Stability of EHI and regularity of MMD spaces

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Abstract

We show that elliptic Harnack inequality is stable under form-bounded perturbations for strongly local symmetric Dirichlet forms on complete locally compact separable metric spaces that satisfy metric doubling property (or equivalently, relative ball connectedness property).

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1 Introduction

Let $X = \{X_t, t \in [0,\infty); \mathbb{P}^x, x \in \mathcal{X}\}$ be a diffusion process on a locally compact separable metric space $(\mathcal{X}, d)$. A function h on a ball $B = B(x,r)$ is *harmonic* if $h(X_{t\wedge\tau_B})$ is a local martingale under $\mathbb{P}^x$ for every $x \in B$; here τ_B is the exit time from B by the process X and the filtration is the minimal augmented filtration generated by X. The (scale-invariant) *elliptic Harnack inequality* (EHI) holds for X if there exist constants $C \ge 1$ and $\delta \in (0,1)$ such that whenever h is non-negative and harmonic on a ball $B = B(x,r)$, then

$$\sup_{B(x,\delta r)} h \le C \inf_{B(x,\delta r)} h. \tag{1.1}$$

If it holds, the EHI is a valuable tool for the study of the process X and its associated heat kernel. A well-known theorem of Moser [Mo1] is that the EHI holds if X is the symmetric diffusion associated with a uniformly elliptic divergence form operator $\mathcal{A} = \mathrm{div}(A(x)\nabla)$. Associated with such a process is the symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R}^d; dx)$, where

$$\mathcal{F} = W^{1,2}(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d; dx) : \nabla f \in L^2(\mathbb{R}^d; dx) \right\}$$

is the Sobolev space on $\mathbb{R}^d$ of order $(1,2)$ and

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^d} \nabla f(x) \cdot A(x) \nabla f(x)\, dx, \quad f \in \mathcal{F}.$$

We say two symmetric Dirichlet forms $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$ on $L^2(\mathbb{R}^d; dx)$ with common domain $\mathcal{F}$ are *comparable* if there exists $C \ge 1$ such that

$$C^{-1}\mathcal{E}^{(1)}(f,f) \le \mathcal{E}^{(2)}(f,f) \le C\mathcal{E}^{(1)}(f,f) \quad \text{for all } f \in \mathcal{F}.$$

Moser's result gives the stability of the EHI, in the sense that if $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$ are comparable symmetric Dirichlet forms on $L^2(\mathbb{R}^d; dx)$, associated with uniformly elliptic divergence form operators $\mathcal{A}_i$, then the EHI holds for $\mathcal{E}^{(2)}$ if and only if it holds for $\mathcal{E}^{(1)}$.

A few years later, Moser [Mo2, Mo3] proved a parabolic Harnack inequality PHI, which holds for non-negative solutions to the heat equation associated with a uniformly elliptic divergence form operator $\mathcal{A}$. In particular, if u is any non-negative solution to the heat equation $\frac{\partial u}{\partial t} = \mathcal{A}u$ in a time-space cylinder $Q = (0, r^2) \times B(x, r)$, then writing $T = r^2, Q_- = (T/4, T/2) \times B(x, \delta r), Q_+ = (3T/4, T) \times B(x, \delta r)$, we have

$$\operatorname*{ess\,sup}_{Q_-} u \le C_P \operatorname*{ess\,inf}_{Q_+} u,$$

where the constants $C_P > 1$ and $\delta \in (0, 1)$ do not depend on x, r or u. Subsequently Grigor'yan and Saloff-Coste in [Gr0, Sal92] gave a characterization of the PHI, and the stability of the PHI follows immediately from this characterization. The methods of these papers are very robust, and this characterization of the PHI was extended to diffusions on locally compact separable metric spaces [St], and to random walks on graphs [De1].

For a number of years the stability of the apparently simpler EHI remained an open problem. Stability on a large class of unbounded spaces (including Riemannian manifolds and graphs) was proved by two of us recently in [BM1]. However, the result there relied on the metric space being geodesic and satisfying some strong local regularity conditions; one key use of this regularity was to ensure the existence of Green's functions.

The natural context for the study of the EHI is that of locally compact separable metric measure spaces with strongly local regular (symmetric) Dirichlet forms, which we call a MMD space. Examples include Riemannian manifolds, the cable systems of graphs [V], as well as various classes of fractals. Not only do MMD spaces provide a common framework for all these examples, but also certain transformations (change of measure, quasisymmetric change of metric) which are natural for MMD spaces but are not so natural for manifolds and graphs. These transformations are key to the argument in [BM1].

This paper has three main goals:

(i) We give a weak sufficient condition (a local Harnack inequality) for a MMD space to have Green's functions. This improves significantly the results of earlier papers, such as [BM1, BM2], which needed some parabolic regularity. In particular, it allows us to drop the Green function assumption ([BM1, Assumption 2.3]) made in [BM1].

(ii) We carry through the program of [BM1] in the context of a MMD space satisfying these weak regularity conditions. In particular, we drop the *bounded geometry* assumption (see [BM1, Assumption 2.5] for its definition) on the MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$, and relax the condition that $(\mathcal{X}, d)$ is a length (or geodesic) space; both are needed in [BM1]. We assume that $(\mathcal{X}, d)$ is a complete metric and satisfies metric doubling—see Definition 1.1, which is equivalent under EHI to the 'relatively ball connectedness' condition (see Definition 5.1). The latter has the advantage that it is preserved by quasisymmetric changes of metric. Example 8.1 shows that some regularity of the metric is needed if we are to have stability of the EHI.

(iii) We do not assume that $(\mathcal{X}, d)$ is a length (or geodesic) space, and the metric spaces $(\mathcal{X}, d)$ may either be of bounded diameter or of infinite diameter.

For a metric space $(\mathcal{X}, d)$, we use $B(x, r)$ to denote the open ball centered at $x \in \mathcal{X}$ with radius r. The closure and the boundary of the ball $B(x, r)$ will be denoted as $\overline{B(x, r)}$ and $\partial B(x, r)$, respectively.

Definition 1.1. *A metric space* $(\mathcal{X}, d)$ *is said to be* metric doubling *(MD) if there exists* $N \geq 2$ *such that for any* $x \in \mathcal{X}$, $R > 0$ *there exist* $z_1, \ldots, z_N \in \mathcal{X}$ *such that* $B(x, R) \subset \cup_{i=1}^N B(z_i, R/2)$.

Remark 1.2. Assouad [Ass] showed that if $(\mathcal{X}, d)$ is (MD), then for every $\alpha \in (0, 1)$, the metric space $(\mathcal{X}, d^\alpha)$ has a bi-Lipschitz embedding into $\mathbb{R}^n$ for some $n \geq 1$. This in particular implies that a metric doubling space $(\mathcal{X}, d)$ is always separable. One can also deduce the separability of a metric doubling space from its definition. Indeed, a metric doubling metric space is totally bounded by [Hei, Exercise 10.17], which implies the separability. Furthermore, if $(\mathcal{X}, d)$ is complete and (MD), then every ball in $(\mathcal{X}, d)$ is relatively compact by the aforementioned totally boundedness property. Consequently, any complete metric doubling space $(\mathcal{X}, d)$ is separable and locally compact.

When $(\mathcal{X}, d)$ is a locally compact metric space, a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, m)$ is said to be *strongly local* if $\mathcal{E}(u, v) = 0$ whenever $u, v \in \mathcal{F}$ have compact support with v being constant in an open neighborhood of $\mathrm{supp}[u]$. See Proposition 2.2 below for its equivalent characterizations.

The main result of this paper is the following stability result on the (scale invariant) EHI. See Definition 4.2 for a precise definition of the EHI.

Theorem 1.3. *Let* $(\mathcal{X}, d)$ *be a complete metric doubling space, and let* m *be a Radon measure on* $\mathcal{X}$ *with full support. Let* $(\mathcal{E}, \mathcal{F})$ *be a strongly local symmetric regular Dirichlet form on* $L^2(\mathcal{X}; m)$. *Suppose that* $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ *satisfies the EHI. Let* $(\mathcal{E}', \mathcal{F})$ *be another strongly local symmetric regular Dirichlet form on* $L^2(\mathcal{X}; m)$ *such that*

$$C^{-1}\mathcal{E}(f, f) \leq \mathcal{E}'(f, f) \leq C\mathcal{E}(f, f) \quad \text{for all } f \in \mathcal{F}.$$

for some $C \geq 1$. *Then* $(\mathcal{X}, d, m, \mathcal{E}', \mathcal{F})$ *satisfies the EHI.*

Theorem 1.3 is established based on characterizations of the EHI given in Theorem 7.9. This stability result is further extended in Theorem 7.11 to strongly local MMD spaces that may have different symmetrizing measures.

Suppose that $(\mathcal{X}, d)$ is a length space. By [BM1, Theorem 3.11], $(\mathcal{X}, d)$ is metric doubling if $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the EHI and has regular Green function. Theorem 1.3 together with Theorems 4.8 and 4.6 readily implies the following corolloary, which substantially improves the main result, Theorem 1.3, of [BM1].

Corollary 1.4. *Suppose that* $(\mathcal{X}, d)$ *is length space, and* m *is a Radon measure on* $\mathcal{X}$ *with full support. Let* $(\mathcal{E}, \mathcal{F})$ *be a strongly local symmetric regular Dirichlet form on* $L^2(\mathcal{X}; m)$. *Suppose that* $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ *satisfies the EHI. Let* $(\mathcal{E}', \mathcal{F})$ *be another strongly local symmetric regular Dirichlet form on* $L^2(\mathcal{X}; m)$ *such that*

$$C^{-1}\mathcal{E}(f, f) \leq \mathcal{E}'(f, f) \leq C\mathcal{E}(f, f) \quad \text{for all } f \in \mathcal{F}.$$

Then $(\mathcal{X}, d, m, \mathcal{E}', \mathcal{F})$ *satisfies the EHI.*

The remainder of this paper is organized as follows. In Section 2, we present definitions and terminology associated with regular symmetric Dirichlet forms as well as some basic facts that will be used in this paper. Existence and regularity of Green functions are given in Section 3 for transient regular Dirichlet forms. The transience condition is removed in Section 4. It is shown there that any strongly local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on a connected locally compact separable metric space $\mathcal{X}$ that satisfies the local EHI is irreducible and has regular Green functions. Various consequences of the EHI are presented in Section 5. In particular, it is shown that for a complete locally compact separable metric space $(\mathcal{X}, d)$, under the EHI, relatively ball connected, metric doubling and quasi-arc connected properties are all mutually equivalent. In Section 6, a good doubling measure μ is constructed on a MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ that satisfies the EHI and is relatively ball connected. This measure relates well with capacities and is a smooth measure with full quasi support on $\mathcal{X}$. It is shown in Section 7 that the Dirichlet form time-changed by the positive continuous additive functional generated by this doubling measure μ is a MMD space $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ that satisfies Poincare inequality PI(Ψ), the cutoff energy inequality CS(Ψ) and a capacity estimate cap(Ψ), where Ψ is a suitable regular scale function. From this we can obtain equivalent characterizations of the EHI in Theorem 7.9, and deduce the stability result of the EHI stated in Theorem 1.3. The scale function Ψ varies both in space and in time; [Te] first studied such location dependent scaling functions in detail. An extension of Theorem 1.3 is given at the end of Section 7 that the second Dirichlet form $\mathcal{E}'$ may have symmetrizing measure μ different from m; see Theorem 7.11. Three examples are given in Section 8. The first example shows that without certain regularity of the metric, the stability of the EHI may fail. The second example is of a strongly local regular Dirichlet form that fails to satisfy the non-scale-invariant Harnack inequality. The third one is of a space which satisfies the EHI and is covered by the results of this paper, but fails to satisfy the local regularity required in [BM1].

2 Preliminaries

In this section, we give definitions of some terminology from Dirichlet form theory that are used in this paper and some basic facts. We refer the reader to [CF, FOT] for more details on the theory of symmetric Dirichlet forms. We use := as a way of definition.

Let $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ be a measurable space and m a σ-finite measure on $\mathcal{X}$. A bilinear form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$ is said to be a *symmetric Dirichlet form* if

(i) $\mathcal{F}$ is a dense linear subspace of $L^2(\mathcal{X}; m)$;

(ii) $\mathcal{E}$ is symmetric and bilinear on $\mathcal{F} \times \mathcal{F}$ such that $\mathcal{E}(f, f) \geq 0$ for every $f \in \mathcal{F}$;

(iii) $\mathcal{F}$ is a Hilbert space with inner product $\mathcal{E}_1(f, g) := \mathcal{E}(f, g) + \int_{\mathcal{X}} f(x)g(x)m(dx)$;

(iv) For every $f \in \mathcal{F}$, $g := (0 \vee f) \wedge 1$ is in $\mathcal{F}$ and $\mathcal{E}(g, g) \leq \mathcal{E}(f, f)$.

A bilinear form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$ satisfying properties (i)-(iii) above is called a *symmetric closed form*. Any symmetric closed form is in one-to-one correspondence with a strongly continuous symmetric contraction semigroup $\{T_t; t \geq 0\}$ on $L^2(\mathcal{X}; m)$. Property (iv) above is called the Markovian property which is equivalent to the corresponding semigroup $\{T_t; t \geq 0\}$ being Markovian; that is, $0 \leq T_t f \leq 1$ for any $f \in L^2(\mathcal{X}; m)$ with $0 \leq f \leq 1$. A real-valued function f is said to be in the *extended Dirichlet space* $\mathcal{F}_e$ if there is an $\mathcal{E}$-Cauchy sequence $\{f_k; k \geq 1\} \subset \mathcal{F}$ so that $\lim_{k\to\infty} f_k = f$ m-a.e. on $\mathcal{X}$, and we define $\mathcal{E}(f, f) = \lim_{k\to\infty} \mathcal{E}(f_k, f_k)$. Clearly, $\mathcal{F} \subset \mathcal{F}_e$. It is known that $\mathcal{F} = \mathcal{F}_e \cap L^2(\mathcal{X}; m)$; see [CF, Theorem 1.1.5(iii)].

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$ is said to be *transient* if there exists a bounded

$g \in L^1(\mathcal{X}; m)$ that is strictly positive on $\mathcal{X}$ so that

$$\int_{\mathcal{X}} |u(x)| g(x) m(dx) \le \mathcal{E}(u,u)^{1/2} \quad \text{for every } u \in \mathcal{F}.$$

Clearly, if $(\mathcal{E}, \mathcal{F})$ is transient, then $(\mathcal{F}_e, \mathcal{E})$ is a Hilbert space. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$ is said to be *recurrent* if $1 \in \mathcal{F}_e$ and $\mathcal{E}(1,1) = 0$. Denote by $\{T_t; t \ge 0\}$ the semigroup on $L^2(\mathcal{X}; m)$ corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{F})$. By Theorem 2.1.5 and Theorem 2.1.8 of [CF], $(\mathcal{E}, \mathcal{F})$ is transient if and only if there is some $L^1(\mathcal{X}; m)$-integrable function g that is strictly positive on $\mathcal{X}$ so that $Gg := \int_0^\infty T_t g \, dt < \infty$ m-a.e. on $\mathcal{X}$; and $(\mathcal{E}, \mathcal{F})$ is recurrent if and only if for any non-negative g on $\mathcal{X}$ with $\int_{\mathcal{X}} g(x) m(dx) < \infty$, $Gg \in \{0, \infty\}$ m-a.e. on $\mathcal{X}$.

Denote by $\mathcal{B}^m(\mathcal{X})$ the completion of the field $\mathcal{B}(\mathcal{X})$ under the measure m. A set $A \in \mathcal{B}^m(\mathcal{X})$ is said to be $\{T_t\}_{t\ge 0}$-*invariant* if $T_t(1_{A^c} f) = 0$ m-a.e. on A for all $t > 0$ and $f \in L^2(\mathcal{X}; m)$. By [CF, Proposition 2.1.6], $A \in \mathcal{B}^m(\mathcal{X})$ is $\{T_t\}_{t\ge 0}$-invariant if and only if $1_A u \in \mathcal{F}$ for every $u \in \mathcal{F}$ and

$$\mathcal{E}(u, v) = \mathcal{E}(1_A u, 1_A v) + \mathcal{E}(1_{A^c} u, 1_{A^c} v) \quad \text{for every } u, v \in \mathcal{F}. \tag{2.1}$$

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$ is said to be *irreducible* if for any $\{T_t\}_{t\ge 0}$-invariant set A, either $m(A) = 0$ or $m(A^c) = 0$. An irreducible Dirichlet form is either transient or recurrent; see [CF, Propositions 2.1.3(iii) and 2.1.6].

A Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$ is said to be *regular* if

(i) $(\mathcal{X}, d)$ is a locally compact separable metric space and m is a Radon measure on $\mathcal{X}$ with full support;

(ii) $\mathcal{F} \cap C_c(\mathcal{X})$ is $\sqrt{\mathcal{E}_1}$-dense in $\mathcal{F}$, where $C_c(\mathcal{X})$ is the space of continuous functions on $\mathcal{X}$ having compact support;

(iii) $\mathcal{F} \cap C_c(\mathcal{X})$ is dense in $C_c(\mathcal{X})$ with respect to the uniform norm $\|f\|_\infty = \sup_{x \in \mathcal{X}} |f(x)|$.

For a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, m)$, an increasing sequence $\{F_k; k \ge 1\}$ of closed subsets of $\mathcal{X}$ is said to be an $\mathcal{E}$-*nest* if $\cup_{k\ge 1} \mathcal{F}_{F_k}$ is $\sqrt{\mathcal{E}_1}$-dense in $\mathcal{F}$, where $\mathcal{F}_{F_k} := \{f \in \mathcal{F} : f = 0 \ m\text{-a.e. on } \mathcal{X} \setminus F_k\}$. A set $N \subset \mathcal{X}$ is said to be $\mathcal{E}$-*polar* if there is an $\mathcal{E}$-nest $\{F_k; k \ge 1\}$ so that $N \subset \mathcal{X} \setminus \cup_{k\ge 1} F_k$. An $\mathcal{E}$-polar set A always has $m(A) = 0$. $\mathcal{E}$-polar sets can also be characterized by using capacity. Given a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$, we can define 1-capacity Cap_1 as follows. For any open subset $U \subset \mathcal{X}$,

$$\mathrm{Cap}_1(U) := \inf\{\mathcal{E}_1(f, f) : f \in \mathcal{F}, \ f \ge 1 \ m\text{-a.e. on } U\} \tag{2.2}$$

with the convention that $\inf \emptyset := \infty$, and for any subset $A \subset \mathcal{X}$,

$$\mathrm{Cap}_1(A) := \inf\{\mathrm{Cap}_1(U) : U \subset \mathcal{X} \text{ open}, U \supset A\}. \tag{2.3}$$

It is known (see [CF, Theorem 1.3.14]) that for a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$, $A \subset \mathcal{X}$ is $\mathcal{E}$-polar if and only if it has zero 1-capacity. A statement depending on $x \in A$ is said to hold $\mathcal{E}$-*quasi-everywhere* ($\mathcal{E}$-q.e. in abbreviation) if there is an $\mathcal{E}$-polar set $N \subset A$ so that the statement is true for every $x \in A \setminus N$. A function f is said to be $\mathcal{E}$-*quasi-continuous* on $\mathcal{X}$ if there is an $\mathcal{E}$-nest $\{F_k; k \ge 1\}$ so that $f\big|_{F_k} \in C(F_k)$ for every $k \ge 1$, where $C(F_k) := \{u : F_k \to \mathbb{R} | u \text{ is continuous}\}$. When there is no possible ambiguity, we often drop $\mathcal{E}$- from

$\mathcal{E}$-quasi-everywhere and $\mathcal{E}$-quasi-continuous. For a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$, every $f \in \mathcal{F}_e$ has an m-version that is quasi-continuous on $\mathcal{X}$, which is unique up to an $\mathcal{E}$-polar set; see [CF, Theorem 2.3.4] or [FOT, Theorem 2.1.7]. We always take a function f in $\mathcal{F}_e$ to be represented by its quasi-continuous version.

Recall that a Hunt process $X = \{X_t, t \geq 0; \mathbb{P}^x, x \in \mathcal{X}\}$ on a locally compact separable metric space $\mathcal{X}$ is a strong Markov process that is right continuous and quasi-left continuous on the one-point compactification $\mathcal{X}_\partial := \mathcal{X} \cup \{\partial\}$ of $\mathcal{X}$. A set $C \subset \mathcal{X}_\partial$ is said to be *nearly Borel measurable* if for any probability measure μ on $\mathcal{X}$ there are Borel sets A_1, A_2 such that $A_1 \subset C \subset A_2$ and

$$\mathbb{P}^\mu(\text{there is some } t \geq 0 \text{ such that } X_t \in A_2 \setminus A_1) = 0.$$

Let m be a Radon measure with full support on $\mathcal{X}$. A Hunt process X is said to be *m-symmetric* if the transition semigroup is symmetric on $L^2(\mathcal{X}; m)$. For an m-symmetric Hunt process X on $\mathcal{X}$, a set $\mathcal{N} \subset \mathcal{X}$ is said to be *properly exceptional* for X if $\mathcal{N}$ is nearly Borel measurable, $m(\mathcal{N}) = 0$ and

$$\mathbb{P}^x(X_t \in \mathcal{X}_\partial \setminus \mathcal{N} \text{ and } X_{t-} \in \mathcal{X}_\partial \setminus \mathcal{N} \text{ for all } t > 0) = 1 \quad \text{for every } x \in \mathcal{X} \setminus \mathcal{N}.$$

In 1971, Fukushima showed that any symmetric regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$ has an m-symmetric Hunt process $X = \{X_t, t \geq 0; \mathbb{P}^x, x \in \mathcal{X}\}$ on $\mathcal{X}$ associated with it in the sense that the transition semigroup of X is a version of the strongly continuous semigroup $\{T_t; t \geq 0\}$ on $L^2(\mathcal{X}; m)$ corresponding to $(\mathcal{E}, \mathcal{F})$, see [FOT, Theorem 7.2.1]. Furthermore, for any non-negative Borel measurable $f \in L^2(\mathcal{X}; m)$ and $t > 0$,

$$P_t f(x) := \mathbb{E}^x[f(X_t)]$$

is a quasi-continuous version of $T_t f$ on $\mathcal{X}$. The Hunt process X associated with a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$ is unique in the following sense (see [FOT, Theorem 4.2.8]): if X' is another Hunt process associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$, then there is a common properly exceptional set outside which these two Hunt processes have the same transition functions. We say the m-symmetric Hunt process X on $\mathcal{X}$ is *transient*, *recurrent*, and *irreducible* if so is its associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$.

In the remainder of this section, $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathcal{X}; m)$ and $X = \{X_t, t \geq 0; \mathbb{P}^x, x \in \mathcal{X}\}$ is the Hunt process associated with it. Let ζ denote the lifetime of X, and $\{\mathcal{F}_t; t \geq 0\}$ be the minimum augmented filtration generated by X.

A subset $\mathcal{N} \subset \mathcal{X}$ is said to be *m-polar* if there is a nearly Borel set $\mathcal{N}_1 \supset \mathcal{N}$ so that $\mathbb{P}^x(\sigma_{\mathcal{N}_1} < \infty) = 0$ for m-a.e. $x \in \mathcal{X}$, where $\sigma_{\mathcal{N}_1} = \inf\{t > 0 : X_t \in \mathcal{N}_1\}$. It is known that a subset $\mathcal{N} \subset \mathcal{X}$ is $\mathcal{E}$-polar if and only if it is m-polar, and any $\mathcal{E}$-polar set is contained in a Borel properly exceptional set for X; see [CF, Theorems 3.1.3 and 3.1.5].

If $(\mathcal{E}, \mathcal{F})$ is irreducible, then (see [CF, Theorem 3.5.6]) for any non-$\mathcal{E}$-polar nearly Borel measurable set A,

$$\mathbb{P}^x(\sigma_A < \infty) > 0 \quad \text{for } \mathcal{E}\text{-q.e. } x \in \mathcal{X}. \tag{2.4}$$

Let D be an open subset of $\mathcal{X}$. The part process X^D of X killed upon exiting D is a Hunt process on D whose associated Dirichlet form $(\mathcal{E}^D, \mathcal{F}^D)$ on $L^2(D; m|_D)$ is regular. Here $m|_D$ is the measure m restricted to the open set D,

$$\mathcal{F}^D = \{f \in \mathcal{F} : f = 0 \ \mathcal{E}\text{-q.e. on } D^c\}, \tag{2.5}$$

and $\mathcal{E}^D = \mathcal{E}$ on $\mathcal{F}^D$; see, e.g., Exercise 3.3.7 and Theorem 3.3.9 of [CF]. It is well known (see, e.g., [CF, Theorem 3.3.8]) that $A \subset D$ is $\mathcal{E}^D$-polar if and only if it is $\mathcal{E}$-polar. Property (2.4) combined with [CF, Proposition 2.1.10] yields the following.

Proposition 2.1. *If $(\mathcal{E},\mathcal{F})$ is irreducible and D^c is not $\mathcal{E}$-polar, then the regular Dirichlet form $(\mathcal{E},\mathcal{F}^D)$ on $L^2(D;m|_D)$ is transient.*

For $u\in\mathcal{F}_e$, the following Fukushima decomposition holds (see [CF, Theorem 4.2.6] or [FOT, Theorem 5.2.2]):

$$u(X_t)-u(X_0)=M_t^u+N_t^u,\quad t\geq 0, \tag{2.6}$$

where M^u is a martingale additive functional of X having finite energy and N^u is a continuous additive functional of X having zero energy. The predictable quadratic variation $\langle M^u\rangle$ of the square-integrable martingale M^u is a positive continuous additive functional of X, whose corresponding Revuz measure is denoted by $\mu_{\langle u\rangle}$. We call $\mu_{\langle u\rangle}$ the energy measure of $u\in\mathcal{F}_e$. It is known that

$$\frac{1}{2}\mu_{\langle u\rangle}(\mathcal{X})\leq\mathcal{E}(u,u)\leq\mu_{\langle u\rangle}(\mathcal{X})\quad\text{for } u\in\mathcal{F}_e.$$

When $(\mathcal{E},\mathcal{F})$ admits no killing inside $\mathcal{X}$, which is equivalent to the Hunt process admitting no killing inside $\mathcal{X}$ (that is, $\mathbb{P}^x(X_{\zeta-}\in\mathcal{X},\zeta<\infty)=0$ for $\mathcal{E}$-q.e. $x\in\mathcal{X}$), we have

$$\mathcal{E}(u,u)=\frac{1}{2}\mu_{\langle u\rangle}(\mathcal{X})\quad\text{for } u\in\mathcal{F}_e. \tag{2.7}$$

When $u\in\mathcal{F}_e$ is bounded, its energy measure $\mu_{\langle u\rangle}$ can be computed by the formula

$$\int_{\mathcal{X}}v(x)\mu_{\langle u\rangle}(dx)=2\mathcal{E}(u,uv)-\mathcal{E}(u^2,v)\qquad\text{for all bounded } v\in\mathcal{F}. \tag{2.8}$$

For general $u\in\mathcal{F}_e$, $\mu_{\langle u\rangle}$ is the increasing limit of $\mu_{\langle u_n\rangle}$ as $n\to\infty$, where $u_n:=(-n)\vee(u\wedge n)\in\mathcal{F}_e$. See (4.3.12)-(4.3.13), and Theorems 4.3.10 and 4.3.11 of [CF] for the above stated properties of $\mu_{\langle u\rangle}$.

The following is taken from Theorem 2.4.3 and Theorem 4.3.4 of [CF].

Proposition 2.2. *The following are equivalent.*

(i) $(\mathcal{E},\mathcal{F})$ *is strongly local (i.e., $\mathcal{E}(u,v)=0$ whenever $u,v\in\mathcal{F}$, the support of u is compact and v is constant on a neighborhood of the support of u);*

(ii) $\mathcal{E}(u,v)=0$ *whenever $u,v\in\mathcal{F}$ with $u(v-c)=0$ m-a.e. on $\mathcal{X}$ for some constant c;*

(iii) *The associated Hunt process X is a diffusion with no killing inside $\mathcal{X}$; that is, there is a Borel properly exceptional set $\mathcal{N}_0\subset\mathcal{X}$ so that for every $x\in\mathcal{X}\setminus\mathcal{N}_0$,*

$$\mathbb{P}^x(X_t\text{ is continuous in } t\in[0,\zeta))=1\text{ and }\quad\mathbb{P}^x(X_{\zeta-}\in\mathcal{X},\zeta<\infty)=0. \tag{2.9}$$

In Theorem 4.7 below, a new criterion for irreducibility will be given for strongly local regular Dirichlet forms.

We use notation $V\Subset D$ for V being a relatively compact open subset of D. For any open set U, we define

$$\mathcal{F}_{\rm loc}^U:=\left\{f\;\middle|\;\begin{array}{l}f\text{ is an } m\text{-equivalence class of } \mathbb{R}\text{-valued Borel measurable}\\ \text{functions on } \mathcal{X} \text{ such that for each } V\Subset U,\text{ there is some}\\ g\in\mathcal{F}\text{ so that } f=g\ m\text{-a.e. on } V.\end{array}\right\}. \tag{2.10}$$

Note that each $f \in \mathcal{F}_{\rm loc}^U$ admits an m-version that is $\mathcal{E}$-quasi-continuous on U, which is unique modulo an $\mathcal{E}$-polar set. We always let a function in $\mathcal{F}_{\rm loc}^U$ be represented by its quasi-continuous version. When $U = \mathcal{X}$, we simply write $\mathcal{F}_{\rm loc}$ for $\mathcal{F}_{\rm loc}^{\mathcal{X}}$.

When the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is strongly local, the energy measure $\mu_{\langle u\rangle}$ has the following strongly local property; see [CF, Proposition 4.3.1 and Theorem 4.3.10(i)].

Proposition 2.3. *Suppose the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is strongly local and D is an open subset of $\mathcal{X}$. Then*

(i) $\mu_{\langle u\rangle}(D) = 0$ *if $u \in \mathcal{F}_e$ and u is constant $\mathcal{E}$-q.e. on D;*

(ii) $\mu_{\langle u\rangle} = \mu_{\langle v\rangle}$ *on D for every $u, v \in \mathcal{F}$ so that $u - v$ is a constant $\mathcal{E}$-q.e. on D.*

Let $\{U_k; k \geq 1\}$ be an increasing sequence of relative compact open subsets whose union is $\mathcal{X}$. For $u \in \mathcal{F}_{\rm loc}$, there is some $u_k \in \mathcal{F}$ so that $u_k = u$ m-a.e. on U_k. Define $\mu_{\langle u\rangle} = \mu_{\langle u_k\rangle}$ on U_k. Since $(\mathcal{E}, \mathcal{F})$ is strongly local, $\mu_{\langle u\rangle}$ is uniquely defined by Proposition 2.3(ii). In view of (2.7), this allows us to extend the definition of $\mathcal{E}$ to $\mathcal{F}_{\rm loc}$ by setting

$$\mathcal{E}(u, u) := \frac{1}{2}\mu_{\langle u\rangle}(\mathcal{X}), \quad u \in \mathcal{F}_{\rm loc}. \tag{2.11}$$

In this paper, we will use time change of Dirichlet form and its associated Hunt process so we need the notion of smooth measure. The following definition is from [CF, Definition 2.3.13].

Definition 2.4. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(\mathcal{X}; m)$. A (positive) Borel measure μ on $\mathcal{X}$ is *smooth* if it satisfies the following conditions:

(a) μ charges no $\mathcal{E}$-polar set;

(b) there exists an $\mathcal{E}$-nest $\{F_k\}$ such that $\mu(F_k) < \infty$ for all $k \geq 1$.

By [CF, Theorem 1.2.14], the above definition of smooth measure is equivalent to that defined in [FOT, p.83]. Clearly every positive Radon measure charging no $\mathcal{E}$-polar set is smooth, as in this case we can take the $\mathcal{E}$-nest $\{F_k\}$ to be the closure of an increasing sequence of relatively compact open sets whose union is $\mathcal{X}$. We say $D \subset \mathcal{X}$ is *quasi open* if there exists an $\mathcal{E}$-nest $\{F_n\}$ such that $D \cap F_n$ is an open subset of F_n in the relative topology for each $n \in \mathbb{N}$. The complement of a quasi open set is called *quasi closed.*

Definition 2.5. (See [CF, Definition 3.3.4] or [FOT, p.190].) Let μ be a smooth Borel measure on $\mathcal{X}$. A set $F \subset \mathcal{X}$ is called a *quasi support* of μ if it satisfies the following:
(a) F is quasi closed and $\mu(\mathcal{X} \setminus F) = 0$.
(b) If $\tilde{F}$ is another set that satisfies (a), then $F \setminus \tilde{F}$ is $\mathcal{E}$-polar.
Such a set F exists by [FOT, Theorem 4.6.3]. We say that μ has *full quasi support* if $\mathcal{X}$ is a quasi support of μ.

We assume in the remaining of this paper that $(\mathcal{E}, \mathcal{F})$ is a symmetric strongly local regular Dirichlet form on $L^2(\mathcal{X}; m)$. We call $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ a metric measure Dirichlet (MMD) space. Sometimes, to emphasize its dependence on the symmetrizing measure, we write $\mathcal{F}^m$ for $\mathcal{F}$. Let $X = \{X_t, t \geq 0; \mathbb{P}^x, x \in \mathcal{X}\}$ be the diffusion process associated with $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$, whose lifetime is denoted as ζ and the shift operators denotes as $\{\theta_t\}_{t\geq 0}$. The one-point compactification of the locally compact metric space $(\mathcal{X}, d)$ is denoted as $\mathcal{X}_\partial := \mathcal{X} \cup \{\partial\}$. For a nearly Borel measurable set $A \subset \mathcal{X}_\partial$, define the stopping times

$$\sigma_A = \inf\{t > 0 : X_t \in A\}, \quad \tau_A = \sigma_{\mathcal{X}_\partial \setminus A} = \inf\{t > 0 : X_t \notin A\},$$

and write τ_x for $\tau_{\{x\}}$. Note that by definition, $\tau_A \leq \zeta$.

Here and in the following, we use the convention that $X_\infty := \partial$, and that any function u defined on a subset of $\mathcal{X}$ is extended to $\{\partial\}$ by taking $u(\partial) = 0$.

The following definition is taken from [Che, Definition 2.1].

