

Cohen-Macaulay modules of noncommutative quadric hypersurfaces

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- (I) Noncommutative quadric hypersurfaces
- (II) Clifford deformations
- (III) Generalized Knörrer's periodicity theorem

(I) Noncommutative quadric hypersurfaces

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- Let A be a connected graded algebra, i.e., $A_0 = \mathbb{k}$.

A is called a **Koszul algebra** if the trivial module \mathbb{k}_A has a graded free resolution

$$0 \longleftarrow \mathbb{k}_A \longleftarrow P_0 \longleftarrow P_1 \longleftarrow \cdots \longleftarrow P_n \longleftarrow \cdots$$

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- **Example.** $A = \mathbb{k}[x, y]$,
 $A = \mathbb{k}_{-1}[x, y]$,
 $A = \mathbb{k}\langle x, y \rangle / (xy - yx + x^2)$.

- A noetherian connected graded algebra A is called an **Artin-Schelter Gorenstein** algebra if
 - (1) $\text{injdim}_A A = \text{injdim} A_A = d < \infty$
 - (2) $\text{Ext}_A^n(\mathbb{k}, A) = 0$ if $n \neq d$, and $\text{Ext}_A^d(\mathbb{k}, A) \cong \mathbb{k}$.

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- A **quantum polynomial algebra** is a Koszul Artin-Schelter regular algebra A such that

(1) $H_A(t) = (1 - t)^{-n}$ for some $n \geq 1$,

(2) A is a domain.

$$H_A(t) = \sum_{n \geq 0} t^n \dim A_n.$$

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 - (1) A_f is a Koszul algebra.
 - (2) If A is of global dimension d , then A_f is an Artin-Schelter Gorenstein algebra of injective dimension $d - 1$.

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 $A' = \mathbb{k}_{-1}[x, y]$, $f = x^2 + y^2$, A'_f

- R a noetherian connected graded algebra.
 $\text{gr } R$, category of finitely generated graded right R -modules
 $\text{tor } R$, full subcategory of $\text{gr } R$ consisting of finite dimensional modules
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$$\text{qgr } R = \text{gr } R / \text{tor } R.$$

- For $M \in \text{gr } R$, let

$$\Gamma(M) = \{m \in M \mid mA \text{ is finite dimensional}\}.$$

The i -th right derived functor of Γ is denoted by $R^i\Gamma$.

For $M \in \text{gr } R$, the *depth* of M is defined to be the number

$$\text{depth}(M) = \min\{i \mid R^i\Gamma(M) \neq 0\}.$$

- Suppose that R is an Artin-Schelter Gorenstein algebra with $\text{injdim} R_R = \text{injdim}_R R = d$.

$M \in \text{gr } R$ is called a **maximal Cohen-Macaulay module** (MCM module) if $\text{depth}(M) = d$.

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- $\text{mcm } R$ the category of all the MCM over R .

$\text{mcm } R$ is a Frobenius category, hence the stable category $\underline{\text{mcm}} R$ is a triangulated category.

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- **Example.** $A = \mathbb{k}[x, y]$, $A' = \mathbb{k}_{-1}[x, y]$, $f = x^2 + y^2$.
 $\underline{\text{mcm}} A_f \cong \underline{\text{mcm}} A'_f \cong D^b(\mathbb{k} \times \mathbb{k})$.

- A fundamental result:

Theorem

Let A be a quantum polynomial algebra and let $f \in A_2$ be a central element. Then there is a finite dimensional algebra $C(A_f)$ such that there is an equivalence of triangulated categories

$$D^b(C(A_f)) \cong \underline{\text{mcm}}A_f.$$

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- **Problems:**

- (1) find a way to compute $C(A_f)$;
- (2) Let $A = \mathbb{k}[x, y]$ and $A' = \mathbb{k}_{-1}[x, y]$, and let $f = x^2 + y^2$. Note that $C(A_f) \cong C(A'_f) \cong \mathbb{k} \times \mathbb{k}$. So, how can we recognize the difference between A_f and A'_f ?

(II) Clifford deformations

Clifford deformation of Koszul algebra

- Let V be a finite dimensional vector space, and let $E = T(V)/(R)$ be a Koszul algebra, where $R \subseteq V \otimes V$.

