

When a weak bialgebra is a weak Hopf algebra

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- 1 Weak bialgebras and weak Hopf algebras
- 2 Minimal weak bialgebras
- 3 The antipode of a weak Hopf algebra

k —a base field of characteristic zero.

A group algebra kG is a Hopf algebra.

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- Is $kG \oplus kG$ still a Hopf algebra?

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A group algebra kG is a Hopf algebra.

- Is $kG \oplus kG$ still a Hopf algebra?
- **No!!** But $kG \oplus kG$ is a **weak Hopf algebra**.

Hopf algebras vs weak Hopf algebras

Definition

A *bialgebra* H is an algebra and a coalgebra such that Δ and ε are algebra maps. In addition, if H has an antipode S , then H is a *Hopf algebra*.

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A weak Hopf algebra is a generalization of a Hopf algebra.

Hopf algebras vs weak Hopf algebras

Definition

A *weak bialgebra* H is an algebra and a coalgebra such that

$$\Delta(ab) = \Delta(a)\Delta(b)$$

$$\varepsilon(abc) = \varepsilon(ab_1)\varepsilon(b_2c) = \varepsilon(ab_2)\varepsilon(b_1c)$$

$$\Delta^2(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1)$$

for $a, b, c \in H$. Further, a k -linear map $S : H \rightarrow H$ is called *antipode* if S satisfies the axioms

$$\textcircled{1} \quad S(h_1)h_2 = 1_1\varepsilon(h_2)$$

$$\textcircled{2} \quad h_1S(h_2) = \varepsilon(h_1)1_2$$

$$\textcircled{3} \quad S(h_1)h_2S(h_3) = S(h)$$

for any $h \in H$. If H has an antipode S , then H is a *weak Hopf algebra*.

Using the sumless Sweedler notation $\Delta(b) = b_1 \otimes b_2$.

$kG_1 \oplus kG_2$ is a weak Hopf algebra

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- $\Delta(g_1 + g_2) = g_1 \otimes g_1 + g_2 \otimes g_2$
- $\varepsilon(g_1 + g_2) = \varepsilon(g_1) + \varepsilon(g_2)$
- $1_H = 1_{kG_1} + 1_{kG_2}$ (H has a “complicated” identity.)
- $S(g_1 + g_2) = g_1^{-1} + g_2^{-1}$

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Look:

$$\Delta(1_H) = 1_{kG_1} \otimes 1_{kG_1} + 1_{kG_2} \otimes 1_{kG_2} \neq 1_H \otimes 1_H = (1_{kG_1} + 1_{kG_2}) \otimes (1_{kG_1} + 1_{kG_2})$$

$$\varepsilon(gh) = \varepsilon(g1_{kG_1})\varepsilon(1_{kG_1}h) + \varepsilon(g1_{kG_2})\varepsilon(1_{kG_2}h) \neq \varepsilon(g)\varepsilon(h), \text{ for } g, h \in G_1 \cup G_2.$$

Hopf algebras vs weak Hopf algebras

$$\begin{aligned}\Delta(1_H) = 1_H \otimes 1_H &\xRightarrow{\text{weaker}} \Delta(1_H) = 1_1 \otimes 1_2 \neq 1_H \otimes 1_H \\ \Delta^2(1_H) &= (1_H \otimes \Delta(1_H))(\Delta(1_H) \otimes 1_H) \\ &= (\Delta(1_H) \otimes 1_H)(1_H \otimes \Delta(1_H))\end{aligned}$$

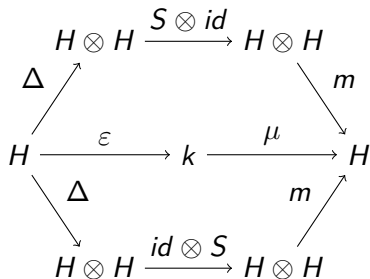
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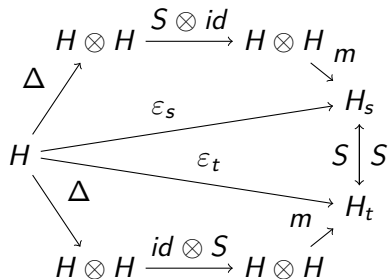
$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b) \xRightarrow{\text{weaker}} \varepsilon(ab) = \varepsilon(a1_1)\varepsilon(1_2b)$$

Hopf algebras vs weak Hopf algebras

Intuitively:



Hopf algebra

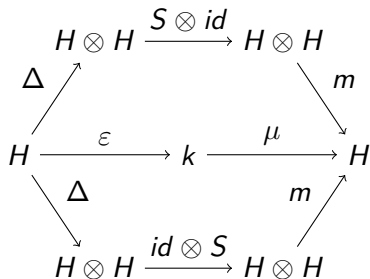


weak Hopf algebra

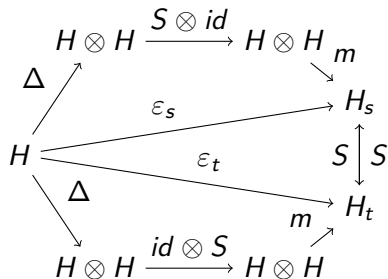
$$H_s := \varepsilon_s(H) = m(S \otimes id)\Delta(H) \quad H_t := \varepsilon_t(H) = m(id \otimes S)\Delta(H)$$

Hopf algebras vs weak Hopf algebras

Intuitively:



Hopf algebra



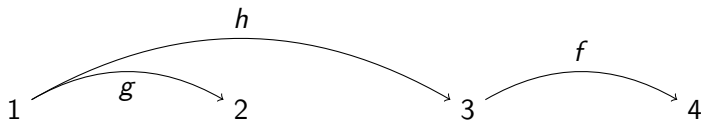
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$$H_s := \varepsilon_s(H) = m(S \otimes id)\Delta(H) \quad H_t := \varepsilon_t(H) = m(id \otimes S)\Delta(H)$$

A weak Hopf algebra H is a Hopf algebra iff $H_t = H_s = k1_H$ iff $\Delta(1_H) = 1_H \otimes 1_H$.

Examples

With the following quiver Q .



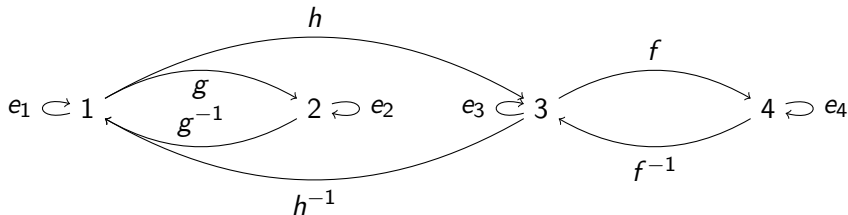
Path algebra of a quiver $Q = (Q_0, Q_1)$ where $Q_0 = \{1, 2, 3, 4\}$ and $Q_1 = \{g, f, h\}$.

Algebra: $1_{kQ} = \sum_{i=1}^4 e_i$, where e_i is the trivial path on i .

Coalgebra: $\Delta(x) = x \otimes x$, $\Delta(e_i) = e_i \otimes e_i$, $\varepsilon(x) = 1$ and $\varepsilon(e_i) = 1$ for all $x \in Q_1$.

Examples

With the following quiver Q .



Groupoid algebra:

$$kG = \frac{kQ}{\langle gg^{-1} - e_2, g^{-1}g - e_1, ff^{-1} - e_4, f^{-1}f - e_3, hh^{-1} - e_3, h^{-1}h - e_1 \rangle}$$

as a quotient weak bialgebra.

Antipode: $S(x) = x^{-1}$ and $S(e_i) = e_i$

for all $x \in \{g, g^{-1}, f, f^{-1}, h, h^{-1}\}$ and $1 \leq i \leq 4$.

Hayashi's face algebra attached to a quiver

For a finite quiver $Q = (Q_0, Q_1)$, $\mathfrak{H}(Q)$ is a weak bialgebra. As a k -algebra,

$$\mathfrak{H}(Q) = \frac{k \langle x_{i,j}, x_{p,q} \mid i, j \in Q_0, p, q \in Q_1 \rangle}{(R)},$$

for indeterminates $x_{i,j}$ and $x_{p,q}$ subject to relations R , given by:

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for indeterminates $x_{i,j}$ and $x_{p,q}$ subject to relations R , given by:

$$\left\{ \begin{array}{l} x_{i,j} x_{k,\ell} = \delta_{i,k} \delta_{j,\ell} x_{i,j} \\ x_{s(p),s(q)} x_{p,q} = x_{p,q} = x_{p,q} x_{t(p),t(q)} \\ x_{p,q} x_{p',q'} = \delta_{t(p),s(p')} \delta_{t(q),s(q')} x_{p,q} x_{p',q'} \end{array} \right.$$

for all $p, p', q, q' \in Q_1$, and $i, j, k, \ell \in Q_0$.