Definition 2.6. Let D be an open subset of $\mathcal{X}$. We say a function u is *harmonic* in D (with respect to the process X) if for every relatively compact open subset U of D, $t \mapsto u(X_{t\wedge\tau_U})$ is a uniformly integrable $\mathbb{P}^x$-martingale for q.e. $x \in U$. We say u is *regular harmonic* in D (with respect to the process X) if $\mathbb{E}^x[|u(X_{\tau_D})|] < \infty$ and $u(x) = \mathbb{E}^x[u(X_{\tau_D})]$ for q.e. $x \in D$.

Remark 2.7. (i) Observe that if $\mathbb{P}^x(\tau_U < \infty) = 1$ for q.e. $x \in U$, then $t \mapsto u(X_{t\wedge\tau_U})$ is a uniformly integrable $\mathbb{P}^x$-martingale for q.e. $x \in U$ if and only if $u(x) = \mathbb{E}^x\left[u(X_{\tau_U})\right]$ for q.e. $x \in U$. Sufficient conditions on $\mathbb{P}^x(\tau_U < \infty) = 1$ for q.e. $x \in U$ are given in Propositions 3.1 and 3.2 below.

(ii) Using the Markov property of X it is clear that any regular harmonic function in D is harmonic in D.

The relation of the above probabilistic notion of harmonicity to the analytic notion of harmonicity has been investigated in [Che] for general symmetric regular Dirichlet forms. In the setting of strongly local symmetric regular Dirichlet forms, one direction becomes much easier to analyze; cf. [Che, Theorem 2.7 and Remark 2.8].

Definition 2.8. Let D be an open subset of $\mathcal{X}$. We say a function u is *$\mathcal{E}$-harmonic* in D if $u \in \mathcal{F}^D_{\rm loc}$ and

$$\mathcal{E}(u, v) = 0 \quad \text{for every } v \in C_c(D) \cap \mathcal{F}. \tag{2.12}$$

Note that as explained in (2.11), the bilinear form $\mathcal{E}(u, v)$ in (2.12) is well defined due to the strong locality of $(\mathcal{E}, \mathcal{F})$. If $h \in \mathcal{F}_e$ is $\mathcal{E}$-harmonic in D and D^c is not $\mathcal{E}$-polar, then it follows immediately from [CF, Theorem 3.4.8] or [FOT, Theorem 4.6.5] that $h(x) = \mathbb{E}^x[h(X_{\tau_D})]$ for q.e. $x \in D$; that is, h is regular harmonic in D. In general, we have the following.

Proposition 2.9. *Suppose D is an open subset of $\mathcal{X}$.*

(i) *Any $\mathcal{E}$-harmonic function in D is harmonic in D. Moreover, it is regular harmonic in any relatively compact open subset U of D so that $\mathcal{X} \setminus U$ is not $\mathcal{E}$-polar.*

(ii) *If u is locally bounded and harmonic in D, then u is $\mathcal{E}$-harmonic in D.*

Proof. (i) This is essentially a particular case of [Che, Theorem 2.7]. For strongly local regular Dirichlet forms, the proof can be much simplified. For reader's convenience, we present a proof here. Suppose that h is an $\mathcal{E}$-harmonic function in D. Let U be any relatively compact open subset of D. Since $h \in \mathcal{F}^D_{\rm loc}$, there is some $f \in \mathcal{F}$ so that $f = h$ m-a.e. and hence q.e. on U as they are all represented by their quasi-continuous versions, and

$$\mathcal{E}(f, v) = \mathcal{E}(u, v) = 0 \quad \text{for every } v \in C_c(U) \cap \mathcal{F}. \tag{2.13}$$

If $U = \mathcal{X}$ q.e., then it follows from (2.13) that $\mathcal{E}(f,f) = 0$, and so, by [Che, Lemma 2.2], $\mathbb{P}^x(f(X_t) = f(X_0) \text{ for all } t \geq 0) = 1$ for q.e. $x \in \mathcal{X}$. Clearly h is harmonic in D in this case. If $\mathcal{X} \setminus U$ has positive $\mathcal{E}$-capacity, by [CF, Theorem 3.4.8] or [FOT, Theorem 4.6.5], $h(x) = f(x) = \mathbb{E}^x[f(X_{\tau_U})] = \mathbb{E}^x[h(X_{\tau_U})]$ for q.e. $x \in U$; that is, h is regular harmonic in U. It follows by the Markov property of X that that $h(X_{t\wedge\tau_U})$ is a uniformly $\mathbb{P}^x$-martingale for q.e. $x \in U$. So h is harmonic in D.

(ii) This is a special case of [Che, Theorem 2.9] so we omit its proof. □

It follows from Proposition 2.9 that a locally bounded measurable function u in D is harmonic in D if and only if it is $\mathcal{E}$-harmonic in D.

Proposition 2.10. *Suppose D is an open subset of $\mathcal{X}$.*

(i) *Any non-negative regular harmonic function u in D is the limit of an increasing sequence of bounded regular harmonic functions $\{u_n; n \geq 1\}$ in D.*

(ii) *Suppose that u is a non-negative harmonic function in D and U is a relatively compact open subset of D. Then u is the increasing limit on U of a sequence of bounded harmonic functions on U.*

Proof. (i) For $n \geq 1$, define $u_n(x) = \mathbb{E}^x\left[(n \wedge u)(X_{\tau_D})\right]$ for $x \in D \setminus \mathcal{N}_0$. Clearly u_n is a bounded regular harmonic function in D that increases to $u(x) = \mathbb{E}^x\left[u(X_{\tau_D})\right]$, $x \in D \setminus \mathcal{N}_0$, as $n \to \infty$.

(ii) Let U be any relatively compact open subset of D. By definition, $t \mapsto u(X_{t\wedge\tau_U})$ is a $\mathbb{P}^x$-uniformly integrable martingale for q.e. $x \in U$. Thus $u(X_{t\wedge\tau_U})$ converges to some random variable $\xi \geq 0$ in $L^1(\mathbb{P}^x)$ and $\mathbb{P}^x$-a.s. for q.e. $x \in U$. In particular, $u(x) = \mathbb{E}^x[\xi]$ for q.e. $x \in U$. For each $n \geq 1$, define $u_n(x) = \mathbb{E}^x[\xi \wedge n]$. Then u_n is bounded and harmonic in U and it increases q.e. to u on U. □

3 Local regularity for transient spaces

Since $(\mathcal{E}, \mathcal{F})$ is strongly local, by Proposition 2.2, its corresponding Hunt process X is a diffusion that admits no killing inside $\mathcal{X}$. Thus there exists a Borel properly exceptional set $\mathcal{N}_0$ so that the Hunt process X, whose lifetime is denoted by ζ, can start from every point in $\mathcal{X} \setminus \mathcal{N}_0$ and (2.9) holds.

It follows that

$$\mathbb{P}^x(X_t = x \text{ for all } t \in [0,\zeta) \text{ and } \zeta < \infty) = 0 \qquad \text{for every } x \in \mathcal{X} \setminus \mathcal{N}_0. \tag{3.1}$$

In the remainder of this section unless otherwise specified, we assume in addition that $(\mathcal{E}, \mathcal{F})$ is transient. In view of [CF, Theorem 3.5.2], by enlarging the Borel properly exceptional set $\mathcal{N}_0$ if needed, we may and do assume that

$$\mathbb{P}^x\left(\zeta = \infty \text{ and } \lim_{t\to\infty} X_t = \partial\right) = \mathbb{P}^x\left(\zeta = \infty\right) \quad \text{for every } x \in \mathcal{X} \setminus \mathcal{N}_0. \tag{3.2}$$

Proposition 3.1. *For any relatively compact open subset $D \subset \mathcal{X}$,*

$$\mathbb{P}^x(\tau_D < \zeta) = 1 \quad \textit{for q.e. } x \in D.$$

Proof. Recall that ∂ is a one-point compactification of $\mathcal{X}$. Since D is a relatively compact open subset of $\mathcal{X}$, $\mathcal{X}\cup\{\partial\}\setminus\overline{D}$ is an open neighborhood of ∂. We know from (2.9) that

$$\mathbb{P}^x(X_{\zeta-}=\partial,\,\zeta<\infty)=\mathbb{P}^x(\zeta<\infty)\quad\text{for } x\in\mathcal{X}\setminus\mathcal{N}_0.$$

This together with (3.2) implies that

$$\mathbb{P}^x\left(\lim_{t\uparrow\zeta}X_t=\partial\right)=1\qquad\text{for any } x\in\mathcal{X}\setminus\mathcal{N}_0.$$

Consequently, we have that $\mathbb{P}^x(\tau_D<\zeta)=1$ for every $x\in D\setminus\mathcal{N}_0$. □

The transience condition on $(\mathcal{E},\mathcal{F})$ can be dropped from Proposition 3.1 if we assume D^c is of positive $\mathcal{E}$-capacity. We emphasize that transience of $(\mathcal{E},\mathcal{F})$ is not assumed in next Proposition.

Proposition 3.2. *Suppose that $(\mathcal{E},\mathcal{F})$ is a symmetric, irreducible, strongly local regular Dirichlet form on $L^2(\mathcal{X};m)$ and $D\subset\mathcal{X}$ is a relatively compact open set of $\mathcal{X}$ so that D^c is not $\mathcal{E}$-polar. Then $\mathbb{P}^x(\tau_D<\infty)=1$ for q.e. $x\in D$.*

Proof. Since $(\mathcal{E},\mathcal{F})$ is an irreducible, it is either transient or recurrent. The conclusion of the proposition follows readily from Proposition 3.1 if $(\mathcal{E},\mathcal{F})$ is transient. When $(\mathcal{E},\mathcal{F})$ is recurrent, the desired conclusion holds as well since X_t in fact visits $\mathcal{X}\setminus D$ infinitely often under $\mathbb{P}^x$ for q.e. $x\in\mathcal{X}$ by [CF, Theorem 3.5.6]. □

Lemma 3.3. $\mathbb{P}^x(\tau_x>0)=0$ *for every $x\in\mathcal{X}\setminus\mathcal{N}_0$.*

Proof. We have by (3.1) and (3.2) that $\tau_x<\zeta$ $\mathbb{P}^x$-a.s. for every $x\in\mathcal{X}\setminus\mathcal{N}_0$. Clearly $X_{\tau_x}=x$ on $\{0<\tau_x<\zeta\}$ since X is a diffusion. Let $A_x:=\{\tau_x>0\}$. Since $\mathbf{1}_{A_x}=\mathbf{1}_{A_x^c}\circ\theta_{\tau_x}$ on $\{0<\tau_x<\zeta\}$, we have by the strong Markov property of X that for $x\in\mathcal{X}\setminus\mathcal{N}_0$,

$$\mathbb{P}^x(A_x)=\mathbb{E}^x\left[\mathbb{P}^{X_{\tau_x}}(A_x^c);0<\tau_x<\zeta\right]=\mathbb{P}^x(A_x^c)\,\mathbb{P}^x(0<\tau_x<\zeta)=(1-\mathbb{P}^x(A_x))\mathbb{P}^x(A_x).$$

It follows that $\mathbb{P}^x(A_x)=0$. □

Let $\mathcal{B}_+(\mathcal{X})$ denote the non-negative Borel measurable functions on $\mathcal{X}$. Denote by $\{P_t;t\geq 0\}$ the transition semigroup of the process X; that is,

$$P_tf(x)=\mathbb{E}^x[f(X_t)],\qquad x\in\mathcal{X}\setminus\mathcal{N}_0,\,t>0,\,f\in\mathcal{B}_+(\mathcal{X}),$$

with the convention that $f(\partial):=0$. Define the Green operator G by

$$Gf(x):=\mathbb{E}^x\int_0^\infty f(X_t)dt=\int_0^\infty\mathbb{E}^x[f(X_t)]dt=\int_0^\infty P_tf(x)dt,\quad x\in\mathcal{X}\setminus\mathcal{N}_0,\,f\in\mathcal{B}_+(\mathcal{X}).$$

Lemma 3.4. *By enlarging the Borel properly exceptional set $\mathcal{N}_0$ if necessary, there is an $L^1(\mathcal{X};m)$-integrable function g_0 that takes values in $(0,1]$ on $\mathcal{X}$ such that*

$$Gg_0(x)\leq 1 \text{ for } x\in\mathcal{X}\setminus\mathcal{N}_0,\quad Gg_0\in\mathcal{F}_e\quad\text{and}\quad\mathcal{E}(Gg_0,Gg_0)\leq 1.\tag{3.3}$$

Proof. By [CF, Theorems 2.1.5(i) and A.2.13(v)], there is an $L^1(\mathcal{X};m)$ function g_1 bounded by 1, strictly positive on $\mathcal{X}$, such that $Gg_1 < \infty$ m-a.e. on $\mathcal{X}$ and $Gg_1 \in \mathcal{F}_e$ with $\mathcal{E}(Gg_1, Gg_1) \leq 1$. Since Gg_1 is excessive and hence finely continuous [CF, Theorems A.2.2], by enlarging the properly exceptional set $\mathcal{N}_0$ if necessary, we may and do assume that $Gg_1(x) < \infty$ for every $x \in \mathcal{X} \setminus \mathcal{N}_0$.

Let $k \geq 1$ and set $f_k = g_1 1_{\{Gg_1 \leq k\}}$ and $T_k = \inf\{t \geq 0 : Gg_1(X_t) \leq k\}$. Since Gg_1 is finely continuous we have $Gf(X_{T_k}) \leq Gg_1(X_{T_k}) \leq k$ on $\{T_k < \infty\}$. Hence by the Markov property

$$Gf_k(x) = \mathbb{E}^x 1_{\{T_k<\infty\}} \int_{T_k}^{\infty} f_k(X_s)ds = \mathbb{E}^x 1_{\{T_k<\infty\}} Gf_k(X_{T_k}) \leq k.$$

Let $g_0 = \sum_{k=1}^{\infty} k^{-1}2^{-k} f_k + 1_{\mathcal{N}_0}$. Then g_0 is strictly positive on $\mathcal{X}$, and

$$Gg_0(x) \leq 1 \quad \text{for every } x \in \mathcal{X} \setminus \mathcal{N}_0,$$

$Gg_0 \in \mathcal{F}_e$ and $\mathcal{E}(Gg_0, Gg_0) \leq \mathcal{E}(Gg_1, Gg_1) \leq 1$ [CF, Theorem 2.1.12-(i)]. □

It follows from (3.3) that for every $x \in \mathcal{X} \setminus \mathcal{N}_0$, $G(x, dy)$, defined by $G(x, A) = G1_A(x)$, is a σ-finite Borel measure on $\mathcal{X}$. By the symmetry of the process X, each P_t is a symmetric operator in $L^2(\mathcal{X};m)$. Hence

$$\int_{\mathcal{X}} g(x)Gf(x)m(dx) = \int_{\mathcal{X}} f(x)Gg(x)m(dx) \qquad \text{for } f, g \in \mathcal{B}_+(\mathcal{X}). \tag{3.4}$$

Definition 3.5. *We say condition* (HC) *holds if there is an $\mathcal{E}$-nest $\{F_n; n \geq 1\}$ consisting of an increasing sequence of compact subsets with $\mathcal{N}_0 \subset \mathcal{X} \setminus \cup_n F_n$ such that if $x_0 \in \mathcal{X}$ and $r \in (0, 1]$, and f has compact support in $B(x_0, 2r)^c$, and satisfies $0 \leq f \leq cg_0$ for some $c > 0$, then $Gf(x)$ is continuous in $B(x_0, r) \cap F_n$ for every $n \geq 1$.*

Condition (HC) is a weaker condition than the condition that bounded harmonic functions in $B(x_0, 2r)$ are continuous in $B(x_0, r)$. We show in Example 8.3 an example of MMD $(\mathcal{X}, d, \mathcal{E}, \mathcal{F}, m)$ that (HC) holds but there is a bounded harmonic function in $B(x_0, 2r)$ that is not continuous at x_0.

Note that it follows from the definition of $\mathcal{E}$-nest in Section 2, if $\{F_n; n \geq 1\}$ is an $\mathcal{E}$-nest, then so is $\{K_n; n \geq 1\}$, where $K_n = \text{supp}[1_{F_n} m]$. Thus without loss of generality, in this paper we always assume that the $\mathcal{E}$-nest in (HC) has the property that $F_n = \text{supp}[1_{F_n} m]$ for every $n \geq 1$. For an $\mathcal{E}$-nest $\{F_n\}$, $\mathcal{N} = \mathcal{X} \setminus \cup_n F_n$ is $\mathcal{E}$-polar and, in particular, has zero m-measure.

Theorem 3.6. *Assume that condition* (HC) *holds with an $\mathcal{E}$-nest $\{F_n\}$. Let $\mathcal{N}$ be a Borel properly exceptional set that contains $\mathcal{X} \setminus \cup_n F_n \supset \mathcal{N}_0$. Then for every $x \in \mathcal{X} \setminus \mathcal{N}$, $G(x, dy)$ is absolutely continuous with respect to m. Consequently, for every $x \in \mathcal{X} \setminus \mathcal{N}$ and $t > 0$, $P_t(x, dy) := \mathbb{P}^x(X_t \in dy)$ is absolutely continuous with respect to m.*

Proof. It follows from (3.4) that $G(x, A) = 0$ m-a.e. on $\mathcal{X}$ for every $A \subset \mathcal{X}$ with $m(A) = 0$ (by taking $f = 1_A$ and $g = 1$). Let g_0 be the strictly positive function from Lemma 3.4. Fix $x_0 \in \mathcal{X} \setminus \mathcal{N}$ and $r > 0$. For $j \geq 1$, let $E_j = \{x \in \mathcal{X} : 2^{-j} < g_0(x) \leq 2^{1-j}\}$. Then the E_j form a partition of $\mathcal{X} \setminus \mathcal{N}_0$. Let $A \subset B(x_0, 2r)^c$ with $m(A) = 0$. Since $1_{A \cap E_j} \leq 2^j g_0$, we have by condition (HC) that for each $k \geq 1$ and $i \geq 1$ the function $x \mapsto G(x, A \cap E_j)$ is continuous and therefore zero on $B(x_0, r) \cap F_k$. Thus $G(x, A \cap E_j) = 0$ for every $x \in B(x_0, r) \setminus \mathcal{N}$. Consequently,

$G(x, A) = \sum_{j=1}^{\infty} G(x, A \cap E_j) = 0$ for every $x \in B(x_0, r) \setminus \mathcal{N}$. In particular, this shows that for every $x_0 \in \mathcal{X} \setminus \mathcal{N}$,

$$G(x_0, dy) \text{ is absolutely continuous with respect to } m(dy) \text{ on } \mathcal{X} \setminus \{x_0\}. \tag{3.5}$$

We claim that $G(x_0, dy)$ is absolutely continuous with respect to $m(dy)$ on $\mathcal{X}$. This is clearly true if $m(\{x_0\}) > 0$. We thus assume $m(\{x_0\}) = 0$ and for $t \geq 0$ define

$$h(x, t) := \mathbb{E}^x \int_t^{\zeta} 1_{\{x_0\}}(X_s)\, ds.$$

We set $h(x) = h(x, 0) = (G1_{\{x_0\}})(x)$, and need to prove that $h(x_0) = 0$.

The function h is harmonic on $\mathcal{X} \setminus \{x_0\}$ and since $m(\{x_0\}) = 0$, we have by (3.4) that $h = 0$ m-a.e. on $\mathcal{X}$. Further, by condition (HC), $h(x) = 0$ on $\mathcal{X} \setminus (\mathcal{N} \cup \{x_0\})$. Thus if $A = \{y : h(y) > 0\}$ then $A \subset \mathcal{N} \cup \{x_0\}$. Let $T = \inf\{s \geq 0 : X_s \notin A\}$; as $\mathcal{N}$ is properly exceptional and (by Lemma 3.3) X leaves x_0 immediately, we have $\mathbb{P}^{x_0}(T = 0) = 1$.

Let $\{\mathcal{F}_t\}_{t \geq 0}$ be the minimum augmented admissible filtration generated by $\{X_t\}_{t \geq 0}$. Let $t > 0$, and set $M_s = h(X_s, t - s)$ for $s \in [0, t]$. By the Markov property of X, for $s \in [0, t]$,

$$\mathbb{E}^x \left[\int_t^{\zeta} 1_{\{x_0\}}(X_r) dr \big| \mathcal{F}_s \right] = \mathbb{E}^{X_s} \int_{t-s}^{\zeta} 1_{\{x_0\}}(X_r) dr = h(X_s, t - s) = M_s.$$

Thus $\{M_s; s \in [0, t]\}$ is a right continuous non-negative martingale. If $\omega \in \{T = 0\}$ then there exists a sequence $s_n \downarrow 0$ such that $X_{s_n}(\omega) \notin A$, and thus

$$0 \leq M_{s_n}(\omega) = h(X_{s_n}, t - s_n) \leq h(X_{s_n}) = 0.$$

Consequently we have $M_0(\omega) = 0$. Thus we have $h(x_0, t) = \mathbb{E}^{x_0} M_0 = 0$. It follows then $h(x_0) = \lim_{t \to 0} h(x_0, t) = 0$.

This together with (3.5) shows that $G(x, dy)$ is absolutely continuous with respect to $m(dy)$ on $\mathcal{X}$ for every $x \in \mathcal{X} \setminus \mathcal{N}$. That $\mathbb{P}^x(X_t \in dy)$ is absolutely continuous with respect to m for every $x \in \mathcal{X} \setminus \mathcal{N}$ and $t > 0$ follows immediately from [CF, Proposition 3.1.11] or [FOT, Theorem 4.2.4]. □

With Theorem 3.6 at hand, we can deduce the following.

Theorem 3.7. *Assume that condition* (HC) *holds with an $\mathcal{E}$-nest $\{F_n\}$. Let $\mathcal{N}$ be a Borel properly exceptional set that contains $\mathcal{X} \setminus \cup_n F_n \supset \mathcal{N}_0$. Then there exists a non-negative jointly $\mathcal{B}(0, \infty) \times \mathcal{B}(\mathcal{X} \times \mathcal{X})$-measurable function $p(t, x, y)$ on $(0, \infty) \times (\mathcal{X} \setminus \mathcal{N}) \times (\mathcal{X} \setminus \mathcal{N})$ such that*

(i) *for every $f \in \mathcal{B}_+(\mathcal{X})$, $x \in \mathcal{X} \setminus \mathcal{N}$ and $t > 0$, $\mathbb{E}^x f(X_t) = \int_{\mathcal{X}} p(t, x, y) f(y) m(dy)$;*

(ii) *$p(t, x, y) = p(t, y, x)$ for every $x, y \in \mathcal{X} \setminus \mathcal{N}$ and $t > 0$;*

(iii) *For every $t, s > 0$ and $x, y \in \mathcal{X} \setminus \mathcal{N}$, $p(t + s, x, y) = \int_{\mathcal{X}} p(t, x, z) p(s, z, y) m(dy)$.*

Consequently, $g(x, y) := \int_0^{\infty} p(t, x, y) dt$, $x, y \in \mathcal{X} \setminus \mathcal{N}$, is a non-negative jointly $\mathcal{B}(\mathcal{X} \times \mathcal{X})$-measurable function on $(\mathcal{X} \setminus \mathcal{N}) \times (\mathcal{X} \setminus \mathcal{N})$ such that

(iv) *$Gf(x) = \int_{\mathcal{X}} g(x, y) f(y) m(dy)$ for every $x \in \mathcal{X} \setminus \mathcal{N}$ and $f \in \mathcal{B}_+(\mathcal{X})$;*

(v) *$g(x,y)=g(y,x)$ for every $x,y\in\mathcal{X}\setminus\mathcal{N}$, and $x\mapsto g(x,y)$ is $X|_{\mathcal{X}\setminus\mathcal{N}}$-excessive for every $y\in\mathcal{X}\setminus\mathcal{N}$.*

(vi) *For every $y_0\in\mathcal{X}\setminus\mathcal{N}$, $x\mapsto g(x,y_0)$ belongs to $\mathcal{F}_{\mathcal{X}\setminus\{y_0\}}$ and is harmonic in $\mathcal{X}\setminus\{y_0\}$. In fact, for any $x_0\in\mathcal{X}\setminus\{y_0\}$ and any $r\in(0,d(x_0,y_0))$, $x\mapsto g(x,y_0)$ is regular harmonic in $B(x_0,r)$.*

Proof. We first show that for each $x\in\mathcal{X}\setminus\mathcal{N}$ and $t>0$, X has a pointwisely defined transition density function $p(t,x,y)$. This part is almost the same as that for [BBCK, Theorem 3.1]. For reader's convenience, we spell out the details here.

By Theorem 3.6, for every $t>0$ and $x\in\mathcal{X}\setminus\mathcal{N}$ there is an integrable kernel $y\mapsto p_0(t,x,y)$ defined on $\mathcal{X}$ such that

$$\mathbb{E}^x\left[f(X_t)\right]=P_tf(x)=\int_{\mathcal{X}}p_0(t,x,y)f(y)\,m(dy)\qquad\text{for every } f\in\mathcal{B}_b(\mathcal{X}).\tag{3.6}$$

We note that $p_0(t,x,y)$ can be chosen to be jointly Borel measurable on $(0,\infty)\times\mathcal{X}\times\mathcal{X}$ by an application of the martingale convergence theorem (see, e.g., [GK, Proposition 5.6] with $H_t(x,y)\equiv\infty$). From the semigroup property $P_{t+s}=P_tP_s$, we have for every $t,s>0$ and $x\in\mathcal{X}\setminus\mathcal{N}$,

$$p_0(t+s,x,y)=\int_{\mathcal{X}}p_0(t,x,z)p_0(s,z,y)m(dz)\qquad\text{for } m\text{-a.e. } y\in\mathcal{X}.\tag{3.7}$$

Note that since P_t is symmetric, we have for each fixed $t>0$,

$$p_0(t,x,y)=p_0(t,y,x)\qquad\text{for } m\text{-a.e. } (x,y)\in\mathcal{X}\times\mathcal{X}.\tag{3.8}$$

For every $t>0$ and $x,y\in\mathcal{X}\setminus\mathcal{N}$, let $s\in(0,t/3)$ and define

$$p(t,x,y):=\int_{\mathcal{X}}p_0(s,x,w)\left(\int_{\mathcal{X}}p_0(t-2s,w,z)p_0(s,y,z)m(dz)\right)m(dw).\tag{3.9}$$

Clearly, $p(t,x,y)$ is jointly Borel measurable on $(0,\infty)\times(\mathcal{X}\setminus\mathcal{N})\times(\mathcal{X}\setminus\mathcal{N})$. By (3.7) and (3.8), the above definition is independent of the choice of $s\in(0,t/3)$. Clearly by (3.8) with $t-2s$ in place of t and (w,z) in place of (x,y), we see that

$$p(t,x,y)=p(t,y,x)\quad\text{for every } x,y\in\mathcal{X}\setminus\mathcal{N}.\tag{3.10}$$

By the semigroup property (3.7), (3.6) and (3.8), we have for any $\phi\geq0$ on $\mathcal{X}$ and $x\in\mathcal{X}\setminus\mathcal{N}$,

$$\begin{aligned}
&\mathbb{E}^x\left[\phi(X_t)\right]\\
&=\int_{\mathcal{X}}\left(\int_{\mathcal{X}}p_0(s,x,w)p_0(t-s,w,y)m(dw)\right)\phi(y)m(dy)\\
&=\int_{\mathcal{X}}\left(\int_{\mathcal{X}}p_0(s,x,w)\left(\int_{\mathcal{X}}p_0(t-2s,w,z)p_0(s,y,z)m(dz)\right)m(dw)\right)\phi(y)m(dy)\\
&=\int_{\mathcal{X}}p(t,x,y)\phi(y)m(dy).
\end{aligned}\tag{3.11}$$

Thus for each $x \in \mathcal{X} \setminus \mathcal{N}$, $p(t,x,y)$ coincides with $p_0(t,x,y)$ for m-a.e. $y \in \mathcal{X}$. For $t,s>0$ and $x,y \in \mathcal{X}\setminus\mathcal{N}$, take $s_0 \in (0, (t\wedge s)/3)$. We have by (3.7)-(3.9)

$$
\begin{aligned}
& p(t+s,x,y) \\
&= \int_{\mathcal{X}} p_0(s_0,x,w)\left(\int_{\mathcal{X}} p_0(t+s-2s_0,w,z)p_0(s_0,y,z)m(dz)\right)m(dw) \\
&= \int_{\mathcal{X}^5} p_0(s_0,x,w)p_0(t-2s_0,w,u_1)p_0(s_0,u_1,u_2)p_0(s_0,u_2,v)p_0(s-2s_0,v,z) \\
&\qquad p_0(s_0,y,z)m(dw)m(du_1)m(du_2)m(dz)m(dv) \\
&= \int_{\mathcal{X}} p(t,x,v)p(s,v,y)m(dv). \qquad (3.12)
\end{aligned}
$$

Define $g(x,y) := \int_0^\infty p(t,x,y)dt$ for $x,y \in \mathcal{X}\setminus\mathcal{N}$. Note that $g : (\mathcal{X}\setminus\mathcal{N})\times(\mathcal{X}\setminus\mathcal{N}) \to [0,\infty]$ is jointly Borel measurable. It follows from (3.11) and Fubini theorem that for every $f \in \mathcal{B}_+(\mathcal{X})$,

$$
Gf(x) := \mathbb{E}^x \int_0^\infty f(X_s)ds = \int_{\mathcal{X}} g(x,y)f(y)m(dy) \quad \text{for every } x \in \mathcal{X}\setminus\mathcal{N}.
$$

Clearly by (3.10), $g(x,y)=g(y,x)$ for every $x,y\in\mathcal{X}\setminus\mathcal{N}$. Note that for each fixed $y \in \mathcal{X}\setminus\mathcal{N}$ and $t>0$, by Fubini theorem and (3.12),

$$
P_t g(\cdot,y)(x) = \int_{\mathcal{X}} p(t,x,z)g(z,y)\,m(dz) = \int_t^\infty p(s,x,y)\,ds \le g(x,y) \quad \text{for every } x \in \mathcal{X}\setminus\mathcal{N},
$$

and $\lim_{t\downarrow 0} P_t g(\cdot,y)(x) = g(x,y)$ (this limit exists since it is non-increasing). This shows that for each fixed $y\in\mathcal{X}\setminus\mathcal{N}$, $x\mapsto g(x,y)$ is an excessive function of $X|_{\mathcal{X}\setminus\mathcal{N}}$.

The proof of (vi) is similar to that for [KW, Proposition 6.2]. Let $y_0 \in \mathcal{X}\setminus\mathcal{N}$. For any $x_0 \in \mathcal{X}\setminus\{y_0\}$, take $0<r<d(x_0,y_0)$ and $0<r_1<d(x_0,y_0)-r$. For any non-negative $f \in C_c(B(y_0,r_1))$, by Fubini theorem and the strong Markov property of X, for each $x \in B(x_0,r)\setminus\mathcal{N}$,

$$
\begin{aligned}
\int_{B(y_0,r_1)} \mathbb{E}^x\left[g(X_{\tau_{B(x_0,r)}},y)\right]f(y)m(dy) &= \mathbb{E}^x\left[(Gf)(X_{\tau_{B(x_0,r)}})\right] \\
&= Gf(x) = \int_{B(y_0,r_1)} g(x,y)f(y)m(dy).
\end{aligned}
$$

Hence for each $x \in B(x_0,r)\setminus\mathcal{N}$,

$$
\mathbb{E}^x\left[g(X_{\tau_{B(x_0,r)}},y)\right] = g(x,y) \quad \text{for } m\text{-a.e. } y \in B(y_0,r_1).
$$

Since $y\mapsto g(z,y)$ is $X|_{\mathcal{X}\setminus\mathcal{N}}$-excessive, it follows from the monotone convergence theorem, Fubini theorem, the fine continuity of $y\mapsto g(z,y)$ (see [CF, Theorem A.2.2] or [FOT, Theorem A.2.5]) and Fatou's lemma that

$$
\begin{aligned}
\mathbb{E}^x\left[g(X_{\tau_{B(x_0,r)}},y_0)\right] &= \lim_{t\downarrow 0} P_t\left(\mathbb{E}^x g(X_{\tau_{B(x_0,r)}},\cdot)\right)(y_0) \\
&\ge \limsup_{t\downarrow 0}\left(P_t g(x,\cdot)1_{B(y_0,r_1)}\right)(y_0) \\
&\ge \mathbb{E}^{y_0}\left[\liminf_{t\downarrow 0} g(x,X_t)1_{B(y_0,r_1)}(X_t)\right] \\
&= g(x,y_0).
\end{aligned}
$$

On the other hand, clearly $\mathbb{E}^x\left[g(X_{\tau_{B(x_0,r)}}, y_0)\right] \le g(x, y_0)$ as $x \mapsto g(x, y_0)$ is excessive for $X|_{\mathcal{X}\setminus\mathcal{N}}$ and hence $\{g(X_t, y_0)\}_{t\ge 0}$ is a right-continuous supermartingale by [CF, proof of Theorem A.2.2]. Thus we have $\mathbb{E}^x\left[g(X_{\tau_{B(x_0,r)}}, y_0)\right] = g(x, y_0)$ for every $x \in B(x_0, r) \setminus \mathcal{N}$; that is, $x \mapsto g(x, y_0)$ is regular harmonic in $B(x_0, r)$. This in particular proves that $x \mapsto g(x, y_0)$ is harmonic in $\mathcal{X}\setminus\{y_0\}$. □

We call the function $g(x, y)$ in Theorem 3.7 the Green function of X.

Remark 3.8. There are gaps in the proofs of the existence of a Green function in [BBK] and [GH, Lemma 5.2]. For details of the gap in [GH], see [BM2, Remark 4.19]. The gap in [BBK] is that it is not proven that the Green's function is the integral kernel of the Green operator (cf. Theorem 3.7 (iv)).

We next give a sufficient condition for (HC).

Definition 3.9. (i) We say that the *(non-scale-invariant) elliptic Harnack inequality* (Ha) holds on $\mathcal{X}$ if for any ball $B = B(x_0, r)$ in $\mathcal{X}$, there are constants $C_B > 1$ and $\delta_B \in (0, 1)$ such that $B(x_0, \delta_B r)$ is relatively compact and that for any non-negative bounded $u \in \mathcal{F}_e$ that is regular harmonic in $B(x_0, r)$,

$$\operatorname{esssup}_{B(x_0,\delta_B r)} u \le C_B \operatorname{essinf}_{B(x_0,\delta_B r)} u. \tag{3.13}$$

Remark 3.10. (i) Note that due to the local compactness of $(\mathcal{X}, d)$, for any $x \in \mathcal{X}$ and $r > 0$, there is some $\delta > 0$ so that $B(x, \delta r)$ is relatively compact and so $0 < m(B(x, \delta r)) < \infty$, as m is a Radon measure with full support on $\mathcal{X}$. So the relative compactness condition on $B(x_0, \delta_B r)$ in (Ha) is always satisfied if we take $\delta_B > 0$ small enough.