A linear map $\theta : R \rightarrow \mathbb{k}$ is called a **Clifford map** if

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- We call $E(\theta)$ a **Clifford deformation** of E .
- Note that a Clifford deformation is a special case of **Poicaré-Birkhoff-Witt deformations**.
- The usual Clifford algebra

$$\mathbb{R}_n^{p,q} = \mathbb{R}\langle x_1, \dots, x_n \rangle / (x_i^2 + 1, x_j^2 - 1 : 1 \leq i \leq p, p+1 \leq j \leq p+q)$$

is a Clifford deformation of the exterior algebra

$$E = \bigwedge \{x_1, \dots, x_n\}.$$

Clifford deformation of Koszul algebra

- Let A be a quantum polynomial algebra.

Proposition

*Let $E = A^!$ be the quadratic dual of the quantum polynomial algebra A . Then E is a **Koszul Frobenius** algebra.*

S.P. Smith, Some finite dimensional algebras related to elliptic curves, in: CMS Conf. Proc., 1996

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- We have the following facts:

Proposition

- Each central element $0 \neq f \in A_2$ is corresponding to a Clifford map θ_f of $E = A^!$.
- The Clifford deformation $E(\theta_f)$ is a **strongly \mathbb{Z}_2 -graded algebra**.
- $C(A_f) \cong E(\theta_f)_0$.

Example

- Let $A = \mathbb{k}\langle x, y, z \rangle / (r_1, r_2, r_3)$, where $r_1 = zx + xz, r_2 = yz + zy, r_3 = x^2 + y^2$. Then A is a quantum polynomial algebra of dimension 3.

f	$C(A_f) = E(\theta_f)_0$
$z^2 + xy + yx + \lambda x^2$	\mathbb{k}^4
$z^2 + xy + yx \pm 2\sqrt{-1}x^2$	$\mathbb{k}[u]/(u^2) \times \mathbb{k}[u]/(u^2)$
z^2	$\mathbb{k}[u, v]/(u^2 - v^2, uv)$
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- Remark.** Let A be a quantum polynomial algebra of dimension 3. If $f \in A_2$ is a central element, $A_f = A/Af$ is called a **noncommutative conic**. The algebras $C(A_f)$ have been classified for noncommutative conics.

H. Hu, Classification of noncommutative conics associated to symmetric regular superpotentials, arXiv:2005.03918.

H. Hu, M. Matsuno, I. Mori, Noncommutative conics in Calabi-Yau quantum planes, arXiv:2104.00221.

Theorem

Let A be a quantum polynomial algebra, and let $f \in A_2$ be a central regular element.

Then $\text{qgr } A_f$ has finite global dimension (i.e., $\text{proj } A_f$ is smooth) if and only if $C(A_f) = E(\theta_f)_0$ is a semisimple algebra.

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(III) Generalizations of Knörrer's periodicity theorem

An example

- **Example.** $A = \mathbb{k}[x, y]$, $A' = \mathbb{k}_{-1}[x, y]$, $f = x^2 + y^2$.

Then $\underline{\text{mcm}}(A_f) \cong \underline{\text{mcm}}(A'_f) \cong D^b(\mathbb{k} \times \mathbb{k})$.

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Then $\underline{\text{mcm}}(A_f) \cong \underline{\text{mcm}}(A'_f) \cong D^b(\mathbb{k} \times \mathbb{k})$.

- Let B be a quantum polynomial algebra and $g \in B_2$ be a central element.

Consider the tensor algebra $B \otimes A$ and $B \otimes A'$, and view $h := g + f$ as an element in $B \otimes A$ (or in $B \otimes A'$, resp.).

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- **Fact:** $\underline{\text{mcm}}(B \otimes A)_h$ is different from $\underline{\text{mcm}}(B \otimes A')_h$!

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- **Fact:** $\underline{\text{mcm}}(B \otimes A)_h$ is different from $\underline{\text{mcm}}(B \otimes A')_h$!
- The reason is the following:

Let E and E' be the Koszul dual of A and A' respectively.

The Clifford deformation of E and that of E' associated to f are very different!

Indeed, $E(\theta_f) \cong \mathbb{M}_2(\mathbb{k})$ and $E'(\theta_f) \cong \mathbb{k}\mathbb{Z}_2 \times \mathbb{k}\mathbb{Z}_2$.

- It seems reasonable to classify the noncommutative quadric hypersurfaces according to the Clifford deformations.

Definition

Let A be a quantum polynomial algebra, and let $f \in A_2$ be a central element. Let E be the Koszul dual of A .