$$1_{\mathfrak{H}(Q)} = \sum_{i,j \in Q_0} x_{i,j}.$$

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for all $p, p', q, q' \in Q_1$, and $i, j, k, \ell \in Q_0$.

$$1_{\mathfrak{H}(Q)} = \sum_{i,j \in Q_0} x_{i,j}.$$

For $a, b \in Q_\ell$, the coalgebra structure is given by

$$\Delta(x_{a,b}) = \sum_{c \in Q_\ell} x_{a,c} \otimes x_{c,b} \quad \text{and} \quad \varepsilon(x_{a,b}) = \delta_{a,b}.$$

Applications of weak Hopf algebras

- 1 Weak Hopf algebras give a description of arbitrary finite index and finite depth II_1 subfactors via a Galois correspondence.
- 2 There is a closed connection between weak Hopf algebras and Von Neumann subfactors.
- 3 Weak Hopf algebras play an important role in the theory of dynamical deformation of Hopf algebras.

Minimal weak bialgebras

Recall that $\Delta(1_H) = \sum 1_1 \otimes 1_2 \neq 1_H \otimes 1_H$ for a given weak bialgebra H .

Definition

A weak sub-bialgebra of H is called *minimal* if it has no proper weak sub-bialgebras, denoted by H_{\min} .

Question: Is H_{\min} possible a weak Hopf algebra?

Yes !

Minimal weak bialgebras

D. Nikshych showed that $H_{\min} = H_s H_t \cong H_s \otimes_{H_t \cap H_s} H_t$.

The following facts are important to help define the antipode of H_{\min} .

- Denote $\Delta(1) = \sum_{i=1}^n w_i \otimes z_i$ (the shortest possible representation).
- H_t is generated by z_1, \dots, z_n and H_s is generated by w_1, \dots, w_n .
- H_s and H_t commute with each other.
- $\varepsilon_s|_{H_t} : H_t \longrightarrow H_s$ and $\varepsilon_t|_{H_s} : H_s \longrightarrow H_t$ are two anti-algebra maps.

Proposition (Wang-Zhang-H)

The minimal weak bialgebra $H_{\min} = H_s H_t \cong H_s \otimes_{H_t \cap H_s} H_t$ is a weak Hopf algebra. A k -linear map S given by

$$\begin{cases} S(w_i) = \varepsilon_t(w_i) \\ S(z_i) = \varepsilon_s(z_i) \end{cases}$$

and extended by $S(w_i z_j) = S(w_i) S(z_j)$ is an antipode for $1 \leq i, j \leq n$.

Remark: Böhm, Nill and Szlachányi showed that if a weak bialgebra has an antipode, then it is unique.

Then for any weak bialgebra H , H_{\min} has a unique antipode S that is given in the previous Proposition.

Minimal weak Hopf algebras

Let $k = \bar{k}$ and $\text{char}(k) = 0$.

Proposition (Nikshych)

Every H_{\min} is completely determined by: (B, A, g) , where B is a f. dim semisimple algebra, $A \subseteq Z(B)$ is a com. subalgebra as follows. As an algebra,

$$H_{\min} = \langle B, B^{\text{op}} \rangle \cong B \otimes_A B^{\text{op}},$$

subject to $b\bar{c} = \bar{c}b, \bar{a} = a$ with $B^{\text{op}} = \{\bar{x} \mid x \in B\}$ for $a \in A$ and $b, c \in B$. As a coalgebra

$$\Delta(b\bar{c}) = \sum b(\overline{ge_1}) \otimes e_2\bar{c}, \quad \varepsilon(b\bar{c}) = \text{Tr}_{\text{reg}}(g^{-1}cb),$$

where $e = \sum e_1 \otimes e_2$ is the unique 2-sided separability element of B and Tr_{reg} is the trace of the regular representation of B and $g \in B$ is a unit s.t $\sum e_1ge_2 = 1$, and the antipode is given by

$$S(b\bar{c}) = g^{-1}cg\bar{b}.$$

Minimal weak Hopf algebras

- ① An element $e \in A \otimes A$ in an algebra A is called a *separability* element if

$$(a \otimes 1)e = e(1 \otimes a)$$

for all $a \in A$ and $m(e) = 1$, where m is the product in A .

- ② A separability element e is 2-sided if $e(a \otimes 1) = (1 \otimes a)e$; if such e exists, it is unique.
- ③ An algebra A having a separable element is called *separable*
- ④ A separable algebra over a field k is finite dimensional. If $\text{char}(k) = 0$, then A is separable if and only if it is semisimple. In this case, A has a 2-sided separability element.

