On pointed Hopf algebras with finite GK-dimension

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I. Preliminaries.

Let $\mathbb{k} = \overline{\mathbb{k}}$ be an algebraically closed field, $\operatorname{char} \mathbb{k} = 0$.

Question. Classify Hopf algebras with finite Gelfand-Kirillov dimension.

Some classical examples:

- A finitely generated commutative Hopf algebra H has finite GK-dim $\iff$ there exists an algebraic group G such that $H \simeq \mathbb{k}[G]$.

- Let $\mathfrak{g}$ be a Lie algebra. Then its enveloping algebra $U(\mathfrak{g})$ has finite GK-dim $\iff$ $\dim \mathfrak{g} < \infty$.

- (Gromov). The group algebra $\mathbb{k}G$ of a finitely generated group G has finite GK-dim $\iff$ G is nilpotent-by-finite ($\exists N \trianglelefteq G$ nilpotent of finite index).

Definition. A Hopf algebra H is **pointed** $\iff$ its simple subcoalgebras are one-dimensional $\iff$ the coradical is a group algebra $\iff$ every simple H-comodule has dim 1.

- $\Bbbk G$, $U(\mathfrak{g})$ are pointed but $\Bbbk[G]$ is pointed iff G is abelian.

- The quantum groups $U_q(\mathfrak{g})$, Lusztig's $\mathcal{U}_q(\mathfrak{g})$, small versions $\mathfrak{u}_q(\mathfrak{g})$, Borel and parabolic subalgebras of them are all pointed.

- The quantum algebra of functions $\Bbbk_q[G]$ on a semisimple algebraic group G is not pointed.

Question. Classify **pointed** Hopf algebras with finite GK-dim.

• There are many results on Hopf algebras with low GK-dim (and good homological properties) by K. A. Brown, N. Ding, K. Goodearl, G. Liu, J. Wu, J. J. Zhang and others.

• A Hopf algebra H is **connected** if pointed and $G(H) = e$.

(Etingof-Gelaki) Connected Hopf algebras with finite GK-dim are quantizations of algebras of functions on a nilpotent Poisson algebraic group.

• A combinatorial approach to graded connected Hopf algebras is due to Zhou-Shen-Lu.

In this talk we will not comment these results and we will follow the aproach through Nichols algebras.

Fix a pointed Hopf algebra H with group G of group-likes.

- Let $\operatorname{gr} H$ be the graded Hopf algebra arising from the coradical filtration of H: $H_0 = \Bbbk G(H) \subset H_1 \subset H_2 \cdots \subset H_n \dots$ Then

$$\operatorname{gr} H \simeq R\#\Bbbk G,$$

- $R = \oplus_{n\geq 0} R^n$ is a Hopf algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$.

- R is connected $(R^0 = \Bbbk)$ and coradically graded (its coradical filtration comes from the grading).

- $V = R^1 \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$; $R' = \Bbbk\langle V\rangle \hookrightarrow R$.

${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ = category of **Yetter-Drinfeld modules** over $\Bbbk G$:

- $V = \oplus_{g \in G} V_g$ is a G-graded vector space;

- V is a left G-module such that $g \cdot V_h = V_{ghg^{-1}}$ (compatibility).

If Γ is an abelian group, then ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ is the category of Γ-graded Γ-modules.

$V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD} \implies V$ braided vector space:

$$c(v \otimes w) = g \cdot w \otimes v, \qquad v \in V_g, w \in V.$$

That is, $c \in GL(V \otimes V)$ satisfies the braid equation

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

A **coradically graded** connected braided Hopf algebra $\mathcal{E} = \bigoplus_{n\in\mathbb{N}_0} \mathcal{E}^n$ with $\mathcal{E}^1 \simeq V$ is called a **post-Nichols** algebra of V.

A graded connected braided Hopf algebra $\mathscr{B} = \bigoplus_{n\in\mathbb{N}_0} \mathscr{B}^n$ **generated in degree 1** i.e. by $\mathscr{B}^1 \simeq V$ is called a **pre-Nichols** algebra of V.

Theorem. $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD} \implies \exists$ unique (up to isomorphism) $\mathscr{B}(V) = \oplus_{n\in\mathbb{N}_0}\mathscr{B}^n(V)$ (graded) Hopf algebra in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ such that

$$\mathscr{B}^0(V) \simeq \Bbbk, \qquad \mathscr{B}^1(V) \simeq V,$$
$$\mathscr{B}(V) = \Bbbk\langle V\rangle, \qquad \text{Prim}(\mathscr{B}(V)) = \mathscr{B}^1(V).$$

In other words, there exists a unique braided Hopf algebra $\mathscr{B}(V)$ which is both post-Nichols and pre-Nichols of V; it is called the **Nichols algebra** of V.

