When a weak bialgebra is a weak Hopf algebra

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- Is $kG \oplus kG$ still a Hopf algebra?
- No!! But $kG \oplus kG$ is a weak Hopf algebra.

Definition

A *bialgebra* H is an algebra and a coalgebra such that Δ and ε are algebra maps. In addition, if H has an antipode S, then H is a *Hopf algebra*.

Hopf algebras vs weak Hopf algebras

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A weak Hopf algebra is a generalization of a Hopf algebra.

Hopf algebras vs weak Hopf algebras

Definition

A *weak bialgebra* H is an algebra and a coalgebra such that

$$\Delta(ab) = \Delta(a)\Delta(b)$$
$$\varepsilon(abc) = \varepsilon(ab_1)\varepsilon(b_2c) = \varepsilon(ab_2)\varepsilon(b_1c)$$
$$\Delta^2(1) = (\Delta(1)\otimes 1)(1\otimes\Delta(1)) = (1\otimes\Delta(1))(\Delta(1)\otimes 1)$$

for $a, b, c \in H$. Further, a k-linear map $S : H \longrightarrow H$ is called *antipode* if S satisfies the axioms

1. $S(h_1)h_2 = 1_1\varepsilon(h1_2)$
2. $h_1S(h_2) = \varepsilon(1_1h)1_2$
3. $S(h_1)h_2S(h_3) = S(h)$

for any $h \in H$. If H has an antipode S, then H is a *weak Hopf algebra*.

Using the sumless Sweedler notation $\Delta(b) = b_1 \otimes b_2$.

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- $\Delta(g_1 + g_2) = g_1 \otimes g_1 + g_2 \otimes g_2$
- $\varepsilon(g_1 + g_2) = \varepsilon(g_1) + \varepsilon(g_2)$
- $1_H = 1_{kG_1} + 1_{kG_2}$ (H has a “complicated” identity.)
- $S(g_1 + g_2) = g_1^{-1} + g_2^{-1}$

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Look:

$$\Delta(1_H) = 1_{kG_1} \otimes 1_{kG_1} + 1_{kG_2} \otimes 1_{kG_2} \neq 1_H \otimes 1_H = (1_{kG_1} + 1_{kG_2}) \otimes (1_{kG_1} + 1_{kG_2})$$

$$\varepsilon(gh) = \varepsilon(g 1_{kG_1})\varepsilon(1_{kG_1} h) + \varepsilon(g 1_{kG_2})\varepsilon(1_{kG_2} h) \neq \varepsilon(g)\varepsilon(h), \text{ for } g, h \in G_1 \cup G_2.$$

Hopf algebras vs weak Hopf algebras

$$\Delta(1_H) = 1_H \otimes 1_H \quad \overset{\text{weaker}}{\Longrightarrow} \quad \begin{aligned} \Delta(1_H) &= 1_1 \otimes 1_2 \neq 1_H \otimes 1_H \\ \Delta^2(1_H) &= (1_H \otimes \Delta(1_H))(\Delta(1_H) \otimes 1_H) \\ &= (\Delta(1_H) \otimes 1_H)(1_H \otimes \Delta(1_H)) \end{aligned}$$

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$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b) \quad \overset{\text{weaker}}{\Longrightarrow} \quad \varepsilon(ab) = \varepsilon(a1_1)\varepsilon(1_2 b)$$

Intuitively:

$$\begin{tikzcd}
 & H \otimes H \arrow[rr, "S \otimes id"] & & H \otimes H \arrow[dr, "m"] & \\
H \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, "\varepsilon"] & & k \arrow[rr, "\mu"] & & H \\
 & H \otimes H \arrow[rr, "id \otimes S"] & & H \otimes H \arrow[ur, "m"'] &
\end{tikzcd}$$

Hopf algebra

$$\begin{tikzcd}
 & H \otimes H \arrow[r, "S \otimes id"] & H \otimes H \arrow[dr, "m"] & \\
 & & & H_s \arrow[dd, shift left, "S"] \\
H \arrow[uur, "\Delta"] \arrow[ddr, "\Delta"'] \arrow[urrr, "\varepsilon_s"] \arrow[drrr, "\varepsilon_t"'] & & & \\
 & & & H_t \arrow[uu, shift left, "S"] \\
 & H \otimes H \arrow[r, "id \otimes S"] & H \otimes H \arrow[ur, "m"] &
\end{tikzcd}$$

weak Hopf algebra

$$H_s := \varepsilon_s(H) = m(S \otimes id)\Delta(H) \qquad H_t := \varepsilon_t(H) = m(id \otimes S)\Delta(H)$$

