \\ K_{i-4} & \xrightarrow{\partial_{i-4}^K} & K_{i-5} \\ \oplus & \vdots & \vdots \end{array}$$

$$\text{Then } F_* : \dots \rightarrow F_i \xrightarrow{\partial_i^F} F_{i-1} \rightarrow \dots \rightarrow F_0 \rightarrow 0$$

is a minimal free resolution of \mathbb{k} over R .

Ongoing / Future work: - Extend this construction $\forall i$ (special cases of R)

THANK YOU! 😊