The structure of Yetter-Drinfeld module in V extends to a structure of Hopf algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ on the tensor algebra $T(V)$.

Analogously $T^c(V) = T(V)$ with a natural comultiplication becomes a Hopf algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ with the quantum shuffle product.

There is a natural map of Hopf algebras in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$

$$\begin{tikzcd}
T(V) \arrow[rr] \arrow[dr] \arrow[rrrr, bend left, "\Omega"] & & \mathscr{B}(V) \arrow[rr] \arrow[dr] & & T^c(V) \\
& \mathscr{B} \arrow[ur, "\pi"'] & & \mathcal{E} \arrow[ur] &
\end{tikzcd}$$

The map Ω is the *quantum symmetrizer* and gives an alternative description of $\mathscr{B}(V) = \mathrm{Im}\ \Omega$.

Summarizing: H pointed Hopf algebra with finitely generated nilpotent-by-finite group G of group-likes

$\rightsquigarrow \operatorname{gr} H \simeq R\#\Bbbk G$, $R = \oplus_{n\geq 0} R^n \rightsquigarrow V = R^1$, $R' = \Bbbk\langle V\rangle \simeq \mathscr{B}(V)$.

$\operatorname{GK-dim} H \geq \operatorname{GK-dim} \operatorname{gr} H \geq \operatorname{GK-dim} R \geq \operatorname{GK-dim} R'$

Question i. Classify $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ such that $\operatorname{GK-dim} \mathscr{B}(V) < \infty$.

Question ii. For such $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ with $\operatorname{GK-dim} \mathscr{B}(V) < \infty$ classify $\mathcal{E}$ post-Nichols of V such that $\operatorname{GK-dim} \mathcal{E} < \infty$.

Question iii. For such V and $\mathcal{E}$ post-Nichols of V classify H such that $\operatorname{gr} H \simeq \mathcal{E}\#\Bbbk G$.

H pointed Hopf algebra with group G. In this talk I will report contributions to:

Question i. Classify $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ such that GK-dim $\mathscr{B}(V) < \infty$,

- when G is abelian,

- when G is finitely generated nilpotent.

Question ii. For such $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ with GK-dim $\mathscr{B}(V) < \infty$ classify $\mathcal{E}$ post-Nichols of V such that GK-dim $\mathcal{E} < \infty$,

- when V is of diagonal type.

II. Nichols algebras over nilpotent groups

Let G be a finitely generated group.

Let $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ = a Yetter-Drinfeld module over $\Bbbk G$, that is

$V = \oplus_{g \in G} V_g$ is G-graded, G acts on V and $g \cdot V_h = V_{ghg^{-1}}$.

Its support is $\{g \in G : V_g \neq 0\}$, a union of conjugacy classes.

Theorem. (A.) If the support of V is an infinite conjugacy class $\mathcal{O}$, then GK-dim $\mathscr{B}(V) \# \Bbbk G = \infty$.

Proof. By Gromov's Theorem, G is nilpotent-by-finite.
Let $K \leq G$ such that for every $g \in G$ there exists a positive integer n such that $g^n \in K$. By a Theorem of Malcev, $[G : K] < \infty$.
Let $x \in \mathcal{O}$. Applying the preceding to $[G : G^x] = \infty$, $\exists g \in G$ such that $x_n = g^n \triangleright x \neq x$ for all $n \in \mathbb{N}$.

Assume that V_x is a finitely generated G^x-module, say by a finite F. Let S be a finite set of generators of G. Then $W = \langle 1, S, g^{\pm 1}, F \rangle$ generates $\mathscr{B}(V)\#\Bbbk G$.

Let $v_0 \in F \backslash 0$, $v_n = g^n v_0 g^{-n}$, $n \in \mathbb{N}$,

$$\Lambda_n = \{v_{i_1} \cdots v_{i_s} : 1 \leq i_1 < \cdots < i_s \leq n,\ 0 \leq s \leq n\}.$$

Then (i) $|\Lambda_n| = 2^n$, (ii) $\Lambda_n \subset W^{(n+1)^2}$, and (iii) Λ_n is linearly independent. Hence $2^n \leq \dim W^{(n+1)^2}$ for all n. $\square$

Lemma. If G is a finitely generated torsion-free nilpotent group, then every non-central conjugacy class is infinite.