Hopf algebras vs weak Hopf algebras

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 & & & H_s \arrow[d, shift left, "S"] \\
H \arrow[uur, "\Delta"] \arrow[urrr, "\varepsilon_s"] \arrow[drrr, "\varepsilon_t"] \arrow[ddr, "\Delta"'] & & & \\
 & & & H_t \arrow[u, shift left, "S"] \\
 & H \otimes H \arrow[r, "id \otimes S"] & H \otimes H \arrow[ur, "m"] &
\end{tikzcd}$$

weak Hopf algebra

$$H_s := \varepsilon_s(H) = m(S \otimes id)\Delta(H) \qquad H_t := \varepsilon_t(H) = m(id \otimes S)\Delta(H)$$

A weak Hopf algebra H is a Hopf algebra iff $H_t = H_s = k1_H$ iff $\Delta(1_H) = 1_H \otimes 1_H$.

Examples

With the following quiver Q.

$$\begin{tikzcd} 1 \arrow[r, bend left, "g"] \arrow[rr, bend left, "h"] & 2 & 3 \arrow[r, bend left, "f"] & 4 \end{tikzcd}$$

Path algebra of a quiver $Q = (Q_0, Q_1)$ where $Q_0 = \{1, 2, 3, 4\}$ and $Q_1 = \{g, f, h\}$.

Algebra: $1_{kQ} = \sum_{i=1}^{4} e_i$, where e_i is the trivial path on i.

Coalgebra: $\Delta(x) = x \otimes x$, $\Delta(e_i) = e_i \otimes e_i$, $\varepsilon(x) = 1$ and $\varepsilon(e_i) = 1$

for all $x \in Q_1$.

Examples

With the following quiver Q.

$$
\begin{tikzcd}
1 \arrow[loop left, "e_1"] \arrow[r, bend left, "g"] \arrow[rr, bend left=40, "h"] & 2 \arrow[l, bend left, "g^{-1}"] \arrow[loop right, "e_2"] & 3 \arrow[loop left, "e_3"] \arrow[ll, bend left=40, "h^{-1}"] \arrow[r, bend left, "f"] & 4 \arrow[l, bend left, "f^{-1}"] \arrow[loop right, "e_4"]
\end{tikzcd}
$$

Groupoid algebra:

$$kG = \frac{kQ}{\langle gg^{-1} - e_2,\ g^{-1}g - e_1,\ ff^{-1} - e_4, f^{-1}f - e_3, hh^{-1} - e_3, h^{-1}h - e_1\rangle}$$

as a quotient weak bialgebra.

Antipode: $S(x) = x^{-1}$ and $S(e_i) = e_i$

for all $x \in \{g, g^{-1}, f, f^{-1}, h, h^{-1}\}$ and $1 \leq i \leq 4$.

Hayashi's face algebra attached to a quiver

For a finite quiver $Q = (Q_0, Q_1)$, $\mathfrak{H}(Q)$ is a weak bialgebra. As a k-algebra,

$$\mathfrak{H}(Q) = \frac{k \langle x_{i,j}, x_{p,q} \mid i, j \in Q_0,\ p, q \in Q_1 \rangle}{(R)},$$

for indeterminates $x_{i,j}$ and $x_{p,q}$ subject to relations R, given by:

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$$\begin{cases} x_{i,j}\ x_{k,\ell} = \delta_{i,k}\delta_{j,\ell}\ x_{i,j} \\ x_{s(p),s(q)}\ x_{p,q} = x_{p,q} = x_{p,q}\ x_{t(p),t(q)} \\ x_{p,q}\ x_{p',q'} = \delta_{t(p),s(p')}\delta_{t(q),s(q')} x_{p,q}\ x_{p',q'} \end{cases}$$

for all $p, p', q, q' \in Q_1$, and $i, j, k, \ell \in Q_0$.

$$1_{\mathfrak{H}(Q)} = \sum_{i,j \in Q_0} x_{i,j}.$$

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for all $p, p', q, q' \in Q_1$, and $i, j, k, \ell \in Q_0$.

$$1_{\mathfrak{H}(Q)} = \textstyle\sum_{i,j \in Q_0} x_{i,j}.$$

For $a, b \in Q_\ell$, the coalgebra structure is given by

$$\Delta(x_{a,b}) = \sum_{c \in Q_\ell} x_{a,c} \otimes x_{c,b} \quad \text{and} \quad \varepsilon(x_{a,b}) = \delta_{a,b}.$$

Applications of weak Hopf algebras

1. Weak Hopf algebras give a description of arbitrary finite index and finite depth II_1 subfactors via a Galois correspondence.
2. There is a closed connection between weak Hopf algebras and Von Neumann subfactors.
3. Weak Hopf algebras play an important role in the theory of dynamical deformation of Hopf algebras.