(ii) If (Ha) holds, then (3.13) holds for any non-negative $u \in \mathcal{F}_e$ that is regular harmonic in $B(x_0, r)$. This is because $u(x) = \mathbb{E}^x\left[u(X_{\tau_{B(x_0,r)}})\right]$ is the increasing limit of $u_n(x) := \mathbb{E}^x\left[(u \wedge n)(X_{\tau_{B(x_0,r)}})\right]$ as $n \to \infty$ q.e. on $\mathcal{X}$, and by [CF, Theorem 3.4.8], $u_n \ge 0$ is a bounded function in $\mathcal{F}_e$ that is regular harmonic in $B(x_0, r)$.

(iii) Note that if property (3.13) holds for a ball $B = B(x_0, r)$ with constants C_B and δ_B then it holds for any larger ball $B(x_0, R)$ with constants C_B and $\delta_B r/R$.

Theorem 3.11. *Assume that* (Ha) *holds and that*

$$\lambda_{\mathcal{X}} := \inf\left\{\mathcal{E}(f, f) : f \in \mathcal{F} \text{ with } \|f\|_{L^2(\mathcal{X};m)} = 1\right\} > 0.$$

Then (HC) *holds. Moreover, the $\mathcal{E}$-nest $\{F_n\}$ in* (HC) *can be taken so that for any compact subset $K \subset \mathcal{X}$ and any $x_0 \in \mathcal{X} \setminus K$, there exist some $r_0 > 0$ with $B(x_0, r_0) \subset K^c$ and a constant $C = C(K, x_0, r_0) > 0$ such that for any $f \in L^1(K; m)$, Gf is continuous on each $K^c \cap F_n$ and*

$$\sup_{x\in B(x_0,r_0)\setminus\mathcal{N}} |Gf(x)| \le C \int_K f(x) m(dx). \tag{3.14}$$

Proof. Since $\lambda_{\mathcal{X}} > 0$, $Gf \in \mathcal{F} \subset L^2(\mathcal{X}; m)$ and $\|Gf\|_{L^2} \leq \lambda_{\mathcal{X}}^{-1} \|f\|_{L^2}$ for every $f \in L^2(\mathcal{X}; m)$.

Under $\lambda_{\mathcal{X}} > 0$ and (Ha), for every $x_0 \in \mathcal{X}$, $r > 0$, and any ball $B(y_0, R) \subset \mathcal{X} \setminus B(x_0, r)$, by Remark 3.10(iii) above and the same argument as that for [GH, (5.10)], we have for any $f \in L^1(\mathcal{X}; m)$ with $f = 0$ on $B(y_0, \delta_{B(y_0,R)} R))^c$,

$$\text{esssup}_{B(x_0,\delta r)} |Gf| \leq \frac{C_{B(x_0,r)} C_{B(y_0,R)}}{\lambda_{\mathcal{X}} \sqrt{m(B(x_0, \delta_{B(x_0,r)} r)) m(B(y_0, \delta_{B(y_0,R)} R))}} \|f\|_{L^1(\mathcal{X};m)}. \tag{3.15}$$

Observe that by the strong Markov property of X, for such f, Gf is regular harmonic in $B(x_0, r)$.

Let $\{x_k; k \geq 1\} \subset \mathcal{X}$ be a dense sequence of points in $\mathcal{X}$, and

$$\Lambda := \big\{ \eta = (x_i, x_j, r_k, r_l) : i, j, k, l \geq 1, r_k \in \mathbb{Q}_+, r_l \in \mathbb{Q}_+ \text{ with } B(x_i, r_k) \cap B(x_j, r_l) = \emptyset \big\}.$$

Let $\delta_{i,k} \in (0, 1)$ be one half of the largest positive constant δ_B in (Ha) for the validity of the Harnack inequality in the ball $B(x_i, r_k)$ and $C_{i,k} > 0$ be the corresponding comparison constant. We select $\delta_{i,k}$ in this way to ensure the uniformality of the constants δ_B when the centers or radius of the balls are close to each other. This uniformality is needed when we do the finite covering of such balls over compact set K in the last step of this proof. Note that Λ is a countable set. For each $\eta = (x_i, x_j, r_k, r_l) \in \Lambda$, let $\{f_p, p \geq 1\}$ be a dense sequence in $C_c(B(x_j, \delta_{j,l} r_l))$ with respect to the supremum norm. Since $f_k \in L^2(\mathcal{X}; m)$, $Gf_k \in \mathcal{F}$ and it is quasi-continuous by the 0-order version of [CF, Proposition 3.1.9] or [FOT, Theorem 4.2.3]. Thus there is an $\mathcal{E}$-nest $\{F_n^{(\eta)}, n \geq 1\}$ consisting of an increasing sequence of compact sets such that $\mathcal{N}_0 \subset \mathcal{X} \setminus \cup_{n=1}^\infty F_n^{(\eta)}$ and Gf_p is continuous on each $F_n^{(\eta)}$ for every integer $p \geq 1$; see [CF, Lemma 1.3.1]. Let $\mathcal{N}_\eta := \mathcal{X} \setminus \cup_{n=1}^\infty F_n^{(\eta)}$, which is $\mathcal{E}$-polar and in particular has zero m-measure, and $C_\eta := \frac{C_{i,k} C_{j,l}}{\lambda_{\mathcal{X}} \sqrt{m(B(x_i, \delta_{i,k} r_k)) m(B(x_j, \delta_{j,l} r_l))}}$. Inequality(3.15) yields that

$$\sup_{x \in B(x_i, \delta_{i,k} r_k) \setminus \mathcal{N}_\eta} |Gf_{k_1}(x) - Gf_{k_2}(x)| \leq C_\eta \|f_{k_1} - f_{k_2}\|_{L^1(B(x_j, \delta_{j,l} r_l); m)}. \tag{3.16}$$

Since $\{f_p, p \geq 1\}$ is uniformly dense in $C_c(B(x_j, \delta_j r_j)$ and m is a Radon measure on $\mathcal{X}$, it follows that Gf is continuous on each $B(x_i, \delta_{i,k} r_k) \cap F_n^{(\eta)}$ and

$$\sup_{x \in B(x_i, \delta_{i,k} r_k) \setminus \mathcal{N}_\eta} |Gf(x)| \leq C_\eta \|f\|_{L^1(B(x_j, \delta_{j,l} r_l); m)}, \tag{3.17}$$

for every $f \in C_c(B(x_j, \delta_j r_j)$. Let D be an open subset of $B(x_j, \delta_j r_j)$. Since 1_D can be approximated pointwise by an increasing sequence of continuous functions with compact support in $B(x_j, \delta_j r_j)$, we have from (3.17) and the monotone convergence theorem that

$$\sup_{x \in B(x_i, \delta_{i,k} r_k) \setminus \mathcal{N}_\eta} G(x, D) \leq C_\eta m(D). \tag{3.18}$$

For any Borel subset $A \subset B(x_j, \delta_j r_j)$, let $\{D_k; k \geq 1\}$ be a decreasing sequence of open subsets of $B(x_j, \delta_j r_j)$ so that $\lim_{k \to \infty} m(D_k \setminus A) = 0$. By (3.18) and the monotone convergence theorem,

$$\sup_{x \in B(x_i, \delta_{i,k} r_k) \setminus \mathcal{N}_\eta} G(x, A) \leq \liminf_{k \to \infty} \sup_{x \in B(x_i, \delta_{i,k} r_k) \setminus \mathcal{N}_\eta} |G(x, D_k))| \leq \liminf_{k \to \infty} C_\eta m(D_k) = C_\eta m(A).$$

The above in particular implies that for each $x \in B(x_i, \delta_{i,k} r_k) \setminus \mathcal{N}_\eta$,

$$G(x, dy) \text{ is absolutely continuous with respect to } m(dy) \text{ on } B(x_j, \delta_j r_j). \tag{3.19}$$

Since $\{f_p, p \geq 1\} \subset C_c(B(x_j, \delta_j r_j)$ is dense in $L^1(B(x_j, \delta_{j,l} r_l); m)$, it follows from (3.16) and (3.19) that Gf is continuous on each $B(x_i, \delta_{i,k} r_k) \cap F_n^{(\eta)}$ and

$$\sup_{x \in B(x_i, \delta_{i,k} r_k) \setminus \mathcal{N}_\eta} |Gf(x)| \leq C_\eta \|f\|_{L^1(B(x_j, \delta_{j,l} r_l); m)},$$

for every $f \in L^1(\mathcal{X}; m)$ with $f = 0$ on $B(x_j, \delta_{j,l} r_l)^c$.

By [CF, Lemma 1.3.1] and its proof, by taking suitable intersections of $F_{n_k}^{(\eta)}$'s, there is an $\mathcal{E}$-nest $\{F_n, n \geq 1\}$ consisting of an increasing sequence of compact subsets of $\mathcal{X}$ such that $\mathcal{N}_0 \subset \mathcal{X} \setminus \cup_{n=1}^\infty F_n$ and that for every $\eta \in \Lambda$, Gf is continuous on each $B(x_i, \delta_{i,k} r_k) \cap F_n$ and

$$\sup_{x \in B(x_i, \delta_{i,k} r_k) \setminus \mathcal{N}} |Gf(x)| \leq C_\eta \|f\|_{L^1(B(x_j, \delta_{j,l} r_l); m)} \tag{3.20}$$

for every $f \in L^1(\mathcal{X}; m)$ with $f = 0$ on $B(x_j, \delta_{j,l} r_l)^c$, where $\mathcal{N} := \mathcal{X} \setminus \cup_n F_n$ which is $\mathcal{E}$-polar.

Let K be a compact subset of $\mathcal{X}$. As $\{x_i\}$ is dense in $\mathcal{X}$ one can deduce from (3.20) by finite covering that for any $f \in L^1(\mathcal{X}; m)$ that vanishes outside K, Gf is continuous on each $K^c \cap F_n$, and for any $x_0 \in K^c$, there is some $r_0 > 0$ with $B(x_0, r_0) \subset K^c$ so that

$$\sup_{x \in B(x_0, r) \setminus \mathcal{N}} |Gf(x)| \leq C(K, B) \|f\|_{L^1(K; m)}.$$

This in particular proves that (HC) holds with the above $\mathcal{E}$-nest $\{F_n\}$ of compact sets. $\square$

Under the assumption of Theorem 3.11, we have by (3.14) that for every compact subset $K \subset \mathcal{X}$, $G(x, K) < \infty$ for $x \in (\mathcal{X} \setminus \mathcal{N}) \setminus K$.

Using a time change argument, we can remove the assumption of $\lambda_{\mathcal{X}} > 0$ in Theorem 3.11.

Theorem 3.12. *Assume that* (Ha) *holds. Then the conclusions of Theorems 3.11 hold. Consequently, the conclusions of Theorems 3.6 and 3.7 hold.*

Proof. Recall that our running assumption is that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$ (or equivalently, its associated Hunt process X) is transient. Let g_0 be as in Lemma 3.4, and $\mu(dx) = g_0(x) m(dx)$. We now make a time change of X via the inverse of the positive continuous additive functional $A_t := \int_0^t g_0(X_s) ds$. That is, let $Y_t = X_{\tau_t}$, where $\tau_t := \inf\{s > 0 : A_s > t\}$. Then Y is μ-symmetric and transient, and its extended Dirichet form is the same as that of X; see [CF, FOT] (since μ and m are mutually absolutely continuous). So the Dirichlet form of Y is $(\mathcal{E}, \mathcal{F}_e \cap L^2(\mathcal{X}; \mu))$ on $L^2(\mathcal{X}; \mu)$. Since Y and X share the same family of harmonic functions, (Ha) holds for Y. We claim that

$$\lambda_{\mathcal{X}}^Y := \inf\{\mathcal{E}(f, f) : f \in \mathcal{F}_e \cap L^2(\mathcal{X}; \mu) \text{ with } \|f\|_{L^2(\mathcal{X}; \mu)} = 1\} \geq 1. \tag{3.21}$$

Denote by $\widetilde{G}$ the Green potential of Y, that is, for $f \geq 0$ on $\mathcal{X}$,

$$\widetilde{G} f(x) := \mathbb{E}^x \int_0^\infty f(Y_t) dt = \mathbb{E}^x \int_0^\infty f(X_{\tau_t}) dt.$$

Using the time change, we see that $\widetilde{G}f(x) = \mathbb{E}^x \int_0^\infty (fg_0)(X_t)dt = G(fg_0)(x)$. In particular, we have $\widetilde{G}1 = Gg_0 \leq 1$. Thus for $u \in L^2(\mathcal{X};\mu)$, by Cauchy-Schwarz and the symmetry of $\widetilde{G}$ with respect to μ,

$$\begin{aligned}\int_{\mathcal{X}} (\widetilde{G}u)^2(x)\mu(dx) &\leq \int_{\mathcal{X}} \widetilde{G}(u^2)(x)\widetilde{G}1(x)\mu(dx) \leq \int_{\mathcal{X}} \widetilde{G}(u^2)(x)\mu(dx) \\ &\leq \int_{\mathcal{X}} u(x)^2\widetilde{G}1(x)\mu(dx) \leq \int_{\mathcal{X}} u(x)^2\mu(dx). \end{aligned} \tag{3.22}$$

Since the spectrum of $\widetilde{G}$ as a symmetric operator from $L^2(\mathcal{X};\mu)$ into itself is the reciprocal of that of the infinitesimal generator of Y, we conclude from (3.22) that $\lambda_{\mathcal{X}}^Y \geq 1$. Alternatively, for any $u \in L^2(\mathcal{X};\mu)$, $\int_{\mathcal{X}} u(x)\widetilde{G}u(x)\mu(dx) \leq \int_{\mathcal{X}} u(x)^2\mu(dx) < \infty$ by (3.22). It follows (cf. [CF, Theorem 2.1.12] or [FOT]) that $\widetilde{G}u \in \mathcal{F}_e \cap L^2(\mathcal{X};\mu)$ with $\mathcal{E}(\widetilde{G}u,\widetilde{G}u) \leq \int_{\mathcal{X}} u(x)\widetilde{G}u(x)\mu(dx)$. Hence for $u \in \mathcal{F}_e \cap L^2(\mathcal{X};\mu)$, we have by (3.22) and the Cauchy-Schwarz,

$$\int_{\mathcal{X}} u^2(x)\mu(dx) = \mathcal{E}(\widetilde{G}u,u) \leq \mathcal{E}(u,u)^{1/2}\mathcal{E}(\widetilde{G}u,\widetilde{G}u)^{1/2} \leq \mathcal{E}(u,u)^{1/2}\,\|u\|_{L^2(\mathcal{X};\mu)}.$$

Consequently,

$$\|u\|_{L^2(\mathcal{X};\mu)} \leq \mathcal{E}(u,u)^{1/2} \quad \text{for every } u \in \mathcal{F}_e \cap L^2(\mathcal{X};\mu).$$

This again proves the claim (3.21).

For the process Y, we can take $g_0^Y = 1$ in the role of g_0 for X in (3.3) as $\widetilde{G}1 = Gg_0 \leq 1$. By Theorem 3.11, (HC) holds for the process Y. Since $\widetilde{G}f = G(fg_0)$ and that an increasing sequence of compact sets $\{F_n\}$ of $\mathcal{X}$ is an $\mathcal{E}$-nest for the process X if and only if it is an $\mathcal{E}$-nest for the process Y in view of [CF, Lemma 3.1.17], we conclude that (HC) as well as the rest of conclusions of Theorem 3.11 hold for the process X. □

Remark 3.13. In Example 8.3, we will given an example of an irreducible strongly local Dirichlet form for which (Ha) fails but (HC) holds. In fact, for this example, there exists a discontinuous positive harmonic function.

4 Green functions

We now drop the hypothesis that $(\mathcal{E},\mathcal{F})$ is transient.

First we introduce two functions $d_{\mathcal{X}}$ and d_c on a locally compact separable metric space $(\mathcal{X},d)$. Let $\{D_k; k \geq 1\}$ be an increasing sequence of relatively compact open subsets with $\cup_{k=1}^\infty D_k = \mathcal{X}$. Define for every $x \in \mathcal{X}$,

$$d_{\mathcal{X}}(x) = \lim_{k\to\infty} \inf_{y\in\mathcal{X}\setminus D_k} d(x,y). \tag{4.1}$$

It is easy to see that

(i) $d_{\mathcal{X}}(x) \in (0,\infty]$ is well defined, independent of the choices of the sequence $\{D_k; k \geq 1\}$ of relatively compact open sets that increases to $\mathcal{X}$;

(ii) either $d_{\mathcal{X}}$ is identically infinite on $\mathcal{X}$ or $d_{\mathcal{X}}(x) < \infty$ for every $x \in \mathcal{X}$ and $|d_{\mathcal{X}}(x) - d_{\mathcal{X}}(y)| \leq d(x,y)$ for every $x,y \in \mathcal{X}$.

We call $d_{\mathcal{X}}$ the distance to the boundary function for the metric space $(\mathcal{X}, d)$.

Remark 4.1. If the metric space $(\mathcal{X}, d)$ is complete, then $d_{\mathcal{X}} \equiv \infty$ on $\mathcal{X}$.

For $x \in \mathcal{X}$, define

$$d_c(x) = \sup\{r > 0 : B(x, r) \text{ is relatively compact}\}. \tag{4.2}$$

Note that $0 < d_c(x) \le d_{\mathcal{X}}(x)$ for every $x \in \mathcal{X}$. Moreover, either d_c is identically infinite on $\mathcal{X}$ or $d_c(x) < \infty$ for every $x \in \mathcal{X}$ and $|d_c(x) - d_c(y)| \le d(x, y)$ for every $x, y \in \mathcal{X}$.

Definition 4.2. For a MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$, we say

(i) the *(scale invariant) elliptic Harnack inequality* (EHI) holds if there exist constants $\delta_H \in (0, 1)$ and $C_H \in (1, \infty)$ so that for any $x \in \mathcal{X}$, $R \in (0, d_{\mathcal{X}}(x))$, and for any nonnegative bounded harmonic function h on a ball $B(x, R)$, one has

$$\operatorname*{ess\,sup}_{B(x,\delta_H R)} h \le C_H \operatorname*{ess\,inf}_{B(x,\delta_H R)} h; \tag{4.3}$$

(ii) $\mathrm{EHI}_{\le 1}$ holds if (4.3) holds for any nonnegative bounded harmonic function on balls $B(x, R)$ with $0 < R < d_{\mathcal{X}}(x) \wedge 1$;

(iii) the *(scale invariant) local elliptic Harnack inequality* $\mathrm{EHI}_{\mathrm{loc}}$ if (4.3) holds for any nonnegative bounded harmonic function on balls $B(x, R)$ with $0 < R < d_c(x) \wedge 1$.

Remark 4.3. (i) Clearly, the EHI implies the $\mathrm{EHI}_{\le 1}$, the $\mathrm{EHI}_{\le 1}$ implies the $\mathrm{EHI}_{\mathrm{loc}}$, and the $\mathrm{EHI}_{\mathrm{loc}}$ implies (Ha).

(ii) If $(\mathcal{X}, d)$ is a geodesic metric space and $\mathrm{EHI}, \mathrm{EHI}_{\le 1}$, or $\mathrm{EHI}_{\mathrm{loc}}$ holds for some value of δ, then the same holds for any other $\delta' \in (0, 1)$ with a constant $C_H(\delta')$.

(iii) Suppose $\mathrm{EHI}_{\mathrm{loc}}$ holds and u is a nonnegative harmonic function on a ball $B(x, R)$ with $0 < R < r_c(x) \wedge 1$. By Proposition 2.10(ii), u is the increasing limit in $B(x, R/2)$ of a sequence of functions $\{u_n; n \ge 1\}$ that is bounded and harmonic in $B(x, R/2)$. It follows that (4.3) holds for u on the ball $B(x, \delta_H R)$, where $\delta'_H = \delta_H/2$. In other words, if $\mathrm{EHI}_{\mathrm{loc}}$ holds, it holds for any non-negative (possibly unbounded) harmonic function.

(iv) If $\mathrm{EHI}_{\mathrm{loc}}$ holds, in view of (iii) above, iterating the condition (4.3) gives a.e. Hölder continuity of harmonic functions, and it follows that any locally bounded harmonic function has a continuous modification [GT12, Lemma 5.2].

(v) By Proposition 2.10(i), if EHI holds, then it holds for any non-negative regular harmonic functions. If every open ball $B(x, r)$ in $(\mathcal{X}, d)$ is relatively compact, by Proposition 2.10(ii) and a similar approximation argument as in (iii) above, by ajusting the value of δ_H is needed, EHI holds (for bounded non-negative harmonic functions) if and only if (4.3) holds for any non-negative harmonic functions.

Let D be an open set of $\mathcal{X}$. Note that if $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the $\mathrm{EHI}_{\mathrm{loc}}$, then so does $(D, d, m|_D, \mathcal{E}, \mathcal{F}^D)$, where $(\mathcal{E}, \mathcal{F}^D)$ is the Dirichlet form for the part process X^D of X killed upon leaving D (see (2.5)). Let D_{diag} denote the diagonal in $D \times D$. For a subset $A \subset \mathcal{X}$, we use $\overline{A}$ to denote its closure and ∂A its boundary.

Definition 4.4. (a) Let D be a non-empty open subset of $\mathcal{X}$ such that D^c is not $\mathcal{E}$-polar. We say that $(\mathcal{E}, \mathcal{F})$ has a *regular Green function* on D if there exists a non-negative $\mathcal{B}(D \times D)$-measurable function $g_D(x, y)$ on $D \times D \setminus D$ with the following properties:

(i) (Symmetry) $g_D(x, y) = g_D(y, x)$ for all $(x, y) \in D \times D \setminus D_{\rm diag}$;

(ii) (Continuity) $g_D(x, y)$ is jointly continuous in $(x, y) \in D \times D \setminus D_{\rm diag}$;

(iii) (Occupation density) There is a Borel properly exceptional set $\mathcal{N}_D$ of $\mathcal{X}$ such that

$$\mathbb{E}^x \int_0^{\tau_D} f(X_s) ds = \int_D g_D(x, y) f(y) \, m(dy) \quad \text{for every } x \in D \setminus \mathcal{N}_D. \tag{4.4}$$

for any $f \in \mathcal{B}_+(D)$

(iv) (Excessiveness) For each $y \in D$, $x \mapsto g_D(x, y)$ is $X^D|_{D \setminus \mathcal{N}_D}$-excessive.

(v) (Harmonicity) For any fixed $y \in D$, the function $x \mapsto g_D(x, y)$ is in $\mathcal{F}_{\rm loc}^{D \setminus \{y\}}$ and for any $x_0 \in D \setminus \{y\}$ and any $r \in (0, d(x_0, y))$, $x \mapsto g_D(x, y)$ is regular harmonic in $B(x_0, r)$.

(vi) (Maximum principles) If $x_0 \in U \Subset D$, then

$$\inf_{U \setminus \{x_0\}} g_D(x_0, \cdot) = \inf_{\partial U} g_D(x_0, \cdot), \qquad \sup_{D \setminus U} g_D(x_0, \cdot) = \sup_{\partial U} g_D(x_0, \cdot). \tag{4.5}$$

We call $g_D(x, y)$ the regular Green function of $(\mathcal{E}, \mathcal{F})$ in D.

(b) We say that $(\mathcal{E}, \mathcal{F})$ has *regular Green functions* if for any bounded, non-empty open set $D \subset \mathcal{X}$ whose complement D^c is not $\mathcal{E}$-polar, $(\mathcal{E}, \mathcal{F})$ has a regular Green function $g_D(x, y)$ on D, where the properly exceptional set $\mathcal{N}_D$ in (iii) can be taken to be a Borel properly exceptional set of X that is independent of D.

Theorem 4.5. *Suppose that the MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ is irreducible and D is a non-empty open subset of $\mathcal{X}$ such that D^c is not $\mathcal{E}$-polar.*

(i) *Assume that $(D, d, m|_D, \mathcal{E}, \mathcal{F}^D)$ satisfies* (Ha). *Then $(\mathcal{E}, \mathcal{F}^D)$ has a Green function $g_D(x, y)$ in the sense that*

(i.a) *$g_D(x, y)$ is a non-negative jointly $\mathcal{B}(D \times D)$-measurable function and there is a Borel properly exceptional set $\mathcal{N}_D$ of X^D such that*

$$\mathbb{E}^x \int_0^{\tau_D} f(X_s) ds = \int_D g_D(x, y) f(y) \, m(dy), \quad x \in D \setminus \mathcal{N}_D,$$

for any $f \in \mathcal{B}_+(D)$;

(i.b) *$g_D(x, y) = g_D(y, x)$ for every $x, y \in D \setminus \mathcal{N}_D$, and $x \mapsto g_D(x, y)$ is $X^D|_{D \setminus \mathcal{N}_D}$-excessive for every $y \in D \setminus \mathcal{N}_D$;*

(i.c) *For every $y_0 \in D \setminus \mathcal{N}_D$, $x \mapsto g(x, y_0)$ is harmonic in $D \setminus \{y_0\}$. Moreover, for any $r \in (0, d(x_0, y_0))$, $x \mapsto g_D(x, y_0)$ is regular harmonic in $B(x_0, r)$.*

(ii) *If $(D, d, m|_D, \mathcal{E}, \mathcal{F}^D)$ satisfies the* $\mathrm{EHI}_{\rm loc}$, *then $(\mathcal{E}, \mathcal{F})$ has a regular Green function on D.*

Proof. Since $(\mathcal{E},\mathcal{F})$ is irreducible and D^c is not $\mathcal{E}$-polar, the regular Dirichlet form $(\mathcal{E},\mathcal{F}^D))$ on $L^2(D;m|_D)$ is transient by Proposition 2.1.

(i) The conclusion of this part follows directly from Theorem 3.12 by replacing $\mathcal{X}$ and X by D and X^D, respectively. Denote the corresponding Borel properly exceptional set in Theorem 3.12 for X^D by $\mathcal{N}_D$. Note that this $\mathcal{N}_D$ has the property that for every $x\in D\setminus\mathcal{N}_D$, the law of X^D_t under $\mathbb{P}^x$ is absolutely continuous with respect to $m|_D$ for every $t>0$. This property will be used in (4.7) as well as at the end of this proof when establishing the excessivieness of $x\mapsto g_D(x,y)$ for every $y\in D$.

(ii) Suppose now that $(D,d,m|_D,\mathcal{E},\mathcal{F}^D)$ satisfies the $\mathrm{EHI}_{\mathrm{loc}}$ in D. Let u be a bounded harmonic function in $B(x_0,2r)\subset D$. Iterating the condition (4.3) yields that there are constants $c_0>0$ and $\beta\in(0,1)$ that depend only $\delta_H(D)$ and $C_H(D)$ in (4.3) (with D in place of $\mathcal{X}$) such that

$$|u(x)-u(y)|\le c_0\|u\|_{L^\infty(B(x_0,2r))}|x-y|^\beta \quad \text{for a.e. } x,y\in B(x_0,r). \tag{4.6}$$

Since (Ha) holds on D, by (i) there is a Green function $g_D(x,y)$ in D. For each fixed $y_0\in D\setminus\mathcal{N}_D$, $x\mapsto g_D(x,y_0)<\infty$ m-a.e. and is harmonic in $D\setminus\{y_0\}$. It follows from the $\mathrm{EHI}_{\mathrm{loc}}$ and Remark 4.3(iii) that $x\mapsto g_D(x,y_0)$ is (essentially) locally bounded in $D\setminus\{y_0\}$. The Hölder estimate (4.6) implies that there is a locally Hölder continuous function $\widetilde g_D(\cdot,y_0)$ on $D\setminus\{y_0\}$ such that $\widetilde g_D(x,y_0)=g_D(x,y_0)$ for m-a.e. $x\in D$. Since $g_D(\cdot,y_0)$ is $X^D|_{D\setminus\mathcal{N}_D}$-excessive, $t\mapsto g_D(X^D_t,y_0)\in[0,\infty]$ is right continuous on $[0,\infty)$ $\mathbb{P}^x$-a.s. for every $x\in D\setminus\mathcal{N}_D$; see, e.g., [CF, Theorem A.2.2]. Since the law of X^D_t under $\mathbb{P}^x$ is absolutely continuous with respect to m_D for every $x\in D\setminus\mathcal{N}_D$, we have for every $x\in D\setminus(\mathcal{N}_D\cup\{y_0\})$, $\mathbb{P}^x$-a.s.,

$$g_D(x,y_0)=\lim_{\mathbb{Q}\ni t\to 0} g_D(X^D_t,y_0)=\lim_{\mathbb{Q}\ni t\to 0}\widetilde g_D(X^D_t,y_0)=\widetilde g_D(x,y_0). \tag{4.7}$$

Thus $g_D(x,y_0)=\widetilde g_D(x,y_0)$ for every $x\in D\setminus(\mathcal{N}_D\cup\{y_0\})$. For each $y_0\in D\setminus\mathcal{N}_D$, we define $\widetilde g_D(y_0,y_0):=g_D(y_0,y_0)$. The above shows that there is a function $\widetilde g_D(x,y)$ defined on $D\times(D\setminus\mathcal{N}_D)$ so that for each $y_0\in D\setminus D\setminus\mathcal{N}_D$, $x\mapsto\widetilde g_D(x,y)$ is locally Hölder continuous on $D\setminus\{y_0\}$ and $\widetilde g_D(x,y_0)=g_D(x,y_0)$ for every $x\in D$.

Since $g_D(x,y)$ is symmetric on $(D\times\mathcal{N}_D)\times(D\times\mathcal{N}_D)$ by (i.b), we have $\widetilde g_D(x,y)=\widetilde g_D(y,x)$ for every $x,y\in D\times\mathcal{N}_D$. We extend the definition of $\widetilde g_D(x,y)$ on $D\times(D\setminus\mathcal{N}_D)$ to $(D\times D)\setminus(\mathcal{N}_D\times\mathcal{N}_D)$ by setting $\widetilde g_D(x,y)=\widetilde g(y,x)$ for $x\in D\setminus\mathcal{N}_D$ and $y\in\mathcal{N}_D$.

Note that $y\mapsto\widetilde g_D(x_0,y)$ is continuous in $D\setminus\{x_0\}$. Clearly we have by (i.b) that $\widetilde g_D(x,y)=g_D(x,y)$ for every $x,y\in D\setminus\mathcal{N}_D$ with $x\neq y$. We next show that such defined $\widetilde g_D(x,y)$ on $(D\times D)\setminus(\mathcal{N}_D\times\mathcal{N}_D)$ is locally jointly Hölder continuous off the diagonal and thus its definition can be continuously extended to $(\mathcal{N}_D\times\mathcal{N}_D)\setminus\{(x,x):x\in\mathcal{N}_D\}$.

Let $x_0,y_0\in D\setminus\mathcal{N}_D$ with $x_0\neq y_0$. There is $r>0$ such that $B(x_0,2r)\cap B(y_0,2r)=\emptyset$. By the $\mathrm{EHI}_{\mathrm{loc}}$ in D, for every $x\in B(x_0,r)\setminus\mathcal{N}_D$ and $y\in B(y_0,r)\setminus\mathcal{N}_D$,

$$\widetilde g_D(x,y)\le C_H(D)\widetilde g_D(x,y_0)\le C_H(D)^2\widetilde g_D(x_0,y_0).$$

It follows from (4.6) that for $x_1,x_2\in B(x_0,r/2)\setminus\mathcal{N}_D$ and $y_1,y_2\in B(y_0,r/2)\setminus\mathcal{N}_D$,

$$\begin{aligned}|\widetilde g_D(x_1,y_1)-\widetilde g_D(x_2,y_2)| &\le |\widetilde g_D(x_1,y_1)-\widetilde g_D(x_2,y_1)|+|\widetilde g_D(x_2,y_1)-\widetilde g_D(x_2,y_2)|\\ &\le C_H(D)^2\widetilde g_D(x_0,y_0)c_0\left(|x_1-x_2|^\beta+|y_1-y_2|^\beta\right).\end{aligned}$$

Consequently $\widetilde g_D(x,y)$ can be extended continuously to $B(x_0,r)\times B(y_0,r)$ and hence to $D\times D\setminus D_{\mathrm{diag}}$ as a locally Hölder continuous function. Clearly, $\widetilde g_D(x,y)=\widetilde g_D(y,x)$ for $x,y\in D$

with $x \neq y$, and for each fixed $y \in D$, $x \mapsto \widetilde{g}_D(x,y)$ is harmonic in $D \setminus \{y\}$. For $w \in \mathcal{N}_D$, we define $\widetilde{g}_D(w,w) := \limsup_{x \neq y \in D, x \to w, y \to w} \widetilde{g}(x,y)$. In summary, we now have a symmetric Borel measurable function $\widetilde{g}_D(x,y) : D \times D \to [0,\infty]$ defined on $D \times D$ so that $\widetilde{g}_D(x,y)$ is locally Hölder continuous on $(D \times D) \setminus D_{\rm diag}$, $\widetilde{g}_D(x,y) = g_D(x,y)$ for $x, y \in D \setminus \mathcal{N}_D$, and for every $y \in D$, $x \mapsto \widetilde{g}_D(x,y)$ is harmonic in $D \setminus \{y\}$. From now on, we take this refined version $\widetilde{g}_D(x,y)$ for the Green function $g_D(x,y)$ in D and drop the tilde from $\widetilde{g}_D(x,y)$.