Corollary. Let G be a finitely generated torsion-free nilpotent group and $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$.
If $\operatorname{GK-dim} \mathscr{B}(V)\#\Bbbk G < \infty$, then $\sup V \subset Z(G)$ (so V 'comes from the abelian case').

Finite nilpotent groups.

A rack is a set $X \neq \emptyset$ with a self-distributive $\triangleright : X \times X \to X$ such that $\varphi_x := x \triangleright \_$ is bijective for all $x \in X$. The main examples are subsets of groups stable under conjugation $x \triangleright y = xy^{-1}$.

A rack X is *abelian* if $x \triangleright y = y$ for all $x, y \in X$.

A finite rack X is *of type C* when there are a decomposable subrack $Y = R \coprod S$ and elements $r \in R$, $s \in S$ such that $r \triangleright s \neq s$, $R = \mathcal{O}_r^{\mathrm{Inn} Y}$, $S = \mathcal{O}_s^{\mathrm{Inn} Y}$, $\min\{|R|, |S|\} > 2$ or $\max\{|R|, |S|\} > 4$.

Theorem. (A., Carnovale, García) A finite rack $\mathcal{O}$ of type C collapses, that is $\dim \mathscr{B}(\mathcal{O}, \mathfrak{q}) = \infty$ for every 2-cocycle $\mathfrak{q}$.

Theorem. A conjugacy class $\mathcal{O}$ in a finite nilpotent group G of **odd** order is either of type C or else an abelian rack.

If $\dim \mathscr{B}(V) < \infty$, then sup V is abelian by the Theorem of the previous slide (again 'reduction to the abelian case').

Conjecture. Let X be a finite rack of type C.
Then $\operatorname{GK-dim} \mathscr{B}(\mathcal{O}, \mathfrak{q}) = \infty$ for every faithful 2-cocycle $\mathfrak{q}$.

Sketch of a proof.
We may assume that G is a finite p-group with p an odd prime.

If $\mathcal{O}$ is not abelian, pick $r, s \in \mathcal{O}$: $r \triangleright s \neq s$. Let $H = \langle r, s \rangle \leq G$.

If $R := \mathcal{O}_r^H \neq S := \mathcal{O}_s^H$, then $\mathcal{O}$ is of type C.

If $R = S$, then it is indecomposable, hence projects onto a simple rack Z. Now Inn R is a p-group, then so is Inn Z, hence $|Z|$ is a power of p. But it is known by [A-Graña] that Inn Z could not be a p-group.

Proposition. (A.) Let G be a finitely generated nilpotent group whose torsion subgroup $T \neq e$ has odd order. Then a finite conjugacy class $\mathcal{O}$ of G is either abelian or else of type C.

The proof makes use of the following result.

Theorem. (Gruenberg). Let G be a finitely generated nilpotent group with torsion $T \neq e$ and $e \neq g \in G$. Then there exists a prime p that divides $|T|$, a finite p-group P and a morphism $\pi : G \to P$ such that $\pi(g) \neq e$.

For the Proposition to have a deep meaning, we need the validity of the Conjecture:

Conjecture. Let X be a finite rack of type C.
Then GK-dim $\mathscr{B}(\mathcal{O}, \mathfrak{q}) = \infty$ for every faithful 2-cocycle $\mathfrak{q}$.

III. Nichols algebras over abelian groups

Let (V, c) be a braided vector space, i.e. $c \in GL(V \otimes V)$ satisfies the braid equation $(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c)$.

A decomposition of V is a family of subspaces $(V_i)_{i \in \mathbb{I}_\theta}$ where

$$V = V_1 \oplus \cdots \oplus V_\theta, \;\; V_i \neq 0, \;\; c(V_i \otimes V_j) = V_j \otimes V_i, \;\; i, j \in \mathbb{I}_\theta, \;\; \theta \geq 2.$$

(V, c) is of **diagonal type** if has a decomposition with all summands of dimension 1, i.e. if $\exists$ a basis $(x_i)_{i \in \mathbb{I}_\theta}$ and a matrix $\mathfrak{q} = (q_{ij})_{i,j \in \mathbb{I}_\theta}$ with entries $\neq 0$, such that

$$c(x_i \otimes x_j) = q_{ij}\, x_j \otimes x_i, \qquad\qquad i, j \in \mathbb{I}_\theta.$$

Fix (V, c) of diagonal type with matrix $\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}_\theta}$. The Nichols algebra $\mathscr{B}(V)$ is $\mathbb{Z}^\theta$-graded, $\deg x_i = \alpha_i$ (canonical basis of $\mathbb{Z}^\theta$).