*If the Clifford deformation $E(\theta_f)$ is a simple \mathbb{Z}_2 -graded algebra, then we call $A_f = A/Af$ is a **simple graded isolated singularity**.*

- Since \mathbb{k} is algebraically closed, there are only two classes of simple \mathbb{Z}_2 -graded algebra:
 - (0) matrix algebras over \mathbb{k} ;
 - (1) matrix algebras over $\mathbb{k}\mathbb{Z}_2$.

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 - (0) matrix algebras over \mathbb{k} ;
 - (1) matrix algebras over $\mathbb{k}\mathbb{Z}_2$.
- If $E(\theta_f)$ is a matrix algebra over \mathbb{k} , then we further call A_f is a **simple graded isolated singularity of 0-type**
If $E(\theta_f)$ is a matrix algebra over $\mathbb{k}\mathbb{Z}_2$, then we further call A_f is a **simple graded isolated singularity of 1-type**.

- Proposition.** Let A and B be quantum polynomial algebras, and let $f \in A_2$ and $g \in B_2$ be central elements. Suppose that $A \otimes B$ is noetherian. Let $h = f + g \in A \otimes B$.
 - If both A_f and B_f are simple graded isolated singularity of **1-type**, then $(A \otimes B)_h$ is a simple graded isolated singularity of **0-type**.
 - If A_f is a simple graded isolated singularity of **1-type** and B is a simple graded isolated singularity of **0-type**, then $(A \otimes B)_h$ is a simple graded isolated singularity of **1-type**.
 - If both A_f and B_f are simple graded isolated singularity of **0-type**, then $(A \otimes B)_h$ is a simple graded isolated singularity of **0-type**.

- A key lemma.

Lemma

Let A and B be quantum polynomial algebras, and let $f \in A_2$ and $g \in B_2$ be central elements. Suppose that $A \otimes B$ is noetherian. Let $h = f + g \in A \otimes B$.

Then we have an isomorphism of \mathbb{Z}_2 -graded algebras

$$E_{(A \otimes B)!}(\theta_h) \cong E_{A!}(\theta_f) \hat{\otimes} E_{B!}(\theta_g),$$

where $\hat{\otimes}$ is the \mathbb{Z}_2 -graded tensor.

Examples

- Let $A = \mathbb{k}[x, y]$, and $f = x^2 + y^2$. Then

$$E(\theta_f) \cong \mathbb{M}_2(\mathbb{k}),$$

where $\mathbb{M}_2(\mathbb{k})$ is viewed as a \mathbb{Z}_2 -graded algebra by setting

$$\mathbb{M}_2(\mathbb{k})_0 = \begin{bmatrix} \mathbb{k} & 0 \\ 0 & \mathbb{k} \end{bmatrix}, \quad \mathbb{M}_2(\mathbb{k})_1 = \begin{bmatrix} 0 & \mathbb{k} \\ \mathbb{k} & 0 \end{bmatrix}.$$

Hence A_f is a simple graded isolated singularity of 0-type.

Examples

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Hence A_f is a simple graded isolated singularity of 0-type.

- Let $A = \mathbb{k}\langle x_1, \dots, x_5 \rangle / (r_1, \dots, r_{10})$, where the generating relations are as follows:

$$\begin{aligned} r_1 &= x_1x_2 - x_2x_1, r_2 = x_1x_3 + x_3x_1, r_3 = x_1x_4 + x_4x_1, \\ r_4 &= x_1x_5 + x_5x_1, r_5 = x_2x_3 - x_3x_2, r_6 = x_2x_4 + x_4x_2, \\ r_7 &= x_2x_5 + x_5x_2, r_8 = x_3x_4 - x_4x_3, r_9 = x_3x_5 + x_5x_3, \\ r_{10} &= x_4x_5 + x_5x_4. \end{aligned}$$

Let $f = x_1^2 + \dots + x_5^2$. Then A_f is a simple graded isolated singularity of 1-type.

- **Remark.** We are unable to find a way to characterize when A_f is a simple graded isolated singularity.

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- **Proposition.** Let A be a quantum polynomial algebra of global dimension n , and let $f \in A_2$ be a central regular element.
 - (i) If A_f is a simple graded isolated singularity of 0-type, then n is even.
 - (ii) If A_f is a simple graded isolated singularity of 1-type, then n is odd.

- We have the following generalized Knörrer's periodicity theorem.

Theorem

Let A and B be quantum polynomial algebras, and let $f \in A_2$ and $g \in B_2$ be central elements. Suppose that $A \otimes B$ is noetherian, and let $h = f + g \in A \otimes B$.