We next show that $g_D(x,y)$ is a regular Green function on D. We already have that the symmetry and continuity properties for $g_D(x,y)$. The occupation density holds for this refined $g_D(x,y)$ as well since $\mathcal{N}_D$ is $\mathcal{E}$-polar so $m(\mathcal{N}_D) = 0$. Thus it remains to show the excessiveness, harmonicity and maximum principle for $g_D(x,y)$ (see Definition 4.4). Suppose U is a relatively compact open subset of D and $x_0 \in U$. Let $r_0 > 0$ be such that $B(x_0, r_0) \subset U$. For every $y \in D \setminus (U \cup \mathcal{N}_D)$ and for every $r \in (0, r_0)$, we have by the symmetry of $g_D(x,y)$ and the strong Markov property of X that

$$\begin{aligned} \int_{B(x_0,r)} g_D(x,y) m(dx) &= \mathbb{E}^y \int_0^{\tau_D} 1_{B(x_0,r)}(X_s) ds = \mathbb{E}^y \mathbb{E}^{X^D_{\sigma_U}} \int_0^{\tau_D} 1_{B(x_0,r)}(X_s) ds \\ &= \int_{B(x_0,r)} \mathbb{E}^y g_D(x, X^D_{\sigma_U}) m(dx). \end{aligned}$$

Dividing both sides by $m(B(x_0,r))$ and then taking $r \to 0$ yields that

$$g_D(x_0, y) = \mathbb{E}^y \left[g_D(x_0, X^D_{\sigma_U}) \right]. \tag{4.8}$$

This together with the continuity of g_D on $D \times \times D \setminus D_{\rm diag}$ shows that

$$\sup_{y \in D \setminus U} g_D(x_0, y) = \sup_{y \in \partial U} g_D(x_0, y). \tag{4.9}$$

Identity (4.8) holds with $B(y_0, r)$ in place of U for any $B(y_0, r) \subset D \setminus \{x_0\}$ by exactly the same argument. Thus harmonicity of Definition 4.4(iii) holds for $g_D(x,y)$.

For each fix $x \in U \setminus \mathcal{N}_D$ and any Borel measurable function $f \geq 0$ on D, by the strong Markov property of X,

$$\int_D g_D(x,y) f(y) m(dy) = \mathbb{E}^x \int_0^{\tau_D} f(X_s) ds \geq \mathbb{E}^x G_D f(X_{\tau_U}) = \int_D \mathbb{E}^x [g_D(X_{\tau_U}, y)] f(y) m(dy).$$

Hence $g_D(x,y) \geq \mathbb{E}^x [g_D(X_{\tau_U}, y)]$ for m-a.e. $y \in D$. Since for every $z \in D$, $y \mapsto g_D(z,y)$ is continuous on $D \setminus \{z\}$, we have by Fatou's lemma that

$$g_D(x,z) \geq \mathbb{E}^x [g_D(X_{\tau_U}, z)] \quad \text{for every } z \in U \setminus \{x\}.$$

Taking $z = x_0$ and by the symmetry of $g_D(x,z)$, we get for every $x \in U \setminus (\mathcal{N}_D \cup \{x_0\})$,

$$g_D(x_0, x) \geq \mathbb{E}^x [g_D(x_0, X_{\tau_U})] \geq \inf_{y \in \partial U} g_D(x_0, y).$$

In the last inequality, we used the fact that $\mathbb{P}^x(\tau_U < \infty) = 1$ due to Proposition 3.2. By the continuity of $x \mapsto g_D(x_0, x)$ on $D \setminus \{x_0\}$, we get

$$\inf_{y \in U \setminus \{x_0\}} g_D(x_0, y) = \inf_{y \in \partial U} g_D(x_0, y).$$

This together with (4.9) establishes the maximum principle for $g_D(x,y)$.

We now show that for every $y \in D$, $x \mapsto g_D(x,y)$ is $X^D|_{D\setminus\mathcal{N}_D}$-excessive. Note that $\mathcal{N}_D$ is a properly exceptional set of X. By (i.b), the above property holds for every $y \in D \setminus \mathcal{N}_D$. For $y \in \mathcal{N}_D$, let $y_n \in D \setminus \mathcal{N}_D$ so that $x_n \to y$ as $n \to \infty$. Let $x \in D \setminus \mathcal{N}_D$. Observe that since y is $\mathcal{E}$-polar, $m(\{y\}) = 0$. As mentioned earlier, the law of X_t^D under $\mathbb{P}^x$ is absolutely continuous with respect to m for every $t > 0$. Thus by the locall Hölder continuity of g_D on $D \times D \setminus D_{\rm diag}$ and the Fatou's lemma, for every $t > 0$,

$$\mathbb{E}^x g(X_t^D, y) \le \liminf_{n\to\infty} \mathbb{E}^x g_D(X_t^D, y_n) \le \liminf_{n\to\infty} g_D(x, y_n) = g_D(x,y). \tag{4.10}$$

On the other hand, since $\mathbb{P}^x(\lim_{t\to 0} X_t^D = x) = 1$ and $x \not\to y$, we have by Fatou's lemma again,

$$g_D(x,y) = \mathbb{E}^x \left[\lim_{t\to 0} g_D(X_t^D, y) \right] \le \liminf_{t\to 0} \mathbb{E}^x g_D(X_t^D, y).$$

This together with (4.10) proves that $x \mapsto g_D(x,y)$ is $X^D|_{D\setminus\mathcal{N}_D}$-excessive for every $y \in \mathcal{N}_D$ and hence for every $y \in D$. This completes the proof that $g_D(x,y)$ is a regular Green function on D. □

Theorem 4.6. *Suppose that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ is irreducible and satisfies the* EHI$_{\rm loc}$. *Then $(\mathcal{E}, \mathcal{F})$ has regular Green functions. Moreover, the Borel properly exceptional set $\mathcal{N}$ of X in the definition of regular Green functions for $(\mathcal{E}, \mathcal{F})$ in Definition 4.4(b) can be taken in such a way that for each $x \in \mathcal{X} \setminus \mathcal{N}$ and every $t > 0$, , $\mathbb{P}^x(X_t \in dy)$ is absolutely continuous with respect to $m(dy)$.*

Proof. Since $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the EHI$_{\rm loc}$, every locally bounded harmonic function is locally Hölder continuous, and $(D, d, m|_D, \mathcal{E}, \mathcal{F}^D)$ satisfies the EHI$_{\rm loc}$ for every non-empty open subset D of $\mathcal{X}$ whose complement D^c is not $\mathcal{E}$-polar. The conclusion of this theorem will follow directly from Proposition 2.1 and Theorem 4.5(ii) if we can show that there is a common properly exceptional set $\mathcal{N}$ of X, independent of D, so that (4.4) of Definition 4.4 for regular Green function $g_D(x,y)$ on D hold for all $x \in D \setminus \mathcal{N}$.

First note that, since $(\mathcal{X}, d)$ is a locally compact separable metric space, there is an increasing sequence of relatively compact open sets $\{D_n; n \ge 1\}$ with $\cup_{n\ge 1} D_n = \mathcal{X}$. Applying Theorem 3.7 to the part process X^{D_n} of X in D, we have for each $n \ge 1$, there is a Borel properly exceptional set $\mathcal{N}_n \subset D_n$, $\mathbb{P}^x(X_t^{D_n} \in dy)$ is absolutely continuous with respect to $m(dy)$ for every $x \in D_n \setminus \mathcal{N}_n$ and $t > 0$. There is a Borel properly exceptional set $\mathcal{N} \supset \cup_{k=0}^\infty \mathcal{N}_k$. Clearly, for each $x \in \mathcal{X} \setminus \mathcal{N}$ and $t > 0$, $\mathbb{P}^x(X_t \in dy)$ is absolutely continuous with respect to $m(dy)$.

Now for any bounded, non-empty open set $D \subset \mathcal{X}$ whose complement D^c is not $\mathcal{E}$-polar, let $g_D(x,y)$ be the regular Green function on D from Theorem 4.5(ii). For $f \in \mathcal{B}_+(D)$, by Theorem 4.5, (4.4) holds for $x \in D$ if $\{x\}$ is not $\mathcal{E}$-polar. For any $x \in D \setminus \mathcal{N}$ and $\{x\}$ is $\mathcal{E}$-polar, we have by the Markov property of X and the Fatou's lemma,

$$\begin{aligned}
\mathbb{E}^x \int_0^{\tau_D} f(X_s) ds &= \lim_{t\to 0} \mathbb{E}^x \left[\mathbb{E}^{X_t^D} \int_0^{\tau_D} f(X_s) ds \right] \\
&= \lim_{t\to 0} \mathbb{E}^x \int_D g_D(X_t^D, y) f(y) m(dy) \\
&\ge \mathbb{E}^x \int_D \liminf_{t\to 0} g_D(X_t^D, y) f(y) m(dy) \\
&= \int_D g_D(x,y) f(y) m(dy),
\end{aligned}$$

where the second equality by Theorem 4.5(ii), while the last equality is due to the fact that $m(\{x\}) = 0$ and the continuity of $g_D(x,y)$ on $D \times D \setminus D_{\rm diag}$. On the other hand, as $m(\mathcal{N}_D) = 0$, we have by Fubini theorem, the symmetry of $g_D(x,y)$ and the $X^D|_{D\setminus\mathcal{N}_D}$-excessiveness of $x \mapsto g_D(x,y)$ for $y \in D \setminus \mathcal{D}$,

$$\begin{aligned}\lim_{t\to 0} \mathbb{E}^x \int_D g_D(X_t^D, y) f(y) m(dy) &= \lim_{t\to\infty} \int_{D\setminus\mathcal{N}_D} \mathbb{E}^x \left[g_D(X_t^D, y) \right] f(y) m(dy) \\ &\le \int_D g_D(x,y) f(y) m(dy)\end{aligned}$$

This proves that

$$\mathbb{E}^x \int_0^{\tau_D} f(X_s) ds = \int_D g_D(x,y) f(y) m(dy)$$

for any $x \in D \setminus \mathcal{N}$. □

We next give a sufficient condition for a strongly local MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ to be irreducible. First we present a characterization of irreducibility for such a Dirichlet form, which in fact holds also for any strongly local quasi-regular Dirichlet forms by using quasi-homeomorphism. See [CF, Theorem 5.2.16] for an irreducible characterization for recurrent Dirichlet forms.

Theorem 4.7. *Let $\mathcal{E}, \mathcal{F})$ be a strongly local regular Dirichlet form on $L^2(\mathcal{X}; m)$. Then the following are equivalent.*

(i) $(\mathcal{E}, \mathcal{F})$ *is irreducible;*

(ii) *If $u \in \mathcal{F}_{\rm loc}$ having $\mathcal{E}(u,u) = 0$, then u is constant $\mathcal{E}$-q.e. on $\mathcal{X}$.*

(iii) *If $u \in \mathcal{F}_{\rm loc} \cap L^\infty(\mathcal{X}; m)$ having $\mathcal{E}(u,u) = 0$, then u is constant $\mathcal{E}$-q.e. on $\mathcal{X}$.*

Proof. (i) ⇒ (ii): Suppose $u \in \mathcal{F}_{\rm loc}$ and $\mathcal{E}(u,u) = 0$. Let $\{U_k; k \ge 1\}$ be an increasing sequence of relative compact open subsets whose union is $\mathcal{X}$. Then for each $k \ge 1$, there is some $u_k \in \mathcal{F}$ so that $u_k = u$ m-a.e. on U_k. By Fukushima's decomposition,

$$u_k(X_t) - u_k(X_0) = M_t^{u_k} + N_t^{u_k}, \quad t \ge 0,$$

where M^{u_k} is a martingale additive functional of X having finite energy and N^{u_k} is a continuous additive functional of X having zero energy. Since $\mu_{\langle u_k \rangle}(U_k) = \mu_{\langle u \rangle}(U_k) = 0$, we have $M_t^{u_k} = 0$ for every $t \in [0, \tau_{U_k}]$ and

$$\mathcal{E}(u_k, \varphi) = 0 \quad \text{for every } \varphi \in \mathcal{F} \cap C_c(U_k).$$

The last display implies by [FOT, Theorem 5.4.1] that $N_t^{u_k} = 0$ for $t \in [0, \tau_{U_k}]$. Consequently, we have for each $k \ge 1$ that almost surely

$$u(X_t) - u(X_0) = u_k(X_t) - u_k(X_0) = 0 \quad \text{for } t \in [0, \tau_{U_k}].$$

As $\lim_{k\to\infty} \tau_{U_k} = \zeta$, we have for quasi-every $x \in \mathcal{X}$, $\mathbb{P}^x$-a.s.,

$$u(X_t) = u(X_0) \quad \text{for every } t \in [0, \zeta). \tag{4.11}$$

For $a \in \mathbb{R}$, define $A_a = \{x \in \mathcal{X} : u(x) > a\}$, which is quasi open. In view of (4.11), $P_t 1_{A_a} \le 1_{A_a}$ m-a.e. on $\mathcal{X}$. Hence by the irreducibility of $(\mathcal{E}, \mathcal{F})$, either $m(A_a) = 0$ or $m(\mathcal{X} \setminus A_a) = 0$. This proves that u is constant m-a.e. and hence $\mathcal{E}$-q.e. on $\mathcal{X}$.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i): Were the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$ not irreducible, there would exist a nearly Borel measurable set A with $m(A) > 0$ and $m(A^c) > 0$ such that for $1_A u \in \mathcal{F}$ for any $u \in \mathcal{F}$ and (2.1) holds. In particular, both 1_A and 1_{A^c} are in $\mathcal{F}_{\rm loc}$. Since $(\mathcal{E}, \mathcal{F})$ is strongly local, by Proposition 2.2,

$$\mathcal{E}(1_A u, 1_{A^c} v) = 0 \quad \text{for every } u, v \in \mathcal{F}.$$

This together with (2.8) gives that for any bounded $u, v \in \mathcal{F}$,

$$\begin{aligned} \int_{\mathcal{X}} v(x) \mu_{\langle 1_A u \rangle}(dx) &= 2\mathcal{E}(1_A u, 1_A uv) - \mathcal{E}(1_A u^2, v) \\ &= 2\mathcal{E}(u, (1_A v)u) - \mathcal{E}(u^2, 1_A v) \\ &= \int_{\mathcal{X}} (1_A v)(x) \mu_{\langle u \rangle}(dx). \end{aligned}$$

This yields

$$\mu_{\langle 1_A u \rangle}(dx) = 1_A(x) \mu_{\langle u \rangle}(dx). \tag{4.12}$$

Let $\{U_k; k \ge 1\}$ be an increasing sequence of relative compact open subsets whose union is $\mathcal{X}$ and $u_k \in \mathcal{F} \cap C_c(\mathcal{X})$ such that $u_k = 1$ on U_k. We have by (4.12) and Proposition 2.3(i) that

$$\mu_{\langle 1_A \rangle}(U_k) = \mu_{\langle 1_A u_k \rangle}(U_k) = \mu_{\langle u_k \rangle}(U_k \cap A) = 0 \quad \text{for each } k \ge 1,$$

and so

$$\mathcal{E}(1_A, 1_A) = \frac{1}{2} \mu_{\langle 1_A \rangle}(\mathcal{X}) = 0.$$

Since 1_A, which is in $\mathcal{F}_{\rm loc}$ and bounded, is not constant m-a.e. on $\mathcal{X}$ this is a contradiction, and proves that that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$ is irreducible. □

Theorem 4.7 in particular implies that irreducibility is invariant under form-bounded perturbations in the following sense. If $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ and $(\mathcal{X}, d, \mu, \mathcal{E}', \mathcal{F})$ are two strongly local regular MMD spaces such that the Radon measure μ does not change $\mathcal{E}$-polar sets and has full quasi support on $\mathcal{X}$, $\mathcal{F} \cap C_c(\mathcal{X}) = \mathcal{F}' \cap C_c(\mathcal{X})$ and there is a constant $C \ge 1$ so that

$$C^{-1} \mathcal{E}(u, u) \le \mathcal{E}'(u, u) \le C \mathcal{E}(u, u) \quad \text{for } u \in \mathcal{F} \cap C_c(\mathcal{X}), \tag{4.13}$$

then $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, m)$ is irreducible if and only if so is $(\mathcal{E}', \mathcal{F}')$ on $L^2(\mathcal{X}, \mu)$. Note that, by [CF, Theorem 5.2.11 and Exercise 3.3.2(iii)], the condition that μ is a smooth measure of $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ with full quasi support on $\mathcal{X}$ ensures that two extended Dirichlet spaces coincide. By [LJ, Proposition 1.5.5(b)], (4.13) implies that

$$C^{-1} \mu_{\langle u \rangle} \le \mu'_{\langle u \rangle} \le C \mu_{\langle u \rangle} \quad \text{on } \mathcal{X} \quad \text{for any } u \in \mathcal{F}_e = \mathcal{F}'_e. \tag{4.14}$$

Theorem 4.8. *Suppose that $(\mathcal{X}, d)$ is connected. If a strongly local $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies (Ha) and any bounded function that is harmonic in a ball is continuous there, then $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ is irreducible.*

Proof. Suppose $u \in \mathcal{F}_{\text{loc}} \cap L^\infty(\mathcal{X}; m)$ and $\mathcal{E}(u,u) = 0$. By the proof of (i) $\Rightarrow$ (ii) part of Theorem 4.7, we know u is harmonic on $\mathcal{X}$ and

$$u(X_t) = u(X_0) \quad \text{for all } t \in [0, \zeta) \tag{4.15}$$

$\mathbb{P}^x$-a.s. for $\mathcal{E}$-q.e. $x \in \mathcal{X}$. By assumption, u has a continuous version; we will use this continuous version and still denote it by u. Since $(\mathcal{E}, \mathcal{F})$ is strongly local, $1 \in \mathcal{F}_{\text{loc}}$ and $\mathcal{E}(1,1) = 0$. Let x_0 be an arbitrary point in $\mathcal{X}$ and denote $u(x_0)$ by a_0. Then $u - a_0 \in \mathcal{F}_{\text{loc}}$ and $\mu_{\langle u - a_0 \rangle} = \mu_{\langle u \rangle}$ by Proposition 2.3(i). Thus

$$\mathcal{E}(u - a_0, u - a_0) = \frac{1}{2}\mu_{\langle u - a_0 \rangle}(\mathcal{X}) = \frac{1}{2}\mu_{\langle u \rangle}(\mathcal{X}) = \mathcal{E}(u,u) = 0.$$

Let $v = |u - a_0|$. By [CF, Theorem 4.3.10], $v \in \mathcal{F}_{\text{loc}}$ and $\mathcal{E}(v,v) = 0$. By the same reasoning as that for u in the above, v is harmonic on $\mathcal{X}$. Since $v(x_0) = 0$, by (Ha) $v(x) = 0$ on $B(x_0, r)$ for some $r > 0$; that is, $u(x) = u(x_0)$ on $B(x_0, r)$ for some $r > 0$. This shows that for any constant $a \in \mathbb{R}$, both $A_a := \{x \in \mathcal{X} : u(x) > a\}$ and its complement $A_a^c = \{x \in \mathcal{X} : u(x) \leq a\}$ are open subsets of $\mathcal{X}$. If u is not a constant, then there is a constant a so that neither A_a nor A_a^c are empty sets. This would contradict to the assumption that $(\mathcal{X}, d)$ is connected. So u must be constant. This establishes the irreducibility of $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ by Theorem 4.7. □

Combining Theorem 4.8 with Theorem 4.6 shows that if $(\mathcal{X}, d)$ is connected and $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the EHI_{loc}, then $(\mathcal{E}, \mathcal{F})$ has regular Green functions.

5 Implications of EHI

Recall the definition of metric doubling from Definition 1.1. We introduce a few related properties. For a metric space $(\mathcal{X}, d)$ and $A \subset \mathcal{X}$, we set $\operatorname{diam}(A) = \sup_{x,y \in A} d(x,y)$. The following definition of relative K ball connectedness is adapted from [GH, Definition 5.5].

Definition 5.1. (i) Let $K > 1$. A metric space $(\mathcal{X}, d)$ is *relatively K ball connected* if for each $\varepsilon \in (0,1)$ there exists an integer $N = N_{\mathcal{X}}(\varepsilon)$ such that if $x_0 \in \mathcal{X}$, $R > 0$ and $x, y \in \overline{B(x_0, R)}$ then there exists a chain of balls $B(z_i, \varepsilon R)$ for $i = 0, \dots, N$ such that $z_0 = x$, $z_N = y$, $B(z_i, \varepsilon R) \subset B(x_0, KR)$ for each i and $d(z_{i-1}, z_i) < \varepsilon R$ for $1 \leq i \leq N$. We write $N_{\mathcal{X}}$ for the integer $N_{\mathcal{X}}(\varepsilon)$ with $\varepsilon = 1/4$. We say also that $(\mathcal{X}, d)$ satisfies the property RBC(K). We say that $(\mathcal{X}, d)$ is *relatively ball connected* if there exists $K > 1$ such that $(\mathcal{X}, d)$ is relatively K ball connected.

(ii) A metric space $(\mathcal{X}, d)$ is said to be *uniformly perfect*, if there exists $C > 1$ such that if $x \in \mathcal{X}$, $r > 0$ and $B(x,r)^c \neq \emptyset$ then $B(x,r) \setminus B(x, r/C) \neq \emptyset$.

(iii) A metric space $(\mathcal{X}, d)$ is said be *L-linearly connected* (for some $L > 1$), if for all $x, y \in \mathcal{X}$, there exists a connected compact set J such that $x, y \in J$ and $\operatorname{diam}(J) \leq L d(x,y)$.

(iv) A *distortion function* is a homeomorphism of $[0, \infty)$ onto itself. Let η be a distortion function. A map $f : (\mathcal{X}_1, d_1) \to (\mathcal{X}_2, d_2)$ between metric spaces is said to be *η-quasisymmetric* or an *η-quasisymmetry*, if f is a homeomorphism and

$$\frac{d_2(f(x), f(a))}{d_2(f(x), f(b))} \leq \eta\left(\frac{d_1(x,a)}{d_1(x,b)}\right)$$

for all triples of points $x, a, b \in \mathcal{X}_1$, $x \neq b$. We say f is a *quasisymmetry* if it is η-quasisymmetric for some distortion function η. We say that metric spaces $(\mathcal{X}_1, d_1)$ and $(\mathcal{X}_2, d_2)$ are *quasisymmetric*, if there exists a quasisymmetry $f : (\mathcal{X}_1, d_1) \to (\mathcal{X}_2, d_2)$. We say that metrics d_1 and d_2 on $\mathcal{X}$ are *quasisymmetric* (or, d_1 is *quasisymmetric* to d_2), if the identity map $\mathrm{Id} : (\mathcal{X}, d_1) \to (\mathcal{X}, d_2)$ is a quasisymmetry.

(v) We say a metric space $(\mathcal{X}, d)$ is *quasi-arc connected*, if there exists a distortion function $\eta : [0, \infty) \to [0, \infty)$ such that for all pairs of distinct points $x, y \in \mathcal{X}$, there exists a subset $J \subset \mathcal{X}$ and an η-quasisymmetry $\gamma : [0, 1] \to J$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Here J is endowed with the metric d and $[0, 1]$ has the Euclidean metric.

The following lemma clarifies some relationships between these conditions.

Lemma 5.2. Let $(\mathcal{X}, d)$ be a complete metric space.

(a) Assume that $(\mathcal{X}, d)$ is relatively K ball connected. Then exists $L > 1$, such that for all $x, y \in \mathcal{X}$, there exists a continuous map $\gamma : [0, 1] \to \mathcal{X}$ such that $\gamma(0) = x$, $\gamma(1) = y$, and $\mathrm{diam}(\gamma([0, 1])) \leq L d(x, y)$. In particular, $(\mathcal{X}, d)$ is L-linearly connected.

(b) If $(\mathcal{X}, d)$ is connected then it is uniformly perfect.

(c) If $(\mathcal{X}, d)$ is relatively ball connected and satisfies metric doubling, then $(\mathcal{X}, d)$ is quasi-arc connected.

(d) If $(\mathcal{X}, d)$ is quasi-arc connected, then $(\mathcal{X}, d)$ is relatively ball connected.

(e) Assume that $(\mathcal{X}, d)$ is relatively K ball connected. If ρ is metric on $\mathcal{X}$ quasisymmetric to d, then $(\mathcal{X}, \rho)$ is also relatively ball connected. In other words, the property of being relatively ball connected is a quasisymmetry invariant.

Proof. (a) Fix $\varepsilon \in (0, 1)$ and let $K, N = N_{\mathcal{X}}(\varepsilon)$ be the constants of relative ball connectivity.

Let $x, y \in \mathcal{X}$ be a pair of distinct points. For each $k \in \mathbb{N}$, we define $\gamma_k : [0, 1] \to \mathcal{X}$ as follows. Let $z_0^{(1)}, z_1^{(1)}, \ldots, z_N^{(1)}$ be a sequence of points in $B(x, K d(x, y))$ such that $d(z_i^{(1)}, z_{i+1}^{(1)}) < \varepsilon d(x, y)$, with $z_0^{(1)} = x, z_N^{(1)} = y$. Let $\gamma_1 : [0, 1] \to \mathcal{X}$ be a piecewise constant function on intervals defined by

$$\gamma_1(t) = z_i^{(1)}, \quad \text{for all } i = 0, 1, \ldots, N-1 \text{ and for all } i/N \leq t < (i+1)/N$$

and $\gamma_1(1) = y$. Similarly, for all $i = 0, \ldots, N-1$, we choose $z_j^{(2)}, j = iN, iN+1, \ldots, iN+N$ such that $z_{iN}^{(2)} = z_i^{(1)}, z_{iN+N}^{(2)} = z_{i+1}^{(1)}, d(z_j^{(2)}, z_{j+1}^{(2)}) < \varepsilon^2 d(x, y)$ and set

$$\gamma_2(t) = z_j^{(2)}, \quad \text{for all } i = 0, 1, \ldots, N^2 - 1 \text{ and for all } i/N^2 \leq t < (i+1)/N^2,$$

with $\gamma_2(1) = y$. We similarly define $\gamma_k : [0, 1] \to \mathcal{X}$ a piecewise constant function on intervals $[j/N^k, (j+1)/N^k)$, $j = 0, 1, \ldots, N^k - 1$. Since for all $t \in [0, 1]$, $d(\gamma_k(t), \gamma_{k+1}(t)) < K\varepsilon^k d(x, y)$, the sequence $\{\gamma_k(t), k \in \mathbb{N}\}$ is Cauchy, and hence converges to say $\gamma(t) \in \mathcal{X}$. This defines a function $\gamma : [0, 1] \to \mathcal{X}$. Note that

$$d(x, \gamma(t)) \leq \sum_{k=0}^{\infty} K\varepsilon^k d(x, y) = K d(x, y)/(1 - \varepsilon).$$

If $|t_1 - t_2| \leq \frac{1}{N^k}$ for some $k \in \mathbb{N}$, we have

$$\begin{aligned} d(\gamma(t_1), \gamma(t_2)) &\leq d(\gamma_k(t_1), \gamma(t_1)) + d(\gamma_k(t_2), \gamma(t_2)) + d(\gamma_k(t_1), \gamma_k(t_2)) \\ &\leq 2\left(\sum_{l=k}^{\infty} K\varepsilon^l d(x,y)\right) + \varepsilon^k d(x,y) \\ &\leq (2K(1-\epsilon)^{-1} + 1)\varepsilon^k d(x,y), \end{aligned}$$

which implies the continuity of γ.

This shows that the image $J = \gamma([0,1])$ is a compact, connected set with $x, y \in J$ with $\operatorname{diam}(J) \leq Ld(x,y)$, where $L = 2K/(1-\varepsilon)$. Therefore $(\mathcal{X}, d)$ is L-linearly connected.
(b) Let $B(x,r)$ be a ball such that $B(x,r)^c \neq \emptyset$ and let $C > 1$. Since $B(x,r)^c$ and $\overline{B(x,r/C)}$ are non-empty disjoint closed sets. Since $(\mathcal{X}, d)$ is connected, $B(x,r) \setminus B(x,r/C) \neq \emptyset$.
(c) By part (a), $(\mathcal{X}, d)$ is linearly connected. By Tukia's theorem ([Mac, Corollary 1.2] and [TV, Theorem 4.9]), $(\mathcal{X}, d)$ is quasi-arc connected.
(d) Let η be the distortion function corresponding to quasi-arc connectedness. Define $K = 1 + \eta(1)$. Let $x, y \in B(x_0, R)$ and $\gamma : [0,1] \to J$ be an η-quasisymmetry such that $\gamma(0) = x, \gamma(1) = y$. For all $t \in [0,1]$,

$$d(x, \gamma(t)) \leq \eta(t) d(x,y) \leq \eta(1) d(x,y).$$

Let $\varepsilon \in (0,1)$ be arbitrary. Let $N \in \mathbb{N}$ be such that $2\eta(2/N)\eta(1) < \varepsilon$ and define $z_i = \gamma(i/N)$ for $i = 0, 1, \ldots, N$. By η-quasisymmetry, we have

$$d(z_i, z_{i+1}) \leq \eta(2/N) d(z_i, w) \leq \eta(2/N)\eta(1) d(x,y) \leq 2\eta(2/N)\eta(1) R < \varepsilon R,$$

where $w = x$ if $i \leq N/2$ and $w = y$ otherwise. This implies that $(\mathcal{X}, d)$ is relatively K ball connected, where $K = 1 + \eta(1)$.
(e) Let $\mathrm{Id} : (\mathcal{X}, \rho) \to (\mathcal{X}, d)$ be a η-quasisymmetry, where $(\mathcal{X}, d)$ is relatively K ball connected. Let $\varepsilon \in (0,1)$ and let $x, y \in \mathcal{X}$ be arbitrary. Chose $\varepsilon' \in (0,1)$ such that

$$\eta(2\varepsilon')\left(\eta(K) + 1\right) < \varepsilon.$$

Choose points $z_0, z_1, \ldots, z_N$ such that $B(z_i, \varepsilon' d(x,y)) \subset B(x, d, Kd(x,y))$ $d(z_i, z_{i+1}) < \varepsilon' d(x,y)$, where $N = N_{(\mathcal{X},d)}(\varepsilon')$ is the constant associated with the relative ball connected property of $(\mathcal{X}, d)$. For any $i = 0, 1, \ldots, N-1$, let $w \in \{x, y\}$ be such that $d(z_i, w) = \max(d(z_i, x), d(z_i, y))$. Since $d(x,y)/2 \leq d(z_i, w)$, we obtain

$$\begin{aligned} \rho(z_i, z_{i+1}) &\leq \eta(d(z_i, z_{i+1})/d(z_i, w))\rho(z_i, w) \leq \eta(2\varepsilon')(\rho(x, z_i) + \rho(x,y)) \\ &\leq \eta(2\varepsilon')\left(\eta(K) + 1\right)\rho(x,y) < \varepsilon\rho(x,y). \end{aligned}$$

Since $\rho(x, z_i) \leq \eta(K)\rho(x,y)$, $(\mathcal{X}, \rho)$ is relatively K_ρ ball connected, where $K_\rho = 2 + 2\eta(K)$. □

Remark 5.3. See [GH, Defintion 5.5] for the definition of *relatively* (ε, K) *ball connected*. It is immediate that if $(\mathcal{X}, d)$ is relatively K ball connected then it is relatively (ε, K) ball connected for any $\varepsilon \in (0,1)$. Conversely it is straightforward to show that if for some $\varepsilon \in (0,1)$, $K > 1$ $(\mathcal{X}, d)$ is relatively (ε, K) ball connected then it is relatively K' ball connected with $K' = 1 + \frac{K}{1-\varepsilon}$.

The main result of this section is the following.

Theorem 5.4. *Let $(\mathcal{X}, d)$ be a complete locally compact separable metric space. Assume that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ is a MMD space that satisfies the EHI. The following are equivalent:*
(a) *$(\mathcal{X}, d)$ is relatively K ball connected for some $K > 1$ and $\overline{B(x,r)}$ is compact for all $x \in \mathcal{X}$ and $r > 0$.*
(b) *$(\mathcal{X}, d)$ satisfies metric doubling.*
(c) *$(\mathcal{X}, d)$ is quasi-arc connected, and $\overline{B(x,r)}$ is compact for all $x \in \mathcal{X}$ and $r > 0$.*

Proof. (b) ⇒ (a). This follows by the argument in [GH, Proposition 5.6]: for any $K > 1 + \delta_H^{-1}$ we obtain relative K-ball connectedness. (The hypothesis of volume doubling there is only used to obtain metric doubling). Note that this implication requires EHI only for $[0,1]$-valued $h \in \mathcal{F} \cap C_c(\mathcal{X})$ only. Since $(\mathcal{X}, d)$ is complete and metric doubling, we know from Remark 1.2 that $\overline{B(x,r)}$ is compact for all $x \in \mathcal{X}, r > 0$.

(c) ⇒ (a), (a) + (b) ⇒ (c) (and so (b) ⇒ (c)) are proved in Lemma 5.2.

The proof of (a) ⇒ (b) needs more preparation and will be given after Lemma 5.17. □

For the proof of (a) ⇒ (b) in Theorem 5.4, we follow [BM1, Section 3]; however it was assumed there that the metric d was geodesic, and some changes are needed to handle the case when we only have that $(\mathcal{X}, d)$ is relatively K ball connected. We now outline these changes.

Definition 5.5. We say that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the condition (HG) if $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ has regular Green functions and there exist constants $C_G, K_G > 0$ such that for any $x_0 \in \mathcal{X}$, $R > 0$ and open set D in $\mathcal{X}$ with $B(x_0, K_G R) \subset D$ and D^c non-$\mathcal{E}$-polar

$$\sup_{y_2 \in D \setminus B(x_0,R)} g_D(x_0, y_2) \le C_G \inf_{y_1 \in \overline{B(x_0,R)} \setminus \{x_0\}} g_D(x_0, y_1). \tag{5.1}$$

Assumption 5.6. Throughout the remainder of this section except for Lemma 5.16, we assume that $(\mathcal{X}, d)$ is a complete locally compact separable metric space, that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the (scale invariant) EHI with constants C_H, δ_H that $(\mathcal{X}, d)$ is relatively K ball connected for some $K \ge 2$, and that $\overline{B(x,r)}$ is compact for all $x \in \mathcal{X}$ and $r > 0$.

Recall that by Lemma 5.2(a), a complete metric space $(\mathcal{X}, d)$ that is relatively K ball connected is connected. Thus under Assumption 5.6, $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ is irreducible by Theorem 4.8 and has regular Green functions by Theorem 4.6. By the maximum principle (4.5) for the regular Green function g_D in Theorem 4.5(ii), we have for any $B = B(x_0, R) \Subset D$,

$$\sup_{D \setminus B} g_D(x_0, \cdot) = \sup_{\partial B} g_D(x_0, \cdot), \qquad \inf_{y \in \overline{B} \setminus \{x_0\}} g_D(x_0, \cdot) = \inf_{\partial B} g_D(x_0, \cdot). \tag{5.2}$$

Proposition 5.7. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 5.6. Then (HG) holds with constants C_G, K_G for $K_G = 2K + 1$, where C_G depends only on $C_H, \delta_H, K_G, N_{\mathcal{X}}$.*

Proof. (a) This follows from the proof of [GH, Lemma 5.7]. (The statement of the result in [GH] has stronger hypotheses, but these are only used to obtain the existence and regularity of the Green function, and prove that $(\mathcal{X}, d)$ is relatively (ε, K) ball connected for some $\varepsilon \in (0,1)$ and $K > 1$.) □

Under Assumption 5.6, $(\mathcal{E}, \mathcal{F})$ has regular Green functions by Theorems 4.8 and 4.6, and (HG) holds with constant $K_G = K + 1$ and $C_G > 1$ by Proposition 5.7.