Theorem. (Kharchenko) $\mathscr{B}(V)$ has a PBW basis:

$\exists$ an ordered S: $\{x_1, \dots, x_\theta\} \subset S \subset \mathscr{B}(V)$ consisting of $\mathbb{Z}^\theta$-homog. elements; and for $s \in S$, $N_s \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, such that

$$\{s_1^{e_1} s_2^{e_2} \dots s_t^{e_t} : t \in \mathbb{N}_0,\ s_1 > \dots s_2 > \dots > s_t,\ 0 < e_j < N_{s_j}, j \in \mathbb{I}_t\}$$

is a basis of $\mathscr{B}(V)$.

- $\Delta_+^{\mathfrak{q}} = (\deg x)_{x\in S}$, S as in the Theorem;
- $\Delta^{\mathfrak{q}} = \Delta_+^{\mathfrak{q}} \cup -\Delta_+^{\mathfrak{q}} \subset \mathbb{Z}^\theta$

Theorem. (Heckenberger) The matrices $\mathfrak{q}$ with $\dim \mathcal{B}(V) < \infty$ (more generally with *finite root system*) are classified.

The classification can be organized in terms of Lie theory.

Conjecture. (A.–Angiono–Heckenberger) GK-dim $\mathcal{B}(V) < \infty \Leftrightarrow$ the root system is finite $\leftrightsquigarrow$ Heckenberger's list.

Partial results:

- (A.–Angiono, Rosso) If all $q_{ii}, q_{ij}q_{ji} \in (\Bbbk - \mathbb{G}_\infty) \cup \{1\}$, $\forall i \neq j \in \mathbb{I}_\theta$ and GK-dim $\mathcal{B}(V) < \infty \Leftrightarrow \mathfrak{q}$ is of finite Cartan type.

- (AAH) If $\mathfrak{a}$ is of affine type, then GK-dim $\mathcal{B}(V) = \infty$.

- The conjecture is true when $\dim V = 2$(AAH) and 3 (Angiono–García Iglesias).

Nichols algebras of blocks

Block $\mathcal{V}(\epsilon,\ell)$: $\exists$ a basis $(x_i)_{i\in\mathbb{I}_\ell}$ such that

$$c(x_i \otimes x_j) = \begin{cases} \epsilon x_1 \otimes x_i, & j = 1 \\ (\epsilon x_j + x_{j-1}) \otimes x_i, & j \geq 2, \end{cases} \qquad i \in \mathbb{I}_\ell.$$

Theorem. (A.–Angiono–Heckenberger)
GK-dim $\mathcal{B}(\mathcal{V}(\epsilon,\ell)) < \infty \iff \ell = 2$ and $\epsilon \in \{\pm 1\}$.

- $\mathcal{B}(\mathcal{V}(1,2)) = \Bbbk\langle x_1, x_2 | x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2\rangle$ **Jordan plane**.

- $\mathcal{B}(\mathcal{V}(-1,2)) = \Bbbk\langle x_1, x_2 | x_1^2, x_2x_{12} - x_{12}x_2 - x_1x_{12}\rangle$ **super Jordan plane**. Here $x_{12} = x_2x_1 + x_1x_2$.

$$\text{GK-dim}\,\mathcal{B}(\mathcal{V}(1,2)) = \text{GK-dim}\,\mathcal{B}(\mathcal{V}(-1,2)) = 2.$$

Nichols algebras of blocks & points

$V = V_1 \oplus \cdots \oplus V_t \oplus V_{t+1} \oplus \cdots \oplus V_\theta$ braided vector space with a decomposition, where $t > 0$,

• $V_1, \dots, V_t \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ are blocks (may assume of dim 2 and $\epsilon^2 = 1$);

•$V_{t+1}, \dots, V_\theta \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ are points (i.e. of dimension 1).

Theorem. (A.–Angiono–Heckenberger) The classification of those V with GK-dim $\mathcal{B}(V) < \infty$ is known. Also the defining relations.