Minimal weak bialgebras

Recall that $\Delta(1_H) = \sum 1_1 \otimes 1_2 \neq 1_H \otimes 1_H$ for a given weak bialgebra H.

Definition

A weak sub-bialgebra of H is called *minimal* if it has no proper weak sub-bialgebras, denoted by H_{min}.

Question: Is H_{min} possible a weak Hopf algebra?

Yes !

Minimal weak bialgebras

D. Nikshych showed that $H_{\min} = H_s H_t \cong H_s \otimes_{H_t \cap H_s} H_t$.

The following facts are important to help define the antipode of $H_{\min}$.

- Denote $\Delta(1) = \sum_{i=1}^{n} w_i \otimes z_i$ (the shortest possible representation).
- H_t is generated by $z_1, \cdots, z_n$ and H_s is generated by $w_1, \cdots, w_n$.
- H_s and H_t commute with each other.
- $\varepsilon_s|_{H_t} : H_t \longrightarrow H_s$ and $\varepsilon_t|_{H_s} : H_s \longrightarrow H_t$ are two anti-algebra maps.

Minimal weak bialgebras

Proposition (Wang-Zhang-H)

The minimal weak bialgebra $H_{\min} = H_s H_t \cong H_s \otimes_{H_t \cap H_s} H_t$ *is a weak Hopf algebra. A k-linear map S given by*

$$\begin{cases} S(w_i) = \varepsilon_t(w_i) \\ S(z_i) = \varepsilon_s(z_i) \end{cases}$$

and extended by $S(w_i z_j) = S(w_i)S(z_j)$ *is an antipode for* $1 \leq i, j \leq n$.

Remark: Böhm, Nill and Szlachányi showed that if a weak bialgebra has an antipode, then it is unique.

Then for any weak bialgebra H, H_{min} has a unique antipode S that is given in the previous Proposition.

Minimal weak Hopf algebras

Let $k = \overline{k}$ and $\mathrm{char}(k) = 0$.

Proposition (Nikshych)

Every H_{min} is completely determined by: (B, A, g), where B is a f. dim semisimple algebra, $A \subseteq Z(B)$ is a com. subalgebra as follows. As an algebra,

$$H_{\mathrm{min}} = \langle B, B^{\mathrm{op}} \rangle \cong B \otimes_A B^{\mathrm{op}},$$

subject to $b\overline{c} = \overline{c}b, \overline{a} = a$ with $B^{\mathrm{op}} = \{\overline{x} \mid x \in B\}$ for $a \in A$ and $b, c \in B$. As a coalgebra

$$\Delta(b\overline{c}) = \sum b(\overline{ge_1}) \otimes e_2\overline{c}, \quad \varepsilon(b\overline{c}) = Tr_{reg}(g^{-1}cb),$$

where $e = \sum e_1 \otimes e_2$ is the unique 2-sided separability element of B and Tr_{reg} is the trace of the regular representation of B and $g \in B$ is a unit s.t $\sum e_1 g e_2 = 1$, and the antipode is given by

$$S(b\overline{c}) = g^{-1}cg\overline{b}.$$

Minimal weak Hopf algebras

1. An element $e \in A \otimes A$ in an algebra A is called a *separability* element if
$$(a \otimes 1)e = e(1 \otimes a)$$
for all $a \in A$ and $m(e) = 1$, where m is the product in A.
2. A separability element e is 2-sided if $e(a \otimes 1) = (1 \otimes a)e$; if such e exists, it is unique.
3. An algebra A having a separable element is called *separable*
4. A separable algebra over a field k is finite dimensional. If $\mathrm{char}(k) = 0$, then A is separable if and only if it is semisimple. In this case, A has a 2-sided separability element.