- (i) If B_g is a simple graded isolated singularity of *0-type*, then there are equivalences of triangulated categories

$$\underline{\text{mcm}}(A \otimes B)_h \cong D^b(\text{mod} E(\theta_f)_0) \cong \underline{\text{mcm}} A_f;$$

- (ii) If B_g is a simple graded isolated singularity of *1-type*, there is an equivalence of triangulated categories

$$\underline{\text{mcm}}(A \otimes B)_h \cong D^b(\text{mod} E(\theta_f)^{\natural}),$$

where $E(\theta_f)^{\natural}$ is the underlying ungraded algebra, and $\text{mod} E(\theta_f)^{\natural}$ is the category of all the finite dimensional modules over $E(\theta_f)^{\natural}$.

Noncommutative Knörrer's periodicity theorem

- In particular, if we take $B = \mathbb{k}[x, y]$ and $g = x^2 + y^2$, then we obtain:

Theorem

Let A be a quantum polynomial algebra and let $f \in A_2$ be a central element. Let $A_f^{\#\#} = A[x, y]/(f + x^2 + y^2)$. Then

$$\underline{\text{mcm}} A_f^{\#\#} \cong \underline{\text{mcm}} A_f.$$

H. Knörrer, Cohen-Macaulay modules on hypersurface singularities I, *Invent. Math.* 88 (1987), 153–164.

A. Conner, E. Kirkman, W. F. Moore, C. Walton, Noncommutative Knörrer periodicity and noncommutative Kleinian singularities, [arXiv:1809.06524](https://arxiv.org/abs/1809.06524).

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An example

- Let A be a quantum polynomial algebra, and let $f \in A_2$ be a central element. Taking a minimal graded projective resolution of \mathbb{k}_{A_f} as follows:

$$\dots \longrightarrow P^{-d} \xrightarrow{\partial^{-d}} P^{-d+1} \xrightarrow{\partial^{-d+1}} \dots \longrightarrow P^0 \longrightarrow \mathbb{k}_{A_f} \longrightarrow 0.$$

Let $\Omega^d(\mathbb{k}_{A_f}) = \ker \partial^{-d+1}$ be the d th syzygy of the trivial module.

We fix a notion as follows:

$$\mathbb{M} := \Omega^d(\mathbb{k}_{A_f})(d).$$

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Theorem

We have $\text{End}_{\text{gr } A_f}(\mathbb{M}) \cong E(\theta_f)_0 \cong C(A_f)$.

J.-W. He, Y. Ye, Pre-resolutions of noncommutative isolated singularities,
arXiv:2005.11873.

An example

- Let $\mathbb{k} = \mathbb{C}$, let $S = \mathbb{k}\langle x, y, z \rangle / (R)$, where $R = \text{span}\{xz + zx, yz + zy, x^2 + y^2\}$.

Let $f = x^2 + z^2 \in S_2$, and $A = S/Sf$.

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- Set $\mathbb{M} = \Omega^2(\mathbb{k}_A)(2)$.

\mathbb{M} is a quotient module of the free module

$m_1A \oplus m_2A \oplus m_3A \oplus m_4A$ with relations:

$$r_1 = m_1x + m_2y + m_3z,$$

$$r_2 = 2m_1x + m_2y + m_3z + m_4z,$$

$$r_3 = m_2z - m_3y + m_4y,$$

$$r_4 = m_1z + m_2z - m_3y + m_4(x + y).$$

An example



$$\mathrm{End}_{\mathrm{gr} A}(\mathbb{M}) = \left\{ \begin{pmatrix} b+d & 0 & a & a \\ 0 & b & c & 0 \\ 0 & -c & b & 0 \\ a & c & d & b+d \end{pmatrix} : a, b, c, d \in \mathbb{k} \right\}.$$

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- We have the following complete set of primitive idempotents in $\text{End}_{\text{gr } A}(\mathbb{M})$:

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2}i & 0 \\ 0 & -\frac{1}{2}i & \frac{1}{2} & 0 \\ 0 & \frac{1}{2}i & -\frac{1}{2} & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2}i & 0 \\ 0 & \frac{1}{2}i & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2}i & -\frac{1}{2} & 0 \end{pmatrix},$$
$$e_3 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad e_4 = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

- We have the following nonprojective nonisomorphic indecomposable MCM modules:

$$\mathbb{M}^1 := A/(y + iz)A, \quad \mathbb{M}^2 := A/(y - iz)A,$$

$$\mathbb{M}^3 := A/(x + z)A, \quad \mathbb{M}^4 := A/(x - z)A.$$

Thank you for your attention!