Corollary 5.8. *(See [BM1, Corollary 3.2].) Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 5.6. Let $K_1 = K + 1$. For any $\delta \in (0, 1/2]$, there exists a positive constant C that depends only on δ and the constants in Assumption 5.6 such that the following holds: for any open set D in $\mathcal{X}$ whose complement D^c is non-$\mathcal{E}$-polar and for any $B(x_0, K_1R) \subset D$,*

$$g_D(x_0, x) \leq C g_D(x_0, y) \quad \text{for } x, y \in B(x_0, R) \setminus B(x_0, \delta R).$$

Proof. Let $x, y \in B(x_0, R) \setminus B(x_0, \delta R)$. Let $\varepsilon = \delta\delta_H/(1 + \delta_H)$; we have $\varepsilon < \delta/2$. We connect y to x_0 by a chain of balls $B(z_i, \varepsilon R)$, $i = 0, 1, \dots N$, with the properties given in Definition 5.1(i) of relatively K ball connected. Let i_0 be the first integer such that $d(z_{i_0}, x_0) < \delta R$. With the definition of ε given above, $g_D(x_0, \cdot)$ is harmonic on $B(z_i, \varepsilon R/\delta_H)$ for $i = 0, \dots, i_0 - 1$, and so we can use the EHI to deduce that $g_D(x_0, z_{i_0}) \leq C_H^N g_D(x_0, y)$. Finally by (HG) we have $g_D(x_0, x) \leq C_G g_D(x_0, z_{i_0})$. □

Lemma 5.9. *(See [BM1, Lemma 3.3].) Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 5.6. Let $K_2 = 2K + 3$. There exists $C_0 > 1$ that depends only on the constants in Assumption 5.6 such that the following holds: Let $x_0 \in \mathcal{X}$, $R > 0$ and let $B(x_0, K_3R) \subset D$, where D is an open set in $\mathcal{X}$ such that D^c is non-$\mathcal{E}$-polar. Then if $x_1, x_2, y_1, y_2 \in B(x_0, R)$ with $d(x_j, y_j) \geq R/4$, then*

$$g_D(x_1, y_1) \leq C_0 g_D(x_2, y_2). \tag{5.3}$$

Proof. Note that for any four numbers $a_i \in [0, R)$, $1 \leq i \leq 4$,

$$\left|(0, R) \setminus \cup_{i=1}^4 [a_i - (R/9), a_i + (R/9)]\right| \geq R - (8R/9) = R/9 > 0$$

so there is some $a_0 \in (0, R)$ so that $|a_0 - a_i| > R/9$ for all $1 \leq i \leq 4$. As $d(x_0, \cdot)$ is continuous and by the RBC(K) property and Lemma 5.2(a), $B(x_0, R)$ contains a connected path from x_0 to $B(x_0, R)^c$, we have $\{d(x_0, x) : x \in B(x_0, R)\} = [0, R)$. Thus there is $a_0 \in (0, R)$ so that $|a_0 - d(x_0, w)| > R/9$ for $w \in \{x_1, x_2, y_1, y_2\}$. Let $z \in B(x_0, R)$ having $d(x_0, z) = a_0$. Now applying Corollary 5.8 to the balls $B(x_1, 2R)$, $B(z, 2R)$ and $B(x_2, 2R)$ with $\delta = 1/18$ consecutively, we get by the symmetry of the Green function $g_D(x, y)$ that

$$g_D(x_1, y_1) \leq C g_D(x_1, z) \leq C^2 g_D(x_2, z) \leq C^3 g_D(x_2, y_2).$$

This establishes the lemma by taking $C_0 = C^3$. □

As in [BM1], we define for an open set $D \subset \mathcal{X}$ with non-$\mathcal{E}$-polar complement:

$$g_D(x, r) = \inf_{y \in \partial B(x,r)} g_D(x, y), \text{ provided } \overline{B(x, r)} \subset D,$$
$$\text{Cap}_D(A) = \inf\{\mathcal{E}(f, f) : f \in (\mathcal{F}^D)_e,\ f \geq 1\ \mathcal{E}\text{-q.e. on } A\}, \quad A \subset D,$$

where $(\mathcal{F}^D)_e = \{u \in \mathcal{F}_e : u = 0\mathcal{E} - \text{q.e. on } \mathcal{X} \setminus D\}$. and by [CF, Theorem 2.4.9] we set $\mathcal{F}_e^D := (\mathcal{F}^D)_e$. We call $\text{Cap}_D(A)$ the relative capacity of A in D. The maximum principle (4.5) implies that $g_D(x, r)$ is non-increasing in r, and an easy application of (HG) gives that if $y \in \partial B(x, r)$ and $\overline{B(x, r)} \cup B(y, K_G r) \subset D$ then

$$g_D(x, r) \leq C_G g_D(y, r). \tag{5.4}$$

Given Proposition 5.7 the proof of the next Lemma is the same as in [BM1, Lemma 3.5].

Lemma 5.10. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 5.6. There is a constant $C_G > 0$ depending only on the constants in Assumption 5.6 such that for any open set D whose complement D^c is non-$\mathcal{E}$-polar and for any $B(x_0, K_G r) \subset D$ where $K_G = 2K + 1$,*

$$g_D(x_0, r) \leq \mathrm{Cap}_D(B(x_0, r))^{-1} \leq C_G g_D(x_0, r). \tag{5.5}$$

Remark 5.11. For any $x \in \mathcal{X}$, $0 < R < \mathrm{diam}(\mathcal{X}, d)/2$, the ball $B(x, R)^c$ is non-$\mathcal{E}$-polar. This is because by the triangle inequality, there exists $z \in \mathcal{X}$ and $0 < r < \mathrm{diam}(\mathcal{X}, d)/2 - R$ so that $B(z, r) \subset B(x, R)^c$. Since m has full support, $m\,(B(x, R)^c) \geq m(B(z, r)) > 0$ and thus $B(x, R)^c$ has positive capacity.

Lemma 5.12. *(See [GH, Lemma 7.1 and 7.4] and [GNY, Lemma 2.5]) Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 6.1. For any sufficiently large $A > 1$, there exist C such that for any ball $B(x, r), n \in \mathbb{N}$ with $A^n r < \mathrm{diam}(\mathcal{X}, d)/A$, denoting $B_k = B(x, A^k r)$, we have*

$$\sum_{i=0}^{n-1} \mathrm{Cap}_{B_{i+1}}(B_i)^{-1} \leq \mathrm{Cap}_{B_n}(B_0)^{-1} \leq C \sum_{i=0}^{n-1} \mathrm{Cap}_{B_{i+1}}(B_i)^{-1}.$$

Proof. The upper bound is contained in [GH, Lemmas 7.1 and 7.4]. The upper bound in [GH] is under the additional volume doubling property assumption but the proof only uses the weaker metric doubling assumption. Alternately, we see that the upper bound follows from (HG) and the constant C can be chosen to C_G for any $K \geq K_G$, where C_G, K_G are as given in Definition 5.5.

The lower bound is a general fact that does not require the EHI – see [GNY, Lemma 2.5]. □

Lemma 5.13. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 5.6. Let $B = B(x_0, R) \subset \mathcal{X}$, and $B_1 = B(x_1, R/(8K_3))$ with $x_1 \in B(x_0, R/(4K_3))$, where $K_3 = K + 2$ and K is as given in Assumption 5.6. There exists $p_0 > 0$ depending only on the constants in Assumption 5.6 such that*

$$\mathbb{P}^y(\sigma_{B_1} < \tau_B) \geq p_0 > 0 \quad \text{for } \mathcal{E}\text{-q.e. } y \in B(x_0, R/(2K_3)). \tag{5.6}$$

Proof. We consider two cases.

(i) Suppose $B(x_0, R)^c$ is non-$\mathcal{E}$-polar. By the maximum principle it is enough to prove this for $y \in \partial B(x_0, R/(2K_3))$. The argument, which uses Corollary 5.8, is the same as in [BM1, Lemma 3.7].

(ii) Now suppose $B(x_0, R)^c$ is $\mathcal{E}$-polar. By Remark 5.11, $R \geq \mathrm{diam}(\mathcal{X}, d)/2$ and $\mathrm{diam}(\mathcal{X}, d) < \infty$. If $R > 8K_3 \,\mathrm{diam}(\mathcal{X}, d)$, then $B = \mathcal{X}$ and (5.6) is obviously true by the compactness of $\mathcal{X}$ and [FOT, Theorem 4.7.1-(iii)]. Therefore, it suffices to consider the case when $R \leq 8K_3 \,\mathrm{diam}(\mathcal{X}, d) < \infty$.

Let $x_0 \in \mathcal{X}, x_1 \in B(x_0, R/(4K_3)), y \in B(x_0, R/(2K_3))$. Let $\varepsilon = 1/(130K_3^2)$ and let $B(z_i, \frac{\varepsilon R}{4K}) = B(z_i, \varepsilon R), 0 \leq i \leq N := N_{\mathcal{X}}(\varepsilon)$ be a chain of balls with $z_0 = y$, $z_N = x_1$, $B(z_i, \frac{\varepsilon R}{4K}) \subset B(x_0, R/4)$ for all each i and $d(z_{i-1}, z_i) < \varepsilon R/(4K)$ for $1 \leq i \leq N$ as in Definition 5.1(i) with $R/(4K)$ in place of R there. Since $8K_3 \varepsilon R \leq 64 K_3^2 \varepsilon \,\mathrm{diam}(\mathcal{X}, d) < \mathrm{diam}(\mathcal{X}, d)/2$, by Case 1 and Remark 5.11, we have

$$\mathbb{P}^w(\sigma_{B(z_i, \varepsilon R)} < \tau_{B(z_{i-1}, 8K\varepsilon R)}) \geq p_0 \quad \text{for } \mathcal{E}\text{-q.e. } w \in B(z_{i-1}, 4\varepsilon R).$$

Since $B(z_i, 8K_3\varepsilon R) \subset B(x_0, R/2)$ for all i, using the strong Markov property, we conclude that

$$\mathbb{P}^y(\sigma_{B_1} < \tau_B) \geq p_0^N > 0 \quad \text{for } \mathcal{E}\text{-q.e.} \quad y \in B(x_0, R/(2K)).$$

By replacing p_0 by p_0^N, we obtain (5.6) in the second case as well. ☐

Remark 5.14. (i) In [BM1, Lemma 3.7], the corresponding result held for $y \in B(x_0, 7R/8)$; we cannot expect that here, since such a point y might not be connected to B_1 by a path inside B.

(ii) Let $B_i = B(z_i, \varepsilon R)$, $0 \leq i \leq n$ be a chain of balls as in Definition 5.1(i). Using this Lemma we have for each i

$$\mathbb{P}^y(\sigma_{B(z_i,\varepsilon R)} < \tau_{B(z_{i-1},8K_3\varepsilon R)}) \geq p_0 \text{ for } \mathcal{E}\text{-q.e.} y \in B(z_{i-1}, 4\varepsilon R).$$

Thus if

$$D = \cup_{i=0}^n B(z_i, 8K_3\varepsilon R),$$

then

$$\mathbb{P}^y(\sigma_{B_n} < \tau_D) \geq p_0^n \text{ for } \mathcal{E}\text{-q.e. } y \in B(z_0, 4\varepsilon R). \tag{5.7}$$

Corollary 5.15. *(See [BM1, Corollary 3.8]). Let $B(x, R) \subset D$, where D is a bounded domain and D^c is non-$\mathcal{E}$-polar. There exist positive constants c and θ that depend only on the constants in Assumption 5.6 such that if $0 < s < r < R/(K + 1)$ then*

$$\frac{g_D(x, r)}{g_D(x, s)} \geq c\Big(\frac{s}{r}\Big)^\theta. \tag{5.8}$$

Proof. This follows easily from Corollary 5.8. ☐

The following Lemma is used to regularize chains of balls obtained by the using the RBC(K) property.

Lemma 5.16. *Suppose that $(\mathcal{X}, d)$ is a metric space satisfying the RBC(K) property. Let $d(x, y) < R$, $\varepsilon \in (0, 1)$ and $\varepsilon R < r < R$. There exists a chain of balls $B(z_i, \varepsilon R)$, $0 \leq i \leq n$ with the following properties:*
(i) *$z_0 = x$, $z_n = y$ and $d(z_{i-1}, z_i) < \varepsilon R$ for $1 \leq i \leq n$;*
(ii) *$B(z_i, \varepsilon R) \subset B(x, KR)$ for $0 \leq i \leq n$;*
(iii) *If $j = \max\{i : z_i \in B(x, r)\}$ then $B(z_i, \varepsilon R) \subset B(x, Kr)$ for $0 \leq i \leq j$;*
(iv) *$n \leq N_{\mathcal{X}}(\varepsilon) + N_{\mathcal{X}}(\varepsilon R/r)$.*

Proof. By the RBC(K) property there exists a chain of balls $B(w_i, \varepsilon R)$, $0 \leq i \leq m_1$ connecting x and y and satisfying the conditions of Definition 5.1(i) with $x_0 = x$. Let $k = \max\{i : w_i \in B(x, r)\}$. Using the RBC($K$) property again for x and w_k, and with ε replaced by $\varepsilon' = \varepsilon R/r$, there exists a chain $B(w_i', \varepsilon R)$, $0 \leq i \leq m_2$ with $B(w_i', \varepsilon R) \subset B(x, Kr)$. Joining the paths $w_0', \ldots, w_{m_2}'$ and $w_{k+1}, \ldots w_{m_1}$ gives a path (z_i) which satisfies the conditions (i)–(iv). ☐

Lemma 5.17. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 5.6. There exists an integer $N \geq 1$ that depends only on the constants in Assumption 5.6 such that if $x_0 \in \mathcal{X}$, $R > 0$ and $B(z_i, R/8)$, $1 \leq i \leq m$, are disjoint balls with $z_i \in B(x_0, R) \setminus B(x_0, R/2)$, then $m \leq N$.*

Proof. This lemma corresponds to [BM1, Lemma 3.10]. In [BM1] the metric d on $\mathcal{X}$ is assumed to be a geodesic distance, and the proof in [BM1] uses this property quite strongly. The proof here is much longer since we only have the weaker property that $(\mathcal{X}, d)$ is relatively K ball connected.

Let $(z_k, 1 \le k \le m)$ satisfy the hypotheses of the Lemma, and write $B_k = B(z_k, R/8)$. Choose $\varepsilon = 1/(720K^2)$, and let $n = N_{\mathcal{X}}(\varepsilon) + N_{\mathcal{X}}(24K\varepsilon)$. For each k we use Lemma 5.16 with $r = R/(24K)$ to find a chain of balls $B(w_{ki}, \varepsilon R)$ with $0 \le i \le n$ connecting z_k to x_0. Note that by taking $w_{ki} = x_0$ for large i if necessary, we can assume that all the chains have length n. We set $l_k = \max\{i : w_{ki} \in B(z_i, R/(24K))\}$, and write $z'_k = w_{kl_k}$.

We now find a subset I of the balls B_i such that the chain $(w_{ik}, 0 \le k \le n)$ associated with one ball does not hit any other ball with index in I.

For $1 \le i, j \le m$ set $a_{ij} = 1$ if $\{w_{ik}, 1 \le k \le n\} \cap B_j \ne \emptyset$, and let $a_{ij} = 0$ otherwise. Let $b_j = \sum_i a_{ij}$. Since each w_{ik} is in at most one ball B_j, we have $\sum_j a_{ij} \le n$, and hence $\sum_j b_j = \sum_i \sum_j a_{ij} \le mn$. Thus if $J = \{j : b_j \le 2n\}$ then $|J| \ge m/2$.

We now consider the collection of balls $(B_i, i \in J)$, and relabel them $B_1, \dots, B_{m_1}$ where $m_1 = |J| \ge m/2$. We now start with the ball B_1, and remove from the collection of balls $B_2, \dots, B_{m_1}$ any ball B_j such that either $a_{1j} = 1$ or $a_{j1} = 1$. Since $1 \in J$ and $a_{11} = 1$, less than $3n$ balls are removed. Set $j_1 = 1$. Let j_2 be the smallest label of a ball which has not been removed; we now repeat the procedure above with this ball, and remove any ball B_i such that $i > j_2$ and $a_{j_2,i} + a_{i,j_2} \ge 1$. We continue until there are no balls left, and write $I = \{j_k, 1 \le k \le m'\}$ for the set of balls which are retained. Since at each step we remove at most $3n - 1$ balls, we have $3nm' \ge \frac{1}{2}m$.

By the construction above we have that

$$w_{ik} \notin \cup_{j \in I \setminus \{i\}} B_j \quad \text{for } i \in I,\ k = 0, \dots, n.$$

For $i \in I$ set $B'_i = B(z_i, \varepsilon R)$, $A_i = B(z'_i, \varepsilon R)$, and let

$$D = B(x_0, 2KR) \setminus \cup_{i \in I} B'_i.$$

We now claim that for $i \in I$

$$\mathbb{P}^y(\sigma_{B'_i} < \tau_{B_i}) \ge p_0^n \quad \text{for } \mathcal{E}\text{-q.e. } y \in 4A_i := B(z'_i, 4\varepsilon R), \tag{5.9}$$

$$\mathbb{P}^{x_0}(\sigma_{A_i} < \tau_D) \ge p_0^n \quad \text{for } \mathcal{E}\text{-q.e. } y \in B(x_0, 4\varepsilon R). \tag{5.10}$$

Both these inequalities follow by chaining the bound in Lemma 5.13, as in Remark 5.14(ii), along a sequence of balls. For (5.9) we use the sequence $B(w_{ij}, \varepsilon R), 0 \le j \le l_i$, and for (5.10) we use $B(w_{ij}, \varepsilon R), l_i \le j \le n$. (We start at $j = n$ and end at $j = l_i$.)

The remainder of the proof is as in [BM1, Lemma 3.10]. Let $F_i = \{\sigma_{A_i} < \tau_D\}$, and

$$Y = \sum_{i \in I} 1_{F_i}$$

be the number of distinct balls A_i hit by $(X_t, 0 \le t \le \tau_D)$. The bound (5.9) implies that if X hits A_i then with probability at least p_0^n it leaves D before it hits any other ball A_j with $j \ne i$. Thus Y is stochastically dominated by a geometric r.v. with mean p_0^{-n}, and so

$$\mathbb{E}^y Y \le p_0^{-n}, \text{ for } \mathcal{E}\text{-q.e. } y \in \mathcal{X}.$$

However, by (5.10) we also have

$$\mathbb{E}^y Y = \sum_{i\in I} \mathbb{P}^y(F_i) \geq |I| p_0^n = m' p_0^n, \text{ for } \mathcal{E}\text{-q.e. } y \in \mathcal{X}.$$

Since $m' \geq \frac{m}{6n}$ as given above, it follows that $m \leq 6np_0^{-2n}$. □

Now we can finish the proof of Theorem 5.4 by giving the

Proof of (a) ⇒ (b) in Theorem 5.4. (i) Suppose that a metric space $(\mathcal{X}, d)$ has the property that there is an integer $N' \geq 1$, so that any ball $B(x, R)$ contains at most N' points that are at distance of at least $R/2$. Given any ball $B(x, R) \subset \mathcal{X}$, take $z_1 \in B(x, R)$, $z_2 \in B(x, R) \setminus B(z_1, R/2)$, and for $k \geq 3$, $z_k \in B(x, R) \setminus \cup_{j=1}^{k-1} B(z_j, R/2)$ if the set is non-empty. By the assumption, we can only proceed this procedure up to some number k_0 no larger than N'. Clearly $\cup_{j=1}^{k_0} B(z_j, R/2) \supset B(x, R)$. Thus $(\mathcal{X}, d)$ is metric doubling. Conversely, suppose $(\mathcal{X}, d)$ is metric doubling with positive integer $N \geq 2$ in Definition 1.1. For any ball $B(x, R)$, applying the definition of (MD) to $B(x, R)$ and to balls with radius $R/2$, and then with radius $R/4$ (to guarantee $x_1, \ldots, x_{N^3} \in B(x, R)$) there are N^3 number of points $x_1, \ldots, x_{N^2}$ in $B(x, R)$ so that $\cup_{j=1}^{N^2} B(x_j, R/4) \supset B(x, R)$. Suppose $\{z_1, \ldots, z_n\}$ are n points in $B(x, R)$ that are at distance of at least $R/2$, then $|\{z_1, \ldots, z_n\} \cap B(x_k, R/4)| \leq 1$ for any $1 \leq k \leq N^3$. Thus $n \leq N^3$. This proves that a metric space $(\mathcal{X}, d)$ is (MD) if and only if there is some constant N' so that any ball $B(x, R)$ contains at most N' points that are at distance of at least $R/2$ from each other.

(ii) Now let $N \geq 1$ be the integer in Lemma 5.17. Let $x_0 \in \mathcal{X}$, $R > 0$, and let $z_i \in B(x_0, R)$, $1 \leq i \leq n$, with the property that the balls $B(z_i, R/8)$ are disjoint. By Lemma 5.17 applied first to $B(x_0, R)$ and then to $B(x_0, R/2)$, there are at most $2N$ of the z_i in $B(x_0, R) \setminus B(x_0, R/4)$. Using the relative K-ball connectivity of $\mathcal{X}$, we can find x_1 such that $R/2 \leq d(x_0, x_1) < 3R/4$ (here we assume without loss of generality that $B(x_0, R/2) \neq \mathcal{X}$; otherwise we can cover $B(x_0, R)$ with $B(x_0, R/2)$). Thus $B(x_0, R/4) \subset B(x_1, R) \setminus B(x_1, R/4)$. So by Lemma 5.17 applied to $B(x_1, R)$, there are at most $2N$ points z_i in $B(x_0, R/4)$. Consequently, $n \leq 4N$. This proves that $(\mathcal{X}, d)$ is (MD) in view of its equivalent characterization given in (i). □

We need to compare the Green function in two domains.

Lemma 5.18. *(See [BM1, Lemma 3.12].) Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 5.6. There exists a constant C_1 that depends only on the constants in Assumption 5.6 such that if $B = B(x_0, R)$, $2B = B(x_0, 2R)$ and $B(x_0, (2 + 1/(128K^2))R)^c$ is non-empty, then*

$$g_{2B}(x, y) \leq C_1 g_B(x, y) \quad \text{for } x, y \in B(x_0, R/(8K)), x \neq y.$$

Proof. Let $a_1 = 1/(8K)$, $a_2 = 1/(4K)$, $a_3 = 1/(2K)$, $\varepsilon = 1/(128K^2)$ and $B_i = B(x_0, a_i R)$. Let $p_1 > 0$. Suppose that for each $y \in B_1$ there exist an open set D with $\overline{B_2} \subset D \subset B$ and a set A such that

$$\mathbb{P}^x(X_{\tau_D} \in A) \geq p_1, \text{ for } \mathcal{E}\text{-q.e. } x \in \overline{B}_2, \tag{5.11}$$

$$\mathbb{P}^w(\tau_{2B} < \sigma_{B_2}) \geq p_1 \text{ for } \mathcal{E}\text{-q.e. } w \in A. \tag{5.12}$$

Let $y \in B_1$. Let $x_1 = x_1(y) \in \partial B_2$ be chosen to maximize $g_{2B}(x', y)$ for $x' \in \partial B_2$. Write $h(w) = \mathbb{P}^w(\tau_{2B} < \sigma_{B_2})$. Then by strong Markov property at τ_D, σ_{B_2} and occupation density formula (4.4) for $\mathcal{E}$-q.e. $z \in D \setminus \{y\}$

$$g_{2B}(z,y) = g_D(z,y) + \mathbb{E}^z g_{2B}(X_{\tau_D}, y) \le g_B(z,y) + \mathbb{E}^z \left[\mathbb{E}^{X_{\tau_D}}[\mathbf{1}_{\{\sigma_{B_2} < \tau_B\}} g_{2B}(X_{\sigma_{B_2}}, y)]\right]$$
$$\le g_B(z,y) + \mathbb{E}^z(1 - h(X_{\tau_D})) g_{2B}(x_1, y).$$

Using (5.11) and (5.12), for $\mathcal{E}$-q.e. $z \in B_2 \setminus \{y\}$ we have

$$g_B(z,y) \ge g_{2B}(z,y)\mathbb{E}^z h(X_{\tau_D}) \ge g_{2B}(z,y) p_1^2.$$

Letting $z \to x_1$ yields

$$g_B(x_1,y) \ge g_{2B}(x_1,y) p_1^2.$$

Then if $x \in B_1 \setminus \{y\}$,

$$g_{2B}(x,y) = g_{B_2}(x,y) + \mathbb{E}^x g_{2B}(X_{\tau_{B_2}}, y) \le g_B(x,y) + g_{2B}(x_1,y)$$
$$\le g_B(x,y) + p_1^{-2} g_B(x_1,y).$$

The first equality above only holds for $\mathcal{E}$-q.e. $x \in B_1 \setminus \{y\}$ but $g_{2B}(x,y) \le g_B(x,y) + p_1^{-2} g_B(x_1,y)$ follows for any $x \in B_1 \setminus \{y\}$ by continuity.

Let x_1' be the point in ∂B_2 which minimizes $g_B(x',y)$. By the maximum principle (4.5), $g_B(x,y) \ge g_B(x_1',y)$. We now apply Corollary 5.8 to the ball $B(y, a_1 R + a_2 R)$ to deduce that $g_B(x_1,y) \le c g_B(x_1',y)$. Combining this with the inequalities above we obtain the bound $g_{2B}(x,y) \le C g_B(x,y)$. (Note that the constant C only depends on p_1 and the constants in Corollary 5.8; it does not depend on y.)

It remains to find $p_1 > 0$ such that there exist sets D and A satisfying (5.11) and (5.12). Let $y_0 \in \partial B(x_0, (2+\varepsilon)R)$. By Lemma 5.16 there exists a sequence $x_0 = z_0, \dots, z_n = y_0$ with $d(z_{i-1}, z_i) < \epsilon R$ for $1 \le i \le n$ such that if $j = \max\{i : z_i \in B_3\}$ then $B(z_i, \varepsilon R) \subset B(x_0, K a_3 R)$ for $0 \le i \le j$. Write $B_i' = B(z_i, \varepsilon R)$. Now let $D = B \setminus \overline{B_j'}$, and $A = \overline{B_j'}$. Recall from Remark 5.14(ii), it suffices to show (5.11) for $x \in B_2$, and (5.12) for $w \in B_j'$.

If $i \ge j$ then $B(z_i, 8K_3\varepsilon R) \cap B_2 = \emptyset$. So we can chain along the sequence of balls $B_j, \dots B_n$ to obtain (5.12) with $p_1 = p_0^n$.

If $0 \le i \le j$ then $d(x_0, z_i) \le K a_3 R$ and so $B(z_i, 8\varepsilon K_3 R) \subset B(x_0, K_3 a_3 R + 8\varepsilon K R) \subset B$. Hence, chaining along this sequence we obtain

$$\mathbb{P}^x(X_{\tau_D} \in A) \ge p_0^j \text{ for } x \in B_0'.$$

To complete the proof of (5.11) we need to extend this estimate to $x \in B_2$.

Let $x_2 \in B_2$. Then there exists a chain of balls $B(w_j, \varepsilon R)$, $0 \le j \le k$ with $w_0 = x_2$, $w_k = x_0$, $d(w_{j-1}, w_j) < \varepsilon R$ for $1 \le j \le k$, and with $B(w_j, \varepsilon R) \subset B(x_0, K a_2 R)$. Since $B(w_j, 8\varepsilon K_3 R) \subset B$, we deduce that

$$\mathbb{P}^x(\sigma_{B_0'} < \tau_B) \ge p_0^k, \quad \text{for } \mathcal{E}\text{-q.e. } \ x \in B(x_2, \varepsilon R).$$

By letting x_2 run over a countable dense subset of B_2, it follows that

$$\mathbb{P}^x(X_{\tau_D} \in A) \ge p_0^{k+n} \quad \text{for } \mathcal{E}\text{-q.e. } x \in B_2.$$

Since k and n only depend on the constants $N_{\mathcal{X}}(\varepsilon)$, this completes the proof of (5.11). $\square$

The following corollary is a direct consequence of Lemmas 5.10 and 5.18 and Remark 5.11.

Corollary 5.19. *(See [BM1, Corollary 3.13].) Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 5.6. There exists C_1 that depends only on the constants in Assumption 5.6 such that for all $A > 8K$ and for all $0 < r < \operatorname{diam}(\mathcal{X}, d)/(6A)$, $x \in \mathcal{X}$,*

$$\operatorname{Cap}_{B(x,2Ar)}(B(x,r)) \leq \operatorname{Cap}_{B(x,Ar)}(B(x,r)) \leq C_1 \operatorname{Cap}_{B(x,2Ar)}(B(x,r)). \tag{5.13}$$

In the following, notations $f \asymp g$, $f \lesssim g$ and $f \gtrsim g$ mean that there are positive constants c_1, c_2 so that $c_1 g \leq f \leq c_2 g$, $f \leq c_2 g$ and $f \geq c_1 g$, respectively, on the common domain of definition of f and g.

Lemma 5.20. *(See [BM1, Lemma 3.14].) Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 5.6.*

(a) *Let D be a bounded domain in $\mathcal{X}$ such that D^c is non-$\mathcal{E}$-polar. Let $x \in \mathcal{X}$ and $r > 0$ be such that $B(x, C_0 r) \subset D$, where $C_0 = 2K + 3$. There exists a constant $C_1 > 0$ such that*

$$\operatorname{Cap}_D(B(x,r)) \leq C_1 \operatorname{Cap}_D(B(y,r)) \quad \text{for } y \in B(x,r).$$

(b) *Let $A > 8K$. There exists a constant $C_2 > 0$ such that*

$$\operatorname{Cap}_{B(x,Ar)}(B(x,r)) \leq C_2 \operatorname{Cap}_{B(y,Ar)}(B(y,r))$$

for $x \in \mathcal{X}, y \in B(x,r)$, $0 < r < \operatorname{diam}(\mathcal{X}, d)/(6A)$.

(c) *Let $A > 8K$ and $A_1 > 0$. There exists a constant $C_3 > 0$ such that*

$$\operatorname{Cap}_{B(x,Ar)}(B(x,r)) \leq C_3 \operatorname{Cap}_{B(y,Ar)}(B(y,r))$$

for $x \in \mathcal{X}, y \in B(x,r)$, $0 < r < \operatorname{diam}(\mathcal{X}, d)/(6A)$. Here the constants C_1, C_2, C_3 depend only on A_1 and the constants in Assumption 5.6.

Proof. (a) As in the proof of Lemma 5.9, choose $z \in B(x,r)$ such that $\min(d(z,x), d(z,y)) \geq r/4$. By Corollary 5.8 and Lemma 5.10, $\operatorname{Cap}_D(B(x,r)) \asymp g_D(x,z)^{-1}$ and $\operatorname{Cap}_D(B(y,r)) \asymp g_D(y,z)^{-1}$. The conclusion now follows from Lemma 5.9.
(b) By Corollary 5.19 and part(a), we have

$$\operatorname{Cap}_{B(x,Ar)}(B(x,r)) \lesssim \operatorname{Cap}_{B(x,2Ar)}(B(x,r)) \asymp \operatorname{Cap}_{B(x,2Ar)}(B(y,r)).$$

Since $B(y, Ar) \subset B(x, 2Ar)$, we have $\operatorname{Cap}_{B(x,2Ar)}(B(y,r)) \leq \operatorname{Cap}_{B(y,Ar)}(B(y,r))$.
(c) The case $A_1 \leq 1$ follows from (b). For $A_1 > 1$, by the RBC(K) condition there exists N such that $x, y \in \mathcal{X}$ with $d(x,y) < A_1 r$, can be connected by a sequence of points $x_0 = x, x_1, \ldots, z_N = y$ with $d(x_{i-1}, x_i) < r$ for $1 \leq i \leq N$, where N depends only on A_1 and the constants in RBC(K) condition. By applying (b) repeatedly, we obtain (c) with $C_3 = C_2^N$, where C_2 is the constant in (b). □

Proposition 5.21. *(See [BM1, Proposition 3.15]) Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 5.6. Let $D \subset \mathcal{X}$ be an open set such that D^c is non-$\mathcal{E}$-polar and let $B(x_0, 2KR) \subset D$. Let $b \geq 24$. Let $F \subset B(x_0, R)$, and suppose there exist disjoint Borel subsets $\{Q_i, 1 \leq i \leq n\}$ of $\mathcal{X}$ with $n \geq 2$ such that*

$$F = \cup_{i=1}^n Q_i$$

and for each i, there exists $z_i \in \mathcal{X}$ so that $B(z_i, R/b) \subset Q_i$. Then there exists $\delta = \delta(\delta_H, b, C_H, K) > 0$ such that

$$\mathrm{Cap}_D(F) \le (1-\delta) \sum_{i=1}^{n} \mathrm{Cap}_D(Q_i).$$

Proof. The proof is similar to that of [BM1, Proposition 3.15]. The only difference is that we use RBC(K) condition and a chaining argument using the EHI along with Lemma 5.13 to obtain the lower bound on the equilibrium potentials h_i for $\mathrm{Cap}_D(Q_i)$.

The proof in [BM1, Proposition 3.15] uses the fact that 0-order equilibrium measure ν_B^D of any $B \subset D$ with $\mathrm{Cap}_D(B) < \infty$ for the part Dirichlet form $(\mathcal{E}, \mathcal{F}^D)$ on D satisfies $\nu_B^D(D) = \mathrm{Cap}_D(B)$. Since this is mentioned in [CF, FOT] under the additional assumption that B is compact, we provide further details on how to verify this equality for an arbitrary Borel set B. Let e_B^D denote the 0-order equilibrium potential of B for $(\mathcal{E}, \mathcal{F}^D)$, so that $\mathcal{E}(e_B^D, u) = \int_D u \, d\nu_B^D$ for any $u \in (\mathcal{F}^D)_e$ by [FOT, Theorem 2.2.5]. Then by [FOT, the 0-order version of Theorem 2.1.5], $\mathrm{Cap}_D(B) = \mathcal{E}(e_B^D, e_B^D) = \int_D e_B^D \, d\nu_B^D$, $\int_D \varphi(1-e_B^D)\, d\nu_B^D = \mathcal{E}(e_B^D, \varphi(1-e_B^D)) = 0$ for any $\varphi \in \mathcal{F} \cap C_c(\mathcal{X})$ with $\varphi\big|_{\mathcal{X}\setminus D}$, hence $\int_D (1-e_B^D)\, d\nu_B^D = 0$, namely $e_B^D = 1$ ν_B^D-a.e., and thus $\mathrm{Cap}_D(B) = \int_D e_B^D \, d\nu_B^D = \nu_B^D(D)$. □

The following lemma is an extension of Corollary 5.19.