Minimal weak Hopf algebras

e.g. $H_s = M_2(k)$ and $H_t = M_2(k)^{\mathrm{op}}$

$e = \frac{1}{2}\sum_{i,j=1}^{2} e_{i,j} \otimes e_{j,i}$ is the 2-sided separability element in H_s and Tr_{reg} is the trace map.
For $n \neq 2 \in k$

$$g = \begin{pmatrix} n & * \\ 0 & 2-n \end{pmatrix},$$

then $\frac{1}{2}\sum_{i,j=1}^{2} e_{i,j} g e_{j,i} = 1$ and one can define non-isomorphic minimal weak Hopf algebras $H_{min} \cong M_4(k)$ for different g.

Question: how many possible minimal weak Hopf algebra structures can be defined on a semisimple algebra B?

Examples of minimal weak bialgebras

1. Groupoid algebra $H = kG$:

$$H_s = H_t = H_{\min} = \oplus_{x \in G_0} k e_x.$$

2. Path algebra $H = kQ$:

$$H_s = H_t = H_{\min} = \oplus_{x \in Q_0} k e_x.$$

3. Hayashi face algebra $H = \mathfrak{H}(Q)$ with $|Q_0| = n$:

$$H_s = \oplus_{i=1}^{n} k \varepsilon_s(x_{i,i}), \;\; H_t = \oplus_{i=1}^{n} k \varepsilon_t(x_{i,i}).$$

Here $H_t \cap H_s = k1_H$ and $H_{\min} = \mathfrak{H}(Q_0) = H_s \otimes_k H_t$.

Minimal weak Hopf algebras

Recall that a direct sum of weak Hopf algebras is still a weak Hopf algebra. So we try to focus on the *indecomposable* weak Hopf algebra H, i.e., $H \neq H_1 \oplus H_2$ where H_1 and H_2 are weak Hopf algebras.

Definition

A weak Hopf algebra H is *coconnected* if and only if $H_s \cap H_t = k$.

Remark: A is indecomposable if $H_t \cap H_s \cap Z(H) = k1$. So $H_{\min}$ is coconnected if and only if it is indecomposable due to

$$H_t \cap H_s \cap Z(H_{\min}) = H_s \cap H_t.$$

Commutative minimal weak Hopf algebras

Proposition (Wang-Zhang-H)

If a commutative $H_{\min}$ is coconnected (indemcomposable), then $H_{\min} \cong \mathfrak{H}(Q_0)$ attached some arrowless quiver Q_0.

Remark: If $|Q_0| = 1$, then $H = \mathfrak{H}(Q_0) = k1_H$.

Commutative minimal weak Hopf algebras

Theorem (Wang-Zhang-H)

A commutative $H_{\min}$ is isomorphic to a direct sum of some Hayashi face algebras $\mathfrak{H}(Q_0)$ attached some arrowless quivers.

Noncommutative minimal weak bialgebra

However, if $H_{\min}$ is noncommutative, then it could be very complicated.

Now we know that every minimal weak bialgebra is a minimal weak Hopf algebra.

Question: When is a general weak bialgebra a weak Hopf algebra?

Generalized inverse in the convolution algebra

In a Hopf algebra H, id is the inverse of S in the conolution algebra $\mathrm{Hom}_k(H, k)$, i.e., $id * S = S * id = \varepsilon$.

Definition (Hayashi)

Let $x^{\pm}$ and $e^{\pm}$ be four elements of an algebra A. We say x^- is an (e^+, e^-)-generalized inverse of x^+ if

$$x^-x^+ = e^+, \quad x^+x^- = e^-$$
$$x^+x^-x^+ = x^+, \quad x^-x^+x^- = x^-.$$

Generalized inverse in the convolution algebra

Remark: we have the following facts:

1. If x^- is an (e^+, e^-)-generalized inverse of x^+, then x^+ is an (e^-, e^+)-generalized inverse of x^-.

2. The (e^+, e^-)-generalized inverse is unique if exists.

3. For a weak Hopf algebra H, $S : H \longrightarrow H$ is an antipode if S is the $(\varepsilon_s, \varepsilon_t)$-generalized inverse of id.

The antipode of a weak Hopf algebra

Denote H_0 the *coradical* coalgebra of a weak bialgebra H.

Theorem (Wang-Zhang-H)

Let H be a weak bialgebra. Suppose that the coradical H_0 is a weak sub-bialgebra of H. Then H_0 is a weak Hopf subalgebra if and only if H is a weak Hopf algebra.