Minimal weak Hopf algebras

e.g. $H_s = M_2(k)$ and $H_t = M_2(k)^{\text{op}}$

$e = \frac{1}{2} \sum_{i,j=1}^2 e_{i,j} \otimes e_{j,i}$ is the 2-sided separability element in H_s and Tr_{reg} is the trace map.

For $n \neq 2 \in k$

$$g = \begin{pmatrix} n & * \\ 0 & 2 - n \end{pmatrix},$$

then $\frac{1}{2} \sum_{i,j=1}^2 e_{i,j} g e_{j,i} = 1$ and one can define non-isomorphic minimal weak Hopf algebras $H_{min} \cong M_4(k)$ for different g .

Question: how many possible minimal weak Hopf algebra structures can be defined on a semisimple algebra B ?

Examples of minimal weak bialgebras

- ① Groupoid algebra $H = kG$:

$$H_s = H_t = H_{\min} = \bigoplus_{x \in G_0} ke_x.$$

- ② Path algebra $H = kQ$:

$$H_s = H_t = H_{\min} = \bigoplus_{x \in Q_0} ke_x.$$

- ③ Hayashi face algebra $H = \mathfrak{H}(Q)$ with $|Q_0| = n$:

$$H_s = \bigoplus_{i=1}^n k\varepsilon_s(x_{i,i}), \quad H_t = \bigoplus_{i=1}^n k\varepsilon_t(x_{i,i}).$$

Here $H_t \cap H_s = k1_H$ and $H_{\min} = \mathfrak{H}(Q_0) = H_s \otimes_k H_t$.

Minimal weak Hopf algebras

Recall that a direct sum of weak Hopf algebras is still a weak Hopf algebra. So we try to focus on the *indecomposable* weak Hopf algebra H , i.e., $H \neq H_1 \oplus H_2$ where H_1 and H_2 are weak Hopf algebras.

Definition

A weak Hopf algebra H is *coconnected* if and only if $H_s \cap H_t = k$.

Remark: A is indecomposable if $H_t \cap H_s \cap Z(H) = k1$. So H_{\min} is coconnected if and only if it is indecomposable due to

$$H_t \cap H_s \cap Z(H_{\min}) = H_s \cap H_t.$$

Proposition (Wang-Zhang-H)

If a commutative H_{\min} is coconnected (indemcomposable), then $H_{\min} \cong \mathfrak{H}(Q_0)$ attached some arrowless quiver Q_0 .

Remark: If $|Q_0| = 1$, then $H = \mathfrak{H}(Q_0) = k1_H$.

Theorem (Wang-Zhang-H)

A commutative H_{\min} is isomorphic to a direct sum of some Hayashi face algebras $\mathfrak{H}(Q_0)$ attached some arrowless quivers.

Noncommutative minimal weak bialgebra

However, if H_{\min} is noncommutative, then it could be very complicated.

When is a weak bialgebra a weak Hopf algebra?

Now we know that every minimal weak bialgebra is a minimal weak Hopf algebra.

Question: When is a general weak bialgebra a weak Hopf algebra?

Generalized inverse in the convolution algebra

In a Hopf algebra H , id is the inverse of S in the convolution algebra $\text{Hom}_k(H, k)$, i.e., $id * S = S * id = \varepsilon$.

Definition (Hayashi)

Let x^\pm and e^\pm be four elements of an algebra A . We say x^- is an (e^+, e^-) -generalized inverse of x^+ if

$$\begin{aligned}x^- x^+ &= e^+, & x^+ x^- &= e^- \\ x^+ x^- x^+ &= x^+, & x^- x^+ x^- &= x^-. \end{aligned}$$

Remark: we have the following facts:

- ① If x^- is an (e^+, e^-) -generalized inverse of x^+ , then x^+ is an (e^-, e^+) -generalized inverse of x^- .
- ② The (e^+, e^-) -generalized inverse is unique if exists.
- ③ For a weak Hopf algebra H , $S : H \longrightarrow H$ is an antipode if S is the $(\varepsilon_s, \varepsilon_t)$ -generalized inverse of id .

The antipode of a weak Hopf algebra

Denote H_0 the *coradical* coalgebra of a weak bialgebra H .

Theorem (Wang-Zhang-H)

Let H be a weak bialgebra. Suppose that the coradical H_0 is a weak sub-bialgebra of H . Then H_0 is a weak Hopf subalgebra if and only if H is a weak Hopf algebra.

Idea

\Leftarrow : $S(H_0) \subseteq H_0$ and so $S|_{H_0}$ is the $(\varepsilon_s, \varepsilon_t)$ -generalized invese of id .