Lemma 5.22. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 5.6 with $K \ge 2$. Let $1 < A_1 \le A_2 < \infty$. There exists a positive constant C_2 that depend only on A_1, A_2, and the constants in Assumption 5.6 such that for all $x \in \mathcal{X}, 0 < r < \mathrm{diam}(\mathcal{X}, d)/(6(A_2 \vee (9K)))$,*

$$\mathrm{Cap}_{B(x,A_2 r)}(B(x,r)) \le \mathrm{Cap}_{B(x,A_1 r)}(B(x,r)) \le C_2 \, \mathrm{Cap}_{B(x,A_2 r)}(B(x,r)).$$

Proof. The estimate $\mathrm{Cap}_{B(x,A_2 r)}(B(x,r)) \le \mathrm{Cap}_{B(x,A_1 r)}(B(x,r))$ follows from domain monotonicity. For the other estimate, by domain monotonicity we may assume $A_2 > 8K$.

Choose $A_3 > 8K$ so that $A_2/A_3 < A_1 - 1$. Then $B(y, A_2 r/A_3) \subset B(x, A_1 r)$ for all $y \in B(x,r)$. By the metric doubling property, there exists $N \in \mathbb{N}$ (depending only on A_3 and the constant associated with metric doubling) such that $y_1, \ldots, y_N \in B(x,r)$ and $\cup_{i=1}^N B(y_i, r/A_3) \supset B(x,r)$. By considering the function $e = \max_{1 \le i \le N} e_i$ where e_i is the equilibrium potential corresponding to $\mathrm{Cap}_{B(y_i, A_2 r/A_3)}(B(y_i, r/A_3))$, we obtain

$$\mathrm{Cap}_{B(x,A_1 r)}(B(x,r)) \le \sum_{i=1}^{N} \mathrm{Cap}_{B(y_i, A_2 r/A_3)}(B(y_i, r/A_3)).$$

By connecting the points x and y_i using a r/A_3 chain and using Lemma 5.20(b), we obtain

$$\mathrm{Cap}_{B(y_i, A_2 r/A_3)}(B(y_i, r/A_3)) \asymp \mathrm{Cap}_{B(x, A_2 r/A_3)}(B(x, r/A_3)),$$

for all $x \in \mathcal{X}$, $r \lesssim \mathrm{diam}(\mathcal{X}, d)$, and $i = 1, \ldots, N$. By Corollary 5.19 and domain monotonicity, we have

$$\mathrm{Cap}_{B(x, A_2 r/A_3)}(B(x, r/A_3)) \asymp \mathrm{Cap}_{B(x, A_2 r)}(B(x, r/A_3)) \le \mathrm{Cap}_{B(x,A_2 r)}(B(x,r)),$$

for all $x \in \mathcal{X}, r \lesssim \mathrm{diam}(\mathcal{X}, d)$. We obtain the desired bound

$$\mathrm{Cap}_{B(x,A_1 r)}(B(x,r)) \lesssim \mathrm{Cap}_{B(x,A_2 r)}(B(x,r))$$

by combining the above three estimates. $\square$

The following lemma is used to compare capacities at different scales.

Lemma 5.23. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 5.6. Let $A_1 > 1$. There exist constants $C_2 > 1$ and $\gamma > 0$ that depend only $A \wedge (8K)$ and on the constants in Assumption 5.6 such that for all $x \in \mathcal{X}$ and $0 < s \le r < \operatorname{diam}(\mathcal{X}, d)/(6(A_2 \vee (9K)))$, we have*

$$C_2^{-1} \left(\frac{r}{s}\right)^{-\gamma} \operatorname{Cap}_{B(x,A_1 s)}(B(x,s)) \le \operatorname{Cap}_{B(x,A_1 r)}(B(x,r)) \le C_2 \left(\frac{r}{s}\right)^{\gamma} \operatorname{Cap}_{B(x,A_1 s)}(B(x,s)).$$

Proof. By Lemma 5.22, we may assume without loss of generality that $A > 8K$. By Remark 5.11, Corollary 5.15, Lemma 5.10 and domain monotonicity, we have

$$\operatorname{Cap}_{B(x,Ar)}(B(x,r)) \asymp g_{B(x,Ar)}(x,r)^{-1} \lesssim \left(\frac{r}{s}\right)^{\theta} g_{B(x,Ar)}(x,s)^{-1} \lesssim \left(\frac{r}{s}\right)^{\theta} \operatorname{Cap}_{B(x,As)}(B(x,s))$$

for all $x \in \mathcal{X}, 0 < s \le r < \frac{\operatorname{diam}(\mathcal{X},d)}{2A}$, where $\theta > 0$ is as given in Corollary 5.15.

For the reverse inequality, we use Corollary 5.19 repeatedly and domain monotonicity to obtain

$$\operatorname{Cap}_{B(x,As)}(B(x,s)) \lesssim \left(\frac{r}{s}\right)^{\theta_1} \operatorname{Cap}_{B(x,Ar)}(B(x,s)) \le \left(\frac{r}{s}\right)^{\theta_1} \operatorname{Cap}_{B(x,Ar)}(B(x,r)),$$

for all $x \in \mathcal{X}, 0 < s \le r < \frac{\operatorname{diam}(\mathcal{X},d)}{6A}$, where $\theta_1 = \log_2 C_1 > 0$, where C_1 is as given in Corollary 5.19. Setting $\gamma = \max(\theta, \theta_1)$, we obtain the desired conclusion. $\square$

6 Good doubling measures

As in [BM1, Section 4] we now use the argument of Volberg and Konyagin [VK] to construct a new measure μ such that $(\mathcal{X}, d, \mu)$ satisfies *volume doubling*; that is, there is a constant $c > 1$ so that $\mu(B(x, 2r)) \le c\,\mu(B(x,r))$ for all $x \in \mathcal{X}$ and $r > 0$. We need further that μ relates well with capacities – see Definition 6.2 below. One key difference from [BM1] is that we do not assume bounded geometry condition on the original MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$. Another difference from [BM1] is that we do not have any cutoff at small length scales. This means that μ need not be absolutely continuous with respect to m, and it is not a priori clear that μ is a smooth measure having full quasi support on $\mathcal{X}$. This property is established in Proposition 6.16 of this section. The key inputs from the previous section are inequalities controlling capacities Corollary 5.19, Lemma 5.20, Proposition 5.21, and Lemma 5.23.

6.1 Construction of a doubling measure

In this section, we often make the following assumption.

Assumption 6.1. We assume that $(\mathcal{X}, d)$ is a complete locally compact separable metric space, and that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the (scale invariant) EHI with constants C_H, δ_H. Furthermore, we assume that the metric space $(\mathcal{X}, d)$ satisfies one (and hence all) of the equivalent conditions in Theorem 5.4.

The following definition is a simplification of [BM1, Definition 4.1]: we do not require absolute continuity with respect to the reference measure m. We do not require the volume doubling property for the measure ν either – this will follow from Lemma 6.3.

Definition 6.2. Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 6.1. Let D be a Borel subset of $\mathcal{X}$. Let $C_0 \in (1, \infty)$, $0 < \beta_1 \leq \beta_2$ and let $A \in [2^{1/3}, \infty)$. Let $I = (0, A^{-4} \operatorname{diam}(D))$. We say a locally finite Borel measure ν on D with full support is $(C_0, A, \beta_1, \beta_2)$-*capacity good* if for all $x \in D$ and $s_1, s_2 \in I$ with $s_1 < s_2$, $0 < \nu(B(x, s_1)) \leq \nu(B(x, s_2)) < \infty$ and

$$C_0^{-1} \left(\frac{s_2}{s_1}\right)^{\beta_1} \leq \frac{\nu(B(x, s_2)) \operatorname{Cap}_{B(x,As_1)}(B(x, s_1))}{\nu(B(x, s_1)) \operatorname{Cap}_{B(x,As_2)}(B(x, s_2))} \leq C_0 \left(\frac{s_2}{s_1}\right)^{\beta_2}. \tag{6.1}$$

Since ν is locally finite, any capacity good measure ν is a Radon measure on D, if D is open in $\mathcal{X}$.

Under Assumption 6.1, we observe by Corollary 5.19 that the second inequality in (6.1) of Definition 6.2 implies the volume doubling property for ν.

Lemma 6.3. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 6.1. Let ν be a $(C_0, A, \beta_1, \beta_2)$-capacity good measure on $\mathcal{X}$. Then it satisfies the volume doubling property.*

Proof. If $\operatorname{diam}(\mathcal{X}, d) = \infty$, then the volume doubling property follows from Lemma 5.23 and domain monotonicity of capacity, since $\operatorname{Cap}_{B(x,As)}(B(x, s))$ and $\operatorname{Cap}_{B(x,2As)}(B(x, 2s)$ are comparable.

In the case $\operatorname{diam}(\mathcal{X}, d) < \infty$, we view $\mathcal{X}$ as the closed ball $\overline{B(x_0, 2 \operatorname{diam}(\mathcal{X}, d))}$ and use Lemma 5.23 to obtain the volume doubling property for balls $B(x, s)$ with $s \lesssim \operatorname{diam}(\mathcal{X}, d)$. The volume doubling property for larger balls follows from a covering argument, the metric doubling property and the fact that $\inf_{x \in \mathcal{X}} \nu(B(x, s)) \gtrsim \nu(B(x_0, s))$ for $s = \frac{1}{3} A^{-4} \operatorname{diam}(\mathcal{X}, d)$ and any $x_0 \in \mathcal{X}$ by RBC(K). □

The following is the main result of this section.

Theorem 6.4 (Construction of a doubling measure)**.** *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 6.1. Then there exist constants $C_0 > 1$, $A > 1$, $0 < \beta_1 \leq \beta_2$ and a Borel measure μ on $\mathcal{X}$ which is $(C_0, A, \beta_1, \beta_2)$-capacity good.*

The proof of Theorem 6.4 requires a preparation of a few results. We begin by adapting the argument in [VK] to construct a measure with the desired properties on a family of compact sets. We then follow [LuS] and obtain μ as a weak* limit of measures defined on an increasing family of compact sets.

The proof uses a family of generalized dyadic cubes, which provide a family of nested partitions of a space. Such a decomposition of space was introduced by Christ [Chr, Theorem 11]. The following is a slight modification of the construction in [KRS, Theorem 2.1]. Since the requirements (g) and (h) are new, we provide some details.

Lemma 6.5. *Let $(\mathcal{X}, d)$ be a complete, metric space satisfying metric doubling and RBC(K) property. Let $x_0 \in \mathcal{X}$ and $A \geq 8$. Then there exists a collection $\{Q_{k,i} : k \in \mathbb{Z}, i \in I_k \subset \mathbb{Z}_+\}$ of Borel sets satisfying the following properties:*

(a) *$\mathcal{X} = \cup_{i \in I_k} Q_{k,i}$ for all $k \in \mathbb{Z}$, and $Q_{k,i} \cap Q_{k,j} = \emptyset$ for all $k \in \mathbb{Z}$ and $i, j \in I_k$ with $i \neq j$.*

(b) *If* $m \leq n$ *and* $i \in I_n$, $j \in I_m$, *then either* $Q_{n,i} \cap Q_{m,j} = \emptyset$ *or* $Q_{n,i} \subset Q_{m,j}$.

(c) *For every* $k \in \mathbb{Z}$ *and* $i \in I_k$, *there exists* $x_{k,i} \in Q_{k,i}$ *such that*

$$B(x_{k,i}, c_A A^{-k}) \subset Q_{k,i} \subset \overline{B(x_{k,i}, C_A A^{-k})},$$

where $c_A = \frac{1}{2} - \frac{1}{A-1}$, *and* $C_A = \frac{A}{A-1}$.

(d) *The sets* $N_k = \{x_{k,i} : i \in I_k\}$, *where* $x_{k,i}$ *are as defined in* (c), *are increasing in* k *and* $x_0 \in N_k$ *for all* $k \in \mathbb{Z}$; *that is* $N_k \subset N_{k+1}$ *for all* $k \in \mathbb{Z}$ *and* $x_0 \in \cap_{k\in\mathbb{Z}} N_k$.

(e) *Properties* (a), (b) and (c) *define a partial order* $\prec$ *on* $\mathcal{I} = \{(k,i) : k \in \mathbb{Z}, i \in I_k\}$ *by inclusion, where* $(k,i) \prec (m,j)$ *if and only if* $k \geq m$ *and* $Q_{k,i} \subset Q_{m,j}$.

(f) *There exists* $C_M = C_M(A) > 0$ *such that, for all* $k \in \mathbb{Z}$ *and for all* $x_{k,i} \in N_k$, *the 'successors'*

$$S_k(x_{k,i}) = \{x_{k+1,j} : (k+1,j) \prec (k,i)\}$$

satisfy

$$|S_k(x_{k,i})| \leq C_M \quad \text{for all } k \in \mathbb{Z}, i \in I_k. \tag{6.2}$$

Furthermore, we have $d(x_{k,i}, y) < A^{-k}$ *for all* $y \in S_k(x_{k,i})$.

(g) *Let*

$$k_0 = \inf\{k \in \mathbb{Z} : |I_k| > 1\}, \tag{6.3}$$

where $|I_k|$ *denotes the cardinality of* I_k. *Then* $k_0 \in \mathbb{Z} \cup \{-\infty\}$ *satisfies*

$$c_A A^{-k_0} \leq \operatorname{diam}(\mathcal{X}, d) \leq 2C_A A^{1-k_0}. \tag{6.4}$$

For all $k \geq k_0$, $k \in \mathbb{Z}$ *and* $i \in I_k$, *we have* $|S_k(x_{k,i})| \geq 2$.

(h) *For all* $k \in \mathbb{Z}$, $Q_{k,0}$ *is compact and* $x_{k,0} = x_0$.

Proof. The sets $Q_{k,j}, k \in \mathbb{Z}, j \in I_k$ are referred to as 'generalized dyadic cubes'. We follow the construction in [KRS] with a minor modification so as to ensure the property (h).

We choose $N_0 \subset \mathcal{X}$ such that $x_0 \in N_0$ and $N_0 = \{x_{0,i} : i \in I_0\}$ is a maximal subset of $\mathcal{X}$ such that $d(x_{0,i}, x_{0,j}) \geq 1$ for all $i \neq j$ with $i, j \in I_0$. For $k > 0$, we define $N_k = \{x_{k,i} : i \in I_k\}$ as a maximal subset of $\mathcal{X}$ such that $N_{k-1} \subset N_k$ and $d(x_{k,i}, x_{k,j}) \geq A^{-k}$ for all distinct $x_{k,i}, x_{k,j} \in N_k$. For $k < 0$, we define $N_k = \{x_{k,i} : i \in I_k\}$ as a maximal set such that $x_0 \in N_k \subset N_{k+1}$, and $d(x_{k,i}, x_{k,j}) \geq A^{-k}$ for all distinct $x_{k,i}, x_{k,j} \in N_k$.

We label the indices I_k such that $x_{k,0} = x_0$ for all $k \in \mathbb{Z}$. For each $(k,i) \in \mathbb{Z} \times \mathbb{Z}_+$ with $i \in I_k$, we pick an element $(k-1,j) \in \mathbb{Z} \times \mathbb{Z}_+$ with $i \in I_{k-1}$ such that

$$d(x_{k,i}, x_{k-1,j}) = \min_{l \in I_{k-1}} d(x_{k,i}, x_{k-1,l}).$$

We define $\prec$ as the smallest partial order that contains the relations $(k,i) \prec (k-1,j)$ for all $(k,i) \in \mathbb{Z} \times \mathbb{Z}_+$ with $i \in I_k$, where $(k-1,j) \in \mathbb{Z} \times \mathbb{Z}_+$ with $i \in I_{k-1}$ is chosen as above.

We relabel the indices I_0 of N_0 such that $0 \in I_0$ remains unchanged and

$$l_1 < l_2 \quad \text{for all } k < 0, l_1 \in \{i \in I_0 : (0,i) \prec (k,0)\} \text{ and } l_2 \in I_0 \setminus \{i \in I_0 : (0,i) \prec (k,0)\}. \tag{6.5}$$

This relabeling exists since $\{i \in I_0 : (0,i) \prec (k,0)\})$ is finite for all $k < 0$ (by the doubling property) and $(k,0) \prec (k-1,0)$ for all $k \le 0$.

Define the sets $Q_{0,i}$ as

$$Q_{0,i} = \overline{\{x_{l,k} : (l,k) \prec (0,i)\}} \setminus \bigcup_{j<i, j \in I_0} Q_{0,j}.$$

For $k < 0$, we define the sets $Q_{k,i}$ inductively as

$$Q_{k,i} = \bigcup_{(k+1,j)\prec(k,i)} Q_{k+1,j},$$

whereas for $k > 0$, we define

$$Q_{k,i} = Q_{k-1,i'} \cap \overline{\{x_{l,j} : (l,j) \prec (k,i)\}} \setminus \bigcup_{j<i, j\in I_k} Q_{k,j}, \quad \text{where } (k,i) \prec (k-1,i').$$

Properties (a)-(e) are contained in [KRS, Theorem 2.1]; (f) is immediate from the above construction and metric doubling.

(g) The estimate $|S_k(x_{k,i})| \ge 2$ relies on the following consequence of RBC(K) (see Lemma 5.2(a)): $r \le \operatorname{diam}(B(x,r)) \le 2r$ for all $B(x,r) \neq \mathcal{X}$. Since $2C_A/c_A = 4A/(A-3) < A$ for all $A \ge 8$, we have

$$\operatorname{diam}(Q_{k,i}) \ge c_A A^{-k} > 2C_A A^{-k-1} \ge \operatorname{diam}(Q_{k+1,j}) \quad \text{for all } k \ge k_0, k \in \mathbb{Z}.$$

Hence $Q_{k,i} \neq Q_{k+1,j}$ for all $k > k_0, i \in I_k, j \in I_{k+1}$, and therefore $|S_k(x_{k,i})| \ge 2$ for all $k \ge k_0$.

Clearly by (c), $\operatorname{diam}(\mathcal{X},d) = \infty$ if and only if $k_0 = -\infty$. If $k_0 \in \mathbb{Z}$, the estimate (6.4) follows from $B(x_0, c_A A^{-k_0}) \subset Q_{k_0,0} \subsetneq \mathcal{X} = Q_{k_0-1,0} \subset \overline{B(x_0, C_A A^{-k_0+1})}$.

(h) By (6.5), $Q_{k,0}$ is closed for all $k \in \mathbb{Z}$, since $Q_{k,0} = \overline{\{x_{l,j} : (l,j) \prec (k,0)\}}$. By (c) and (MD), $Q_{k,0}$ is compact for all $k \in \mathbb{Z}$. □

We fix a family

$$\{Q_{k,i} : \, k \in \mathbb{Z}, i \in I_k \subset \mathbb{Z}_+\},$$

of generalized dyadic cubes as given by Lemma 6.5, and define the nets N_k and successors $S_k(x)$ as in the lemma.

Definition 6.6. We define the *predecessor* $P_k(x)$ of $x \in N_k$ to be the unique element of N_{k-1} such that $x \in S_{k-1}(P_k(x))$. Note that for $x \in N_k$, $S_k(x) \subset N_{k+1}$ whereas $P_k(x) \in N_{k-1}$. For $x \in \mathcal{X}$, we denote by $Q_k(x)$ the unique $Q_{k,i}$ such that $x \in Q_{k,i}$.

Let $k_0 \in \mathbb{Z} \cup \{-\infty\}$ be as defined in (6.3). For $k \in \mathbb{Z}$ with $k \ge k_0 + 2$, denote by $c_k(x)$ the relative capacity

$$c_k(x) = \operatorname{Cap}_{B(x,A^{-k+1})}(Q_k(x)). \tag{6.6}$$

The following lemma provides useful estimates on c_k. Note that if $k \ge k_0 + 2$, then

$$A^{-k+1} \le A^{-k_0-1} \le c_A^{-1} A^{-1} \operatorname{diam}(\mathcal{X},d) = \frac{2(A-1)}{(A-3)A} \operatorname{diam}(\mathcal{X},d).$$

Lemma 6.7 (Relative capacity estimates for generalized dyadic cubes)**.** *Let $(\mathcal{X},d,m,\mathcal{E},\mathcal{F})$ be a MMD space that satisfies Assumption 6.1. There exists $A_0 \ge 8$ such that the following hold.*

(a) *For all* $A \geq A_0$*, there exists* $C_1 > 0$ *such that for all* $k \geq k_0+2, x, y \in \mathcal{X}$ *with* $d(x,y) \leq 4A^{-k}$*, we have*

$$C_1^{-1} c_k(y) \leq c_k(x) \leq C_1 c_k(y). \tag{6.7}$$

(b) *For all* $A \geq A_0$*, there exists* $C_1 = C_1(A) > 0$ *such that for all* $k \geq k_0 + 2, x \in N_k$ *and* $y \in S_k(x)$*, we have*

$$C_1^{-1} c_k(x) \leq c_{k+1}(y) \leq C_1 c_k(x). \tag{6.8}$$

(c) *For all* $A \geq A_0$*, there exists* $C_1 = C_1(A) > 0$ *such that for all* $x \in \mathcal{X}$ *and* $s < \operatorname{diam}(\mathcal{X}, d)/A^4$*,*

$$C_1^{-1} c_k(x) \leq \operatorname{Cap}_{B(x,As)}(B(x,s)) \leq C_1 c_k(x) \tag{6.9}$$

where $k \in \mathbb{Z}$ *is the unique integer such that* $A^{-k} \leq s < A^{-k+1}$*.*

Proof. We use domain monotonicity of capacity along with Corollary 5.19, and Lemma 5.23 to show first (6.9) with $s = A^{-k}$ for all $k \geq k_0 + 2$ and $x \in \mathcal{X}$ (with C_1 independent of A) and then Lemma 5.20 to obtain the above estimates. For (c), note that $A^{-k} \leq s < \operatorname{diam}(\mathcal{X}, d)/A^4 \leq 2C_A A^{-k_0-3} < A^{-k_0-2}$ implies $k \geq k_0 + 3$. □

We record one more estimate regarding the subadditivity of c_k, which will play an essential role in ensuring (6.1) and follows from Proposition 5.21 and domain monotonicity of capacity.

Lemma 6.8. *([BM1, Lemma 4.6]) Let* $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ *be a MMD space that satisfies Assumption 6.1. There exists* $A_0 \geq 4, \delta = \delta(A) \in (0,1)$ *such that for all* $k \in \mathbb{Z}$*,* $k \geq k_0 + 2$*,* $A \geq A_0$*, for all* $x \in N_k$*, we have*

$$c_k(x) \leq (1-\delta) \sum_{y \in S_k(x)} c_{k+1}(y).$$

Henceforth, we fix an $A \geq 8$ large enough such that the conclusions of Lemmas 6.7 and 6.8 hold.

We need the following modification of [VK, Lemma, p. 631], which was stated in [BM1, Lemma 4.7] without a proof. For the reader's convenience, we provide its full proof below.

Lemma 6.9. *Let* $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ *be a MMD space that satisfies Assumption 6.1. Let* $c_k(\cdot)$*,* $k \geq k_0 + 2$*,* $k \in \mathbb{Z}$ *denote the capacities of the corresponding generalized dyadic cubes as defined in* (6.6)*. There exists* $C > 1$ *satisfying the following. Let* $k \geq k_0 + 2$*,* $k \in \mathbb{Z}$*, and let* μ_k *be a probability measure on* N_k *such that*

$$\frac{\mu_k(e')}{c_k(e')} \leq C^2 \frac{\mu_k(e'')}{c_k(e'')} \quad \text{for all } e', e'' \in N_k \text{ with } d(e', e'') \leq 4A^{-k}. \tag{6.10}$$

Then there exists a probability measure μ_{k+1} *on* N_{k+1} *such that the following hold.*

(1) *For all* $g', g'' \in N_{k+1}$ *with* $d(g', g'') \leq 4A^{-k-1}$ *we have*

$$\frac{\mu_{k+1}(g')}{c_{k+1}(g')} \leq C^2 \frac{\mu_{k+1}(g'')}{c_{k+1}(g'')}. \tag{6.11}$$

(2) *Let* $\delta \in (0,1)$ *be the constant in Lemma 6.8. For all points* $e \in N_k$ *and* $g \in S_k(e)$*,*

$$C^{-1} \frac{\mu_k(e)}{c_k(e)} \leq \frac{\mu_{k+1}(g)}{c_{k+1}(g)} \leq (1-\delta) \frac{\mu_k(e)}{c_k(e)}. \tag{6.12}$$

(3) *The construction of the measure μ_{k+1} from the measure μ_k can be regarded as the transfer of masses from the points of N_k to those of N_{k+1}, with no mass transferred over a distance greater than $(1+4/A)A^{-k}$.*

Proof. By the metric doubling property

$$\sup_{k\in\mathbb{Z}} \sup_{x\in N_k} |S_k(x)| = S < \infty, \tag{6.13}$$

where $S_k(x)$ is as defined in Lemma 6.5(f). We choose

$$C = C_1 S,$$

where C_1 is chosen such that (6.7), (6.8) and (6.9) hold. Let $k \geq k_0 + 2$, $k \in \mathbb{Z}$, and let μ_k be any probability measure on N_k such that (6.10) holds.

The transfer of mass is accomplished in two steps. In the first step we distribute the mass $\mu_k(e)$ to all its successors $S_k(e)$ such that the mass of $g \in S_k(e)$ is proportional to $c_{k+1}(g)$; that is

$$f_1(g) = \frac{c_{k+1}(g)}{\sum_{g'\in S_k(e)} c_{k+1}(g')}\mu_k(e),$$

for all $e \in N_k$ and $g \in S_k(e)$.

By (6.13), Lemma 6.7 and Lemma 6.8, we have

$$C^{-1}\frac{\mu_k(e)}{c_k(e)} \leq \frac{f_0(g)}{c_{k+1}(g)} \leq (1-\delta)\frac{\mu_k(e)}{c_k(e)}, \tag{6.14}$$

for all points $e \in N_k$ and $g \in S_k(e)$. If the measure f_1 on N_{k+1} satisfies condition (1) of the Lemma, we set $\mu_{k+1} = f_1$. This is the desired measure. Condition (2) is satisfied by (6.14), and (3) is obviously satisfied by Lemma 6.5(f). The second step is not necessary in this case.

But if f_0 does not satisfy condition (1) of the Lemma, then we proceed as follows at the second step. Let $p_1, \ldots, p_T$ be the indexed pairs of points $\{g', g''\}$ with $g', g'' \in N_{k+1}$ and $0 < d(g', g'') \leq 4A^{-k-1}$. Take the pair $p_1 = \{g_1', g_1''\}$. If $\frac{f_0(g_1')}{c_{k+1}(g_1')} \leq C^2 \frac{f_0(g_1'')}{c_{k+1}(g_1'')}$ and $\frac{f_0(g_1'')}{c_{k+1}(g_1'')} \leq C^2 \frac{f_0(g_1')}{c_{k+1}(g_1')}$, then we set $f_1 = f_0$. Assume one of the inequalities is violated, say $\frac{f_0(g_1')}{c_{k+1}(g_1')} > C^2 \frac{f_0(g_1'')}{c_{k+1}(g_1'')}$. Then we construct a measure f_1 from f_0 such that

$$\begin{aligned} f_1(g_1') &= f_0(g_1') - \alpha_1, \\ f_1(g_1'') &= f_0(g_1'') + \alpha_1, \\ f_1(g) &= f_0(g), \quad g \neq g_1', g_1'', \end{aligned}$$

where $\alpha_1 > 0$ is chosen such that

$$\alpha_1 \left(\frac{C^2}{c_{k+1}(g_1'')} + \frac{1}{c_{k+1}(g_1')} \right) = \frac{f_0(g_1')}{c_{k+1}(g_1')} - C^2 \frac{f_0(g_1'')}{c_{k+1}(g_1'')}.$$

It is clear that $\frac{f_1(g_1')}{c_{k+1}(g_1')} = C^2 \frac{f_1(g_1'')}{c_{k+1}(g_1'')}$.

The next step is the construction of a measure f_2 from f_1 in exactly the same way that f_1 was constructed from f_0. Here we consider the pair p_2. A measure f_3 is next constructed from f_2 and so on. We claim that $\mu_{k+1} = f_T$ is the desired measure in the lemma. If $\mathcal{X}$ is non-compact,

$\mu_{k+1}(g) := \lim_{j\to\infty} \to f_j(g), g \in N_{k+1}$ (the existence of this limit is an easy consequence of metric doubling property).

We first verify that for all $e \in N_k$, for all $g \in S_k(e)$ and for all $s = 0, 1, \ldots, T$, we have

$$C^{-1}\frac{\mu_k(e)}{c_k(e)} \le \frac{f_s(g)}{c_{k+1}(g)} \le (1-\delta)\frac{\mu_k(e)}{c_k(e)}. \tag{6.15}$$

By (6.14), it is clear that (6.15) holds for $s = 0$. We now show (6.15) by induction. Suppose (6.15) holds for $s = j$, we will verify it for $s = j+1$. Let $p_{j+1} = \{g', g''\}$, $e' = P_{k+1}(g'), e'' = P_{k+1}(g'')$. If $f_j = f_{j+1}$, there is nothing to prove. But if $f_{j+1} \neq f_j$, then assume, say, that

$$\frac{f_j(g')}{c_{k+1}(g')} > C^2\frac{f_j(g'')}{c_{k+1}(g'')}. \tag{6.16}$$

By (6.16) and the construction, we have

$$f_{j+1}(g') < f_j(g'), \quad f_{j+1}(g'') > f_j(g''). \tag{6.17}$$

Therefore by the induction hypothesis (6.15) for $s = j$ and (6.17), we have

$$\frac{f_{j+1}(g')}{c_{k+1}(g')} \le (1-\delta)\frac{\mu_k(e')}{c_k(e')}, \quad \frac{f_{j+1}(g'')}{c_{k+1}(g'')} \ge C^{-1}\frac{\mu_k(e')}{c_k(e')}.$$

Therefore it suffices to verify that

$$\frac{f_{j+1}(g')}{c_{k+1}(g')} \ge C^{-1}\frac{\mu_k(e')}{c_k(e')}, \quad \frac{f_{j+1}(g'')}{c_{k+1}(g'')} \le (1-\delta)\frac{\mu_k(e'')}{c_k(e'')}. \tag{6.18}$$

Suppose the first inequality in (6.18) fails to be true, then by construction, (6.17) and the induction hypothesis (6.15) for $s = j$, we have

$$C^{-1}\frac{\mu_k(e')}{c_k(e')} > \frac{f_{j+1}(g')}{c_{k+1}(g')} = C^2\frac{f_{j+1}(g'')}{c_{k+1}(g'')} > C^2\frac{f_j(g'')}{c_{k+1}(g'')} \ge C\frac{\mu_k(e'')}{c_k(e'')}, \tag{6.19}$$

which implies $\frac{\mu_k(e')}{c_k(e')} > C^2\frac{\mu_k(e'')}{c_k(e'')}$. However $\frac{\mu_k(e')}{c_k(e')} \le C^2\frac{\mu_k(e'')}{c_k(e'')}$, by the assumption on μ_k, since

$$d(e', e'') \le d(e', g') + d(g', g'') + d(e'', g'') \le 2A^{-k} + 4A^{-k-1} \le 4A^{-k}.$$

This proves the first inequality in (6.18). The proof of the second inequality in (6.18) is similar. Indeed, assume to the contrary that $\frac{f_{j+1}(g'')}{c_{k+1}(g'')} > (1-\delta)\frac{\mu_k(e'')}{c_k(e'')}$; then we have

$$(1-\delta)\frac{\mu_k(e')}{c_k(e')} \ge \frac{f_j(g')}{c_{k+1}(g')} > \frac{f_{j+1}(g')}{c_{k+1}(g')} = C^2\frac{f_{j+1}(g'')}{c_{k+1}(g'')} > C^2(1-\delta)\frac{\mu_k(e'')}{c_k(e'')}, \tag{6.20}$$

which again implies $\frac{\mu_k(e')}{c_k(e')} > C^2\frac{\mu_k(e'')}{c_k(e'')}$. Therefore (6.15) follows by induction. In particular, $\mu_{k+1} = f_T$ satisfies condition (2) of the lemma.

We now verify condition (1) for $\mu_{k+1} = f_T$. For this, it suffices to prove the following assertion: if

$$C^{-2}\frac{f_j(g'')}{c_{k+1}(g'')} \le \frac{f_j(g')}{c_{k+1}(g')} \le C^2\frac{f_j(g'')}{c_{k+1}(g'')} \tag{6.21}$$

holds for a pair of points $g', g'' \in N_{k+1}$ such that $0 < d(g', g'') \leq 4A^{-k-1}$, then the same inequalities hold when f_j is replaced by f_{j+1}.

We now prove this. If $p_{j+1} = \{g', g''\}$, then $f_{j+1} = f_j$ and there is nothing to prove. If $\{g', g''\} \cap p_{j+1} = \emptyset$, then again there is nothing to prove. Let $p_{j+1} = \{g_1, g_2\}$. Without loss of generality, we assume $p_{j+1} \cap \{g', g''\} = \{g_1\}$ where $g_1 = g''$ and $f_j(g'')/c_{k+1}(g'') > C^2 f_j(g_2)/c_{k+1}(g_2)$. Then

$$\frac{f_{j+1}(g'')}{c_{k+1}(g'')} = C^2 \frac{f_{j+1}(g_2)}{c_{k+1}(g_2)}, \quad f_{j+1}(g'') < f_j(g''), \quad f_{j+1}(g') = f_j(g'). \tag{6.22}$$

Therefore, only the second inequality in (6.21) can fail for f_{j+1}. Suppose that this happens, that is

$$\frac{f_{j+1}(g')}{c_{k+1}(g')} > C^2 \frac{f_{j+1}(g'')}{c_{k+1}(g'')}. \tag{6.23}$$

Let $e' = P_{k+1}(g')$ and $e_2 = P_{k+1}(g_2)$. Then by (6.23), (6.22) and (6.15)

$$(1-\delta)\frac{\mu_k(e')}{c_k(e')} \geq \frac{f_{j+1}(g')}{c_{k+1}(g')} > C^2 \frac{f_{j+1}(g'')}{c_{k+1}(g'')} = C^4 \frac{f_{j+1}(g_2)}{c_{k+1}(g_2)} \geq C^3 \frac{\mu_k(e_2)}{c_k(e_2)}, \tag{6.24}$$

which implies that $\frac{\mu_k(e')}{c_k(e')} > C^2 \frac{\mu_k(e_2)}{c_k(e_2)}$. However since $d(e', e_2) \leq d(e', g') + d(g', g'') + d(g_1, g_2) + d(g_2, e_2) \leq 2(A^{-k} + 4A^{-k-1}) \leq 4A^{-k}$, we have a contradiction and hence (6.23) is false. This shows (6.21) for the case $f_j(g'')/c_{k+1}(g'') > C^2 f_j(g_2)/c_{k+1}(g_2)$. The case $f_j(g'')/c_{k+1}(g'') < C^{-2} f_j(g_2)/c_{k+1}(g_2)$ is analyzed similarly and therefore the assertion given by (6.21) is proved. It remains to observe that this assertion proves condition (1) of the lemma for the measure $\mu_{k+1} = f_T$. Along the path from f_0 to f_T, we "correct" the measure at all pairs of points where condition (1) is violated, and the assertion given by (6.21) shows that once a pair is corrected, it remains corrected when further changes are made.