Idea

$\Longleftarrow$: $S(H_0) \subseteq H_0$ and so $S|_{H_0}$ is the $(\varepsilon_s, \varepsilon_t)$-generalized invese of id.
$\Longrightarrow$: Let S_0 be the antipode of H_0. Extending S_0 to be a map $g : H \longrightarrow H$. Then show that $g * (\varepsilon_t + \sum_{n \geq 1} (\varepsilon_t - id * g)^n)$ is the $(\varepsilon_s, \varepsilon_t)$-generalized inverse of id.

The antipode of a weak Hopf algebra

Corollary (Wang-Zhang-H)

Let H be a weak bialgebra and suppose that B is a weak Hopf subalgebra. If H_0 is a weak sub-bialgebra of B, then H is a weak Hopf algebra.

Remark: Any f.dim sub-bialgebra of a Hopf algebra is a Hopf sub-algebra.

Theorem (Schauenburg, 2002)

Let H be a weak Hopf algebra, and B a f.dim weak sub-bialgebra of H. TFAE:

1. *B is a weak Hopf sub-algebra.*
2. *The right H_s-module B is isomorphic to the H_t-module B obtained by restricting the left H_t-module B along ε_t.*

The antipode of a weak Hopf algebra

Theorem (Wang-Zhang-H)

Let H be a pointed weak bialgebra. Then H is a weak Hopf algebra if and only if H_0 is a groupoid algebra.

As a consequence, we have

Proposition (Wang-Zhang-H)

If H is a cocommutative weak bialgebra over an algebraically closed field k, then H_0 is a groupoid algebra if and only if H is a weak Hopf algebra.

A path algebra kQ is a cocommutative and pointed weak bialgebra. As a consequence, we have

Corollary (Wang-Zhang-H)

1. *A path algebra kQ of a quiver $Q = (Q_0, Q_1)$ with both $|Q_1|$ and $|Q_0|$ greater than one is not a weak Hopf algebra.*
2. *The quotient kQ/I is a weak Hopf algebra if and only if kQ/I is a groupoid algebra, where I is a bi-ideal of kQ.*

Hayashi face algebras

$\mathfrak{H}(Q)$ is not a weak Hopf algebra structure in general. But it is possible to construct a weak Hopf algebra on a Hayashi face algebra $\mathfrak{H}(Q)$.

Example: Let Q be the quiver as below:

$$\begin{tikzcd} & 3 \arrow[dl, "p_3"'] & \\ 1 \arrow[rr, "p_1"] & & 2 \arrow[ul, "p_2"'] \end{tikzcd}$$

$$K = \frac{\langle \mathfrak{H}(Q), \mathfrak{H}(Q^{\text{op}}) \rangle}{\left\langle \ x_{i,j} - y_{i,j} \ \middle| \ x_{i,j} \in \mathfrak{H}(Q)_0, \ \ y_{i,j} \in \mathfrak{H}(Q^{op})_0 \ \right\rangle}$$

is a weak bialgebra, where kQ^{op} is the *opposite* algebra of kQ by reserving of all of the arrows in kQ.

We will show that

$$H = K/R$$

is a weak Hopf algebra, where

$$R = \left\langle \begin{array}{c} x_{p,q'}x_{p^*,q^*} - \delta_{q',q}x_{s(p),s(q)} \\ x_{q,p}x_{(q')^*,p^*} - \delta_{q',p}x_{s(q),s(p)} \\ x_{p^*,q^*}x_{p',q} - \delta_{p,p'}x_{t(p),t(q)} \\ x_{q^*,(p')^*}x_{q,p} - \delta_{p,p'}x_{t(q),t(p)} \end{array} \middle| \; p,q,p',q' \in Q_l \text{ for each } l \in \mathbb{N} \right\rangle$$

and with anitpode as below

$$\begin{cases} S(x_{p_i,p_j}) = x_{p_j^*,p_i^*} \quad \text{for} \quad i,j \in Q_0 \\ S(x_{p_j^*,p_i^*}) = x_{p_i,p_j} \quad \text{for} \quad i,j \in Q_0 \\ S(x_{i,j}) = x_{j,i} \quad \text{for} \quad i,j \in Q_0. \end{cases}$$

Future directions

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2. Construct a weak Hopf algebra from a general weak bialgebra.

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1. Is an Artinian weak Hopf algebra finite-dimensional ?
2. Construct a weak Hopf algebra from a general weak bialgebra.
3. Investigate the classification of finite-dimensional semisimple weak Hopf algebras.

Thank You!