\Rightarrow : Let S_0 be the antipode of H_0 . Extending S_0 to be a map $g : H \rightarrow H$. Then show that $g * (\varepsilon_t + \sum_{n \geq 1} (\varepsilon_t - id * g)^n)$ is the $(\varepsilon_s, \varepsilon_t)$ -generalized inverse of id .

The antipode of a weak Hopf algebra

Corollary (Wang-Zhang-H)

Let H be a weak bialgebra and suppose that B is a weak Hopf subalgebra. If H_0 is a weak sub-bialgebra of B , then H is a weak Hopf algebra.

Remark: Any f.dim sub-bialgebra of a Hopf algebra is a Hopf sub-algebra.

Theorem (Schauenburg, 2002)

Let H be a weak Hopf algebra, and B a f.dim weak sub-bialgebra of H . TFAE:

- ① *B is a weak Hopf sub-algebra.*
- ② *The right H_s -module B is isomorphic to the H_t -module B obtained by restricting the left H_t -module B along ε_t .*

The antipode of a weak Hopf algebra

Theorem (Wang-Zhang-H)

Let H be a pointed weak bialgebra. Then H is a weak Hopf algebra if and only if H_0 is a groupoid algebra.

The antipode of a weak Hopf algebra

As a consequence, we have

Proposition (Wang-Zhang-H)

If H is a cocommutative weak bialgebra over an algebraically closed field k , then H_0 is a groupoid algebra if and only if H is a weak Hopf algebra.

The antipode of a weak Hopf algebra

A path algebra kQ is a cocommutative and pointed weak bialgebra. As a consequence, we have

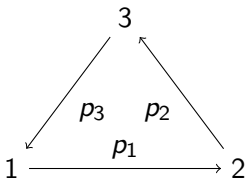
Corollary (Wang-Zhang-H)

- 1 *A path algebra kQ of a quiver $Q = (Q_0, Q_1)$ with both $|Q_1|$ and $|Q_0|$ greater than one is not a weak Hopf algebra.*
- 2 *The quotient kQ/I is a weak Hopf algebra if and only if kQ/I is a groupoid algebra, where I is a bi-ideal of kQ .*

Hayashi face algebras

$\mathfrak{H}(Q)$ is not a weak Hopf algebra structure in general. But it is possible to construct a weak Hopf algebra on a Hayashi face algebra $\mathfrak{H}(Q)$.

Example: Let Q be the quiver as below:



$$K = \frac{\langle \mathfrak{H}(Q), \mathfrak{H}(Q^{\text{op}}) \rangle}{\left\langle x_{i,j} - y_{i,j} \mid x_{i,j} \in \mathfrak{H}(Q)_0, y_{i,j} \in \mathfrak{H}(Q^{\text{op}})_0 \right\rangle}$$

is a weak bialgebra, where kQ^{op} is the *opposite* algebra of kQ by reserving of all of the arrows in kQ .

We will show that

$$H = K/R$$

is a weak Hopf algebra, where

$$R = \left\langle \begin{array}{l} x_{p,q'} x_{p^*,q^*} - \delta_{q',q} x_{s(p),s(q)} \\ x_{q,p} x_{(q')^*,p^*} - \delta_{q',p} x_{s(q),s(p)} \\ x_{p^*,q^*} x_{p',q} - \delta_{p,p'} x_{t(p),t(q)} \\ x_{q^*,(p')^*} x_{q,p} - \delta_{p,p'} x_{t(q),t(p)} \end{array} \middle| p, q, p', q' \in Q_l \text{ for each } l \in \mathbb{N} \right\rangle$$

and with antipode as below

$$\begin{cases} S(x_{p_i,p_j}) = x_{p_j^*,p_i^*} & \text{for } i,j \in Q_0 \\ S(x_{p_j^*,p_i^*}) = x_{p_i,p_j} & \text{for } i,j \in Q_0 \\ S(x_{i,j}) = x_{j,i} & \text{for } i,j \in Q_0. \end{cases}$$

Future directions

- 1 Is an Artinian weak Hopf algebra finite-dimensional ?

Future directions

- ① Is an Artinian weak Hopf algebra finite-dimensional ?
- ② Construct a weak Hopf algebra from a general weak bialgebra.

Future directions

- ① Is an Artinian weak Hopf algebra finite-dimensional ?
- ② Construct a weak Hopf algebra from a general weak bialgebra.
- ③ Investigate the classification of finite-dimensional semisimple weak Hopf algebras.

Thank You!