It remains to verify condition (3). Note that by Lemma 6.5(f), there was a mass transfer over a distance of at most A^{-k} while passing from μ_k to f_0. Therefore it suffices to verify that while passing from f_0 to $f_T = \mu_{k+1}$ there is a transfer over a distance of at most $4A^{-k-1}$.

We will now verify this. It suffices to verify that there are no pairs $p_l = \{g_1, g_2\}$, $p_n = \{g_2, g_3\}$, $l, n \in \mathbb{Z} \cap [1, T], l \neq n$, such that mass is transferred from g_1 to g_2 (in the transition from f_{l-1} to f_l) and then mass is transferred from g_2 to g_3 (in the transition from f_{n-1} to f_n). Assume the opposite. First note that the assertion given by (6.21) can be modified as follows. If the second inequality in (6.21) is true for f_j it remains true for f_{j+1}. The same argument as before goes through. Using this modified version of the assertion, and the assumption that there are a mass transfer from g_1 to g_2 followed by a mass transfer from g_2 to g_3, we have

$$\frac{f_0(g_1)}{c_{k+1}(g_1)} > C^2 \frac{f_0(g_2)}{c_{k+1}(g_2)}, \quad \frac{f_0(g_2)}{c_{k+1}(g_2)} > C^2 \frac{f_0(g_3)}{c_{k+1}(g_3)}. \tag{6.25}$$

If $e_1 = P_{k+1}(g_1)$, $e_3 = P_{k+1}(g_3)$, then

$$d(e_1, e_3) \leq d(e_1, g_1) + d(g_1, g_2) + d(g_2, g_3) + d(g_3, e_3) \leq 2r(A^{-k} + 4A^{-k-1}) \leq 4A^{-k}.$$

Consequently by assumption, $\mu_k(e_1)/c_k(e_1) \leq C^2 \mu_k(e_3)/c_k(e_3)$. However the inequalities (6.25) and (6.15) imply the opposite inequality $\mu_k(e_1)/c_k(e_1) > C^2 \mu_k(e_3)/c_k(e_3)$. We have arrived at the desired contradiction and the proof of the lemma is complete. □

Remark 6.10. Lemma 6.9 and its proof above remain valid, if N_k is replaced by any $M_k \subset N_k$ and N_{k+1} by $M_{k+1} = \bigcup_{y \in M_k} S_k(y)$.

We now adapt the method in [VK] to construct the doubling measure.

Proposition 6.11 (Measure in a cube)**.** *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 6.1. . There exist constants C_0, $A > 1$ and $0 < \beta_1 \le \beta_2$ such that for any integer $l \ge k_0 - 1$, there exists a $(C_0, A, \beta_1, \beta_2)$-capacity good measure $\nu = \nu_l$ on $Q_{l,0}$.*

Proof. Choose $A \ge 8$ large enough such that the conclusions of Lemmas 6.9, 6.7, 6.8 hold. Set $M_k = N_k \cap Q_{l,0}$ for $k \ge l+3$, so that $M_{k+1} = \bigcup_{y \in M_k} S_k(y)$ (cf. Remark 6.10). Let μ_{l+3} be the probability measure on M_{l+3} such that μ_{l+3} is proportional to c_{l+3}; that is

$$\mu_{l+3}(x) = \frac{c_{l+3}(x)}{\sum_{y \in M_{l+3}} c_{l+3}(y)}, \quad \text{for all } x \in M_{l+3}.$$

We use Lemma 6.9 and Remark 6.10 to inductively construct probability measures μ_k on M_k for all $k \ge l+3$. We define the measure $\nu = \nu_l$ as a weak (sub-sequential) limit of the measures μ_k as $k \to \infty$ (the existence of such a limit follows from the compactness of $Q_{l,0}$). We claim that

$$\nu \text{ is } (C_0, A, \beta_1, \beta_2)\text{-capacity good for some } C_0, A, \beta_1, \beta_2 > 0. \tag{6.26}$$

For each $x \in Q_{l,0}$ and $k \ge l+3$, let $e_{x,k} \in M_k$ be the unique point in M_k such that $e_{x,k} \in Q_k(x)$, so that by Lemma 6.5(c),

$$d(x, e_{x,k}) \le C_A A^{-k}.$$

If $s < A^{-4} \operatorname{diam}(Q_{l,0}) \le A^{-4} 2 C_A A^{-l} \le A^{-l-3}$, then $s < A^{-l-3}$. In order to show (6.26), we prove the following two-sided estimate on measure of balls: there exists $C_2 \ge 1$ such that

$$C_2^{-1} \mu_n(e_{x,n}) \le \nu(B(x,s) \cap Q_{l,0}) \le C_2 \mu_n(e_{x,n}), \quad \text{for all } x \in Q_{l,0}, s < A^{-l-3}, \tag{6.27}$$

where n is the unique integer such that $A^{-n-1} \le s < A^{-n}$.

Note that, by Lemma 6.9(3) the mass from $e \in M_k$ travels a distance of at most

$$(1 + 4A^{-1}) \sum_{l=k}^{\infty} A^{-l} = C_3 A^{-k}, \quad \text{where } C_3 := (1 + 4A^{-1})(1 - A^{-1})^{-1} \le \tfrac{12}{7}. \tag{6.28}$$

Therefore, none of the mass outside $M_n \cap B(x, (1 + C_3)A^{-n})$ falls in $B(x,s)$, and therefore

$$\nu(B(x,s)) \le \mu_n \left(M_n \cap B(x, (1 + C_3)A^{-n}) \right) \quad \text{for all } x \in Q_{l,0}, s \in (0, A^{-l-3}). \tag{6.29}$$

By the triangle inequality, if $e \in M_n \cap B(x, (1 + C_3)A^{-n})$, then

$$d(e, e_{x,n}) \le d(e, x) + d(x, e_{x,n}) \le (1 + C_3)A^{-n} + C_A A^{-n} \le 4A^{-n}.$$

Therefore by (6.29), (6.10), (6.7), and the metric doubling property, we obtain the upper bound $\nu(B(x,s)) \lesssim \mu_n(e_{x,n})$ in (6.27).

For the lower bound in (6.27), using (6.28), we have that for all $x \in Q_{l,0}$ and for all $s < A^{-l-3}$ with $A^{-n-1} \le s < A^{-n}, n \in \mathbb{Z}$, the mass from $e_{x,n+2}$ travels a distance of at most $C_3 A^{-n-2} \le \frac{12}{7} A^{-n-2}$ from $e_{x,n+2}$. Since $d(x, e_{x,n+2}) \le C_A A^{-n-2} \le \frac{8}{7} A^{-n-2}$, we have that the mass from $e_{x,n+2}$ stays within

$$B(x, 3A^{-n-2}) \subset B(x, \frac{3}{A} s) \subset B(x, s/2).$$

Therefore

$$\nu(B(x,s)) \geq \mu_{n+2}(e_{x,n+2}). \tag{6.30}$$

By (6.12) and (6.8), we obtain that $\mu_{n+2}(e_{x,n+2})$ and $\mu_n(P_{n+1}(P_{n+2}(e_{x,n+2})))$ are comparable, where P_{n+1}, P_{n+2} denote the predecessor as given in Definition 6.6. By the triangle inequality, we obtain that $d(e_{x,n}, P_{n+1}(P_{n+2}(e_{x,n+2}))) \leq 4A^{-n}$, and therefore by (6.7) and (6.10), we obtain that $\mu_n(P_{n+1}(P_{n+2}(e_{x,n+2})))$ and $\mu_n(e_{x,n})$ are comparable. Combining the above with (6.30), we obtain the lower bound $\nu(B(x,s)) \gtrsim \mu_n(e_{x,n})$ in (6.27). This completes the proof of (6.27).

Next, we obtain (6.26) from (6.27). Let $0 < s_1 < s_2 < A^{-4}\operatorname{diam}(Q_{l,0})$. Let $n_1, n_2 \in \mathbb{Z}$ be such that $A^{-n_i-1} \leq s_i < A^{-n_i}$ for $i = 1, 2$. For $x \in Q_{l,0}$, let $x_{n_i} \in M_{n_i}$ be the unique point in M_{n_i} such that $x_{n_i} \in Q_{n_i}(x)$. By (6.27) and Lemma 6.7(a),(b),(c), we have

$$\frac{\nu(B(x,s_2))\operatorname{Cap}_{B(x,As_1)}(B(x,s_1))}{\nu(B(x,s_1))\operatorname{Cap}_{B(x,As_2)}(B(x,s_2))} \asymp \frac{\mu_{n_2}(x_{n_2})c_{n_1}(x)}{\mu_{n_1}(x_{n_1})c_{n_2}(x)} \asymp \frac{\mu_{n_2}(x_{n_2})c_{n_1}(x_{n_1})}{\mu_{n_1}(x_{n_1})c_{n_2}(x_{n_2})}.$$

Next, by using Lemma 6.9(2), we obtain

$$(1-\delta)^{n_2-n_1} \lesssim \frac{\nu(B(x,s_2))\operatorname{Cap}_{B(x,As_1)}(B(x,s_1))}{\nu(B(x,s_1))\operatorname{Cap}_{B(x,As_2)}(B(x,s_2))} \asymp \frac{\mu_{n_2}(\tilde{x}_{n_2})c_{n_1}(x_{n_1})}{\mu_{n_1}(x_{n_1})c_{n_2}(\tilde{x}_{n_2})} \lesssim C_4^{n_1-n_2},$$

where $C_4 > 1$ is the constant C in Lemma 6.9 and $\delta \in (0,1)$ is as in Lemma 6.8. The desired estimate (6.26) follows by setting $\beta_1 = -\log(1-\delta)/\log A$ and $\beta_2 = \log C_4/\log A$. □

We are now in the position to give the

Proof of Theorem 6.4. The compact case follows by choosing $l = k_0 - 1$ in Proposition 6.11.

It suffices to consider the non-compact case. For $l \leq -1, l \in \mathbb{Z}$ let ν_l be the measure given by Proposition 6.11 on $Q_{l,0}$, and choose $a_n > 0$ so that

$$a_l \nu_l(B(x_0,1)) = 1, \quad \text{for all } l \in \mathbb{Z}, l < 0.$$

A compactness argument similar to that in [LuS] yields the existence of a measure ν which is a sub-sequential weak* limit of the sequence of measures $a_l\nu_l$ as $l \to -\infty$, bounded on compacts, such that it is $(C_0, A, \beta_1, \beta_2)$-capacity good. □

6.2 A criterion for smoothness of measure

In this section, we will provide a useful sufficient condition for a doubling measure to be smooth. The definition of a smooth measure is given in Definition 2.4.

The following lemma follows immediately from [CF, Theorem 3.3.8] or [FOT, Theorem 4.4.3] and the countable subadditivity for capacities.

Lemma 6.12. *Let $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F})$ be a MMD space. Let $\{B_i : i \in I\}$ be a countable family of open balls such that $\cup_{i\in I} B_i = \mathcal{X}$. Let $U \subset \mathcal{X}$ be a Borel set. Then U has zero capacity for $(\mathcal{E}, \mathcal{F})$ if and only if $U_i := U \cap B_i$ has zero capacity for the part Dirichlet form $(\mathcal{E}^{B_i}, \mathcal{F}^{B_i})$ for all $i \in I$.*

Proposition 6.13. *Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies the EHI and Assumption 6.1. Let μ be a capacity good measure. Then μ is a smooth Radon measure.*

Proof. By Theorem 4.6, $(\mathcal{E},\mathcal{F})$ has regular Green functions. Denote by $\mathcal{N}$ the corresponding Borel properly exceptional set of $\mathcal{X}$ for the regular Green functions of $(\mathcal{E},\mathcal{F})$.

Let A denote the constant in Lemma 5.12. Let $B = B(x_0, r)$ denote any ball such that $r \le \operatorname{diam}(\mathcal{X}, d)/A^2$. For $x \in \mathcal{X}, s < \operatorname{diam}(\mathcal{X}, d)/A^2$, we set $\Psi(x,s) = \frac{\mu(B(x,s))}{\operatorname{Cap}_{B(x,As)}(B(x,s))}$.

We will show that $x \mapsto \int_B g_B(x,y)\,\mu(dy)$ is bounded uniformly in $B \setminus \mathcal{N}$. Note that $\int_B g_B(x,y)\mu(dy)$ is well defined for every $x \in B\setminus\mathcal{N}$ in view of the definition (see Definition 4.4) of regular Green function on B. By Lemma 6.3, the measure μ satisfies the reverse volume doubling property (RVD) [Hei, Exercise 13.1]: that is there exists $c_0 > 0$ such that there exist $c_0 \in (0,1), C_3 \in (1,\infty), \alpha \in (0,\infty)$ such that

$$\mu(B(x,R)) \ge c_0 \left(\frac{R}{r}\right)^{\alpha} \mu(B(x,r)), \quad \text{for all } x \in \mathcal{X}, 0 < r \le R < \operatorname{diam}(\mathcal{X},d)/C_3. \tag{6.31}$$

In particular by letting $r \to 0$ in the above equation. we obtain $\mu(\{x\}) = 0$ for all $x \in \mathcal{X}$.

Fix $x \in B \setminus \mathcal{N}$ and set $B_i = B(x, A^{1-i} d(x,y)), A_i = B_i \setminus B_{i+1}$ for $i \in \mathbb{N}_{\ge 0}$.

$$\begin{aligned}
\int_B g_B(x,y)\,\mu(dy) &\le \int_B g_{B(x,4r)}(x,y)\,\mu(dy) \text{ (by domain monotonicity)}\\
&\le \sum_{i=1}^{\infty} \int_{B\cap A_i} g_{B(x,Ar)}(x,y)\,\mu(dy) \quad \text{(since } \mu(\{x\}) = 0 \text{ by (RVD))}\\
&\lesssim \sum_{i=0}^{\infty} \int_{B\cap A_i} \operatorname{Cap}_{B_0}(B_{i+1})^{-1}\,\mu(dy) \quad \text{(by (HG), (4.5), Lemma 5.10)}\\
&\lesssim \sum_{i=0}^{\infty} \int_{B\cap A_i} \sum_{j=0}^{i} \operatorname{Cap}_{B_j}(B_{j+1})^{-1}\,\mu(dy) \quad \text{(by Lemma 5.12)}\\
&\lesssim \sum_{j=0}^{\infty} \operatorname{Cap}_{B_j}(B_{j+1})^{-1} \sum_{i=j}^{\infty} \int_{B\cap A_i} d\mu \lesssim \sum_{j=0}^{\infty} \operatorname{Cap}_{B_j}(B_{j+1})^{-1}\mu(B_j)\\
&\lesssim \sum_{j=0}^{\infty} \Psi(x, 2^{2-j}r) \lesssim \Psi(x_0, r). \quad \text{(by (6.1), Lemmas 6.3 and 5.20(b)).}
\end{aligned} \tag{6.32}$$

We claim that $\mu|_B$ is a smooth measure on B. Suppose not. Then there is a compact subset $K \subset B$ which is $\mathcal{E}^B$-polar (which is equivalent to being $\mathcal{E}$-polar) so that $\mu(K) > 0$. Let $h(x) := \int_K g_B(x,y)\mu(dy)$. By (6.32), $h(x)$ is bounded on $B \setminus \mathcal{N}$. By Lemma 5.10 and the maximum principle (4.5), $g_B(x,y) > 0$ for $x \in B$ and $y \in K$ with $0 < d(x,y) < r_1$ for some $r_1 > 0$. Thus $\{x \in B\setminus\mathcal{N} : h(x) > 0\}$ has positive $\mathcal{E}$-capacity. On the other hand, by the definition of regular harmonic function, $x \mapsto g_D(x,y)$ is harmonic in $B \setminus \{y\}$ for every $y \in B$ by Theorem 4.5(ii). So by (6.32) and Fubini's theorem, $h(x) = \int_K g_B(x,y)\mu(dy)$ is a bounded harmonic function in $B \setminus K$. Since K is $\mathcal{E}$-polar, h is bounded harmonic in B. Let $\{D_n; n \ge 1\}$ be an increasing sequence of relatively compact open sets that increases to B and that $K \subset D_n$ for every $n \ge 1$. By Remark 2.7 and Proposition 3.2, $h(x) = \mathbb{E}^x[h(X_{\tau_{D_n}})]$ for $x \in D_n \setminus \mathcal{N}$. Hence for every $x \in B \setminus \mathcal{N}$,

$$h(x) = \lim_{n\to\infty} \mathbb{E}^x[h(X_{\tau_{D_n}})] = \lim_{n\to\infty} \int_K \mathbb{E}^x g_D(X_{\tau_{D_n}}, y)\mu(dy). \tag{6.33}$$

For every $y_0 \in B$, as $x \mapsto g_B(x,y)$ is $X^B|_{B\setminus\mathcal{N}_B}$-excessive, where $\mathcal{N}_B$ is a Borel properly exceptional set for X^B appearing in Definition 4.4(iv) for g_B. Thus $t \mapsto g_B(X_t^B, y_0)$ is a non-negative

$\mathbb{P}^x$-supermartingale for $x \in B \setminus \mathcal{N}_B$, and consequently, $\lim_{n\to\infty} g_B(X_{\tau_{D_n}}, y_0)$ exists $\mathbb{P}^x$-a.s. By the maximum principle (4.5), EHI and Hölder estimate (4.6), $\lim_{n\to\infty} g_B(X_{\tau_{D_n}}, y)$ is uniformly Hölder continuous in $y \in B(y_0, r_2)$ for some $r_2 > 0$. For any bounded compactly support function $\varphi \geq 0$ on D, by the strong Markov property and bounded convergence theorem

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}^x \int_D g_B(X_{\tau_{D_n}}, y)\varphi(y)\, m(dy) &= \lim_{n\to\infty} \mathbb{E}^x \left(\mathbb{E}^x \left[\int_0^{\tau_B} \varphi(X_s)\, ds \circ \theta_{\tau_{D_n}} \right] \right) \\ &= \lim_{n\to\infty} \mathbb{E}^x \int_{\tau_{D_n}}^{\tau_B} \varphi(X_s) ds = 0.\end{aligned}$$

Thus by the Fatou's lemma and the Hölder regularity mentioned above, we conclude that $\lim_{n\to\infty} g_B(X_{\tau_{D_n}}, y_0) = 0$ for every $y_0 \in B$ $\mathbb{P}^x$-a.s. Observe that $\{g_D(X_{\tau_{D_n}}, y); n \geq 1, y \in K\}$ are uniformly bounded random variables by the maximum principle (4.5). Thus we conclude from (6.33) by the bounded convergence theorem that

$$h(x) = \int_K \mathbb{E}^x \left[\lim_{n\to\infty} g_D(X_{\tau_{D_n}}, y) \right] \mu(dy) = 0 \quad \text{for every } x \in B \setminus \mathcal{N}_B.$$

This contradicts to the fact that $\{x \in B \setminus \mathcal{N} : h(x) > 0\}$ has positive $\mathcal{E}$-capacity. We have thus proved that μ is a smooth Radon measure. □

A smooth measure μ on $\mathcal{X}$ uniquely determines a positive continuous additive functional $A^\mu = \{A_t^\mu; t \geq 0\}$ of X. It can be used to define a time-changed process $Y_t := X_{\tau_t}$, where

$$\tau_t := \inf\{r > 0 : A_r^\mu > t\}.$$

Let $S(\mu)$ denote the quasi support of μ (see Definition 2.5) and F be the topological support of μ. Clearly $S(\mu) \subset F$ $\mathcal{E}$-q.e. and $\mu(F \setminus S(\mu)) = 0$. Suppose μ is a smooth Radon measure. Then the time-changed process Y, after possibly modification on a Borel properly exceptional set for X, is an μ-symmetric Hunt process on F and its associated Dirichlet form $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ on $L^2(F;\mu)$ is regular. Moreover,

$$\begin{aligned}\mathcal{F}^\mu &= \left\{ \phi \in L^2(F, \mu) : \phi = u \quad \mu\text{-a.e. for some } u \in \mathcal{F}_e \right\}, \\ \mathcal{E}^\mu(\phi, \phi) &= \mathcal{E}(H_{S(\mu)}u, H_{S(\mu)}u), \quad \text{for } \phi \in \mathcal{F}^\mu, \text{ and an arbitrary } u \in \mathcal{F}_e \text{ with } \phi = u \quad \mu\text{-a.e.},\end{aligned} \tag{6.34}$$

where $\mathcal{F}_e$ is the extended Dirichlet space of $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ and $H_{S(\mu)}u(x) = \mathbb{E}^x u(X_{\sigma_{S(\mu)}})$ for $x \in \mathcal{X}$. See [CF, Theorem 5.2.13] or [FOT, Theorem 5.1.5 and Theorem 6.2.1]. The Dirichlet form $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ is called the *trace Dirichlet form* of $(\mathcal{E}, \mathcal{F})$ on $L^2(F, \mu)$. If μ has full quasi support, then $\mathcal{F}_e^\mu = \mathcal{F}_e$ by [CF, Corollary 5.2.12] and (6.34) can be simplified to

$$\mathcal{F}^\mu = \mathcal{F}_e \cap L^2(\mathcal{X}, \mu), \quad \mathcal{E}^\mu(u, u) = \mathcal{E}(u, u) \quad \text{for all } u \in \mathcal{F}_e \tag{6.35}$$

Remark 6.14. The above mentioned properties for time-changed processes and Dirichlet forms in fact hold for any smooth measure μ rather than just smooth Radon measures except that the time-changed process is a right process instead of being a Hunt process on F and the trace Dirichlet form $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ is quasi-regular on $L^2(S(\mu); \mu)$ instead of being regular on $L^2(F; \mu)$. See [CF, Theorem 5.2.7].

Recall the definition of quasi support of a smooth measure from Definition 2.5. In this work, we are interested in smooth measures with full quasi support as defined below.

Definition 6.15 (Admissible smooth measures)**.** Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space. We say that a smooth Radon measure μ on $\mathcal{X}$ is *admissible* if μ has full quasi support. In particular, the time-changed Dirichlet form is given by (6.35).

Proposition 6.16. *Suppose that $(\mathcal{X}, d)$ is relatively K ball connected for some $K > 1$ and $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space that satisfies Assumption 6.1. Let μ be a $(C_0, A_1, \beta_1, \beta_2)$-capacity good (hence smooth) measure on $\mathcal{X}$ for some $C_0, A > 1$ and $0 < \beta_1 < \beta_2$. Then μ is admissible.*

Proof. Let $\mathcal{N}$ be a Borel properly exceptional set for the Hunt process X associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$ so that the conclusion of Theorem 4.6 holds. Denote by $S(\mu)$ a quasi support of μ. As discussed in [BK, Proposition 2.6], it suffices to show that

$$\mathbb{P}^x(\sigma_{S(\mu)} = 0) = 1 \qquad \text{for quasi every } x \in \mathcal{X}. \tag{6.36}$$

For the reader's convenience, we recall why (6.36) implies that μ has full quasi support. By [FOT, Theorem 4.6.1(i)] we may assume that $S(\mu)^c$ is nearly Borel and finely open, by adjusting $S(\mu)$ on a set of capacity zero. Then since $S(\mu)^c$ is nearly Borel and finely open, for any $x \in S(\mu)^c \backslash \mathcal{N}$ we have $\mathbb{P}^x(\sigma_{S(\mu)} > 0) = 1$, which by (6.36) implies that $S(\mu)^c$ has capacity zero.

Note that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ is irreducible by Theorem 4.8. Let $x_0 \in \mathcal{X} \setminus \mathcal{N}$. Let $t > 0$ and $\varepsilon > 0$ be arbitrary. Applying Lemma 3.3 to the part process $X^{B(x_0,R_0)}$ of X killed upon leaving a ball $B(x_0, R_0)$ whose complement has positive capacity, we can choose $r = r(x, t, \varepsilon)$ so that $\mathbb{P}^x(T < t) > 1 - \varepsilon$ for $\mathcal{E}$-q.e. $x \in B(x_0, r_0)$, where $T = \tau_{B(x,r)}$. By applying Lemma 3.3 to countably many such balls $B(x_0, r_0)$, we conclude that

$$\mathbb{P}^x(T < t) > 1 - \varepsilon, \quad \text{for } \mathcal{E}\text{-q.e. } x \in \mathcal{X}, \text{ where } T = \tau_{B(x,r)}, r = r(x, t, \varepsilon). \tag{6.37}$$

By decreasing $r = r(x, t, \varepsilon)$ if necessary, we may assume that $0 < r < \operatorname{diam}(\mathcal{X}, d)/A^4$, where A is the constant in capacity good condition. Since we will use Proposition 5.7, by increasing A if necessary we assume that $A \geq 2K + 1$, where K is the constant in Assumption 5.6. This increase in A is possible due to Lemma 5.22. Fixing $r = r(x, t, \varepsilon)$ as above, we define

$$\begin{aligned} K_1 &= \left(B(x, A^{-1}r) \setminus B(x, A^{-2}r)\right) \cap S(\mu), \\ A_1 &= \{\sigma_{K_1} < T\}. \end{aligned}$$

We show that there exists a constant $c_0 \in (0, 1)$ that depends only on the constants associated with Assumption 6.1), and capacity good condition such that

$$\mathbb{P}^x(A_1) \geq c_0 \quad \text{for } \mathcal{E}\text{-q.e. } x \in \mathcal{X}. \tag{6.38}$$

Let e denote the equilibrium measure for K_1 such that $e(\overline{K_1}) = \operatorname{Cap}_B(K_1)$, where $B = B(x, r)$. To prove (6.38), we observe that

$$\mathbb{P}^z(\sigma_{K_1} < \tau_B) = \int_{\overline{K_1}} g_B(z, y)\, e(dy) \quad \text{for } \mathcal{E}\text{-q.e. } z \in B. \tag{6.39}$$

To obtain (6.39), we use [FOT, Theorem 4.3.3 and the 0-order version of Exercise 4.2.2] to conclude that both sides of (6.39) are quasi-continuous versions of the 0-order equilibrium potential for K_1 with respect to the part Dirichlet form on B. We would like to use (6.39) for $z = x$, but z could belong to the exceptional set associated with (6.39). To this end, we note that both sides of (6.39) are $X^B|_{B \backslash \mathcal{N}}$-excessive from Theorem 4.6 and [CF, Lemma A.2.4(ii)] respectively.

By the absolute continuity property of $X|_{\mathcal{X}\setminus\mathcal{N}}$ from Theorem 4.6 and [CF, Lemma A.2.17(iii)], we conclude that

$$\mathbb{P}^z(\sigma_{K_1} < \tau_B) = \int_{\overline{K_1}} g_B(z,y)\, e(dy) \quad \text{for every } z \in B \setminus \mathcal{N}. \tag{6.40}$$

It is crucial that the properly exceptional set $\mathcal{N}$ does not depend on B. By (6.40) and (4.5),

$$\mathbb{P}^x(A_1) = \int_{\overline{K_1}} g_B(x,y)\, e(dy) \geq g_B(x, A^{-1}r)\, \mathrm{Cap}_B(K_1), \quad \text{for } \mathcal{E}\text{-q.e. } x \in \mathcal{X}. \tag{6.41}$$

By (6.32), (6.1) and Lemma 5.10, there exists $C_1 > 0$ such that,

$$\int_{B(y,r/A)} g_B(y,z)\, \mu(dz) \leq C_1 g_B(x, r/A)\mu(B(x,r/A)) \quad \text{for all } y \in B(x,r). \tag{6.42}$$

Using the fact that $(\mathcal{X}, d)$ is relatively ball connected, for any $n \geq 1$, there exists $y \in \mathcal{X}$ such that $B(y, A^{-n-1}r(A-1)/3) \subset B(x, A^{-n}r) \setminus B(x, A^{-n-1}r)$. Since μ is a doubling measure and $\mu(S(\mu)^c) = 0$, we obtain

$$\mu(K_1) = \mu\left(B(x, A^{-1}r) \setminus B(x, A^{-2}r)\right) \geq \mu\left(B(y, A^{-2}r(A-1)/3)\right) \gtrsim \mu\left(B(x, A^{-1}r)\right). \tag{6.43}$$

We recall the following inequality for capacity: for any Radon measure ν on B with $\int_B g_B(\cdot, z)\, \nu(dz) \leq 1$ q.e. on B and $\nu(B \setminus K_1) = 0$,

$$\nu(K_1) \leq \mathrm{Cap}_B(K_1).$$

See [FOT, p.441, Solution to Exercise 2.2.2] and note also [FOT, Exercise 4.2.2]. By considering the measure $\nu(\cdot) = \mu(K_1 \cap \cdot)/(C_1 g_B(x, r/A)\mu(B(x, r/A)))$, (6.42) and the above inequality, we obtain

$$\begin{aligned} \mathrm{Cap}_B(K_1)^{-1} &\leq \nu(K_1)^{-1} = C_1 g_B(x, r/A)\mu(B(x, r/A))/\mu(K_1) \\ &\lesssim g_B(x, r/A), \quad \text{(by (6.43))}. \end{aligned} \tag{6.44}$$

Combining (6.41) and (6.44) establishes the claim (6.38). Choosing $\varepsilon = c_0/2$, we obtain for $\mathcal{E}$-q.e. $x \in \mathcal{X}$

$$\begin{aligned} \mathbb{P}^x(\sigma_{S(\mu)} \leq t) &\geq \mathbb{P}^x(\sigma_{K_1} < T) - \mathbb{P}^x(T \geq t) \quad (\text{since } \{\sigma_{K_1} < T\} \subset \left\{\sigma_{S(\mu)} \leq t\right\} \cup \{T \geq t\}) \\ &> c_0 - \varepsilon = \tfrac{1}{2}c_0 \quad \text{(by (6.37) and (6.38))}. \end{aligned}$$

Since $t > 0$ is arbitrary, the Blumenthal 0-1 law [CF, Lemma A.2.5] gives $\mathbb{P}^x(\sigma_{S(\mu)} = 0) = 1$. □

7 Quasisymmetry and stability

Although the assumption that all MMD spaces are strongly local is in force in this section, we remark that Lemma 7.1 and Lemma 7.5(a) in fact hold for general Dirichlet forms as well.

The following is a straightforward consequence of the definition of quasisymmetry.

Lemma 7.1. *([BM1, Lemma 5.3]) Let $(\mathcal{X}, d_1, \mu, \mathcal{E}, \mathcal{F}^\mu)$ be a MMD space and let d_2 be a metric on $\mathcal{X}$ quasisymmetric to d_1. If $(\mathcal{X}, d_2, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the EHI, then so does $(\mathcal{X}, d_1, \mu, \mathcal{E}, \mathcal{F}^\mu)$.*

The next definition is a slight modification of [BM1, Definition 5.4], the change being made so that it applies to both compact and non-compact spaces.

Definition 7.2. We say that a function $\Psi : \mathcal{X} \times [0,\infty) \to [0,\infty)$ on a metric space $(\mathcal{X}, d)$ is a *regular scale function* if $\Psi(x,0) = 0$ for all $x \in \mathcal{X}$ and there exist constants $C_1, \beta_1, \beta_2 > 0$ such that, for all $x, y \in \mathcal{X}$ and finite $0 < s \le r \le \operatorname{diam}(\mathcal{X}, d)$, we have with $R := d(x,y)$, $\Psi(y,s) > 0$ and

$$C_1^{-1}\Big(\frac{r}{R\vee r}\Big)^{\beta_2}\Big(\frac{R\vee r}{s}\Big)^{\beta_1} \le \frac{\Psi(x,r)}{\Psi(y,s)} \le C_1\Big(\frac{r}{R\vee r}\Big)^{\beta_1}\Big(\frac{R\vee r}{s}\Big)^{\beta_2}. \tag{7.1}$$

Given a regular scale function Ψ on $(\mathcal{X}, d)$, we now define a metric d_Ψ. This is proved as in [BM1] – the proof there still works when $\operatorname{diam}(\mathcal{X}, d) < \infty$.

Proposition 7.3. *([BM1, Proposition 5.7]) Let Ψ be a regular scale function on a metric space $(\mathcal{X}, d)$. There exists a metric $d_\Psi : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ satisfying the following properties:*

(a) *There exist $C, \beta > 0$ such that for all $x, y \in \mathcal{X}$,*

$$C^{-1}\Psi(x, d(x,y)) \le d_\Psi(x,y)^\beta \le C\Psi(x, d(x,y)). \tag{7.2}$$

(b) *d and d_Ψ are quasisymmetric.*

(c) *Assume in addition that $(\mathcal{X}, d)$ (or equivalently $(\mathcal{X}, d_\Psi)$) is uniformly perfect. Fix $A > 1$. Let B_Ψ and B denote metric balls in $(\mathcal{X}, d_\Psi)$ and $(\mathcal{X}, d)$ respectively. If $x \in \mathcal{X}$ and $r, s > 0$ satisfy, either $B_\Psi(x,s) \subset B(x,r) \subset B_\Psi(x, As) \subsetneq \mathcal{X}$ or $B(x,r) \subset B_\Psi(x,s) \subset B(x, Ar) \subsetneq \mathcal{X}$, then there is a constant $C_1 > 1$ (which does not depend on $x \in \mathcal{X}, r > 0, s > 0$) such that*

$$C_1^{-1}s^\beta \le \Psi(x,r) \le C_1 s^\beta, \tag{7.3}$$

where $\beta > 0$ is as given in (7.2).