Up to the Conjecture on diagonal type...

However this is not the end of the story since there are indecomposable Yetter-Drinfeld modules that are not blocks. We call them **pale blocks**

Theorem. (A.–Angiono–Heckenberger) The classification of those $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of **dimension 3** with one pale block such that GK-dim $\mathcal{B}(V) < \infty$ is known.

Theorem. (A.–Angiono–Moya) The classification of those $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of **dimension 4** with at least one pale block such that GK-dim $\mathcal{B}(V) < \infty$ is known.

In both Theorems we present the defining relations and the explicit PBW-basis (that exists by a general result of Ufer).

IV. Eminent pre-Nichols algebras.

The set $\mathfrak{Post}(V)$ of isomorphism classes of post-Nichols algebras of V is partially ordered with $T^c(V)$ maximal and $\mathscr{B}(V)$ minimal.

The set $\mathfrak{Pre}(V)$ of isomorphism classes of pre-Nichols algebras of V is partially ordered with $T(V)$ minimal and $\mathscr{B}(V)$ maximal.

$$\begin{tikzcd}
& & \mathscr{B}(V) \arrow[dll, hook] \arrow[d, hook] \arrow[drr, hook] & & \\
\mathcal{E} \arrow[drr, hook] & \dots \quad \dots & \mathcal{E}' \arrow[d, hook] & \dots \quad \dots & \mathcal{E}'' \arrow[dll, hook] \\
& & T(V) & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & T(V) \arrow[dll, two heads] \arrow[d, two heads] \arrow[drr, two heads] & & \\
\mathscr{B} \arrow[drr, two heads] & \dots \quad \dots & \mathscr{B}' \arrow[d, two heads] & \dots \quad \dots & \mathscr{B}'' \arrow[dll, two heads] \\
& & \mathscr{B}(V) & &
\end{tikzcd}$$

Question ii. For such $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ with GK-dim $\mathscr{B}(V) < \infty$ classify $\mathcal{E}$ post-Nichols of V such that GK-dim $\mathcal{E} < \infty$.

$\mathfrak{Post}_{\mathsf{fGK}}(V) \leq \mathfrak{Post}(V)$ = post-Nichols algebras of V with finite GK-dim.

Lemma. (AAH) $\mathscr{B}$ a pre-Nichols algebra of V, $\mathcal{E} = \mathscr{B}^d$ the graded dual of $\mathscr{B}$. Then GK-dim $\mathcal{E} \leq$ GK-dim $\mathscr{B}$. If $\mathcal{E}$ is finitely generated, then the equality holds.

Question ii bis. For such $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ with GK-dim $\mathscr{B}(V) < \infty$ classify $\mathscr{B}$ pre-Nichols of V such that GK-dim $\mathscr{B} < \infty$.

$\mathfrak{Pre}_{\mathsf{fGK}}(V) \leq \mathfrak{Pre}(V)$ = pre-Nichols algebras of V with finite GK-dim.

Question ii bis. For such $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ with $\operatorname{GK-dim} \mathscr{B}(V) < \infty$ classify $\mathscr{B}$ pre-Nichols of V such that $\operatorname{GK-dim} \mathscr{B} < \infty$.

A pre-Nichols algebra is *eminent* if it is a minimum in $\mathfrak{Pre}_{\mathsf{fGK}}(V)$; i.e. a pre-Nichols algebra $\widehat{\mathscr{B}}$ is eminent if

- $\operatorname{GK-dim} \widehat{\mathscr{B}} < \infty$;

- if $\mathscr{B}$ is a pre-Nichols algebra of V with $\operatorname{GK-dim} \mathscr{B} < \infty$, then there exists a morphism of pre-Nichols algebras $\widehat{\mathscr{B}} \twoheadrightarrow \mathscr{B}$.

Warning: Eminent pre-Nichols algebras do not always exist (e.g. if the braiding is the usual transposition).

Theorem. (A.–Sanmarco) A braided vector space of Cartan diagonal type admits an eminent pre-Nichols algebra, that turns out to be the distinguished pre-Nichols algebra introduced by Angiono, except for very few exceptions in low rank.

Remarks: 1) When the braided vector space of Cartan type is the one at the origin of quantum groups at roots of one, the distinguished pre-Nichols algebra is the positive part of the Kac–De Concini–Procesi quantum group.

2) The Theorem was extended to other braided vector spaces of diagonal type by Angiono, Campagnolo and Sanmarco.

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