We now introduce Poincaré, cutoff energy inequalities, and capacity bounds with respect to a regular scale function Ψ on $(\mathcal{X}, d)$. This is again a slight modification of [BM1, Definition 5.8 and 5.13], so as to include both bounded and unbounded spaces. Recall that a *cutoff function* φ for $B_1 \subset B_2$ is any function $\varphi \in \mathcal{F}^\mu$ such that $0 \le \varphi \le 1$ in $\mathcal{X}$, $\varphi \equiv 1$ in an open neighbourhood of $\overline{B_1}$, and $\operatorname{supp}\varphi \subset B_2$. Recall also that $\mu_{\langle f\rangle}$ is the energy measure of $f \in \mathcal{F}^\mu$; see Section 2

Definition 7.4. Let Ψ be a regular scale function on $(\mathcal{X}, d)$, and $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ a MMD space.

(i) We say that $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the *Poincaré inequality* PI(Ψ), if there exists constants $C, A_1, A_2 \ge 1$ such that for all $x \in \mathcal{X}$, $R \in (0, \operatorname{diam}(\mathcal{X}, d)/A_2)$ and $f \in \mathcal{F}^\mu$, $\mu(B(x,R)) < \infty$ and

$$\int_{B(x,R)} (f - \overline{f})^2 \, d\mu \le C\Psi(x,R)\, \mu_{\langle f\rangle}(B(x, A_1 R)), \tag{PI(Ψ)}$$

where $\overline{f} = \frac{1}{\mu(B(x,R))}\int_{B(x,R)} f\, d\mu$.

(ii) We say that $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the *cutoff energy inequality* CS(Ψ), if there exist $C_1, C_2 > 0, A_1, A_2 > 1$ such that the following holds. For all $R \in (0, \operatorname{diam}(\mathcal{X}, d)/A_2)$, $x \in \mathcal{X}$ with $B_1 = B(x,R)$ and $B_2 = B(x, A_1 R)$, there exists a cutoff function φ for $B_1 \subset B_2$ such that for any $u \in \mathcal{F}^\mu \cap L^\infty$,

$$\int_{\mathcal{X}} u^2 d\mu_{\langle\varphi\rangle} \le C_1 \mu_{\langle u\rangle}(B_2 \setminus B_1) + \frac{C_2}{\Psi(x,R)}\int_{B_2\setminus B_1} u^2\, d\mu. \tag{CS(Ψ)}$$

(iii) We say that $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the *capacity estimate* $\mathrm{cap}(\Psi)$ if there exist positive constants $C_1, A_1, A_2 > 1$ such that for all $R \in (0, \mathrm{diam}(\mathcal{X}, d)/A_2)$ and $x \in \mathcal{X}$

$$C_1^{-1} \frac{\mu(B(x,R)}{\Psi(x,R)} \le \mathrm{Cap}_{B(x,A_1R)}(B(x,R)) \le C_1 \frac{\mu(B(x,R)}{\Psi(x,R)}. \tag{$\mathrm{cap}(\Psi)$}$$

If $\Psi(r) = r^\beta$, we denote $\mathrm{PI}(\Psi), \mathrm{CS}(\Psi), \mathrm{cap}(\Psi)$ by $\mathrm{PI}(\beta), \mathrm{CS}(\beta), \mathrm{cap}(\beta)$ respectively.

The following lemma shows that the Poincaré and cutoff energy inequalities take a much simpler form with respect to the metric d_Ψ.

Lemma 7.5. *([BM1, Lemma 5.9]) Let $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ be a uniformly perfect MMD space and let Ψ be a regular scale function. Let d_Ψ be the metric constructed in Proposition 7.3 with $\beta > 0$ as given in* (7.2)*. Then*

(a) $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies $\mathrm{PI}(\Psi)$ if and only if $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies $\mathrm{PI}(\beta)$.

(b) $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies $\mathrm{CS}(\Psi)$ if and only if $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies $\mathrm{CS}(\beta)$.

The following comparison of annuli follows readily from the definition.

Lemma 7.6. *([MT, Lemma 1.2.18]) Let the identity map $\mathrm{Id} : (\mathcal{X}, d_1) \to (\mathcal{X}, d_2)$ be an η-quasisymmetry for some distortion function η. Then for all $A > 1, x \in \mathcal{X}, r > 0$, there exists $s > 0$ such that, with B_i denoting balls in $(\mathcal{X}, d_i)$*

$$B_2(x, s) \subset B_1(x, r) \subset B_1(x, Ar) \subset B_2(x, \eta(A)s). \tag{7.4}$$

In (7.4)*, s can be defined as*

$$s = \sup \{0 \le s_2 < 2\,\mathrm{diam}(\mathcal{X}, d_1) : B_2(x, s_2) \subset B_1(x, r)\}$$

Moreover, for all $A > 1$, $x \in \mathcal{X}$ and $r > 0$, there exists $t > 0$ such that

$$B_1(x, r) \subset B_2(x, t) \subset B_2(x, At) \subset B_1(x, A_1 r), \tag{7.5}$$

where $A_1 = 1/\eta^{-1}(A^{-1})$. In (7.5)*, t can be defined as*

$$t = A^{-1} \sup \{0 \le r_2 < 2A\,\mathrm{diam}(\mathcal{X}, d_2) : B_2(x, Ar_2) \subset B_1(x, A_1 r)\}.$$

The following is an analogue of Lemma 7.5 for the capacity estimate $\mathrm{cap}(\Psi)$.

Lemma 7.7. *Let $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ be a MMD space that satisfies the EHI and let Ψ be a regular scale function. Suppose that $(\mathcal{X}, d)$ is complete and that μ satisfies the volume doubling property on $(\mathcal{X}, d)$. Let d_Ψ be the metric constructed in Proposition 7.3 with $\beta > 0$ as given in* (7.2)*. Then $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies $\mathrm{cap}(\Psi)$ if and only if $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies $\mathrm{cap}(\beta)$.*

Proof. Let B and B_Ψ denote balls in the metrics d and d_Ψ respectively. By Lemma 7.1, $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ also satisfies the EHI. Let the identity map $\mathrm{Id} : (\mathcal{X}, d_\Psi) \to (\mathcal{X}, d)$ be an η-quasisymmetry. Note that μ satisfies the volume doubling property with respect to the metric d and d_Ψ.

Let $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfy $\mathrm{cap}(\Psi)$. Set $A_1 = \eta(2)$ and choose $\varepsilon \in (0, \frac{1}{2}]$ so that $\eta(4\varepsilon) \le \frac{1}{2A_1}$. By Lemma 5.22, we may assume that

$$\mathrm{Cap}_{B(x,A_1r)}(B(x,r)) \asymp \frac{\mu(B(x,r))}{\Psi(x,r)}, \quad \text{for all } x \in \mathcal{X}, 0 < r \lesssim \mathrm{diam}(\mathcal{X}, d). \tag{7.6}$$

By Lemma 7.6 and Proposition 7.3(c), for all $x \in \mathcal{X}$, $0 < s < \varepsilon \operatorname{diam}(\mathcal{X}, d_\Psi)$, there exists $r > 0$ such that $B(x,r) \subset B_\Psi(x,s) \subset B_\Psi(x,2s) \subset B(x,\eta(2)r) \neq \mathcal{X}$, since $r < \operatorname{diam}(B_\Psi(x,s),d) < \eta(4\varepsilon)\operatorname{diam}(\mathcal{X},d)$ by [Hei, Proposition 10.8] and $s^\beta \asymp \Psi(x,r)$. By the volume doubling property $\mu(B(x,r)) \asymp \mu(B_\Psi(x,s))$. By domain monotonicity and (7.6), we have

$$\operatorname{Cap}_{B_\Psi(x,2s)}(B_\Psi(x,s)) \geq \operatorname{Cap}_{B(x,A_1 r)}(B(x,r)) \asymp \frac{\mu(B(x,r))}{\Psi(x,r)} \asymp \frac{\mu(B_\Psi(x,s))}{s^\beta}, \tag{7.7}$$

for all $x \in \mathcal{X}, 0 < s \lesssim \operatorname{diam}(\mathcal{X}, d_\Psi)$.

Set $A_2 = 1/\eta^{-1}(A_1^{-1})$. By Lemma 7.6 and Proposition 7.3(c), for all $x \in \mathcal{X}, s \in (0,(2A_2)^{-1}\operatorname{diam}(\mathcal{X},d_\Psi))$, there exists $r > 0$ such that $B_\Psi(x,s) \subset B(x,r) \subset B(x,A_1 r) \subset B_\Psi(x,A_2 s) \neq \mathcal{X}$ and $\Psi(x,r) \asymp s^\beta$. By the volume doubling property, $\mu(B(x,r)) \asymp \mu(B_\Psi(x,s))$. By Lemma 5.22, [Hei, Proposition 10.8], domain monotonicity and (7.6), we have

$$\begin{aligned}\operatorname{Cap}_{B_\Psi(x,2s)}(B_\Psi(x,s)) &\asymp \operatorname{Cap}_{B_\Psi(x,A_2 s)}(B_\Psi(x,s)) \\ &\leq \operatorname{Cap}_{B(x,A_1 r)}(B(x,r)) \asymp \frac{\mu(B(x,r))}{\Psi(x,r)} \asymp \frac{\mu(B_\Psi(x,s))}{s^\beta}\end{aligned} \tag{7.8}$$

for all $x \in \mathcal{X}, 0 < s \lesssim \operatorname{diam}(\mathcal{X}, d_\Psi)$. By (7.7) and (7.8), $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies cap(β).

The converse follows from a similar argument. □

We will now apply these results in the context of a change of measure on a MMD space. Let $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ be a MMD space which satisfies the EHI and satisfy one (and hence all) of the three equivalent conditions in Theorem 5.4. Let $(\mathcal{E}, \mathcal{F}_e)$ be its corresponding extended Dirichlet space, and μ be the measure constructed in Theorem 6.4. By Propositions 6.13 and 6.16, μ is a positive Radon measure charging no set of capacity zero and possessing full quasi-support. Let $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ denote the time-changed Dirichlet space with respect to μ as defined in (6.34). We have $\mathcal{F}^\mu = \mathcal{F}_e \cap L^2(\mathcal{X},\mu)$, $\mathcal{E}^\mu(f,f) = \mathcal{E}(f,f)$ for all $f \in \mathcal{F}^\mu$, and $\mathcal{F}^\mu_e = \mathcal{F}_e$ (cf. [CF, Theorems 5.2.2, (5.2.17) and Corollary 5.2.12]). Moreover, the Dirichlet form $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ on $L^2(\mathcal{X};\mu)$ shares the same quasi notions as the original Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X};m)$; see [CF, Theorem 5.2.11].

Theorem 7.8. *Let $(\mathcal{X}, d)$ be complete and metric doubling. Suppose that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ is a MMD space which satisfies the EHI. Let μ be a $(C_0, A, \beta_1, \beta_2)$-capacity good measure with $A \geq 2^{1/4}$. Denote $D = \operatorname{diam}(\mathcal{X}, d)$. Then the function $\Psi : \mathcal{X} \times [0,\infty) \to [0,\infty)$ defined by $\Psi(x,0) = 0$ and*

$$\Psi(x,r) = \begin{cases} \frac{\mu(B(x,r))}{\operatorname{Cap}_{B(x,r/A^4)}(B(x,r/A^5))} & \text{if } 0 < r < D, \\ \frac{\mu(B(x,D))}{\operatorname{Cap}_{B(x,D/A^4)}(B(x,D/A^5))} & \text{if } r \geq D \text{ and } D < \infty, \end{cases} \tag{7.9}$$

is a regular scale function on $(\mathcal{X}, d)$. Furthermore, the MMD space $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the Poincaré inequality PI(Ψ)*, the cutoff energy inequality* CS(Ψ) *and the capacity estimate* cap(Ψ)*.*

Proof. By volume doubling (Lemma 6.3) and Lemma 5.20(c), there exists $C_2 > 0$ such that for all finite $0 < r \leq D$ and for all $x, y \in \mathcal{X}$ with $d(x,y) \leq r$, we have

$$C_2^{-1}\Psi(x,r) \leq \Psi(y,r) \leq C_2 \Psi(x,r). \tag{7.10}$$

Let $x, y \in \mathcal{X}$ and set $R := d(x,y)$. If $R \le r$ the inequalities in (7.1) are immediate from (6.1) and (7.10). If $s \le r < R$, then writing

$$\frac{\Psi(x,r)}{\Psi(y,s)} = \frac{\Psi(x,r)}{\Psi(x,R)} . \frac{\Psi(y,R)}{\Psi(y,s)} . \frac{\Psi(x,R)}{\Psi(y,R)},$$

and bounding each of the three terms on the right using (6.1) and (7.10) give (7.1). Thus Ψ is a regular scale function.

By Lemma 5.23, the MMD space $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies cap(Ψ).

Let d_Ψ and $\beta > 0$ be as given by Proposition 7.3. By Lemma 7.7, the MMD space $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies cap(β). By Lemma 7.1 and Proposition 7.3(b), $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the EHI.

By Lemma 5.2 the space $(\mathcal{X}, d_\Psi)$ is uniformly perfect, and hence the measure μ on $(\mathcal{X}, d_\Psi)$ satisfies the reverse volume doubling property (RVD) as defined in (6.31) [Hei, Exercise 13.1].

Therefore by [GHL, Theorem 1.2], since $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the EHI and cap(β), it satisfies PI(β) and CS(β). We now conclude using Lemma 7.5. □

The following gives equivalent characterization of the EHI for a MMD space $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$.

Theorem 7.9. *Let $(\mathcal{X}, d)$ be a complete, metric doubling, connected metric space with a strongly local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$. The following are equivalent:*

(a) *$(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the EHI.*

(b) *There exist an admissible smooth doubling Radon measure μ on $(\mathcal{X}, d)$ and a regular scale function Ψ such that the time-changed MMD space $(\mathcal{X}, d, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies the Poincaré inequality* PI(Ψ) *and the cutoff energy inequality* CS(Ψ)*.*

(c) *There exist an admissible smooth doubling Radon measure μ on $(\mathcal{X}, d)$, a metric d_Ψ on $\mathcal{X}$ that is quasisymmetric to d, and $\beta > 0$, such that the time-changed MMD space $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies Poincaré inequality* PI(β) *and the cutoff energy inequality* CS(β) *for some $\beta > 0$.*

Proof. (a) ⇒(b) This is immediate from Lemma 6.3, Propositions 6.13, 6.16, Theorems 6.4 and 7.8.

(b)⇒(c) By Lemma 5.2(b), $(\mathcal{X}, d)$ is uniformly perfect. Let d_Ψ and $\beta > 0$ be as given by Proposition 7.3. Quasisymmetry of d_Ψ follows from Proposition 7.3(b). Then PI(β) and CS(β) for $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ follow from Lemma 7.5.

(c)⇒(a) By Lemma 5.2(b), $(\mathcal{X}, d_\Psi)$ is uniformly perfect. Thus μ satisfies reverse volume doubling property [Hei, Exercise 13.1]. Since $(\mathcal{X}, d)$ is metric doubling, so is $(\mathcal{X}, d_\Psi)$ [Hei, Theorem 10.18]. So by [BM1, Proposition 5.11 and Remark 5.12], we obtain the condition $(\mathrm{Gcap}_\le)_\beta$ in [GHL]. Then by the implication $(\mathrm{Gcap}_\le)_\beta$ plus PI(β) to the EHI in [GHL, Theorem 1.1], we obtain the EHI for $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$. Since d_Ψ and d are quasisymmetric, the desired EHI follows from Lemma 7.1. □

Remark 7.10. (i) Note that conditions (b) and (c) in the Theorem above do not include the requirement that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the conditions (HC) or (Ha) introduced in Section 3. (It would be undesirable to include (Ha) or (HC), since we do not know if they are stable.) Thus (b) or (c) does not immediately give the existence of Green's functions;

however the existence of regular Green functions does follow from the implications (b), (c) $\Rightarrow$ (a) and Theorems 4.8 and 4.6

The proof in [GHL] that $(\mathrm{Gcap}_{\leq})_\beta$ plus PI(β) implies the EHI does not require the existence of Green's functions.

(ii) The result that (a) implies (c) in Theorem 7.9 can be sharpened as follows. If $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the EHI then for any $\beta > 2$ there exists a metric d_Ψ on $\mathcal{X}$ that is quasisymmetric to d, and an admissible smooth doubling Radon measure μ such that the time-changed MMD space $(\mathcal{X}, d_\Psi, \mu, \mathcal{E}, \mathcal{F}^\mu)$ satisfies Poincaré inequality PI(β) and the cutoff energy inequality CS(β). The condition $\beta > 2$ is sharp in the sense that any β in property (c) necessarily satisfies $\beta \geq 2$ and there are examples for which $\beta = 2$ is not possible. These results are contained in [KM].

Proof of Theorem 1.3. The condition that $\mathcal{E}(f, f) \asymp \mathcal{E}'(f, f)$ for all $f \in \mathcal{F}$ implies that the associated energy measures satisfy $\mu_{\langle f \rangle} \asymp \mu'_{\langle f \rangle}$; see (4.14). Hence the conditions PI(Ψ) and CS(Ψ) hold for $\mathcal{E}'$ by Theorems 5.4 and 7.9, and therefore the implication (b) $\Rightarrow$ (a) in Theorem 7.9 implies that the EHI holds for $\mathcal{E}'$. □

The following is an extension of Theorem 1.3, where the symmetrizing measures for the Dirichlet forms may be different.

Theorem 7.11. *Let $(\mathcal{X}, d)$ be a complete, doubling metric space, and let m be a Radon measure on $\mathcal{X}$ with full quasi support. Let $(\mathcal{E}, \mathcal{F})$ be a strongly local regular Dirichlet form on $L^2(\mathcal{X}; m)$. Suppose that $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ satisfies the EHI. Let μ be a smooth Radon measure of $(\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ with full quasi support on $\mathcal{X}$, and $(\mathcal{E}', \mathcal{F}')$ be another strongly local regular Dirichlet form on $L^2(\mathcal{X}; \mu)$ such that $\mathcal{F} \cap C_c(\mathcal{X}) = \mathcal{F}' \cap C_c(\mathcal{X})$ and*

$$C^{-1}\mathcal{E}(f, f) \leq \mathcal{E}'(f, f) \leq C\mathcal{E}(f, f) \quad \text{for all } f \in \mathcal{F} \cap C_c(\mathcal{X}). \tag{7.11}$$

Then $(\mathcal{X}, d, \mu, \mathcal{E}', \mathcal{F}')$ satisfies the EHI.

Proof. Let X be the Hunt process associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$. Since μ is a smooth Radon measure with full quasi-support, its associated positive continuous additive functional A_t is strictly increasing up to the lifetime of X. Thus its time-changed process $Y_t := X_{\tau_t}$, with $\tau_t := \inf\{r > 0 : A_t > t\}$, has the same family of harmonic functions as that of X. By (6.35), the Dirichlet form $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ of the time-changed process Y is regular on $L^2(\mathcal{X}; \mu)$ and has the property that $\mathcal{F}^\mu = \mathcal{F}_e \cap L^2(\mathcal{X}; \mu)$, $\mathcal{F}^\mu_e = \mathcal{F}_e$ and $\mathcal{E}^\mu = \mathcal{E}$ on $\mathcal{F}_e$. Moreover, $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ is strongly local and satisfies the EHI. Since both m and μ are Radon, any $f \in \mathcal{F}$ that has compact support is in $\mathcal{F}^\mu$ and, since $\mathcal{F} = \mathcal{F}_e \cap L^2(\mathcal{X}, m)$,

$$\mathcal{F} \cap C_c(\mathcal{X}) = (\mathcal{F}_e \cap L^2(\mathcal{X}; m)) \cap C_c(\mathcal{X}) = (\mathcal{F}^\mu_e \cap L^2(\mathcal{X}; \mu)) \cap C_c(\mathcal{X}) = \mathcal{F}^\mu \cap C_c(\mathcal{X}).$$

Since $\mathcal{F}' \cap C_c(\mathcal{X}) = \mathcal{F}^\mu \cap C_c(\mathcal{X})$ is dense in $\mathcal{F}'$ and $\mathcal{F}^\mu$ with respect to the Hilbert norms $\sqrt{\mathcal{E}'_1}$ and $\sqrt{\mathcal{E}^\mu_1}$, respectively, where

$$\mathcal{E}'_1(u, u) := \mathcal{E}'(u, u) + \int_{\mathcal{X}} u(x)^2 \mu(dx) \quad \text{and} \quad \mathcal{E}^\mu_1(u, u) := \mathcal{E}^\mu(u, u) + \int_{\mathcal{X}} u(x)^2 \mu(dx),$$

we have by (7.11) that $\mathcal{F}' = \mathcal{F}^\mu$ and

$$C^{-1}\mathcal{E}^\mu(f,f) \le \mathcal{E}'(f,f) \le C\mathcal{E}^\mu(f,f) \quad \text{for all } f \in \mathcal{F}^\mu.$$

The desired conclusion of the theorem now follows from Theorem 1.3 applied to the MMD space $(\mathcal{X}, d, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$. □

Remark 7.12. The stability results of this paper, Theorem 1.3 and Theorem 7.11, hold for the $\text{EHI}_{\le 1}$ as well. We now indicate the needed modifications. All of the results of Section 5 extend easily under the assumption $\text{EHI}_{\le 1}$ except that the conclusions only hold for balls of small enough radii. The main difference is in the construction of the measures ν_l in Proposition 6.11. Instead of the initial condition on M_{l+3} for the inductive construction using Lemma 6.9, we set the initial condition on M_1 to the uniform probability measure on M_1, where M_1 is as given in the generalized dyadic decomposition of $Q_{l,0}$. Then the weak* subsequential limit as in the proof of Theorem 6.4 will be a capacity good measure (only at small enough scales using the same argument). However, this property is enough so that our construction gives a smooth measure with full quasi support. All the results used in Section 7 (for example, [GHL, Theorem 1.2]) will also admit local versions. As noted in [GT12, Remark 4.6], [GT12, Proof of Theorem 4.2] cannot be localized, but [GK, Theorems 6.2 and 7.3] give a localization of it. Although there is no clear reference in the literature for these results, a careful reading of the proofs in the literature shows that these local versions do hold, with essentially the same proof.

8 Examples

Example 8.1. The following example uses the instability of the Liouville property given in Lyons [Lyo] to show that without some regularity of the metric the EHI is not stable.

We begin by describing briefly Lyons's example. Let $\Gamma = (\mathbb{V}_\Gamma, E_\Gamma)$ be the free group with two generators a and b, and let $\mathbb{V} = \mathbb{V}_\Gamma \times \{0,1\}$. Lyons constructed two sets of symmetric conductances $\{a^{(i)}_{xy}, x, y \in \mathbb{V}\}$, $i = 1,2$, on Γ such that if $E_i = \{(x,y) : a^{(i)}_{xy} > 0\}$ then $E_1 = E_2$. Denote $E_G = E_1 = E_2$, and let $\mathbb{G} = (\mathbb{V}, E_G)$ be the associated graph. The two sets of conductances have the following additional properties:

(1) For each $x \in \mathbb{V}$, $4 \le \left|\{y : a^{(i)}_{xy} > 0\}\right| \le 8$, so the graph $\mathbb{G}$ has uniformly bounded vertex degree.

(2) There exists $p_0 \in (0,1)$ so that $a^{(i)}_{xy} \in \{0\} \cup [p_0, 1]$ for all x, y, $i = 1,2$. Thus the conductances $a^{(i)}_{xy}$ are uniformly bounded above and below on the graph $\mathbb{G}$.

Define the quadratic forms, for $f : \mathbb{V} \to \mathbb{R}$,

$$\mathcal{E}^{(i)}(f,f) = \tfrac{1}{2} \sum_{x\in\mathbb{V}} \sum_{x\in\mathbb{V}} a^{(i)}_{xy} (f(y) - f(x))^2, \quad i = 1,2. \tag{8.1}$$

In view of (2) above we have

$$p_0 \mathcal{E}^{(1)}(f,f) \le \mathcal{E}^{(2)}(f,f) \le p_0^{-1} \mathcal{E}^{(1)}(f,f) \quad \text{for any function } f \text{ on } \mathbb{V}. \tag{8.2}$$

Let d be the graph distance on $\mathbb{V}$ and m be the measure on $\mathbb{V}$ which assigns mass 1 to each vertex x. In view of (8.2), there exists a common linear subspace $\mathcal{F} \subset L^2(\mathbb{V}, m)$ such that $(\mathcal{E}^{(i)}, \mathcal{F})$ is a symmetric regular Dirichlet form on $L^2(\mathbb{V}, m)$ for $i = 1,2$.

Recall that the strong Liouville property (SLP) holds (for a MMD space) if every positive harmonic function is constant, and the Liouville property (LP) holds if every bounded harmonic function is constant. Lyons constructed $\{a_{xy}^{(i)}\}$ for $i = 1, 2$ so that the SLP holds for the MMD space $(\mathbb{V}, d, m, \mathcal{E}^{(1)}, \mathcal{F})$ while the LP fails for the MMD $(\mathbb{V}, d, m, \mathcal{E}^{(2)}, \mathcal{F})$. Thus Lyons example shows that neither SLP nor LP are stable.

Let $(\mathcal{X}, \tilde{d})$ be the cable system for the graph $\mathbb{G}$: each edge $e \in E_G$ is replaced by a copy of $[0, 1]$ – see for example [V] for details of the construction. Let μ be the measure which assigns a copy of Lebesgue measure to each cable. The metric $\tilde{d}$ is the unique length metric on $\mathcal{X}$ which equals Euclidean distance on each cable. Write $(\tilde{\mathcal{E}}^{(i)}, \tilde{\mathcal{F}})$ for the (regular and strongly local) Dirichlet forms on $\mathcal{X}$ associated with the cable system. (For further details of this construction see [V, BM1].)

Now let $d'(x, y) = 1 \wedge \tilde{d}(x, y)$ for $x, y \in \mathcal{X}$, and write $B'(x, r)$ for balls with respect to the metric d'. Then $(\mathcal{X}, d')$ is locally compact and complete, and $(\mathcal{X}, d', \mu, \tilde{\mathcal{E}}^{(i)}, \mathcal{F})$, $i = 1, 2$, are MMD spaces. Note that (MD) fails for this space. The instability of the SLP and LP for the graph $\mathbb{G}$ extends to the cable systems, so that the MMD space $(\mathcal{X}, d', \mu, \tilde{\mathcal{E}}^{(1)}, \mathcal{F})$ satisfies the SLP while $(\mathcal{X}, d', \mu, \tilde{\mathcal{E}}^{(2)}, \mathcal{F})$ fails to satisfy the LP.

We now consider the EHI for balls $B'(x, r/2) \subset B'(x, r)$. Note first that if $r \le 2$ then since each vertex of $\mathbb{V}$ has between 4 and 8 neighbours, the EHI follows from the local Harnack inequality for both $(\mathcal{X}, d', \mu, \tilde{\mathcal{E}}^{(1)}, \mathcal{F})$ and $(\mathcal{X}, d', \mu, \tilde{\mathcal{E}}^{(2)}, \mathcal{F})$. If $r > 2$ then $B'(x, r) = B'(x, r/2) = \mathcal{X}$, and so if h is positive and $\mathcal{E}^{(1)}$-harmonic on $B'(x, r)$ then h is constant, and thus the EHI holds for the MMD space $(\mathcal{X}, d', \mu, \tilde{\mathcal{E}}^{(1)}, \mathcal{F})$. On the other hand, as the LP fails for $\mathcal{E}^{(2)}$ there exists a non-constant bounded $\mathcal{E}^{(2)}$-harmonic function h on $\mathcal{X}$, which we can normalize so that $\inf_{\mathcal{X}} h = 0$, $\sup_{\mathcal{X}} h = 1$. It is thus clear that the EHI for the MMD space $(\mathcal{X}, d', \mu, \tilde{\mathcal{E}}^{(2)}, \mathcal{F})$ fails for every ball $B'(x, r/2) \subset B'(x, r)$ with $r > 2$.

Remark 8.2. It would be interesting to have an example of strongly local MMD space that does not have (MD) property for which EHI fails but for which all balls are relatively compact. It does not seem easy to modify the example above to give this.

Example 8.3. We give an example of a strongly local irreducible MMD space where harmonic functions may be discontinuous and (Ha) fails. However the condition (HC) does hold. The space consists of three parts: the closure of a domain in $\mathbb{R}^2$, the standard Sierpinski gasket, and a line segment. Let $\mathcal{X}_1$ be the compact Sierpinski gasket, with vertices $A_1 = (0, 0)$, $A_2 = (1, 0)$ and $A_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $\mathcal{X}_2 = [0, 1] \times [-1, 0]$ a unit closed square, and let $\mathcal{X}_3$ be a smooth curve outside $\mathcal{X}_1 \cup \mathcal{X}_2$ that connects the vertex A_3 of the Sierpinski gasket with the point $A_4 = (1, -1/2)$ at the middle of the right side of the square $\mathcal{X}_2$. We identify $\mathcal{X}_3$ with a closed line segment of length $l > 1$.

Let $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_3$, equipped with Euclidean metric inherited from $\mathbb{R}^2$. Clearly, $(\mathcal{X}, d)$ is a compact separable metric space. Let m_1 be the measure on $\mathcal{X}_1$ which assigns mass 3^{-n} to each triangle of side 2^{-n}, and for $j = 2, 3$, let m_j be Lebesgue measure on $\mathcal{X}_j$. Let m be the measure on $\mathcal{X}$ such that $m|_{\mathcal{X}_i} = m_i$ for each i. Clearly, m is a finite measure on $\mathcal{X}$.

Let $(\mathcal{E}^{(1)}, \mathcal{F}^{(1)})$ be the strongly local Dirichlet form on $L^2(\mathcal{X}_1, m_1)$ associated with the standard diffusion on the Sierpinski gasket – see [Kig, Chapter 3]. Let $(\mathcal{E}^{(2)}, \mathcal{F}^{(2)})$ be the Dirichlet form associated with reflecting Brownian motion on $\mathcal{X}_2$, and let $(\mathcal{E}^{(3)}, \mathcal{F}^{(3)})$ be the Dirichlet form associated with Brownian motion on $\mathcal{X}_3$, with reflection at the two endpoints. Following [Kum] we can construct a strongly local Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, m)$ such that $\{f|_{\mathcal{X}_i}, f \in \mathcal{F}\} = \mathcal{F}^{(i)}$ for $i = 1, 2, 3$.

Let $X = \{X_t, t \geq 0; \mathbb{P}^x, x \in \mathcal{X}\}$ be the diffusion process associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$. The diffusion X is conservative since $1 \in \mathcal{F}$ and $\mathcal{E}(1,1) = 0$. The diffusion X on $\mathcal{X}$ behaves as follows:

(i) when X_t is inside $\mathcal{X}_1$, it behaves like Brownian motion on the Sierpinski gasket $\mathcal{X}_1$ until it reaches the vertex A_3 or the bottom K;

(ii) when X_t is inside $\mathcal{X}_2$, it behaves like two-dimensional Brownian motion in $\mathcal{X}_2$ reflected on $\partial\mathcal{X}_2 \setminus \mathcal{X}_1$;

(iii) when X_t is inside $\mathcal{X}_3$, it behaves like one-dimensional Brownian motion reflected at the end point A_4;

(iv) when X_t is at the vertex A_3, it has positive probability to enter either $\mathcal{X}_1$ and $\mathcal{X}_3$; when X_t is at the Cantor set K, it has positive probability to enter either $\mathcal{X}_1$ and $\mathcal{X}_2$; when the X_t is at A_4, it gets reflected into $\mathcal{X}_3$.

Note that single point $A_4 \in \mathcal{X}_2 \cap \mathcal{X}_3$ is polar for reflected Brownian motion in $\mathcal{X}_2$. Thus the process X starting from $\mathcal{X}_2 \setminus \{A_4\}$ can only enter $\mathcal{X}_3$ through the Sierpinksi gasket $\mathcal{X}_1$ via vertex A_3.

For any $r \in (0, 1/2)$, let $h(x) = \mathbb{P}^x(\tau_{B(A_4,r)} \in B(A_4, r) \cap \mathcal{X}_2)$. Clearly h is harmonic in the ball $B(A_4, r)$, $h(x) = 1$ for $x \in B(A_4, r) \cap \mathcal{X}_2 \setminus \{A_4\}$ and $h(x) = 0$ on $B(A_4, r) \cap \mathcal{X}_3$. Thus h does not satisfy the non-scale-variant Harnack inequality. In other words, (Ha) fails for this strongly local Dirichlet form $(\mathcal{E}, \mathcal{F})$.

Note that the point A_4 is of positive capacity and $(\mathcal{E}, \mathcal{F})$ is irreducible. On the other hand, the part Dirichlet $(\mathcal{E}, \mathcal{F}^{B(A_4,r)})$ on $L^2(B(A_4, r), m|_{B(A_4,r)})$ is not irreducible for any $r \in (0, 1/2]$; the space $B(A_4, r)$ has two disjoint invariant sets $B(A_4, r) \cap \mathcal{X}_2)$ and $B(B_4, r) \cap \mathcal{X}_3$. (This example also shows that a strongly local regular Dirichlet form does not need to be irreducible even though the underlying metric space is connected.)

Let

$$G_k = \{x \in \mathcal{X}_2 : 0 < |x - A_4| < 1/k\} \cup [0,1] \times (-1/k, 0),$$

and set $F_k = \mathcal{X} \setminus G_k$. Then (F_k) is an $\mathcal{E}$-nest consisting of compact sets with $\cup_{k=1}^{\infty} F_k = \mathcal{X}$. The function h given above is continuous on each set G_k, and using known properties of the diffusion X on the spaces $\mathcal{X}_i$ one can verify that (HC) holds.

Example 8.4. To give a concrete example of an irreducible strongly local MMD space that fits the setting of Theorem 1.3 but fails to satisfy the local regularity of [BM1] in the compact setting, consider $\mathcal{X}$ to be the join of Vicsek tree (compact) with the unit interval $[0, 1]$, where the symmetrizing measure m is given by the Hausdorff measure on each of the pieces. The space $\mathcal{X}$ satisfies the relatively ball connected condition. We take $(\mathcal{E}, \mathcal{F})$ to be the strongly local regular Dirichlet form on $L^2(\mathcal{X}; m)$ obtained by combining the Dirichlet form associated with Brownian motion on $(0, 1]$ with the Dirichlet form associated with the diffusion on the Vicsek tree, in a similar fashion to the previous example. The argument in [De2] can be adapted to show that this example satisfies the EHI. This example is essentially due to Delmotte [De2].

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